Head Start to

College Algebra and Trigonometry

A Preparatory Workshop for
MAT 1275CO College Algebra and Trigonometry
with Corequisite

Prepared by

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Head Start to College Algebra and Trigonometry

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In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to College Algebra and Trigonometry before the actual College Algebra and Trigonometry course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The College Algebra and Trigonometry workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Precalculus requirement.

The College Algebra and Trigonometry workshop meets four days, three hours a day, for a total of 15 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Module I

The Graph and the Equation of a Line

Negative numbers appeared in the 7th century in the work of Indian mathematician Brahmagupta (598 - 670) who used the ideas of 'fortunes' and 'debts' for positive and negative. By this time a system based on place-value was established in India, with zero being used in the Indian number system. Brahmagupta used a special sign for negatives and stated the rules for dealing with positive and negative quantities as follows:

A debt minus zero is a debt.
A fortune minus zero is a fortune.
Zero minus zero is a zero.
A debt subtracted from zero is a fortune.
A fortune subtracted from zero is a debt.
The product of zero multiplied by a debt or fortune is zero.
The product of zero multiplied by zero is zero.
The product or quotient of two fortunes is one fortune.
The product or quotient of two debts is one fortune.
The product or quotient of a debt and a fortune is a debt.
The product or quotient of a fortune and a debt is a debt.

https://nrich.maths.org/5961, retrieved September 14, 2021.

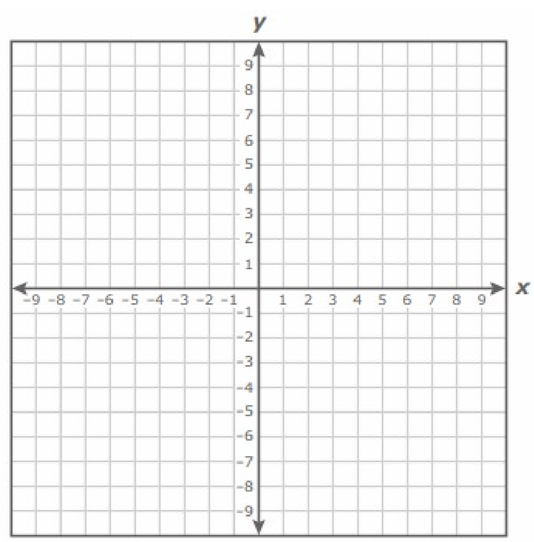
1.1 The Graph and the Equation of a Line

Ordered pairs are represented by the notation (x, y)

Exercise 1: Plot each ordered pair on a rectangular coordinate system and identity the location of each ordered pair.

a) $(6,3)$ b) $(-4,2)$ c) $(-1,-2)$ d) $(5,-3)$

e) $(0,0)$ f) $(0,4)$ g) $(-5,0)$ h) $(0,-1)$



Linear Equation in Two Variables

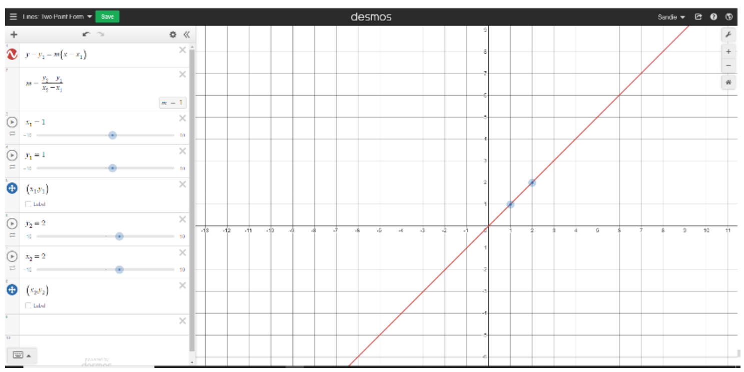
A linear equation in two variables is an equation of the form $Ax + By = C$

The Slope of a Line

The slope, m, of the line between two points (x_1, y_1) and (x_2, y_2) is given by

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Use the Desmos link, https://www.desmos.com/calculator/iutexvvyoc to sketch the graph by entering the values of (x_1, y_1) and (x_2, y_2).



Exercise 2: For each problem below, find the slope of the line through the given pair of points algebraically using the slope formula.

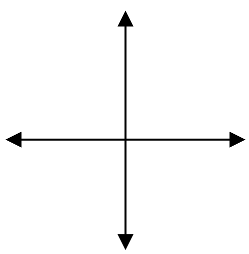
a) Find the slope of the line through the points $(-2,5)$ and $(-8,1)$. Sketch the graph on Desmos.

b) Find the slope of the line through the points $(-4,2)$ and $(-1,1)$. Sketch the graph on Desmos.

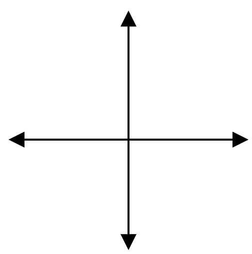
c) Find the slope of the line through the points $(-3,-6)$ and $(2,-6)$. Sketch the graph on Desmos.

d) Find the slope of the line through the points $(-5,5)$ and $(-5,1)$. Sketch the graph on Desmos.

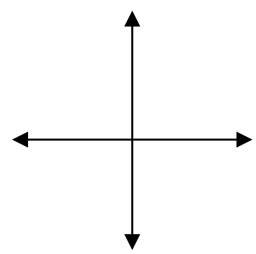
What can you summarize about the slope of a line? Graph a line for each given slope.



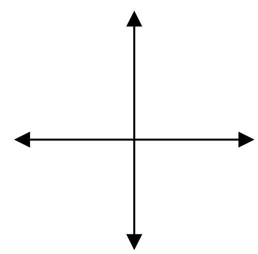
Positive Slope



Negative Slope



Zero Slope



Slope Undefined

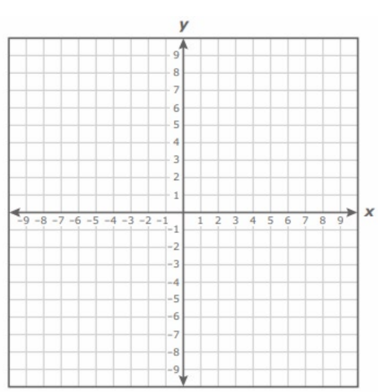
Special Lines: The Vertical and Horizontal Lines

The graph of $x = a$ is a vertical line through $(a, 0)$

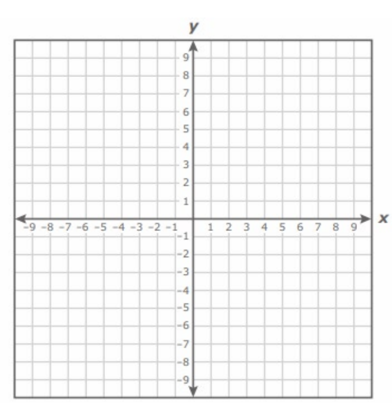
The graph of $y = a$ is a horizontal line through $(0, a)$

Exercise 3: Graph the vertical and the horizontal lines below.

Graph $x = 3$



Graph $y = -1$



The Equation of a Line

Slope-Intercept Form of the Equation of a Line

The equation of any line with slope m and y-intercept b is given by $y = mx + b$

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}} \qquad\qquad b = y - \text{intercept}$$

Point-Slope Form of the Equation of a Line

The equation of the line through (x_1, y_1) with slope m is given by $y - y_1 = m(x - x_1)$

Use the link for Desmos activity: https://www.desmos.com/calculator/2rnqgoa6a4

Use the sliders to change the values of m and b

What happens when you change m? What do you notice about the line when m is positive, when m is negative, when m is zero?

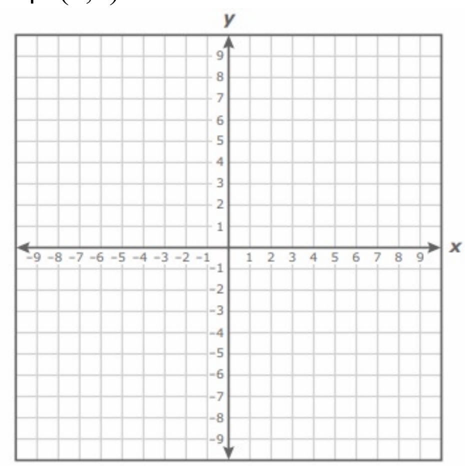
____

____

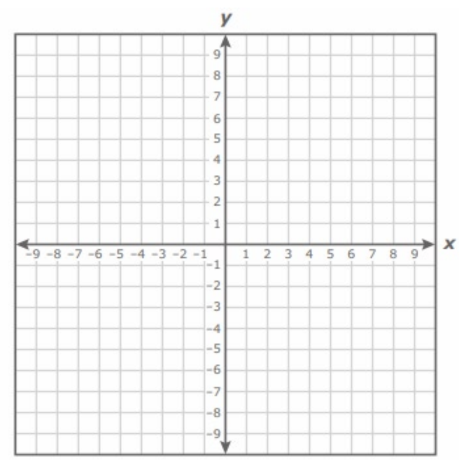
What happens when you change b?

____

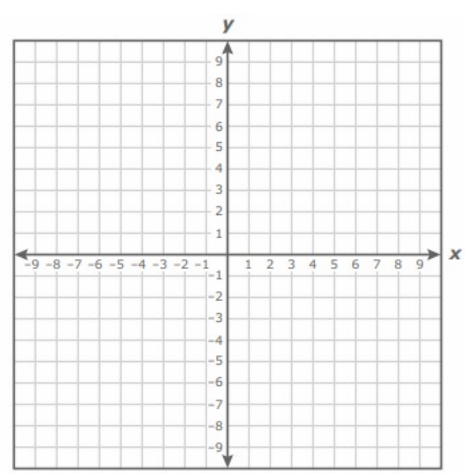
Exercise 4: Find the equation of the line with slope $\frac{2}{3}$ and y-intercept $(0,4)$.
Express the equation in $y = mx + b$ form. Graph.



Exercise 5: Give the slope and y-intercept for the line $y = -2x + 3$. Graph.



Exercise 6: Give the slope and y-intercept for the line $x + 2y = 4$.
Graph. (First express the equation in $y = mx + b$ form.)



There is at least one error in each problem below. Find where the error(s) occurs and provide the correct work.

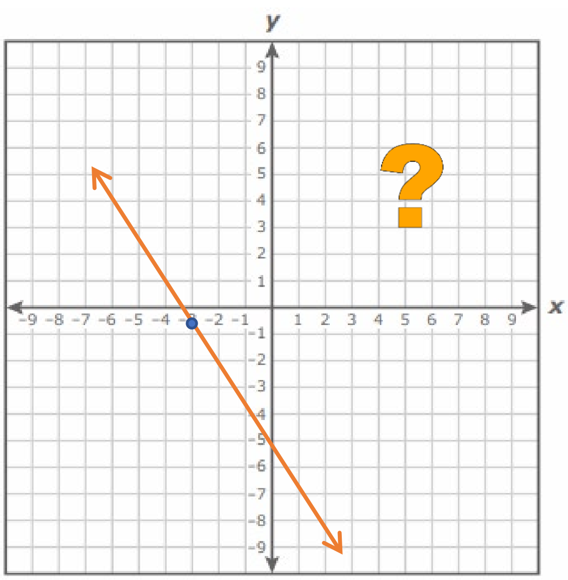
The slope and y-intercept of $3x - 2y = 6$ is $-\frac{3}{2}$ and 6

Find the correct slope and y-intercept of

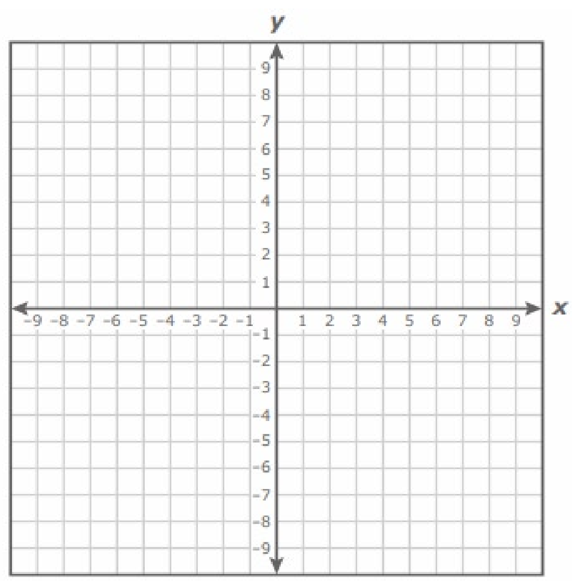
$3x - 2y = 6$

Find the errors #2:

The graph of $3x - 2y = 6$ is



Sketch the correct graph of $3x - 2y = 6$.

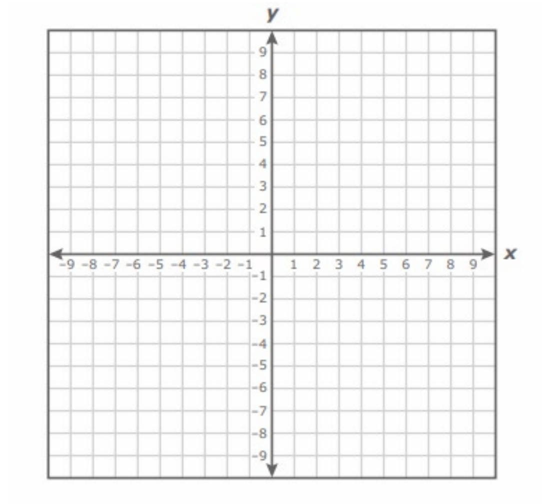


Given slope m and one point (x_1, y_1), find the equation of the line.

Step 1: Use the point-slope form $y - y_1 = m(x - x_1)$ to set up the equation of the line

Step 2: Simplify Step 1 and put it in the slope-intercept form $y = mx + b$.

Exercise 7: Find the equation of the line with slope $-\frac{1}{4}$ and passes through the point $(-8, 5)$. Express the equation in the slope-intercept $y = mx + b$ form. Graph.



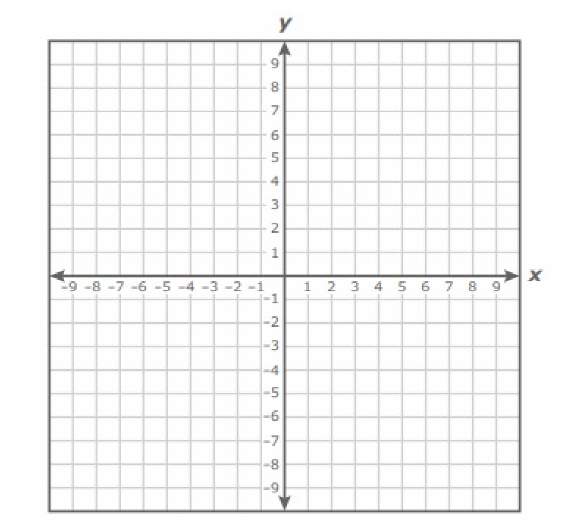
Given two points (x_1, y_1) and (x_2, y_2), find the equation of the line through the two points.

Step 1: Use the slope formula m = $\frac{y_2 - y_1}{x_2 - x_1}$ to find the slope

Step 2: Use the point-slope form $y - y_1 = m(x - x_1)$ to set up the equation of the line

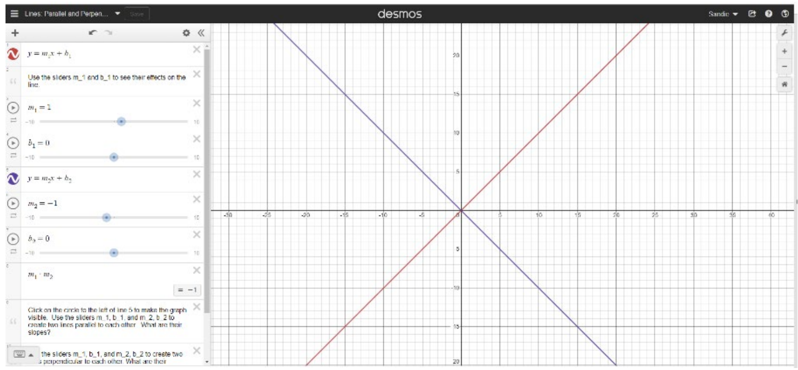
Step 3: Simplify Step 2 and put it in the slope-intercept form $y = mx + b$.

Exercise 8: Find the equation of the line that passes through the points $(-1, -6)$ and $(1,2)$. Express the equation in the slope-intercept $y = mx + b$ form. Graph.



Parallel and Perpendicular Lines

Use the link for the Desmos activity: https://www.desmos.com/calculator/vlpe8pffe8



Use the sliders to create two lines parallel to each other.
Have you observed any relation between the slopes of two parallel lines?

____

____

Use the sliders to create two lines perpendicular to each other.
Have you observed any relation between the slopes of two perpendicular lines?

____

____

If $y = m_1x + b_1$ and $y = m_2x + b_2$ are two non-vertical lines, they are parallel if they have the same slope, or $m_1 = m_2$. They are perpendicular if their slopes are negative reciprocal of each other, or $m_1 \cdot m_2 = -1$

Given the equation of a line $y = -2x - 5$, write an equation of a line that is

parallel to $y = -2x - 5$: ____

perpendicular to $y = -2x - 5$: ____

Desmos: Graph all three lines in Desmos.

Exercise 9: Show whether the lines are parallel, perpendicular, or neither.

a) $y = -2x - 5$
$y = -2x + 7$

b) $5x + 3y = 8$
$15x - 9y = 5$

c) l_1 passes the points $(-4, -3)$ and $(2,5)$
l_2 passes the points $(3,1)$ and $(-5,2)$

SECTION 1.1 SUPPLEMENTARY EXERCISES

1. Find the slope and the equation of the line through the given points

   a) $(8,-4)$ and $(-1,-6)$

   b) $(2,-4)$ and $(-5,3)$

   c) $(2,5)$ and $(2,-5)$

   d) $(3,6)$ and $(-1,6)$

   e) $(3,1)$ and $(5,6)$

2. a) If the slope passing through the points $(-4,-1)$ and $(x,2)$ is $-\frac{3}{5}$, find x.

   b) If the slope passing through the points $(-2,-3)$ and $(x,4)$ is $\frac{1}{3}$, find x.

   c) If the slope passing through the points $(-1,5)$ and $(0,y)$ is -7, find y.

   d) If the slope passing through the points $(-8,-4)$ and $(-4,y)$ is $-\frac{5}{4}$, find y.

3. Use the point-slope form to find the equation of a line:

   a) slope $=-\frac{2}{3}$ through $(-1,3)$

   b) slope $=4$ through $(3,-2)$

   c) slope $=-3$ through $(0,5)$

   d) through $(1,-3)$ and $(5,5)$

   e) through $(-3,2)$ and $(-5,-2)$

   f) through $(-1,0)$ and $(0,4)$

   g) through $(-2,4)$ and $(3,-6)$

4. Write in slope-intercept form equation of the following lines and sketch both lines in one graph.
   Can you point out the relationship of these two lines?

   a) The line pass the point $(6,-1)$ and has a slope of $-\frac{3}{2}$.

b) The line pass the point $(3,-2)$ and has a slope of $\frac{2}{3}$.

5. Determine whether the lines are parallel, perpendicular, or neither.

a) $\begin{array}{l} y-3=4x \\ y=-\frac{1}{4}x+2 \end{array}$

b) $\begin{array}{l} 2x+5y=6 \\ y-\frac{1}{4}x=2 \end{array}$

c) $\begin{array}{l} 6x+2y=7 \\ 3x=6-y \end{array}$

d) $\begin{array}{l} x+3y=-9 \\ y=3x+4 \end{array}$

e) $\begin{array}{l} 14x+2y=1 \\ -\frac{21}{2}x-\frac{3}{2}y=4 \end{array}$

f) $\begin{array}{l} 5x-6=y \\ y=-\frac{2}{3}x+3 \end{array}$

6. Write an equation of a line passing through

a) the point $(4,5)$ and parallel to $2x-y=4$

b) the point $(\frac{1}{2},2)$ and perpendicular to $y+5=2x$

c) the point $(3,-6)$ and parallel to $8x-y=4$

d) the point $(-1,-1)$ and perpendicular to $x+3y=9$

e) the point $(3,5)$ and parallel to $y=7$

f) the point $(3,5)$ and perpendicular to $y=7$

g) the point $(4,-5)$ and parallel to $x=5$

h) the point $(4,-5)$ and perpendicular to $x=5$

i) the point $(-2,4)$ and parallel to $7x+y=9$

j) the point $(-1,\frac{5}{2})$ and perpendicular to $y=3x+8$

1.2 Solving Systems of Linear Equations

Systems of Two Linear Equations

A pair of equations $\begin{cases} 3x+2y=4 \\ x-y=3 \end{cases}$ is called **a system of linear equations**.

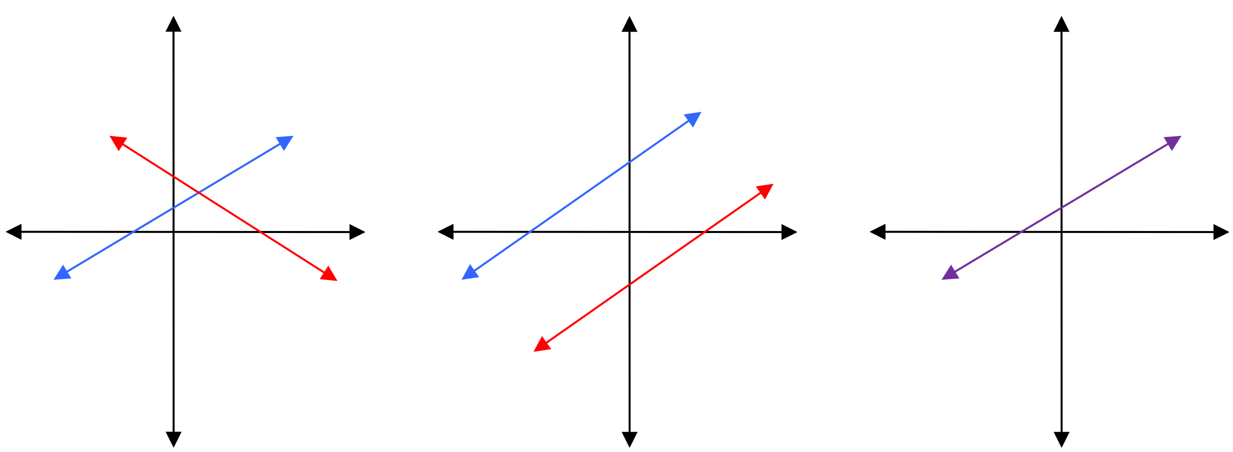
Determining Whether an Ordered Pair is a Solution

A solution of a system of two equations in two variables is an ordered pair (x, y) that makes both equations true.

Exercise 1: Determine whether the ordered pair $(2,-1)$ is a solution of the system $\begin{cases} 3x+2y=4 \\ x-y=3 \end{cases}$

Enter both equations in Desmos. Find and label the intersection.

Possible Solutions to Systems of Two Linear Equations



How many solutions are there in each system above?

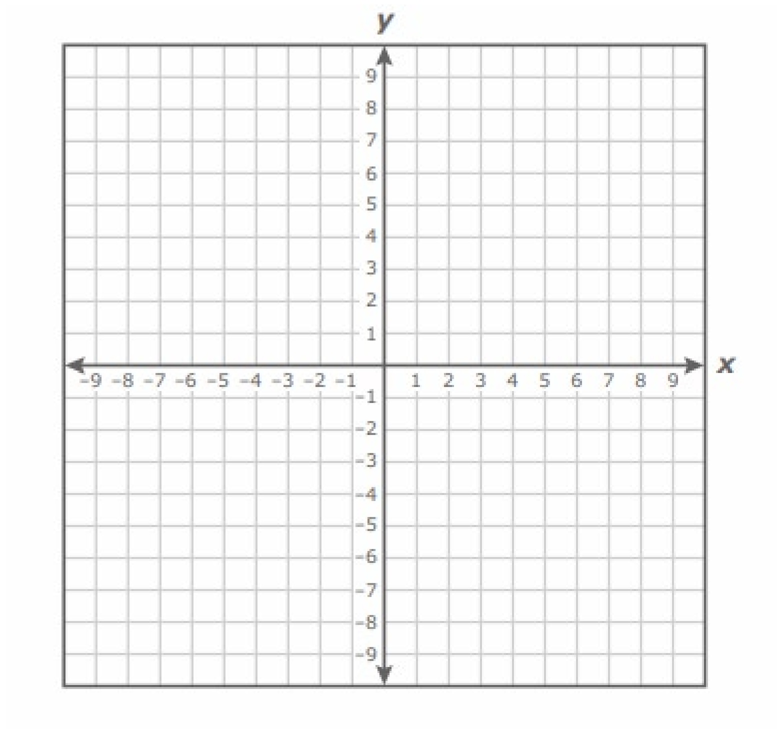
____________________________   ____________________________   ____________________________

Methods of Solving a System of Linear Equations

I. The Graphing Method

Graph each line on the same set of axes. Find the intersection of the two lines.

Exercise 2: Solve the system of linear equations by graphing $\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$



II. The Substitution Method

Solve for either the x or the y in either equation. Substitute it into the other equation. Solve.

Exercise 3: Solve the system of linear equations by the substitution method $\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$

III. The Addition-Elimination Method

Add two equations to eliminate either x or y from the equation. Solve the subsequent equation.

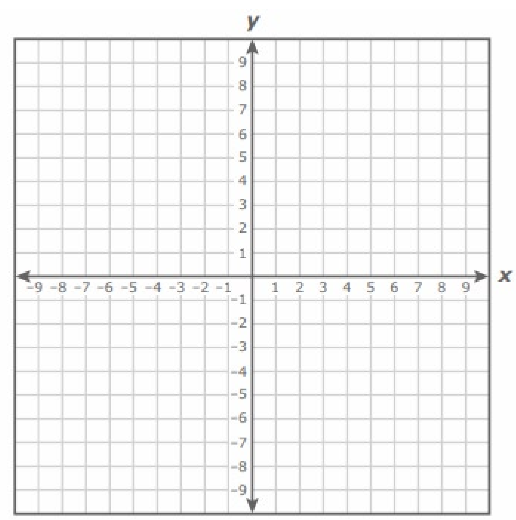
Exercise 4: Solve the system of linear equations by the addition-elimination method $\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$

Exercise 5: Solve the system of linear equations using the Addition-Elimination method $\begin{cases} 2x - 3y = 8 \\ -3x + y = -5 \end{cases}$

Exercise 6: Solve the system of linear equations using the Addition-Elimination method $\begin{cases} 4x - 3y = 9 \\ 3x + 2y = 11 \end{cases}$

Exercise 7: Solve the system of linear equations **graphically** and **algebraically**. What do you notice about the solution set? $\begin{cases} x + 2y = 4 \\ 2x + 4y = 6 \end{cases}$

a. Graphically

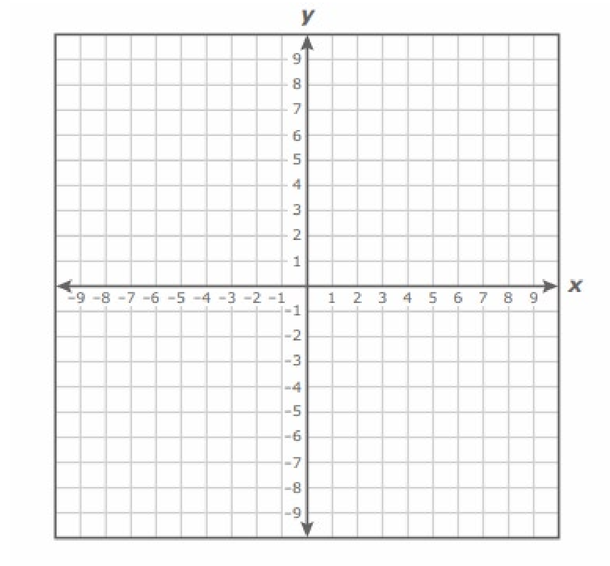


b. Algebraically (Either the Substitution or the Addition-Elimination Method)

$$\begin{cases} x + 2y = 4 \\ 2x + 4y = 6 \end{cases}$$

Exercise 8: Solve the system of linear equations **graphically** and **algebraically**. What do you notice about the solution set? $\begin{cases} 3x + y = 2 \\ 6x + 2y = 4 \end{cases}$

a. Graphically



b. Algebraically (Either the Substitution or the Addition-Elimination Method)

$$\begin{cases} 3x + y = 2 \\ 6x + 2y = 4 \end{cases}$$

SECTION 1.2 SUPPLEMENTARY EXERCISES

1. a) Determine whether the ordered pair $(-5,1)$ is a solution of the system $\begin{cases} x+2y=-3 \\ x-y=-4 \end{cases}$

   b) Determine whether the ordered pair $(3,-1)$ is a solution of the system $\begin{cases} \frac{3}{2}x+\frac{1}{2}y=4 \\ \frac{2}{3}x-y=-8 \end{cases}$

2. Solve the system of linear equations by the graphing, substitution and elimination methods.

a) $\begin{cases} x+y=3 \\ 2x-2y=10 \end{cases}$ b) $\begin{cases} x+y=5 \\ 3x-y=3 \end{cases}$ c) $\begin{cases} 3x-2y=8 \\ x+4y=12 \end{cases}$

3. Solver the system of linear equations by any method.

a) $\begin{cases} 2x+y=5 \\ 3x-y=5 \end{cases}$ b) $\begin{cases} 4x+3y=-7 \\ 2x-y=-1 \end{cases}$ c) $\begin{cases} 2x+2y=-2 \\ 2x-y=4 \end{cases}$

d) $\begin{cases} 3x-4y=5 \\ 6x-8y=8 \end{cases}$ e) $\begin{cases} x+y=4 \\ 3x+3y=12 \end{cases}$ f) $\begin{cases} y=-\frac{1}{3}x+8 \\ x=3y \end{cases}$

g) $\begin{cases} 2x+3y=23 \\ y=x+1 \end{cases}$ h) $\begin{cases} x=y+5 \\ 3x+7y=115 \end{cases}$ i) $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=6 \\ \frac{1}{5}x+\frac{1}{8}y=1 \end{cases}$

j) $\begin{cases} 5x-6y=18 \\ 10x+12y=-6 \end{cases}$ k) $\begin{cases} \frac{1}{2}x-\frac{1}{3}y=1 \\ \frac{1}{3}x-\frac{1}{4}y=1 \end{cases}$

Module II - Factoring

*Schwartzman (1994) defines **Power**: "From old French word **poier**, from the Vulgar Latin **potere** a variant of classic Latin **posse** meaning 'to be able.' The Indo-European root is **poti**, meaning 'powerful [as in]: Lord.' If one is able to do many things one is considered powerful. A powerful person typically has a large number of possessions (a word derived from 'posse') and a large amount of money. In algebra when even a small number like 2 (two) is multiplied by itself a number of times, the result becomes very large quickly; metaphorically speaking, the result is powerful. If the term 'power' is used precisely, it refers to the result of multiplying a number by itself a certain number of times."*

Swartzman, S. (1994).The words of mathematics: An etymological dictionary of mathematical terms used in English. USA: The Mathematical Association of America.

Section 2.1 Factoring Review

Multiplication of Polynomials

Multiplying two Monomials

1. **Multiple the coefficients.**
2. **Add the exponents**

Multiplying a Monomial by a Polynomial

1. **Use the Distributive Property:** $a(b+c) = ab + ac$
2. **Multiply each monomial term**

Exercise 1: Multiply the monomials $(5x^3y^6)(-3xy^4)$

Exercise 2: Multiply using the Distributive Property

a) $2p^3(4p^7-5)$

b) $9m^3n^6(-6m^2n^2+8m^2n+1)$

The Greatest Common Factor

The <u>greatest common factor (GCF)</u> for a polynomial is the largest monomial that divides each term of the polynomial.

Factoring the greatest common factor of a polynomial:

1. **Determine the greatest common factor**
2. **Write the answer in factored form.**

The GCF factoring process is the reverse of the Distributive Property

Multiply

The Distributive Property: $\boldsymbol{a(b+c)=ab+ac}$

Factor/Divide

The GCF factoring: $\boldsymbol{ab+ac=a(b+c)}$

Exercise 3: Factor the greatest common factor and express the answer in factored form. Check by multiplying using the Distributive Law.

a) Factor $5p^3 + 15p^2 - 30p$

GCF: ______________________________

Answer in factored form: ____

Check: ____

b) Factor $9a^3b^4 - 6a^2b^3 + 3ab^2$

GCF: ______________________________

Answer in factored form: ____

Check: ____

c) Factor $5(x + y) - 6x(x + y)$

GCF: ______________________________

Answer in factored form: ____

Check: ____

Factoring by Grouping

Before factoring by grouping, first we factor out GCF from all four terms.

Steps in factoring by grouping (Assume there is no GCF)

1. Group pairs of terms and factor each pair.
2. If there is a common binomial factor, then factor it out.
3. If there is no common binomial factor, then interchange the middle two terms and repeat the process over. If there is still no common binomial, then the polynomial cannot be factored.

Factor $4x + 6y + 2xy + 3y^2$ by grouping

Step 1: Group the pairs of terms	$(\mathbf{4x + 6y}) + (\mathbf{2xy + 3y^2})$
Step 2: Factor the GCF from each pair	$\mathbf{2}(\mathbf{2x + 3y}) + \mathbf{y}(\mathbf{2x + 3y})$
Step 3: Factor the common binomial factor	$(\mathbf{2x + 3y})(\mathbf{2 + y})$

Exercise 4: Factor by grouping. Follow the steps in the table

a) Factor by grouping: $56 + 21k + 8h + 3hk$

	Show work here
Step 1: Group the pairs of terms	
Step 2: Factor the GCF from each pair	
Step 3: Factor the common binomial factor	

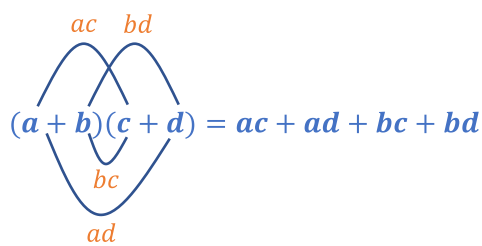
b) Factor by grouping: $5x^2 + 40x - xy - 8y$

	Show work here
Step 1: Group the pairs of terms. Be careful with the signs.	
Step 2: Factor the GCF from each pair. Be careful with the signs.	
Step 3: Factor the common binomial factor	

Factoring Trinomials

Multiplying Two Binomials

The FOIL (Front-Outer-Inner-Last) Method: $(a+b)(c+d) = ac + ad + bc + bd$



Exercise 5: Multiply the binomials and combine like terms

a) $(x+2)(x+3)$ b) $(x+2)(x-3)$

Factoring Trinomials with Lead Coefficients of 1

Since the product of two binomials is often a trinomial, it is expected that many trinomials will factor as two binomials. For example, to factor x^2+5x+6, we must find two binomials $x+a$ and $x+b$ such that

$$\begin{aligned} x^2+5x+6 &= (x+a)(x+b) \\ &= x^2+bx+ax+ab \\ &= x^2+ax+bx+ab \\ &= x^2+(a+b)x+ab \end{aligned}$$

Matching the coefficient of each term, one can see the binomials must satisfy product $ab=6$ and the sum $a+b=5$. Since $2\cdot 3=6$ and $2+3=5$, we find our binomials $x^2+5x+6=(x+2)(x+3)$.

Similarly, if we want to factor x^2-x-6, we will consider $x^2-x-6=(x+a)(x+b)$ where the product $ab=-6$ and the sum $a+b=-1$. Considering all possible factorizations of -6, we find

$$(2)(-3) = -6 \text{ and } 2+(-3) = -1, \text{ thus,}$$

$$x^2-x-6=(x+2)(x-3).$$

Steps to factoring trinomials with lead coefficient of 1

1. Write the trinomial in descending powers.
2. List the factorizations of the constant (third) term of the trinomial.
3. Pick the factorization where the sum of the factors is the coefficient of the middle term.
4. Check by multiplying the binomials.

Exercise 6: Factor the trinomial with lead coefficient 1 by checking for the correct pair of product and sum

a) Factor $x^2 - 8x + 12$

Answer in factored form: ____

Check: ____

b) Factor $a^2 + 5a + 6$

Answer in factored form: ____

Check: ____

c) Factor $y^2 - 9y - 36$

Answer in factored form: ____

Check: ____

d) Factor $x^2 + 14xy + 45y^2$

Answer in factored form: ____

Check: ____

e) Factor $x^2 - 9xy + 18y^2$

Answer in factored form: ____

Check: ____

f) Factor $a^2 + 7ab - 8b^2$

Answer in factored form: ____

Check: ____

Factoring Trinomials with Lead Coefficients other than 1

Method 1: Trial and error

1. List the factorizations of the third term of the trinomial.
2. Write them as two binomials and determine the correct combination where the sum of the outer product, ad, and the inner product, bc, is equal to the middle term of the trinomial.

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d}) = \boldsymbol{ac}+\boldsymbol{ad}+\boldsymbol{bc}+\boldsymbol{bd}$$

bc

ad

Method 2: Factoring by grouping

1. Form the product ac.
2. Find a pair of numbers whose product is ac and whose sum is b.
3. Rewrite the polynomial to be factored so that the middle term bx is written as the sum of the two terms whose coefficients are the two numbers found in step
4. Factor by grouping.

Exercise 7: Factor each trinomial below

a) $2h^2 - 5h - 3$

b) $2h^2 - h - 3$

c) $2h^2 - 5h - 12$

d) $3k^2 - 14k - 5$

Factoring Special Products

Exercise 8: Multiply the binomials and combine like terms.
Can you come up with a formula for this type of multiplication?

____

a) $(x+3)(x-3)$

b) $(5x+8)(5x-8)$

Exercise 9: Multiply the binomials and combine like terms.
Can you come up with a formula for this type of multiplication?

____

a) $(x-3)^2$

b) $(2x+5)^2$

Many trinomials can be factored by using special product formulas.

Difference of Two Squares (DOTS): $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

Exercise 10: Factor each binomial below. Check if it is a difference of two squares.

a) $x^2 - 49$

b) $25 - b^2$

c) $36x^2 - 16y^2$

d) $100t^2 - 49r^2$

e) Factor completely $16x^4 - 81$

f) $x^2 + 36$

What have you noticed about the sum of two squares? ________________________________

Perfect Square Trinomial $\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$

$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$

Exercise 11: Factor each trinomial below. Check that the first and the third terms are perfect square, and the middle term is $2ab$, then apply the perfect square formula

a) $t^2 - 8t + 16$

b) $p^2 + 10t + 25$

c) $16a^2 - 40a + 25$

d) $9b^2 + 42b + 49$

e) $16y^2 - 72y + 81$

Mixed Factoring/Factoring Completely

To factor a polynomial, **first factor the greatest common factor**, then consider the number of terms in the polynomial.

I. **Two terms**: Determine if the binomial is a difference of two squares.
   If it is a difference of two squares, then $a^2 - b^2 = (a+b)(a-b)$

II. **Three Terms**: Determine if the trinomial is a perfect square trinomial.
   a) If the trinomial is a perfect square, then
   $$a^2 + 2ab + b^2 = (a+b)^2$$
   $$a^2 - 2ab + b^2 = (a-b)^2$$
   b) If the trinomial $ax^2 + bx + c$ is not a perfect square, then
      i) if the leading coefficient $a = 1$, then check the product = c and sum = b to determine the correct combination.
      ii) if the leading coefficient $a > 1$, then use trial and error or the ac-method.

III. **Four terms**: Try to factor by grouping.

Exercise 12: Factor each polynomial **completely**. This may mean factoring GCF and/or factoring in two or more steps.

a) Factor completely $3x^4 - 3x^3 - 36x^2$

b) Factor completely $20a - 5a^3$

c) Factor completely $16a^5b - ab$

d) Factor completely $8x^2 - 24x + 18$

e) Factor completely $16x^3y - 40x^2y^2 + 25xy^3$

f) Factor completely $12x^3 + 11x^2 + 2x$

g) Factor completely $7z^2w^2 - 10zw^2 - 8w^2$

SECTION 2.1 SUPPLEMENTARY EXERCISES

1. a^2-5a

2. $25y^3-15y^2$

3. $5ab^2-15a^3b$

4. $3p^6q^2-24pq^3$

5. $2b^4c^5-14b^3c^3$

6. $16x-4x^3$

7. y^3-81y

8. $49-9t^2$

9. $64w^2-81$

10. $36x^2y-9y$

11. $4a^2-121$

12. $36x^2-25y^2$

13. $a^2-4a-12$

14. b^2-6b+5

15. $x^2+7x+10$

16. $m^2+9mn+18n^2$

17. $x^2+9xy-36y^2$

18. $-6x^2-9x+15$

19. m^3-4m^2-21m

20. $24+5n-n^2$

21. $2x^3-2x^2y-12xy^2$

22. $2x^3y+x^2y^2-6xy^3$

23. $am-5a+2bm-10b$

24. $15x-12ax+10y-8ay$

25. $7b-2bd+21c-6cd$

26. $15ax+4by+10ay+6by$

Section 2.2 Solving Quadratic Equations by Factoring

Solving a Quadratic Equation by Factoring

Zero Factor Theorem: If $ab = 0$ then $a = 0$ or $b = 0$ or both.

Solving Quadratic Equations by Factoring

1. **Put the equation in standard form: $ax^2 + bx + c = 0$.**
2. **Factor completely.**
3. **Use the zero-product rule, set each factor containing the variable equal to zero and solve for x. Note: Do not solve for the constant factor.**

Exercise 1: Solve the equations by factoring

a) Solve: $x^2 - 4x - 32 = 0$

b) Solve: $36 - 49y^2 = 0$

c) Solve: $a^3 - 6a^2 + 5a = 0$

d) Solve: $2x^3 + 9x^2 = 5x$

SECTION 2.2 SUPPLEMENTARY EXERCISES

1. Solve the following equations by factoring:

a. $x^2 + 6x + 8 = 0$

b. $2t^2 + 3t - 14 = 0$

c. $x^2 - 10x + 25 = 0$

d. $w^2 + 14w + 55 = 6$

e. $t^2 - 5t = 24$

f. $2y^3 + 12y^2 - 32y = 0$

g. $k^2 - 25 = 0$

h. $16x^2 = 49$

i. $z^2 - 9z = 0$

j. $-6x^2 - x + 12 = 0$

k. $5m^2 + 20m = 6 - 9m$

l. $(y+5)^2 - 4 = 0$

m. $(n-3)(3n-2) - 8n = 0$

n. $10x^2 = 27x - 18$

Module III

Radicals and the Square Root Property

Radicals, specifically square roots, date back as far as c.1650 B.C., from the time of Egypt's Middle Kingdom. The Rhind Papyrus makes references to square roots since they are tied to the diagonals of squares and rectangles; often applicable in the construction of a temple. The "Rhind Papyrus" is named after Henry Rhind, a Scottish lawyer, who purchased it in Egypt in 1858. It was placed in the British Museum in London, England, in 1864 and is still there today; a fragment is also in the collection of the Brooklyn Museum on Eastern Parkway. The Rhind Papyrus, according to the British Museum website, is a "list of practical problems encountered in administrative and building works. The text contains 84 problems concerned with numerical operations, practical problem-solving, and geometrical shapes."

Robins, G. & Shute, C. (1990). The Rhind Mathematical Papyrus: An ancient Egyptian text. New York, NY: Dover. The British Museum; downloaded on 8/1/ 2011, from http://www.britishmuseum.org/explore/highlights/highlight_objects/aes/r/rhind_mathematical_papyrus.aspx

Section 3.1: Simplifying Radicals

Simplifying Radicals

Exercise 1: Simplify each radical below

a. $\sqrt{9}$
b. $\sqrt{18}$
c. $\sqrt{25}$
d. $\sqrt{75}$
e. $\sqrt{49}$
f. $\sqrt{98}$
g. $\sqrt{100}$
h. $\sqrt{500}$

List at least 12 numbers that are perfect square:

____

Exercise 2: Simplify each radical below

a. $\sqrt{x^2}$
b. $\sqrt{x^3}$
c. $\sqrt{x^4}$
d. $\sqrt{x^5}$
e. $\sqrt{x^6}$
f. $\sqrt{x^7}$
g. $\sqrt{x^{98}}$
h. $\sqrt{x^{99}}$

What pattern have you noticed? Write a rule for simplifying a radical of the form $\sqrt{x^n}$

When n is even, $\sqrt{x^n} =$

When n is odd, $\sqrt{x^n} =$

Exercise 3: Simplify each radical

a) $\sqrt{12x^6y^3}$

b) $3x\sqrt{20x^3y^6z^9}$

Find the error in each problem. Justify your answers. Provide the correct answers.

Find the errors	What are the errors?	Corrections
$\sqrt{x^9} = x^3$		
$\sqrt{48} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12}$		
$\sqrt{16} = \sqrt{4} = 2$		
$5 + 2\sqrt{x} = 7\sqrt{x}$		
$\frac{2+\sqrt{6}}{2} = 1 + \sqrt{3}$		

Section 3.2 Solving Quadratic Equations by the Square Root Property

The Square Root Property

If $x^2 = a$ and $a \geq 0$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Sometimes it is indicated as $x = \pm\sqrt{a}$

Solving a Quadratic Equation by the Square Root Property

Isolate the square and then apply the Square Root Property

Exercise 1: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $x^2 = 9$

b) $x^2 - 54 = 0$

Exercise 2: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $(a - 5)^2 = 16$

b) $(4t - 3)^2 - 18 = 0$

Section 3.3 Solving Quadratic Equations by Completing the Square

To solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square, we first divide the equation by the leading coefficient a. The procedure below will assume $a = 1$ and the quadratic equation is of the form $x^2 + bx + c = 0$

Solving a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square:

Steps
1. Write the left side in the form $x^2 + bx$ and move the constant term to the right side.
2. Add a constant, $\left(\frac{b}{2}\right)^2$, that will complete the square. The constant is added to both sides of the equation.
3. Write the left side as a binomial squared.
4. Apply the Square Root Property.
5. Solve for x and simplify.

Example: Solve by completing the square: $x^2 + 8x + 4 = 0$

Step 1: Write the left side in the form $x^2 + bx$ and move the constant term to the right side.	$x^2 + 8x = -4$
Step 2: Add a constant $\left(\frac{8}{2}\right)^2 = 16$ to both sides of the equation.	$x^2 + 8x + 16 = -4 + 16$
Step 3: Write the left side as a binomial squared.	$(x + 4)^2 = 12$
Step 4: Apply the Square Root Property.	$\sqrt{(x+4)^2} = \pm\sqrt{12}$ $x + 4 = \pm\sqrt{12}$
Step 5: Solve for x and simplify.	$x = -4 \pm \sqrt{12}$ $x = -4 \pm 2\sqrt{3}$

Exercise 1: Solve by completing the square. $y^2 - 6y - 8 = 0$

Exercise 2: Solve by completing the square. $2w^2 + 12w - 14 = 0$

SECTION 3.2 SUPPLEMENTARY EXERCISES

1. Solve the following equations by using the square root Property:

a. $x^2 = 100$

b. $s^2 - 12 = 0$

c. $y^2 + 7 = 88$

d. $(y-3)^2 - 16 = 0$

e. $(y+7)^2 - 25 = 0$

f. $(y-1)^2 - 12 = 0$

g. $(2k-1)^2 - 9 = 0$

f. $9(2m-3)^2 + 8 = 449$

SECTION 3.3 SUPPLEMENTARY EXERCISES

1. Solve the following equations by completing the square:

a. $p^2 - p = 6$

b. $x^2 + 10x - 7 = 0$

c. $y^2 - 3y - 6 = 0$

d. $6k^2 + 17k + 5 = 0$

e. $5m^2 - 10m + 3 = 0$

f. $3x^2 + 4 = 8x$

g. $6w^2 + 12x = 48$

h. $x^2 - 10x + 26 = 8$

Module IV

Quadratic Equations and Graphs

The ancient Egyptians were interested in the solutions of a quadratic equation in order to determine the dimension of a floor plan for a desired area and perimeter of a given shape. If a rectangular room, for example, they would be interested in the solutions to

$$L \cdot W = A$$
$$2L + 2W = P$$

which we can now solve algebraically with a quadratic equation. Without the sophisticated number system nor the method of quadratic equations, the ancient Egyptians developed a lookup table of standard dimensions and sizes for different shapes. Engineers would find the most fitting design based on the table developed. Unfortunately, there were limitations using the table and errors from incoherent and erroneous reproduction of the tables, which led to flawed engineering.

Hell, D. (2004). History behind Quadratic Equations. Retrieved on June 22,2011.

Section 3.1: Solving Quadratic Equations by the Quadratic Formula

The quadratic $ax^2 + bx + c = 0$ can be solved in many ways:

I. Factor the quadratic and then use the Zero Factor Theorem (Section 2.2)
2. The Square root property (Section 3.2)
3. Completing the square (Section 3.3)
4. The Quadratic Formula (Section 4.1)

The Quadratic Formula

A quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1: Solve using the quadratic formula

a) $x^2 - 5x - 8 = 0$

b) $x^2 + 6x + 2 = 0$

c) $2x^2 - 4x = 3$

<u>How many errors can you find?</u>

Find where the errors occur and provide the correct solution

Solve using the quadratic formula $x^2 - 4x = 16$

Find the errors (at least six)

$a = 1,\ b = -4,\ c = 16$

$$x\ = \frac{-4 \pm \sqrt{4^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-64}}{2}$$

$$= \frac{-4 \pm \sqrt{-48}}{2}$$

$$= -4 \pm \sqrt{-24}$$

$$= -4 \pm 2\sqrt{6}$$

The two solutions are:

$x = -2\sqrt{6}$ or $x = -6\sqrt{6}$

Provide the correct solution here

Exercise 2: Which is the best method for solving $4x^2 - 4x = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

Exercise 3: Which is the best method for solving $x^2 - 63 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

Exercise 4: Which is the best method for solving $x^2 - 6x - 9 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

Exercise 5: Which is the best method for solving $2x^2 - 6x - 9 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

SECTION 4.1 SUPPLEMENTARY EXERCISES:

Solve the following equations by using the quadratic formula:

1. $y^2 + 10y + 9 = 0$

2. $s^2 - s = 30$

3. $m^2 - 81 = 0$

4. $6x^2 + 2x + 3 = 0$

5. $10x^2 - 13x - 3 = 0$

6. $4x^2 + 6x + 1 = 0$

7. $m^2 + 5m = 2$

8. $2k^2 + 9k = -7$

9. $(z-2)(z+4) + 6 = 0$

10. $(3x-7)(x+5) = -31$

11. $11n^2 - 4n(n-2) = 6(n+3)$

12. $x(x-3) = -10x - 7$

Section 4.2 Graph of Quadratic Equations - Parabolas

The graph of a **quadratic equation of** the form $y = ax^2 + bx + c$ where $a \neq 0$ is a parabola (U-shaped). The parabola has either a maximum or a minimum point, called the **vertex**. Often times, it is more convenient to express the quadratic equation in the vertex form.

Method 1: The Vertex Form

The Vertex Form of a Quadratic equation is $\boldsymbol{y = a(x-h)^2 + k}$, where $\boldsymbol{(h, k)}$ is the vertex.

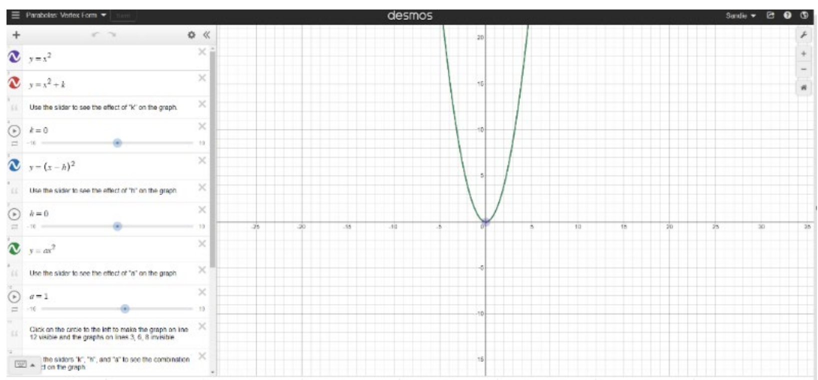
Method 2: The Vertex Formula

The Vertex Formula of a quadratic equation of the form $\boldsymbol{y = ax^2 + bx + c}$:

x-coordinate of the vertex is $\boldsymbol{-\frac{b}{2a}}$

Substitute x-coordinate to find the y-coordinate of the vertex

Use the Desmos link below to see the effect of changing h, k and a on the parabola.
https://www.desmos.com/calculator/0txid19ts5



How does the graph change when you change h? When h is positive? When h is negative?

____

____

How does the graph change when you change k? When k is positive? When k is negative?

____

____

How does the graph change when you change a? When a is positive? When a is negative? When $|a| > 1$? When $|a| < 1$?

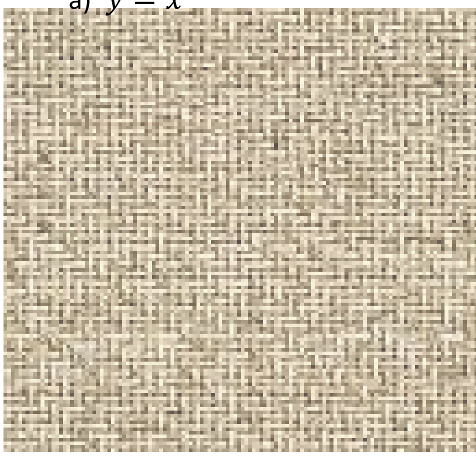
____

____

____

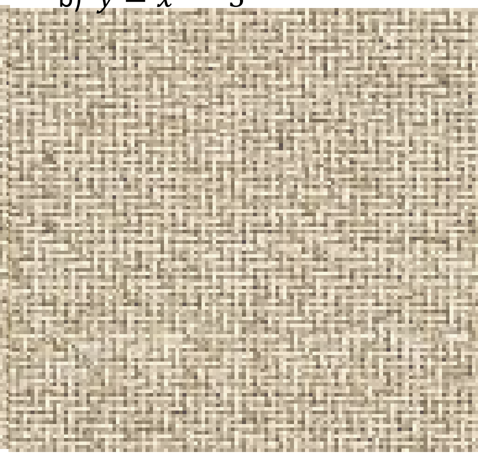
Exercise 1: For each function: (i) Identify the vertex; (ii) sketch the graph.

a) $y = x^2$



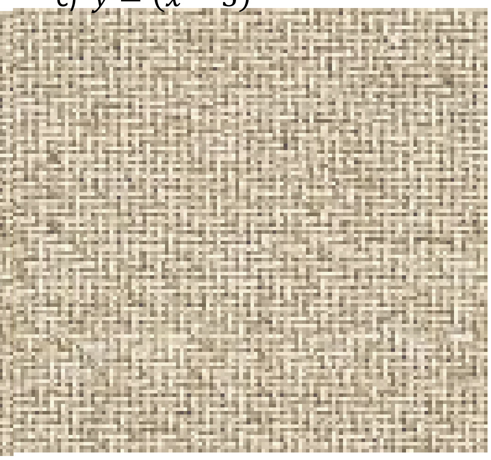
The vertex is _____________

b) $y = x^2 - 3$



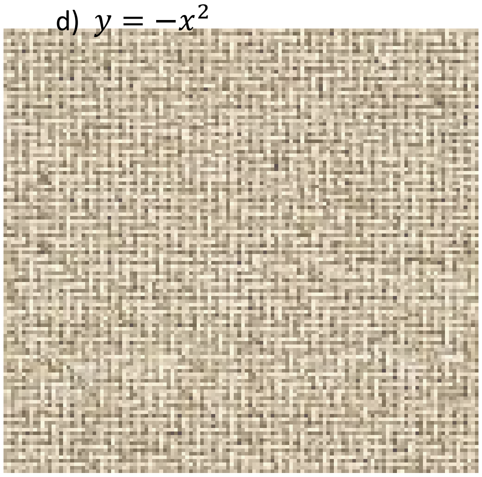
The vertex is _____________

c) $y = (x-3)^2$



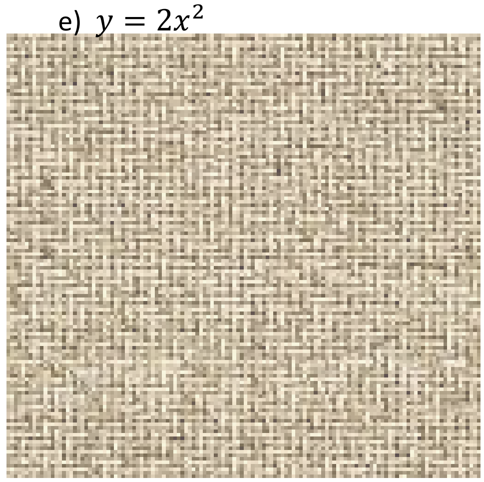
The vertex is _____________

d) $y = -x^2$



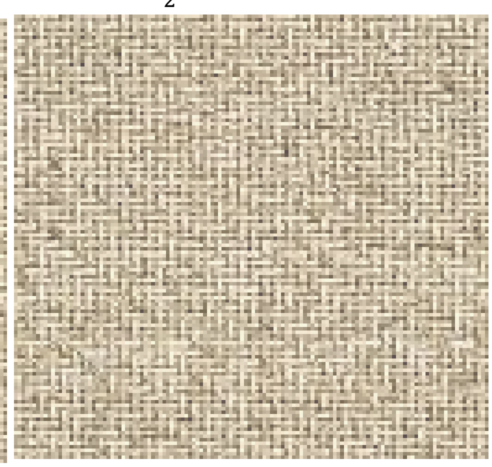
The vertex is _____________

e) $y = 2x^2$



The vertex is _____________

f) $y = \frac{1}{2}x^2$



The vertex is _____________

Exercise 2: Use the **Vertex Form** to identify the vertex. Sketch the graph. Find the roots (solutions) of the equation by setting $y = 0$ and solve.

a) $y = (x+2)^2 - 4$

The vertex is _______________

The roots are ________________________

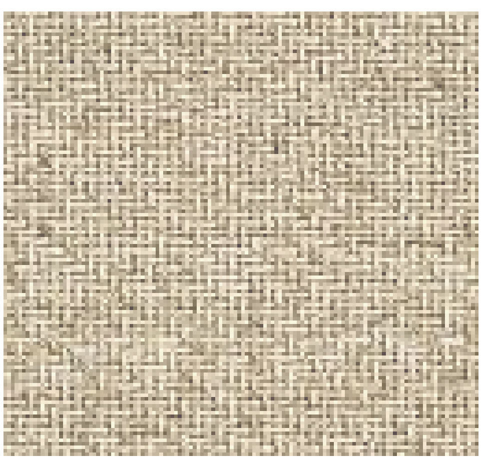
b) $y = -(x+5)^2 + 6$

The vertex is _______________

The roots are ________________________

Exercise 3: Use the **Vertex Formula** to find the vertex. Sketch the graph. Find the roots (solutions) of the equation by setting $y = 0$ and solve.

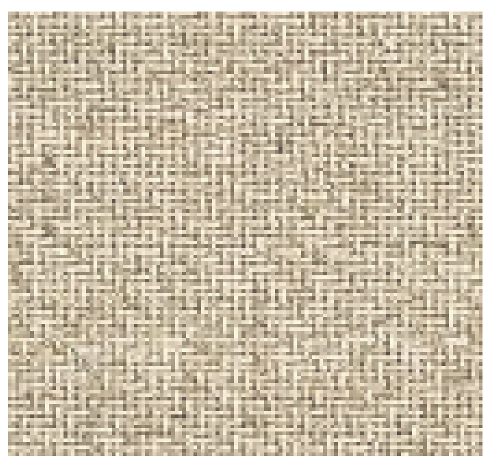
a) $y = x^2 + 4x - 5$



The vertex is _______________

The roots are ________________________

b) $y = -x^2 - 2x - 5$



The vertex is _______________

The roots are ________________________

SECTION 4.2 SUPPLEMENTARY EXERCISES

1. Identify at least three features of the graph of a quadratic function:

____

____

____

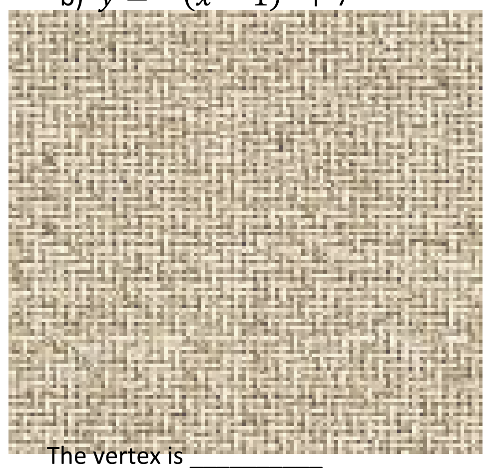
2. For each function, (i) identify the vertex; (ii) sketch the graph; (iii) find the roots (solutions) of the function.

a) $y = (x-3)^2 - 1$

The vertex is __________

The roots are __________________

b) $y = -(x-1)^2 + 7$



The vertex is __________

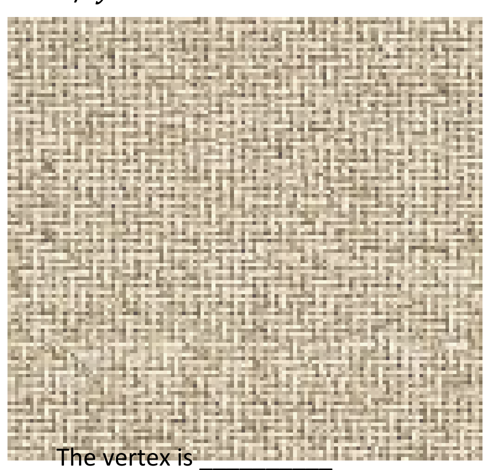
The roots are __________________

c) $y = \frac{1}{4}(x-5)^2 + 4$

The vertex is __________

The roots are __________________

d) $y = 2x^2 + 4x - 5$



The vertex is __________

The roots are __________________

Module V

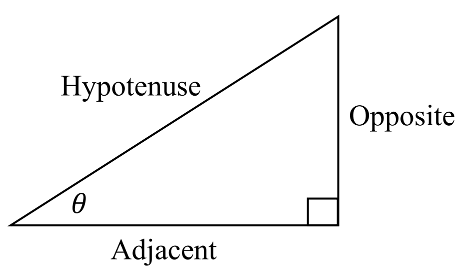
Trigonometry

*The development of Trigonometry spans all cultures. The word is derived from combining two Greek words; **trigonon** which means "triangle" and **metron** "to measure." Around three millennia ago, the Babylonian number system of base sixty (sexagesimal) promoted the idea of 360 degrees in a circle, 60 minutes in a degree, and sixty seconds in a minute. This concept led to having sixty minutes in an hour.*

Willers, M. (2009). Algebra: The x and y of everyday math. New York, NY: Fall River Press.

Section 5.1 Definition of Trigonometry

Defining Trigonometric Functions



$$\sin\theta = \frac{\text{Side Opposite } \theta}{\text{Hypotenuse}} = \frac{\text{O}}{\text{H}}$$ **SOH** (**S**ine theta is **O**pposite over **H**ypotenuse)

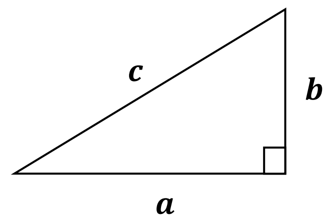
$$\cos\theta = \frac{\text{Side Adjacent to } \theta}{\text{Hypotenuse}} = \frac{\text{A}}{\text{H}}$$ **CAH** (**C**osine theta is **A**djacent over **H**ypotenuse)

$$\tan\theta = \frac{\text{Side Opposite } \theta}{\text{Side Adjacent to } \theta} = \frac{\text{O}}{\text{A}}$$ **TOA** (**T**angent theta is **O**pposite over **A**djacent)

The Pythagorean Theorem

For any right triangle with legs a, b and hypotenuse c, the square of the hypotenuse is equal to the sum of squares of the two legs, or

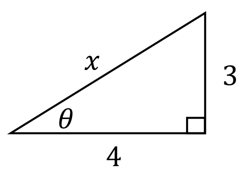
$$a^2 + b^2 = c^2$$



Exercise 1. For each right triangle below,

a) Use the Pythagorean Theorem to determine the value of x.

b) Use the definition of $\sin\theta$, $\cos\theta$, $\tan\theta$ to find their values in simplest fraction form.

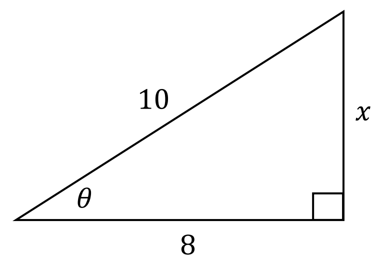


a) Find x

b) $\sin\theta$ =

$\cos\theta$ =

$\tan\theta$ =



a) Find x

b) $\sin\theta$ =

$\cos\theta$ =

$\tan\theta$ =

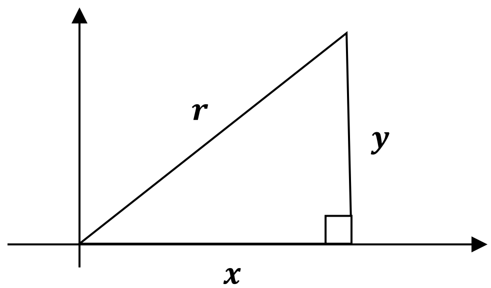
Compare $\sin\theta$, $\cos\theta$, $\tan\theta$ for the two triangles above, what can you say about their values?

____

Observation 1: The two triangles in (i) and (ii) are similar. (Their corresponding sides are proportional by the same ratio.)

Observation 2: Trigonometric ratios for similar right triangles are equal.

Defining the Other Trigonometric Functions



$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y}$

The Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$ $\qquad$ $\csc\theta = \dfrac{1}{\sin\theta}$

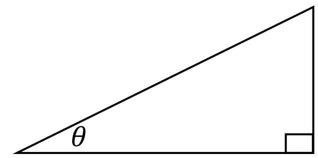
$\cos\theta = \dfrac{1}{\sec\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{1}{\cot\theta}$ $\qquad$ $\cot\theta = \dfrac{1}{\tan\theta}$

The Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Exercise 1: Given $\sin\theta = \frac{5}{13}$ find the other five trigonometric functions.

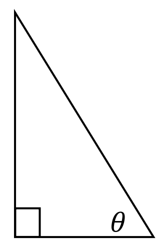


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Exercise 2: Given $\tan\theta = \frac{5}{2}$ find the other five trigonometric functions.

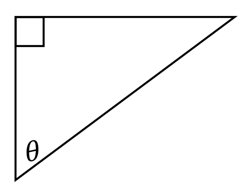


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

Exercise 3: Given $\cos\theta = \frac{\sqrt{7}}{4}$, find the other five trigonometric functions.



$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

SECTION 5.1 SUPPLEMENTARY EXERCISES

1. In a right triangle, if $\cos\theta = \frac{7}{24}$ find the other five trigonometric functions.

2. In a right triangle, if $\sin\theta = \frac{\sqrt{7}}{4}$ find the other five trigonometric functions.

3. In a right triangle, if $\tan\theta = \frac{3}{2}$ find the other five trigonometric functions.

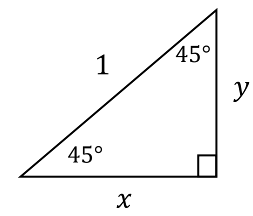
4. In a right triangle, if $\cos\theta = \frac{\sqrt{5}}{5}$ find the other five trigonometric functions.

5. In a right triangle, if $\tan\theta = \frac{\sqrt{3}}{5}$ find the other five trigonometric functions.

Section 5.2 The Special Right Triangles ($45° - 45° - 90°$ and $30° - 60° - 90°$)

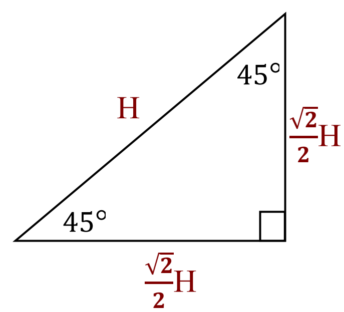
Special Right Triangles: The $45° - 45° - 90°$ triangle

Exercise 1: a) Find x and y. (Hint: This is an isosceles triangle.)
b) Determine $\sin 45°$, $\cos 45°$, and $\tan 45°$.



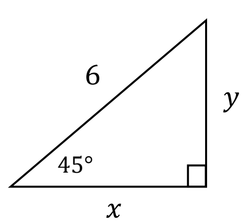
The $45° - 45° - 90°$ triangle has following relationship between the sides:

- **The two legs are equal.**
- **The leg is $\frac{\sqrt{2}}{2}$ times the hypotenuse.**

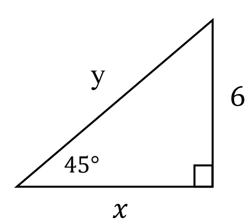


Exercise 2: For each triangle below, find the missing sides x and y.

a)

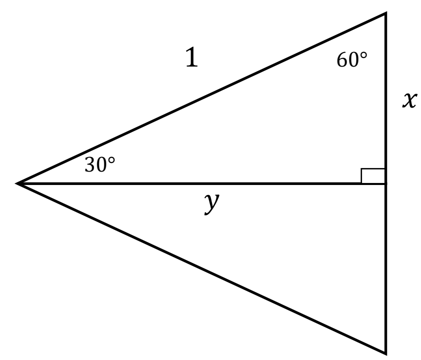


b)



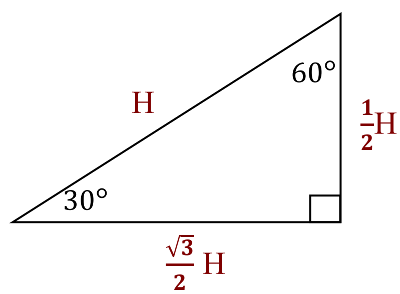
Special Right Triangles: The $30° - 60° - 90°$ triangle

Exercise 3: a) Find x and y. (Hint: Two $30° - 60° - 90°$ triangles form an equilateral triangle.)
b) Determine $\sin 30°$, $\cos 30°$, and $\tan 30°$.
c) Determine $\sin 60°$, $\cos 60°$, and $\tan 60°$.



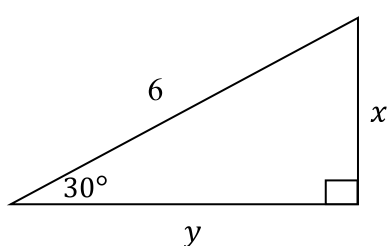
The $30° - 60° - 90°$ triangle has following relationship between the sides:

- **The shorter leg (side opposite $30°$) is $\frac{1}{2}$ of the hypotenuse.**
- **The longer leg (side opposite $60°$) is $\frac{\sqrt{3}}{2}$ times the hypotenuse.**

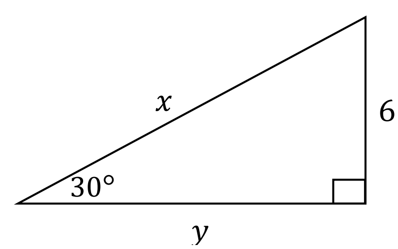


Exercise 4: For each triangle below, find the missing sides x and y.

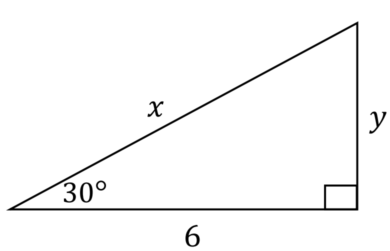
a)



b)



c)



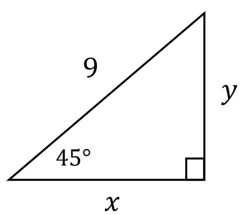
SECTION 5.2 SUPPLEMENTARY EXERCISES

1. Evaluate.

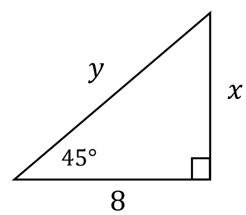
$\sin 30^\circ =$	$\sin 45^\circ =$	$\sin 60^\circ =$
$\cos 30^\circ =$	$\cos 45^\circ =$	$\cos 60^\circ =$
$\tan 30^\circ =$	$\tan 45^\circ =$	$\tan 60^\circ =$
$\csc 30^\circ =$	$\csc 45^\circ =$	$\csc 60^\circ =$
$\sec 30^\circ =$	$\sec 45^\circ =$	$\sec 60^\circ =$
$\cot 30^\circ =$	$\cot 45^\circ =$	$\cot 60^\circ =$

2. Find the missing sides x and y.

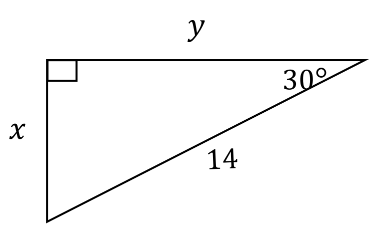
a)



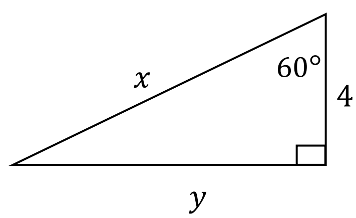
b)



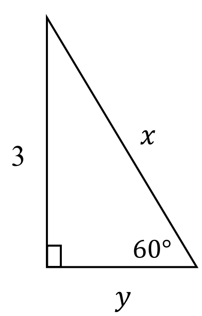
c)



d)



e)



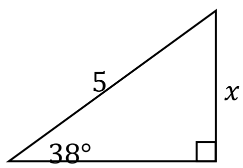
Section 5.3 Solve Using Right Triangle Trigonometry

Choose the appropriate trigonometric equation, substitute the values, use calculator to solve.

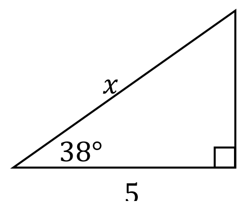
$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$

Exercise 1: Solve for x. Express answer to the nearest tenths.

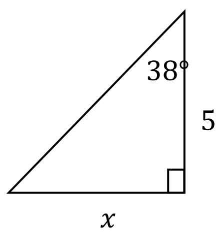
a)



b)

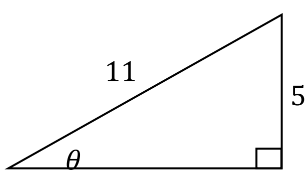


c)

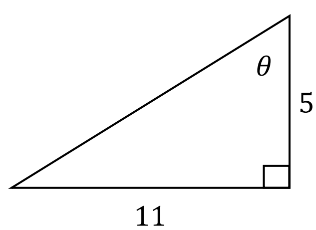


Exercise 2: Solve for θ. Express answer to the nearest tenths of a degree.

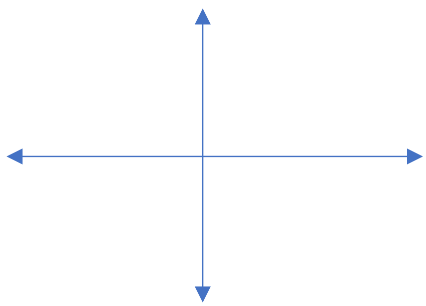
a)



b)



Exercise 3: Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(24, 7)$. Sketch the angle on the axes.



$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

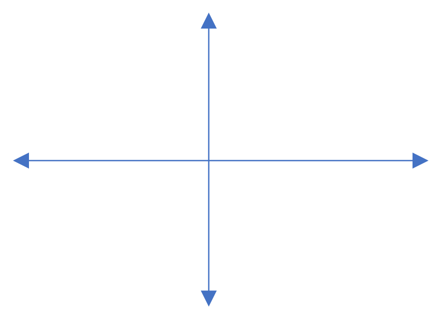
$\tan\theta =$ $\cot\theta =$

Exercise 4: Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-5,12)$:

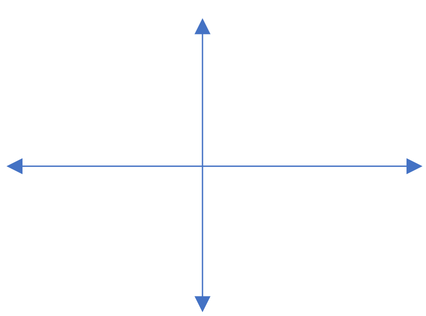
$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$



Exercise 5: Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-\sqrt{7},-\sqrt{3})$:



$\sin\theta =$ $\csc\theta =$

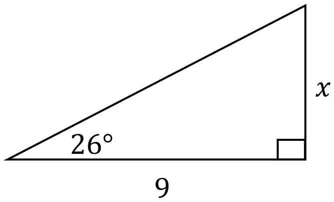
$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

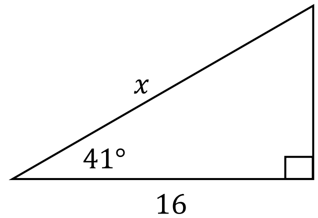
SECTION 5.3 SUPPLEMENTARY EXERCISES

1. Solve for x

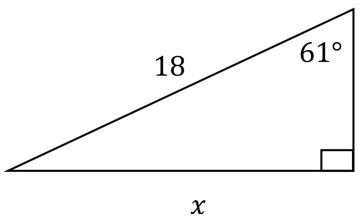
a)



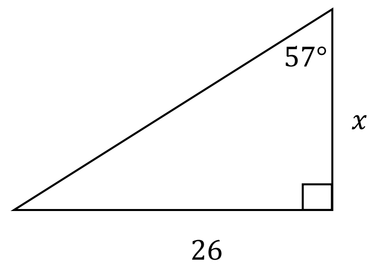
b)



c)

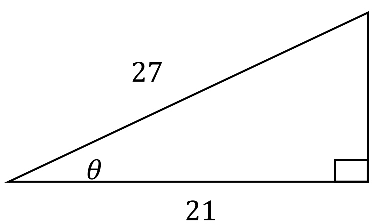


d)

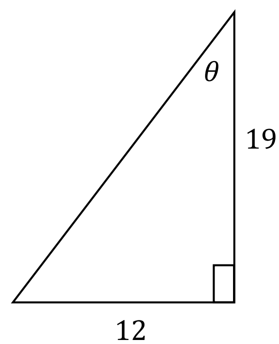


2. Solve for θ. Express answer to the nearest tenths of a degree.

a)



b)



3. Find the values of the trigonometric functions of the angle θ with its terminal side passing through the following points:

a) $(1,1)$
b) $(-4,3)$
c) $(\sqrt{5},-2)$
d) $(6,-\sqrt{10})$
e) $(-\sqrt{6},-\sqrt{7})$
f) $(\sqrt{3},\sqrt{6})$
g) $(-5,-12)$
h) $(\sqrt{7},-12)$
i) $(\frac{5}{2},7)$
j) $(-3\sqrt{3},\sqrt{5})$
k) $(-3,-11)$

Additional Resource A1: Signed Number Review

Rule for Combining Signed Numbers:

(1) To find the number value:

If two numbers have like signs, add their absolute values.
If two numbers have different signs, subtract their absolute values.

(2) To determine the sign:

The sign of the result is the sign of the number with larger absolute value.

Definition: The absolute value of a number is the number without its sign.

What happens when double signs appear in front of a number?

Change double signs to a single sign using the table below. Then combine the signed numbers following the rule for combining signed numbers.

If the double signs are	The single sign becomes
+ +	+
− −	+
+ −	−
− +	−

Rule for Multiplying or Dividing Two Signed Numbers:

(1) To find the number value: multiply or divide the two numbers.
(2) To determine the sign of the result, use the table below:

If the two signs are	The sign of the result is
+ +	+
− −	+
+ −	−
− +	−

The Order of Operations

The order of operations in the sequence of

P E M D A S (Please Excuse My Dear Aunt Sally)

1st Parenthesis
2nd Exponents
3rd Multiplication and Division from left to right
4th Addition and Subtraction from left to right

Note: The order for multiplication and division is from left to right of the expression, whichever first. Same for addition and subtraction.

Exercise 1: Change all double signs to a single sign and then combine the numbers.

a) $(-1) + (-9) =$

b) $10 + (-4) =$

c) $(+6) - (+13) =$

d) $-15 - (-12) =$

e) $15 + (-6) - (-8) - (+12) =$

f) $-(-9) - (+14) + (10) - (7) =$

Exercise 2: Evaluate using the rule for multiplying or dividing signed numbers

a) (9)(-16)

b) (-8)(0)(-3)

c) (-4)(-6)(-5)

d) $\dfrac{-98}{-7}$

e) $\dfrac{-12}{0}$

f) $\dfrac{0}{+7}$

Exercise 3: Evaluate using the order of operations

a) $\dfrac{-6-(-12)}{-3}$

b) $-3[2-(1-8)]$

c) $-5^2-(-2)^3$

d) $\dfrac{25-2(-5)}{-2-5}$

There is at least one error in each problem below. Find where the error(s) occurs and provide the correct work.

Find the errors	What are the errors?	Corrections
$-6-(-12)=-18$		
 $-5^2 = 25$		
$-(-5)^2 = 25$		
$4 \cdot 3^2 - 10$ $= \quad 144 - 10 \quad = \quad 134$		
$10 - 4 \cdot (-3)$ $= \quad 6 \cdot (-3)$ $= \quad -18$		

Additional Resource A2: Properties of Exponents

Instead of writing out long multiplication of the same number, for example, 2×2×2 , a symbolic representation of this idea was developed, so that axaxa=a^3. Rene Descartes introduced this idea in 1637 in his book "La Geometrie," so that any value multiplied by itself a certain number of times could be represented as a^2, a^3, or a^4. Descartes was both a philosopher and a mathematician, and is famous for, among other things, the line, Cogito, ergo sum, which in English is I think, therefore I am.

Cajori, F. (2007).A history of mathematical notations. Volume 1, 2, 335-339. Chicago, IL: Open Court Publishing Co.

Properties of Exponents

Let x be a real number and m , n integers, the properties of exponents are summarized below.

		Examples
Product Rule	$x^m \cdot x^n = x^{m+n}$	$y^8 \cdot y^3 =$
Quotient Rule	$\frac{x^m}{x^n} = x^{m-n}$ where $(x \neq 0)$	$\frac{a^7}{a} =$
Zero Exponent	$x^0 = 1$ where $(x \neq 0)$	$w^0 =$
Power Rule	$(x^m)^n = x^{m \cdot n}$	$(b^6)^2 =$
Power of a Product	$(x \cdot y)^n = x^n y^n$	$(r^4 t)^3 =$
Power of a Quotient	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ where $(y \neq 0)$	$\left(\frac{p^9}{q^2}\right)^5 =$
Negative Exponent	$x^{-n} = \frac{1}{x^n}$ where $(x \neq 0)$	$h^{-3} =$

Simplify the following exponents and express your answers in positive exponents

Exercise 1: Which exponent property is used to simplify the expressions? ___________________________

a) $(5x^7)(-2x^8)$

b) $\left(5x^5y^{-3}\right)\left(6x^8y^5\right)$

Exercise 2: Which exponent property is used to simplify the expressions? ___________________________

a) $\dfrac{-24a^3}{8a}$

b) $\dfrac{-27x^6y^4}{18xy^2}$

Exercise 3: Which combination exponent properties are used to simplify the expressions? ___________

____

a) $\left(3s^4t^7\right)^2$

b) $\left(\dfrac{2p^6}{3q^2}\right)^3$

Exercise 4: Which exponent property is used to simplify the expressions? __________________________

a) $\left(-9a^5b^3c\right)^0$

b) $-(9a^5b^3c)^0$

Exercise 5: Which exponent property is used to simplify the expressions? __________________________

a) b^{-5}

b) $\dfrac{1}{b^{-7}}$

Exercise 6: Use combination properties to simplify the expressions

a) $\dfrac{\left(12m^{13}n^7\right)\left(6m^{-11}n^{-5}\right)}{\left(9m^{-7}n^3\right)}$

b) $\left(2x^{-11}y^2\right)^5$

c) $\dfrac{\left(4v^{-7}w^{-9}\right)^3}{\left(8v^3w^{-4}\right)^2}$

There is at least one error in each problem below. Find where the error(s) occurs and provide the correct work.

Find the errors	What are the errors?	Corrections
$3^3 = 9$ $3^{-3} = -9$		
$a^2 \cdot a^3 = a^6$		
$x^2 + x^2 = x^4$		
$\dfrac{x^5}{x^{-2}} = x^3$		
$(2x^5)^4 = 8x^{20}$		
$\dfrac{x^2 y^3}{xy^3} = xy$		

SUPPLEMENTARY EXERCISES

Simplify.

1. $\left(3x^6y\right)\left(-2xy^7\right)$

2. $(6a^3b^3)(-9ab^{-3})$

3. $\dfrac{p^6qr^7}{p^3q^4r}$

4. $\dfrac{-8m^4n^6}{-4m^4n^2}$

5. $\dfrac{-15u^6v^2}{(-12u^3v)^2}$

6. $(-3a^4b^3)^4$

7. $\left(6g^4h\right)^2$

8. $\left(2x^5y^2\right)^{-4}(4x^4y)^2$

9. $(5x^7y^0z^{-1})^2(2xy^{-5})^3(2y^3z^2)^3$

10. $\dfrac{\left(3a^3b^6\right)^{-2}}{\left(2a^6b\right)^{-3}}$

11. $\left(\dfrac{16x^3y^6z^6}{-4xy^{-5}z^{-2}}\right)^{-2}$

12. $\left(\dfrac{e^6f^{-3}}{e^{-3}f^4}\right)^8$

13. $\left(\dfrac{b^{-3}c}{a^2b^{-9}c^5}\right)^{-5}$

14. $\dfrac{\left(3x^8y^{-2}\right)^{-3}\left(2x^6y^{-2}\right)^2}{\left(2x^{-2}y^7\right)^{-5}}$

15. $\dfrac{\left(3a^3b^6\right)^{-2}}{\left(2a^6b\right)^{-3}}$

16. $\left(\dfrac{16x^3y^6z^6}{-4xy^{-5}z^{-2}}\right)^{-2}$

17. $\left(\dfrac{e^6f^{-3}}{e^{-3}f^4}\right)^8$

18. $\left(\dfrac{b^{-3}c}{a^2b^{-9}c^5}\right)^{-5}$

19. $\dfrac{\left(3x^8y^{-2}\right)^{-3}\left(2x^6y^{-2}\right)^2}{\left(2x^{-2}y^7\right)^{-5}}$

20. $\dfrac{(5a^{-2}b^6c^3)^3(25a^3b^4)^{-4}}{15a^{12}c^5}$

Additional Resource A3: Addition, Subtraction, Multiplication, and Division of Radicals

Adding and Subtracting Radicals

Exercise 4: Simplify each radical term, then combine "like radicals" terms by adding or subtracting their coefficients.

a) $2\sqrt{50}-6\sqrt{125}+\sqrt{98}+3\sqrt{80}$

b) $\sqrt{4x^7y^5}+9x^2y^2\sqrt{x^3y}-5xy\sqrt{x^5y^3}$

Multiplying Radicals

Exercise 5: **Multiplying two radicals of the form:** $(a\sqrt{x})\cdot(b\sqrt{y})=ab\sqrt{xy}$. Simplify the answer.

a) $(-2\sqrt{3})(-4\sqrt{6})$

b) $\left(-2x\sqrt{6x^6y^5}\right)\cdot\left(5y\sqrt{8x^3y^5}\right)$

Exercise 6: Multiply the radical expressions using the Distributive Law $a(b+c)=ab+ac$. Simplify the answer.

c) $\sqrt{5}(4+\sqrt{5})$

d) $\sqrt{6}(\sqrt{12}+\sqrt{24})$

Exercise 7: Multiply the radical expressions using **FOIL**. Simplify the answer.

FOIL (Front-Outer-Inner-Last) **or** $(a+b)(c+d)=ac+ad+bc+bd$.

e) $(2+\sqrt{3})(4+\sqrt{3})$

f) $(2\sqrt{5}-9)^2$

g) $(\sqrt{7}-5)(\sqrt{7}+5)$

h) $(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})$

Helpful Radical Multiplication Rule:

Square of a Radical: $\left(\sqrt{a}\right)^2 = \left(\sqrt{a}\right)\left(\sqrt{a}\right) = a$

Product of radical conjugates: $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$

Or $\left(\sqrt{a}+b\right)\left(\sqrt{a}-b\right) = \left(\sqrt{a}\right)^2 - (b)^2 = a - b^2$

Dividing Radicals by Rationalizing the Denominators

Strategy for rationalizing denominators with radicals:
If the denominator has one term, multiply the numerator and the denominator by the radical:

Examples a) $\frac{2}{\sqrt{a}} = \frac{2}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{2\sqrt{a}}{a}$

b) $\frac{4}{3\sqrt{a}} = \frac{4}{3\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{4\sqrt{a}}{3a}$

If the denominator has two terms, multiply the numerator and the denominator by the conjugate of the denominator:

Examples a) $\frac{3}{\sqrt{a}+\sqrt{b}} = \frac{3}{\sqrt{a}+\sqrt{b}} \cdot \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3\left(\sqrt{a}-\sqrt{b}\right)}{a-b}$

b) $\frac{5}{\sqrt{a}-b} = \frac{5}{\sqrt{a}-b} \cdot \frac{\sqrt{a}+b}{\sqrt{a}+b} = \frac{5\left(\sqrt{a}+b\right)}{a-b^2}$

Exercise 8: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}}$

b) $\frac{3}{2\sqrt{x}}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}}$

d) $\frac{\sqrt{6}}{5-\sqrt{2}}$

SUPPLEMENTARY EXERCISES

1. Simplify the expressions.
    a) $\sqrt{48}$

    b) $\sqrt{20} \cdot \sqrt{5}$

    c) $\sqrt{\frac{64}{25}}$

    d) $\sqrt{\frac{13}{49}}$

2. Rationalizing the denominator.

    a) $\sqrt{\frac{5}{6}}$

    b) $\frac{9}{\sqrt{2}}$

    c) $\sqrt{\frac{36}{7}}$

    d) $\frac{5}{\sqrt{2}-\sqrt{3}}$

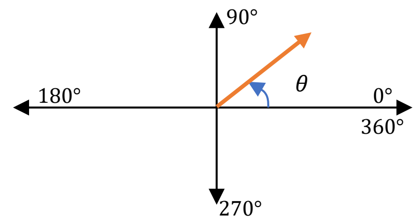
    d) $\frac{\sqrt{7}}{\sqrt{7}+4}$

    d) $\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}-\sqrt{3}}$

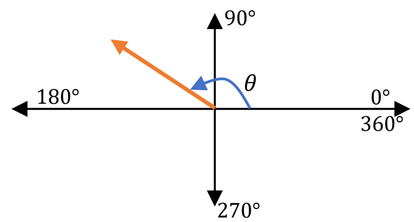
Additional Resource A4: The Unit Circle

An angle θ in standard form has its initial side on the positive x-axis (measured counterclockwise from 0°) to the terminal side of θ.

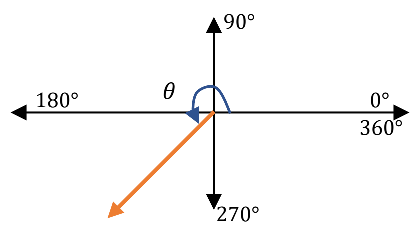
Quadrant I contains angles between ______________________



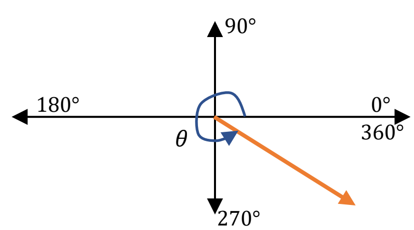
Quadrant II contains angles between ______________________



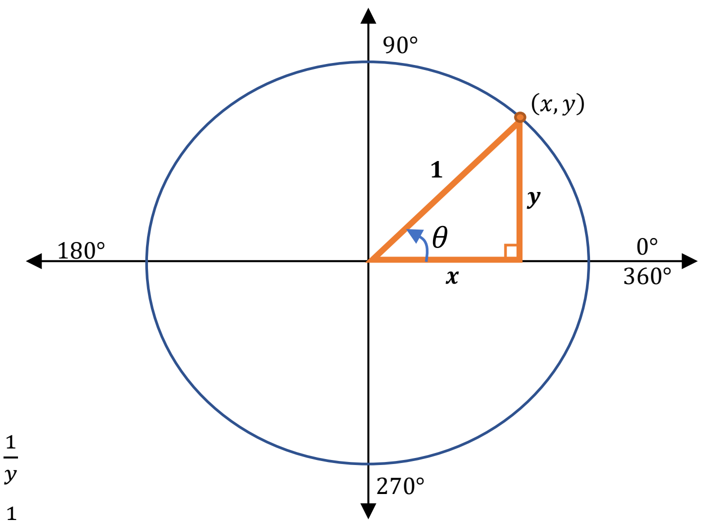
Quadrant III contains angles between ______________________



Quadrant IV contains angles between ______________________



Defining Trigonometric functions on the Unit Circle:



$\sin\theta = y$ $\qquad\qquad$ $\csc\theta = \frac{1}{y}$

$\cos\theta = x$ $\qquad\qquad$ $\sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad\qquad$ $\cot\theta = \frac{x}{y}$

Quadrant II $(90° < \theta < 180°)$

S **Sine is positive**

$\sin\theta$, $\csc\theta$ are positive
$\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are negative

Quadrant I $(0° < \theta < 90°)$

A **All are positive**

$\sin\theta$, $\csc\theta$, $\cos\theta$, $\sec\theta$, $\tan\theta$, $\cot\theta$ are all positive

Quadrant III $(180° < \theta < 270°)$

T **Tangent is positive**

$\tan\theta$, $\cot\theta$ are positive
$\sin\theta$, $\csc\theta$, $\cos\theta$, $\sec\theta$ are negative

Quadrant IV $(270° < \theta < 360°)$

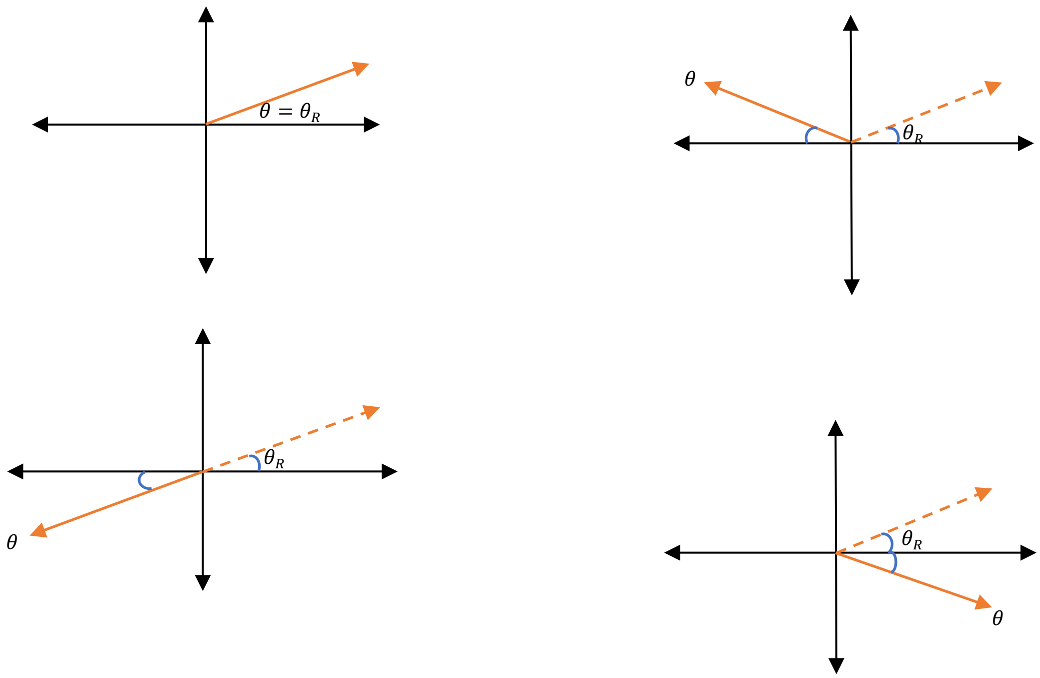
C **Cosine is positive**

$\cos\theta$, $\sec\theta$ are positive
$\sin\theta$, $\csc\theta$, $\tan\theta$, $\cot\theta$ are negative

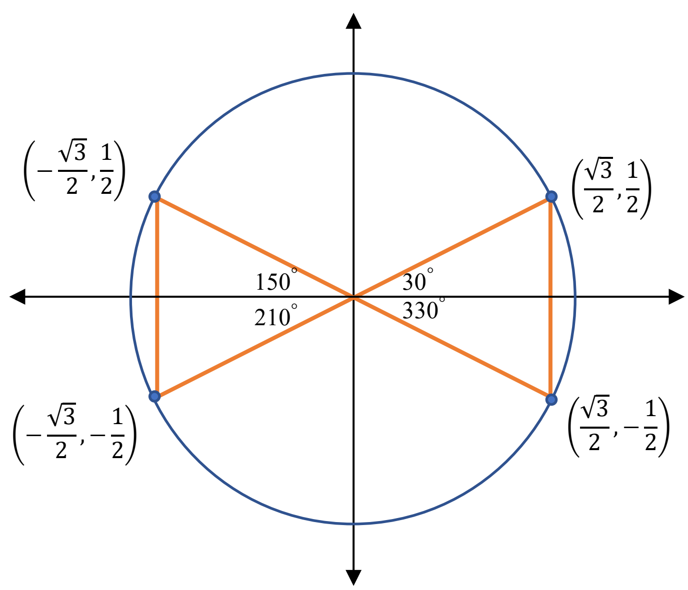
Reference Angle

The **reference angle** θ_R for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis.

Angle and its reference angle:

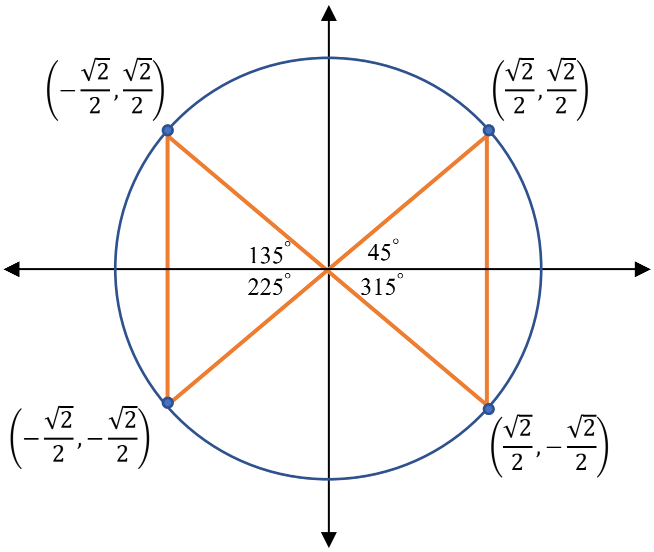


Use a bow-tie to relate angles with 30° reference angle.
What can you say about their coordinates on the unit circle? ______________________________________



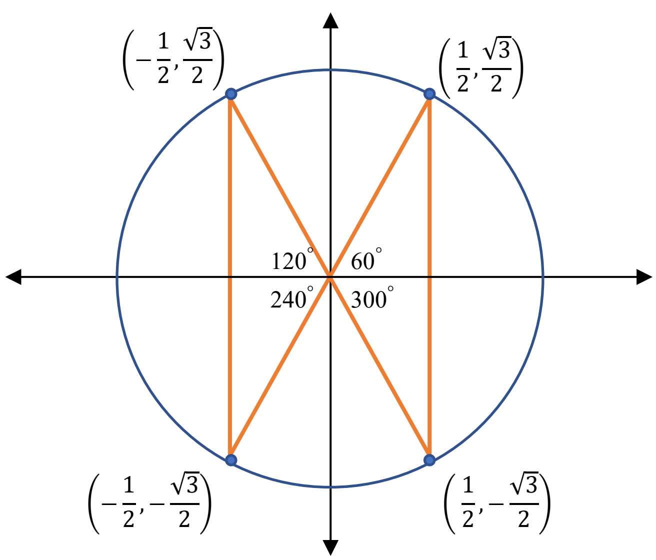
Use a bow-tie to relate angles with 45° reference angle

What can you say about their coordinates on the unit circle? ____



Use a bow-tie to relate angles with 60° reference angle

What can you say about their coordinates on the unit circle? ____

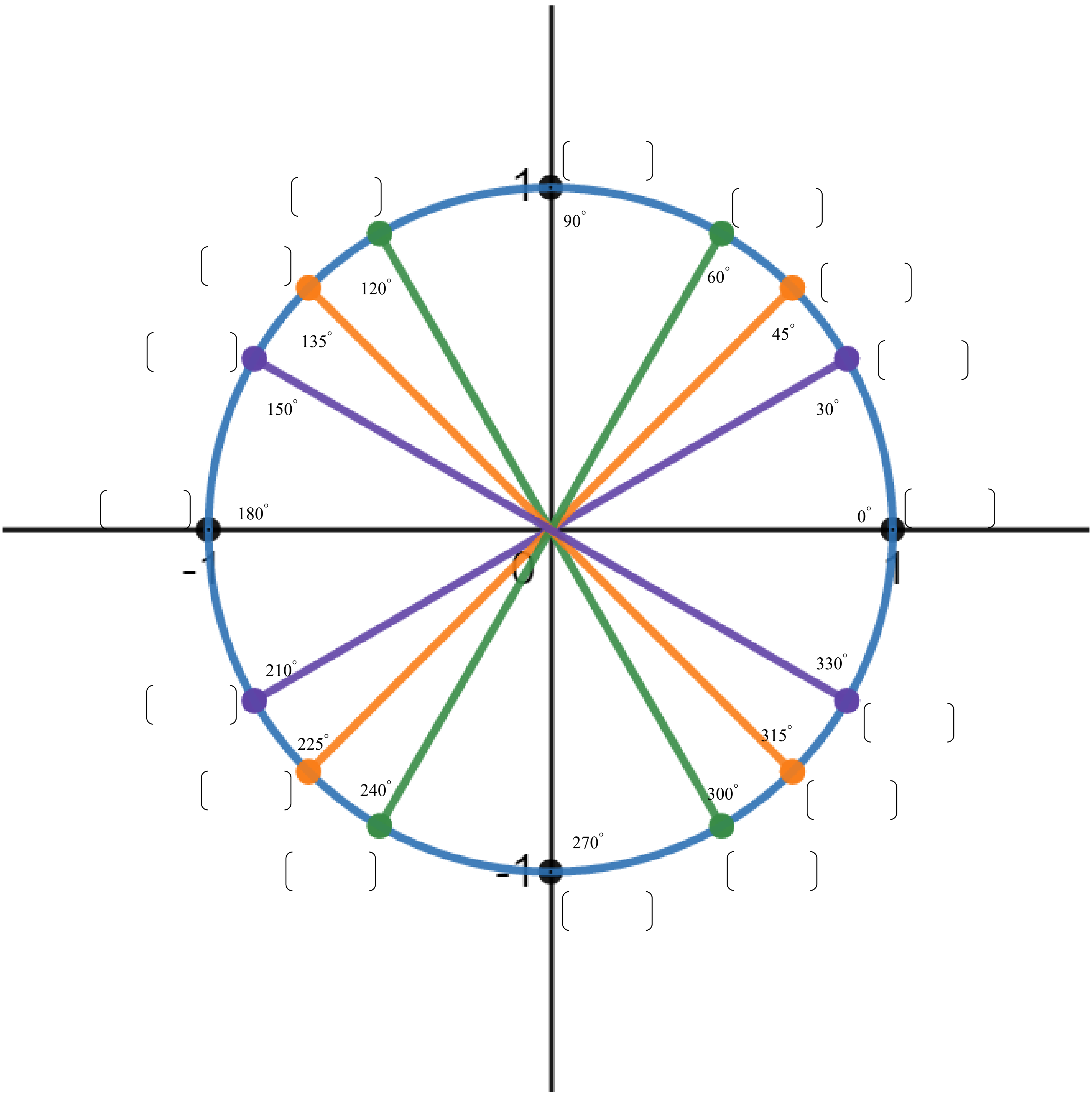


The Unit Circle and the Special Angles

Color code the reference angles:

Purple: Reference angle 30°
Orange: Reference angle 45°
Green: Reference angle 60°

Write the coordinates of the points on the unit circle in the brackets below.



Try both Desmos activities below to see how the coordinates on the unit circle change with angles.

Desmos: The Unit Circle 1
https://www.desmos.com/calculator/bydhfz5b9d

Desmos: The Unit Circle 2
https://www.desmos.com/calculator/p6cy3bxkkn

Exercise 1: Fill out the tables below for each given θ and the trigonometric function. Use the unit circle, the symmetry relation of the reference angles, and the definition of the trigonometric functions on the unit circle to determine the answers. Do not use the calculator.

Angles between $0°$ and $360°$				
	Quadrant of θ (I, II, III, or IV)	Sign (+ or –)	Reference angle ($30°, 45°$, or $60°$)	Value
$\sin 150°$				$\sin 150° =$
$\tan 240°$				$\tan 240° =$
$\cos 135°$				$\cos 135° =$
$\sin 300°$				$\sin 300° =$

Negative angles (angles which measure clockwise)				
	Quadrant of θ (I, II, III, or IV)	Sign (+ or –)	Reference angle ($30°, 45°$, or $60°$)	Value
$\sin(-60°)$				$\sin(-60°) =$
$\cos(-210°)$				$\cos(-210°) =$
$\tan(-135°)$				$\tan(-135°) =$

Angles greater than $360°$				
	Quadrant of θ (I, II, III, or IV)	Sign (+ or –)	Reference angle ($30°, 45°$, or $60°$)	Value
$\sin 390°$				$\sin 390° =$
$\cos 495°$				$\cos 495° =$
$\tan 780°$				$\tan 780° =$

Exercise 2: Find the value of the trigonometric functions of other special angles.

a) $\sin 0°$

b) $\cos 90°$

c) $\tan 180°$

d) $\sin 270°$

e) $\cos 360°$

f) $\tan 450°$

Exercise 3: Find the angles θ in the interval $0° \leq \theta < 360°$ for each problem below. Use the unit circle to find the angles.

a) If $\sin\theta = \frac{\sqrt{3}}{2}$, find all possible values of θ in the interval $0° \leq \theta < 360°$.

b) If $\cos\theta = \frac{\sqrt{2}}{2}$, find all possible values of θ in the interval $0° \leq \theta < 360°$.

c) If $\sin\theta = -\frac{1}{2}$, find all possible values of θ in the interval $0° \leq \theta < 360°$.

d) If $\tan\theta = -1$, find all possible values of θ in the interval $0° \leq \theta < 360°$.

SUPPLEMENTARY EXERCISES

1. Fill in the blanks:

    a) $\sin\theta$ and $\csc\theta$ is positive in Quadrants _____ and _____

    $\sin\theta$ and $\csc\theta$ is negative in Quadrants _____ and _____

    b) $\cos\theta$ and $\sec\theta$ is positive in Quadrants _____ and _____

    $\cos\theta$ and $\sec\theta$ is negative in Quadrants _____ and _____

    c) $\tan\theta$ and $\cot\theta$ is positive in Quadrants _____ and _____

    $\tan\theta$ and $\cot\theta$ is negative in Quadrants _____ and _____

2. Find the value of the given trigonometric function without using a calculator.

    a) $\cos 150°$
    b) $\sin 240°$
    c) $\tan 225°$
    d) $\csc 300°$
    e) $\cot 120°$
    f) $\sec 135°$
    g) $\cos(-240°)$
    h) $\tan(-150°)$
    i) $\cot(-210°)$
    j) $\csc(-60°)$
    k) $\sec(-135°)$
    l) $\sin 480°$
    m) $\tan 405°$
    n) $\sec 750°$
    o) $\csc 90°$
    p) $\cot 180°$
    q) $\sec(-90°)$