New York City College of Technology

HEAD START
TO
QUANTITATIVE REASONING

A Preparatory Workshop for MAT 1190 Quantitative Reasoning

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In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to Quantitative Reasoning before the actual Quantitative Reasoning course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The Quantitative Reasoning workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Precalculus requirement.

The Quantitative Reasoning workshop meets four days, three hours a day, for a total of 12 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Section 1: Signed Number Review

Rule for Combining Signed Numbers:

(1) To find the number value:

If two numbers have like signs, add their absolute values.
If two numbers have different signs, subtract their absolute values.

(2) To determine the sign:

The sign of the result is the sign of the number with larger absolute value.

Definition: The absolute value of a number is the number without its sign.

What happens when double signs appear in front of a number?

Change double signs to a single sign using the table below. Then combine the signed numbers following the rule for combining signed numbers.

If the double signs are	The single sign becomes
+ +	+
− −	+
+ −	−
− +	−

Rule for Multiplying or Dividing Two Signed Numbers:

(1) To find the number value: multiply or divide the two numbers.
(2) To determine the sign of the result, use the table below:

If the two signs are	The sign of the result is
+ +	+
− −	+
+ −	−
− +	−

The Order of Operations

The order of operations must be performed in the sequence of

P E M D A S (Please Excuse My Dear Aunt Sally)
1st Parenthesis
2nd Exponents
3rd Multiplication and Division from left to right
4th Addition and Subtraction from left to right

Note: The order for multiplication and division is from left to right of the expression, whichever first. Same for addition and subtraction.

Exercise 1: Change all double signs to a single sign and then combine the numbers.

a) $(-1) + (-9) =$

b) $10 + (-4) =$

c) $(+6) - (+13) =$

d) $-15 - (-12) =$

e) $15 + (-6) - (-8) - (+12) =$

f) $-(-9) - (+14) + (10) - (7) =$

Exercise 2: Evaluate using the rule for multiplying or dividing signed numbers

a) (9)(-16)

b) (-8)(0)(-3)

c) (-4)(-6)(-5)

d) $\dfrac{-98}{-7}$

e) $\dfrac{-12}{0}$

f) $\dfrac{0}{+7}$

Exercise 3: Evaluate using the order of operations

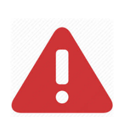
a) $\dfrac{-6-(-12)}{-3}$

b) $-3[2-(1-8)]$

c) $-5^2 - (-2)^3$

d) $\dfrac{25-2(-5)}{-2-5}$

There is at least one error in each problem below. Find where the error(s) occurs and provide the correct work.

Find the errors	What are the errors?	Corrections
$-6-(-12)=-18$		
$-5^2 = 25$ 		
$-(-5)^2 = 25$		
$4 \cdot 3^2 - 10$ $= 144 - 10 = 134$		
$10 - 4 \cdot (-3)$ $= 6 \cdot (-3)$ $= -18$		

Section 2: Linear Equations

Solving Linear Equations

The addition property of an equation: If $a = b$ then $a + c = b + c$

The subtraction property of an equation: If $a = b$ then $a - c = b - c$

The multiplication property of an equation: If $a = b$ then $a \cdot c = b \cdot c$

The division property of an equation: If $a = b$ then $a \div c = b \div c$ $\quad c \neq 0$

To solve a linear equation, one can use the addition, subtraction, multiplication, and division property of an equation to solve (or isolate) the variable.

Example 1: Solve for x: $x - 5 = -12.$

Apply the addition property to solve for x:

$$x - 5 + 5 = -12 + 5$$
$$x = -7$$

Example 2: Solve for x: $4x = 3x + 9.$

Apply the subtraction property to solve for x:

$$4x - 3x = 3x - 3x + 9$$
$$x = 9$$

Example 3: Solve for x: $\frac{x}{2} = 9.$

Apply the multiplication property to solve for x:

$$\frac{x}{2} \cdot 2 = 9 \cdot 2$$
$$x = 18$$

Example 4: Solve for x: $5x = -35.$

Apply the division property to solve for x:

$$\frac{5x}{5} = \frac{-35}{5}$$
$$x = -7$$

Example 5: Solve for x: $2x + 7 = 13.$

Apply combination of properties to solve for x:

$$2x + 7 - 7 = 13 - 7$$
$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

Steps in solving linear equations:

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Exercise 1:

a) If $4x + 1 = 5$, solve for x and check the solution.

b) If $7y - 2 = 2y + 8$, solve for y and check the solution.

c) If $-4(x - 3) = -12$, solve for x and check the solution. First use the distributive property to remove the parentheses.

Multiply

The Distributive Property: $\boldsymbol{a(b + c) = ab + ac}$

Exercise 2: Solve each equation below

a) Solve for t: $2t + 10 = 4t - 30$

b) Solve for t: $6(t - 1) = 2t - 3$

c) Solve for x: $12(x-2)-10(x+7)=14$

d) Solve for p: $5p-(p+4)=4+4(p+2)$

Exercise 3: City A's public school population is growing much faster than the schools can handle. With an increase of new students each year, overcrowding has become a major problem. The growth in student population can be approximated by the equation

$$P = 25{,}140x + 956{,}100$$

where P is the city's school population and x is the year starting in 2010. In the equation substitute x=1 for 2010, x=2 for 2011, and so on.

a) Find the number of students in current year. (What value of x do you substitute?)

b) Predict the number of students in 2030.

c) In what year did the student population equal to 1,308,060?

Exercise 4: The mathematical model

$$S = .55x - 34.5$$

approximates the Broadway tickets sold from 1999 – 2005. In the model, $x = 99$ corresponds to the 1999 – 2000 season, $x = 100$ corresponds to the 2000 – 2001 season, and so on. S is the number of tickets sold in millions.

a) Based on this model, how many tickets were sold in the 2003 – 2004 season?

b) In what season did the number of tickets sold reach 20.5 million

Exercise 5: Jane works as a waitress. Today, she worked an 8 hour shift, and was paid $200, including $108 gratuity.

a) How much is Jane's base paid per hour?

b) On Saturday, Jane worked 10 hours and received $247 gratuity. How much did Jane received in total, including gratuity?

Solving Linear Equations Containing Fractions

Steps in solving linear equations containing fractions:

1. Determine the Least Common Denominator (LCD) of all denominators in the equation. Remember, the denominator is 1 if a term does not show a denominator.
2. Multiply each term of the equation by the LCD. Use the LCD to eliminate the denominators.
3. Solve the simplified linear equation.

Example 6: Solve for b : $\frac{4b}{5} - 7 = \frac{b}{10}$

LCD = 10

Multiply each term by the LCD: $\frac{4b}{5} \cdot 10 - 7 \cdot 10 = \frac{b}{10} \cdot 10$

Eliminate the denominators: $8b - 70 = b$

Solve: $b = 10$

Exercise 6:

a) Solve for y : $\frac{4-2y}{3} = \frac{3}{4} - \frac{5y}{6}$

b) Solve for x: $\frac{7x}{3} + 5 = \frac{4x}{8} + 10$

c) Solve for x: $-\frac{1}{3}x + 3 = \frac{1}{4}x - 4$

Exercise 7: A father left $\frac{1}{2}$ of his estate to his son, $\frac{1}{3}$ of his estate to his granddaughter, and the remaining \$6,000 to charity. What was his total estate?

Exercise 8: In order to get a C for her sociology course, Betsy needs at least a 70% average. On exam 1 she scored 78% and on exam 2 she scored 68%. What is the lowest she can get on the last exam?

Exercise 9: In a charity triathlon, Mark ran half the distance, swam a quarter of the distance, and bike the remaining 15 miles. What was the total distance of the race?

SECTION 2 SUPPLEMENTARY EXERCISES:

Solve the following linear equations

1. $9x - 25 = 4x$
2. $7y + 2 = 11y - 3$
3. $2 + 4w = -2(3 + 7w)$
4. $3(1 + 2a) = 7a - 3$
5. $2(w + 3) - 14 = 6(32 - 3w)$
6. $5(a + 2) - 4(a + 1) = 3$
7. $6(p - 4) = 8(p - 3)$
8. $5 - 4y = 7 - 2(3y - 1)$
9. $6m - 2(m - 3) = 4 + 4(m + 1)$
10. $-3(x - 1) + 8(x - 3) = 11x + 7$
11. $2 - 5(a - 1) - (3a + 2) = -(2a - 6) - (3a - 5)$

Solve the following linear equations containing fractions

12. $\frac{5}{2} + \frac{3}{4}p = -4$
13. $\frac{1}{6}(5 - b) = \frac{1}{4}(2 + b)$
14. $\frac{4}{5}y - \frac{y}{4} = \frac{1}{2}$
15. $-\frac{3}{7} - \frac{d}{3} = -\frac{2d - 3}{7}$
16. $\frac{w}{4} + \frac{11}{12} = \frac{1}{2} - \frac{w}{6}$
17. $\frac{7 + m}{6} - \frac{3m - 1}{2} = \frac{5 - 2m}{5}$
18. $\frac{1 - 3t}{3} - t = \frac{3t - 1}{9} + \frac{1}{4}$
19. If the perimeter of a triangular flower bed is 15 feet with two sides the same and the third side 3 feet longer, what are the measures of the three sides of the flower bed?
20. A math class containing 57 students was divided into two sections. One section has three more students than the other. How many students were in each section?
21. On his first four tests, Allen got 65%, 72%, 85%, and 90%. He wants a B in the course, which would mean his overall average would have to be between 80% and 89%. What range of scores does Allen need on his last test so he can get a B in the course?

Section 3: Ratio and Proportion

A **ratio** is a comparison of two quantities using division.

A **proportion** is an equation in which two ratios are stated to be equal.

Exercise 1: Solve each proportion.

a) $\frac{x-6}{12} = \frac{1}{3}$

b) $\frac{4}{x-3} = \frac{16}{x-2}$

Exercise 2: The American Dietetic Association reported that 31 out of every 100 people want to lose weight. If 384 students were surveyed at random in the student union, how many would want to lose weight?

Exercise 3: Under normal conditions, 1.5 feet of snow will melt into 2 inches of water. After a monster snowstorm, there were 3.5 feet of snow. How many inches of water will there be when the snow melts?

Exercise 4: In New York City, approximately 2 out of 3 people are fully vaccinated with COVID-19 vaccines*. Suppose a restaurant has a capacity of 240 customers when it is full. If the restaurant operates at 50% capacity, how many of the restaurant's customers might not have been fully vaccinated? (*Data retrieved on November 8, 2021 shows the % of fully vaccinated NYC residents is 68%. https://www1.nyc.gov/site/doh/covid/covid-19-data-vaccines.page)

Section 4: Working with Formulas

Exercise 1: The area of a triangle has the formula: $A = \frac{bh}{2}$

a) Solve the formula for height h.

b) Find the height h using the formula in (a) if the area of the triangle is 30 and the base is 10.

Exercise 2: The formula for the volume of a rectangle solid is $V = wlh$.

a) Solve the formula for h.

b) Determine the height h, for various volume, width, and length in the table on next page.

Volume (V)	Width (w)	Length (l)	Height (h)
640	8	8	
2000	25	16	

Exercise 3: Most countries in the world use the Celsius system for temperature. Countries that still use the Fahrenheit system include Bahamas, Palau, Belize, the Cayman Islands, the Federated States of Micronesia, the Marshall Islands, and the United States and its territories such as Puerto Rico, the U.S. Virgin Islands, and Guam. (https://www.worldatlas.com/articles/countries-that-use-fahrenheit.html)

a) The formula converting Fahrenheit to Celsius is $C = \frac{5}{9}(F - 32)$. Find the formula for converting Celsius to Fahrenheit.

b) Determine the equivalent temperature in Celsius or Fahrenheit.

Celsius	Fahrenheit
You are traveling in Europe and the hotel thermometer indicates $25°C$.	
	The oven is in Celsius. How would you set the oven temperature if you want $350°F$?

SECTION 3 & 4 SUPPLEMENTARY EXERCISES:

1. A basketball team played 32 games and won 4 more games than it lost. Find the number of games the team won.
2. The electrical resistance for a conductor can be found by the formula $R = \frac{KL}{d^2}$. Solve the formula for L.
3. The formula for earned run average in baseball is $E = \frac{R}{I} \times 9$. Solve the formula for I.
4. The formula for slugging percentage in baseball is $P = \frac{S+2D+3T+4H}{A}$. Solve the formula for H.

Section 5: Working with Percents

Percent means hundredths, or per hundred. That is, $1\% = \frac{1}{100}$.

Converting a percent to a decimal:
In order to change a percent to a decimal, drop the % sign and move the decimal point two places to the left.

Converting a decimal to a percent:
To change a decimal to a percent, move the decimal point two places to the right and add the percent sign.

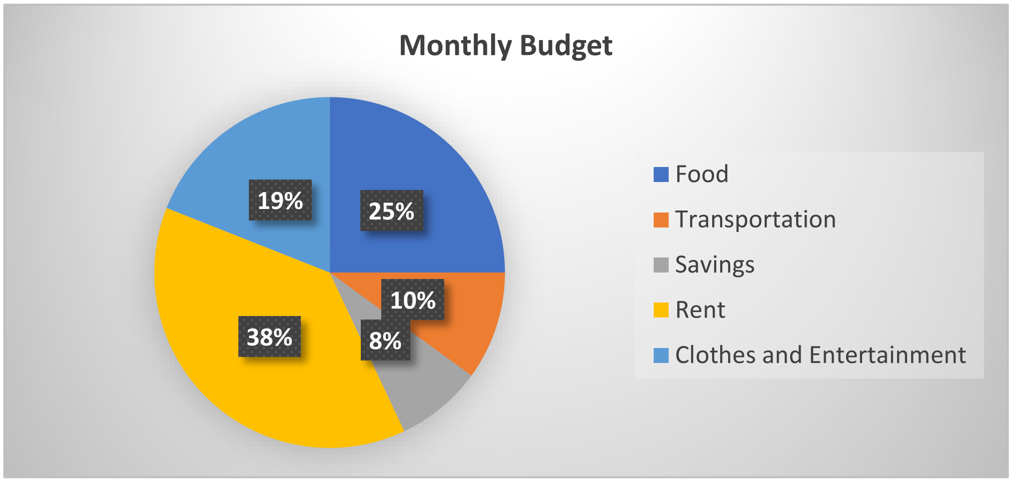
Converting a fraction to a percent:
To change a fraction to a percent, first change the fraction to a decimal, then change the decimal to a percent.

Exercise 1: In the following table, determine the equivalent percent, decimal, or fraction. Simplify wherever possible.

Percent	Decimal	Fraction
0.5 %		
	1.2	
		$\frac{2}{3}$
1 ¼ %		
		$\frac{1}{8}$

Exercise 2: Suppose Katherine brings home $4000 income per month.

a) How much does she spend on rent?
b) What is her total monthly expenses (everything other than the savings)?
c) If she spends $100 less on clothes and entertainment and puts the $100 into savings, then what % of the monthly budget is for clothes and entertainment? What % is for savings?



Exercise 3: You've had your eye on a luggage set that sells for $159.99, waiting for a sale. Finally, success! To make room for next year's model, the price has been reduced by 40%. You also have a 20% off coupon you can use on top of that. If the sales tax is 8.875%, how much will you pay for the luggage set?

Exercise 4: Go to the link to access the US Census website https://www.census.gov/popclock/ for US and world population. Also go to the link to access Our World in Data COVID-19 vaccination tracker https://ourworldindata.org/covid-vaccinations.

a) What is the estimated US population as of today? Why is it an estimate? What are some of the reasons that cause it to be not exact?
b) What is the percent of US population vaccinated with COVID-19 vaccines as of today?
c) How many people in the US have been vaccinated with COVID-19 vaccines as of today?

Exercise 5: A salesperson receives a base salary of \$30,000 and a commission of 10% of the total sales for the year.

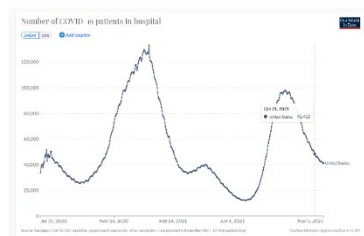
a) If the salesperson sells \$150,000 worth of merchandise, what is his total income for the year, including his base salary?
b) In the second year, he received a 5% of promotion of his base salary. At the end of the year, he received total income of \$52,500, including his base salary. How much merchandise did he sell?

Percent Increase or Percent Decrease

$$\frac{\%}{100} = \frac{\textbf{Final amount} - \textbf{Original amount}}{\textbf{Original amount}}$$

Exercise 6: Go to the link below to access the Our World in Data COVID-19 tracker:
https://ourworldindata.org/grapher/current-covid-patients-hospital?country=USA

a) How many COVID-19 patients were hospitalized on January 14, 2021?
b) How many COVID-19 patients were hospitalized on June 27, 2021?
c) What is the percent decrease in the number of COVID-19 patients from Jan. 14 to June 27 of 2021?
d) What is the percent increase in the number of COVID-19 patients from June 27 to September 1 of 2021?



Exercise 7: A department store advertised that certain merchandise was reduced 25%. Also, an additional 10% discount would be given to the first 100 customers on a specific day. The advertisement then stated that this amounted to a 35% reduction in the price of an item. Is the advertiser being honest?

SECTION 5 SUPPLEMENTARY EXERCISES:

1. Express each as a percent.

a) 0.63 b) 1.56 c) $1\frac{1}{4}$ d) $\frac{5}{16}$

2. Express each as a decimal.

a) 6% b) 75.6% c) 275% d) 0.05% e) $66\frac{1}{3}\%$

1. The sales tax in Atlanta, Georgia, is 8%. Find the amount of tax and the total cost of a portable DVD player on sale for $149.

2. If a sales clerk receives a 7% commission on all sales, find the commission the clerk receives on the sale of a computer system costing $1,799.99.

3. You bought $129 in clothes at a department store. You can choose between coupons that offer $20 off your entire purchase or 25% off. Which will save you more money? By how much?

4. A 20-inch flat panel computer monitor is on sale for 249.99. it was reduced $80.00 from the original price. Find the percent reduction in price.

5. The average teachers' and superintendents' salaries in a school district in western Pennsylvania was $50,480. Five years later, the new average was $54,747. Find the percent increase.

6. In 2012 the population of the town of Oak Creek was 23,258. In 2013, the population increased to 23,632. Find the percent increase in the population.

Section 6: Simple Interest & Compound Interest

Formulas for Computing Simple Interest and Future Value

Simple Interest

$$I = Prt \quad \text{(Interest = Principal * rate * time)}$$

I is the interest charged or earned
P is the starting balance of the account (also called initial deposit, or principal)
r is the annual interest rate on the principal
t is the amount of time in years

Future Value

$$A = P + I \quad \text{or} \quad A = P + Prt = P(1 + rt)$$

A is the total amount of principal plus interest after t years (also called the future value)

Exercise 1:
a) What is the simple interest on a $12,000 loan for 5 years at 7%? What is the future value of the loan after 5 years?

b) What is the simple interest on the same loan but for only 3 months? What is the future value after 3 months?

Exercise 2: The Banker's Rule treats every month like it has 30 days, so it uses 360 days in a year.

a) Use Banker's rule that there are 360 days in a year, what is the simple interest on a $2,200 loan at 7% interest for 100 days?

b) Use the actual 365 days in a year, what is the simple interest for the same loan?

c) If this is an investment, would you prefer (a) or (b)? If this is a loan, would you prefer (a) or (b)? Why or why not?

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Exercise 3: Jane had to borrow $8,000 over the course of 6 years to complete her education. The interest paid was $4,046.40. Find the rate.

Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A is the total amount of principal plus interest after t years (or the **future value**)
P is the starting balance (or the initial deposit, or principal)
r is the annual interest rate in decimal form
n is the number of compounding periods in one year

If the interest is
compounded annually, use $n = 1$;
compounded semiannually, use $n = 2$;
compounded quarterly, use $n = 4$;
compounded monthly, use $n = 12$;
compounded daily, use $n = 365$;

See how compounding works: Suppose $1,000 is compounded quarterly at 12% rate. This means interest is calculated at 3% every quarter.

	Balance at the beginning of the compounding period	**Balance at the end of the compounding period**
1st Quarter	$1,000	$1,000 + $30 = $1,030
2nd Quarter	$1,030	
3rd Quarter		
4th Quarter		

Exercise 4: If there is a $1,000 balance on a credit card, how much is the balance after one year at 12% interest rate if the interest is calculated as below. Assume there is no payment made and there is no late fee penalty for one year.

a) Simple interest

b) The interest is compounded monthly.

c) The interest is compounded daily.

d) Check your credit card statement to determine the interest rate (annual percentage rate). Use the rate to calculate the total amount you would have to pay on a $1,000 balance after one year.

Exercise 5: Samantha invested some money at 4.5% over 3 years, compounded monthly. The balance in her account at the end of 3 years is $2,860.62.

a) What was the initial principal amount of her investment?

b) What was the total amount of interest earned?

SECTION 6 SUPPLEMENTARY EXERCISES:

1. Find the future value of a loan that has principal $800, interest rate 4%, over a 5-year period,
   a) at simple interest
   b) if the interest is compounded quarterly

2. Find the future value of a loan that has principal $1350, interest rate 3.38%, over 2 ½ years,
   a) at simple interest
   b) if the interest is compounded monthly

3. Mary Beck earned $216 in simple interest on a savings account at 2% over 3 years. Find the principal amount on the account.

4. Matt's Appliance Store borrowed $6,200 for 3.5 years for repairs. The rate was 6.25% compounded monthly.
   a) What was the total payment on the loan?
   b) What was the total interest charged on the loan?

Section 7: Probability

Basic Probability

A **probability experiment** is a process that leads to well-defined results called outcomes.
An **outcome** is the result of a single trial of a probability experiment.
A **sample space S** is the set of all possible outcomes of a probability experiment.
An **event E** is any subset of the sample space of a probability experiment.

Note:

1. Probability an event is never negative and never greater than one: $0 \leq P(E) \leq 1$
2. When an event is impossible, its probability is zero. When an event is certain to occur, its probability is one.
3. The sum of the probabilities of all outcomes is one.

Empirical Probability is based on observed frequency usually in an experiment; it is written as a ratio of the number of times a particular event has occurred to the total number of trials.

$$P(E) = \frac{\text{Observed frequency of the specific event (f)}}{\text{Total number of trials (n)}} = \frac{f}{n}$$

Theoretical Probability, or classical probability, is determined on the basis of reasoning without actually performing experiments. When the outcomes are equally likely, it is the ratio of the number of outcomes to the total number of possible outcomes.

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of possible outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

For example, weather forecast is based on empirical probability. By saying there is a “70% chance of rain,” we are making the claim based on our observations of similar weather conditions in the past, that 7 out of 10 times it rains.

On the other hand, to say there is a “50% chance of getting a HEAD or a TAIL in a coin toss,” we are making the claim based on theoretical probability, not based on observed frequency. In actual coin toss, we may get HEADS 49 times and TAILS 51 times out of 100 tosses, but we wouldn’t say the probability of getting a HEAD is 49%.

Exercise 1: An airline company has a record of running late on its flights 145 times out of 700 flights per week. What is the probability a flight by this company will be late? Is it based on empirical probability or theoretical probability? (Answer: Empirical, based on observed frequency)

Exercise 2: In a math class of 21 women and 15 men, if one person is selected at random to come to the board to show the solution to a problem, what is the probability that the student is a woman? Is it based on empirical probability or theoretical probability? (Answer: Theoretical, based on likelihood; there is no actual experiment to see how many women get selected)

Most of the examples in this section will be based on theoretical probability.

Exercise 3: If we spin only once on the spin wheel, determine the probability of an event below. Assume there is equal probability for landing on any slot and the events are random.

a) Probability of a simple event:
   i. What is the probability of landing on red? ______________________________
   ii. What is the probability of landing on 500? ______________________________

b) Probability of a compound event:
   i. What is the probability of landing on either green or 250? ______________________
   ii. What is the probability of landing on 250 that is green? ______________________
   iii. What is the probability of landing on a number that is 500 or greater? ______________

c) Probability of a complement:
   i. What is the probability of landing on a color that is not red? ____________________
   ii. What is the probability of landing on a number that is not 500? __________________

Probability of a compound event: "OR" probability

If A and B are events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Probability of a complement: "NOT" probability

A complement of an event means the event does <u>not</u> occur. If $\bar{A}$ is a complement of event A, then

$$P(\bar{A}) = P(\text{Not } A) = 1 - P(A)$$

Exercise 4: A deck of cards has 52 cards. See picture below.

13 diamonds ♦ 13 hearts ♥

13 clubs ♣ 13 spades. ♠

If one card is selected at random, determine the probability of the following event:

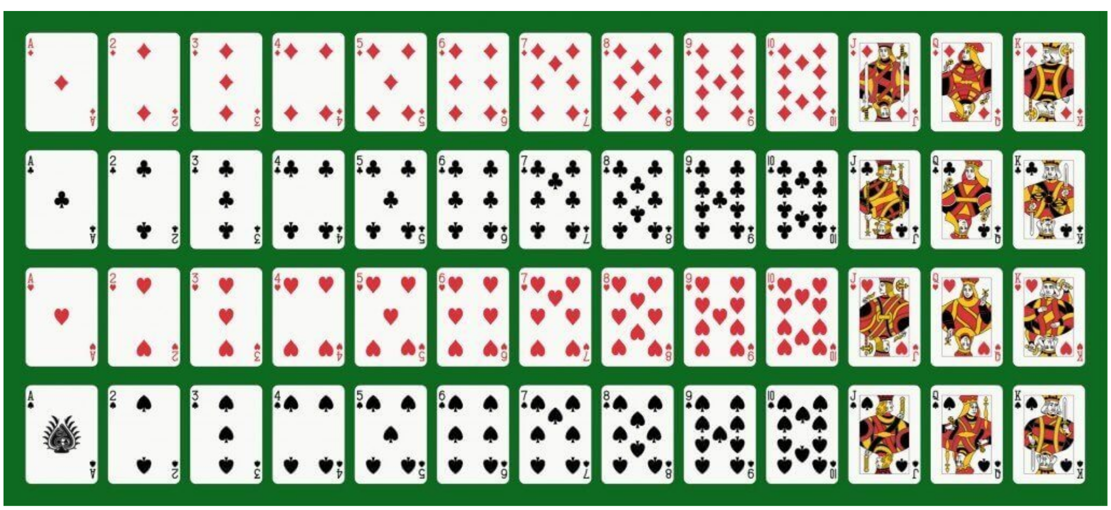
a) the card selected is a diamond? ______________________________

b) the card selected is a face card (Jack, Queen, or King)? ______________________________

c) the card selected is a diamond or a face card? ______________________________

d) the card selected is not a face card? ______________________________

e) the card selected is either red or not a face card? ______________________________



Exercise 5: A 2-lb bag of Hershey's miniatures contains 33 milk chocolate bars, 27 Krackel bars, 26 Mr. Goodbars, and 19 Special Dark bars.

a) Set up a frequency distribution.

Frequency Distribution Table	
Type of candy bar	Frequency

b) Find the probability that a bar chosen at random is

i. a Mr. Goodbar ____

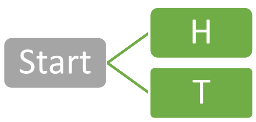
ii. a Krackel or Special Dark _________________________________

iii. Not a milk chocolate bar _________________________________

Probability using a Tree Diagram

Tossing a coin once gives simple events of either HEAD or TAIL. If we want to consider the sample space of tossing a coin two or more times, it becomes more complicated. We can use a tree diagram to represent a sequence of events to show the outcomes after two tosses, after three tosses, etc. In multiple tosses, the order of events matters. For example, first toss HEAD and the second toss TAIL is indicated by HT, which is different from first toss TAIL and second toss HEAD indicated by TH.

The Tree Diagram for tossing a coin once

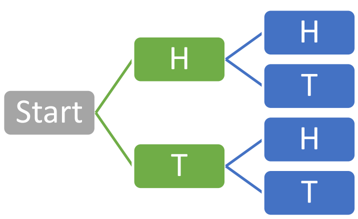


H

T

Sample Space = {H, T}

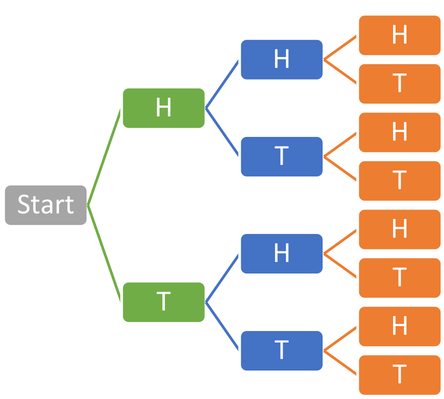
The Tree Diagram for tossing a coin two times



HH
HT
TH
TT

Sample Space = {HH, HT, TH, TT}

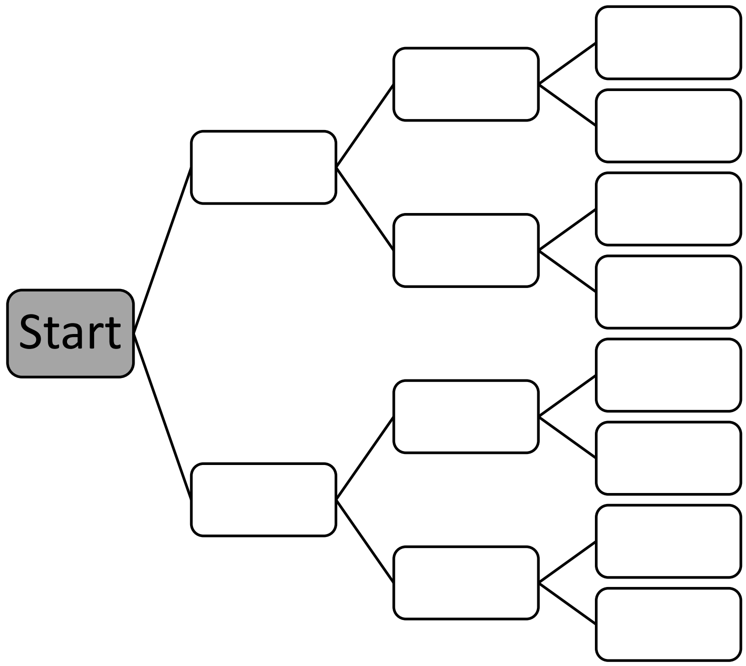
The Tree Diagram for tossing a coin three times



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Sample Space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Exercise 6: Draw a tree diagram for a couple who want to have three children. List the sample space.



Sample Space

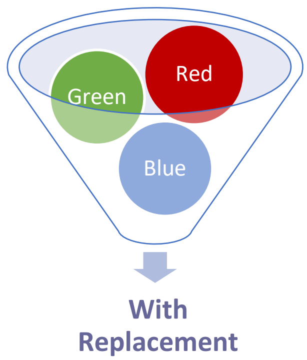
a) What is the probability the family will have first child boy, second child boy, and third child girl?

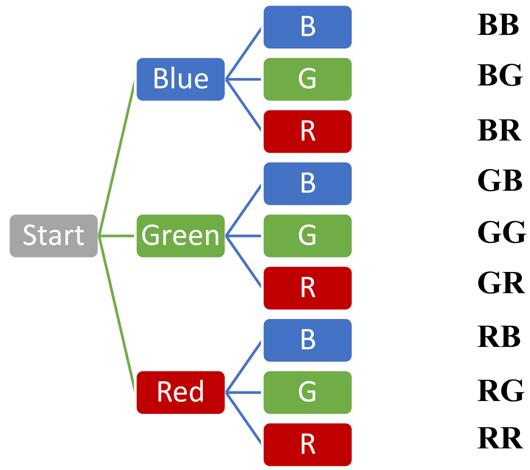
b) What is the probability the family will have two boys and one girl in any order?

c) What is the probability the family will have two boys and one girl, and the girl is not the oldest?

d) What is the probability the family has no girl?

The Tree Diagram for selecting two balls with replacement





Exercise 7: Two balls are selected **with** replacement.

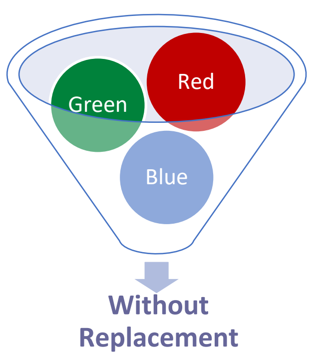
a) What is the probability the first ball selected is blue, second is red?______________________________

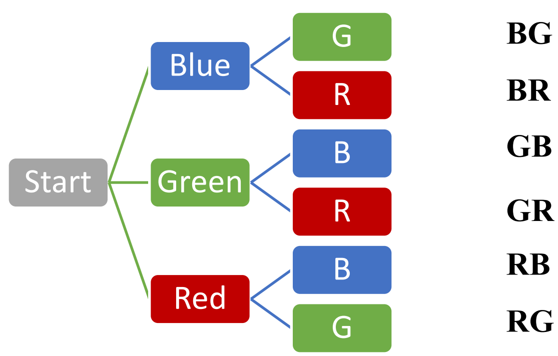
b) What is the probability one ball is blue and the other ball is red (in either order)?_________________

Does it matter about the order? Why or why not? ____

c) What is the probability both balls are red? ____

The Tree Diagram for selecting two balls without replacement





Exercise 8: Two balls are selected **without** replacement.

a) What is the probability the first ball selected is blue, second is red?______________________________

b) What is the probability one ball is blue and the other ball is red (in either order)?_________________

Does it matter about the order? Why or why not? ____

c) What is the probability both balls are red? ____

Probability of Independent Events & Conditional Probability

Probability of Independent Events:

If events A and B are independent events, that is the probability of event B occurring is the same whether or not event A occurs, then the probability of event A and event B is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

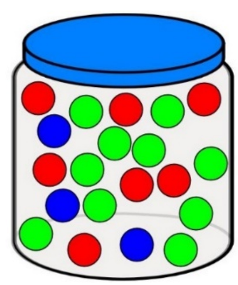
When the balls are selected one at a time **with** replacement, each selection is considered an independent event, we use the probability formula above.

Conditional Probability:

The probability of the event B occurs, given that event A has happened, is represented as $P(B|A)$. The conditional probability of event A and event B is

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

When the balls are selected one at a time **without** replacement, the selections are not independent events. The selection of the second ball depends on the selection of the first ball, hence we use conditional probability.



Exercise 9: A jar has 20 balls: 3 blue, 10 green, and 7 red. What is the probability for the following selections of balls? With replacement? Without replacement?

	With Replacement	Without Replacement
a) Blue and Green	$P(\text{Blue and Green})$ $= P(\text{Blue}) \cdot P(\text{Green})$ $= \frac{3}{20} \cdot \frac{10}{20} = \frac{3}{40}$	$P(\text{Blue and Green})$ $= P(\text{Blue}) \cdot P(\text{2nd Green} \mid \text{1st Blue})$ $= \frac{3}{20} \cdot \frac{10}{19} = \frac{3}{38}$

	With Replacement	**Without Replacement**
b) Two Green Balls		
c) Three Green Balls		
d) Blue, Green, Red in this order		
e) Blue, Green, Red in any order		

SECTION 7 SUPPLEMENTARY EXERCISES:

1. Each number from one to twelve is written on a card and the twelve cards are placed in a box. If a card is selected at random, find the probability that the number on the card is
   a) 7
   b) An odd number
   c) A number less than four
   d) A number greater than seven

2. Suppose that in a certain game, it's equally likely that you will win, lose, or tie. Find the probability of
   a) Losing twice in a row.
   b) Winning at least once in two tries.
   c) Having the same outcome twice in a row.

3. A couple has two children.
   a) Draw a tree diagram for a couple having two children.
   b) Find the probability that
      i. Both children are girls.
      ii. At least one child is a girl.
      iii. Both children are of the same gender.

4. On the Price Is Right game show, a contestant spins a wheel with numbers 1 through 7, with equally sized regions for each of these numbers. If the contestant spins once, find the probability that the number is
   a) 6
   b) An even number
   c) A number greater than 4
   d) A number less than 8
   e) A number greater than 7

5. In a math class of seven women and nine men, if one person is selected at random to come to the board to show the solution to a problem, what is the probability that the student is a man?

6. In a survey, 16% of male college students said that they lie sometimes to get a woman to go out on a date with them. If a male college student is chosen at random, find the probability that he does not lie to get a date with a woman.

7. On one college campus with 5,300 students, 31% are freshman, 28% are sophomores, 26% are juniors, and the rest are seniors. If a student is randomly chosen from the campus phone book to win a $500 gift card to the bookstore, find the probability that
   a) The student isn't a freshman.
   b) The student is a senior.
   c) The student is a sophomore or junior.

Section 8: Statistics

Measure of Central Tendency

Statistics is the science of collecting, organizing, analyzing & interpreting numerical facts.

For a set of data, we are interested in knowing about its
Measures of central tendency: denote the 'average' value in distribution
Measures of variability: denote the 'spread' of scores in distribution

Measures of Central Tendency

- ***Mode:*** The mode is the element of the data set that occurs most frequently.
- ***Median:*** The median is halfway point of a data set when it's arranged in order.
  Note: The median is the value in the middle if the number of data is odd; the median is the mean of the two middle values if the number of data is even.
- ***Mean*** $(\bar{x})$ ***:*** The mean is the sum of the data values divided by the number of values. Also called the average, the mean is the most frequently used measure of central tendency.

$$\bar{x} = \frac{\sum x}{n}$$

Exercise 1: According to US Census, the top 10 most populous cities in United States and their population density are listed in the table below. Data retrieved from the US Census website: https://www.census.gov/popclock/

Most Populous Cities in the US		
City	**Population**	**Density: Population per square mile**
New York City, NY	8,336,817	27,754.20
Los Angeles, CA	3,979,576	8,486.00
Chicago, IL	2,693,976	11,848.50
Houston, TX	2,320,268	3,624.30
Phoenix, AZ	1,680,992	3,247.20
Philadelphia, PA	1,584,064	11,796.80
San Antonio, TX	1,547,253	3,189.50
San Diego, CA	1,423,851	4,369.30
Dallas, TX	1,343,573	3,954.80
San Jose, CA	1,021,795	5,746.60

a) Find the central tendency mean, median, and mode of the Population.
b) Find the central tendency mean, median, and mode of the Population Density.
c) Create a City and Population graph
d) Create a City and Density graph

Of the three measures of central tendency in this module: mode, median, and mean, which one makes more sense in looking at the population or density data? What may be the strengths or weaknesses with each measure of central tendency?

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Comparing the graphs:

- **City** and **Population**
- **City** and **Density**

What is the purpose of each graph? How are they similar? How are they different? What is the best way to organize data? Why?

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Measure of Variability: Standard Deviation

While mean, median, and mode are measures of central tendency, standard deviation is the measure of variability.

Exercise 2: Find the mean, median, and mode for each of the following data set.

Data set 1: **7, 7, 7, 7** Mean ($\bar{x}$) = Median= Mode=

Data set 2: **6, 7, 7. 8** Mean ($\bar{x}$) = Median= Mode=

Data set 3: **1, 7, 7, 13** Mean ($\bar{x}$) = Median= Mode=

What have you noticed about the measures of central tendency for data sets 1, 2, and 3?

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Standard Deviation (*s*)

Standard deviation is a measure of variation, or the spread of a data set, based on measuring how far each data value is different from the mean.

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

Where x represents the individual value in the data set and $\bar{x}$ represents the mean of the data set.

Exercise 3: Find the standard deviation for the three data sets in Exercise 2.

1) Find the standard deviation of 7, 7, 7, 7

Mean ($\bar{x}$) = standard deviation $s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} =$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
7		
7		
7		
7		
Σx=	$\Sigma(x-\bar{x})$ =	$\Sigma(x-\bar{x})^2$ =

2) Find the standard deviation of 6, 7, 7. 8

$\bar{x}$ = $s =$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
7		
7		
7		
7		
Σx=	$\Sigma(x-\bar{x})=$	$\Sigma(x-\bar{x})^2=$

3) Find the standard deviation of 1, 7, 7, 13

$\bar{x}$ = $s =$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
7		
7		
7		
7		
Σx=	$\Sigma(x-\bar{x})=$	$\Sigma(x-\bar{x})^2=$

How would you use standard deviation to describe the data set 1, 2, and 3?

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Interpreting standard deviation

Small standard deviation means the data set is
- clustered around the mean
- small variability
- more homogeneous

Large standard deviation means the data set is
- more spread out
- big variability
- more heterogeneous.

Is it better to have a small standard deviation or a large standard deviation? It depends on the goal. For example, a small standard deviation is good in quality control, since the data set is close to the mean. A large standard deviation is good if variation is the goal.

Standard deviation in weather forecast

If the standard deviation on a 5-day weather forecast is large, what do you think it means about the forecast? Explain.

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The large standard deviation in weather forecast means either the weather fluctuates wildly, or the forecast may not be reliable. Imagine a 5-day forecast that is 80, 70, 60, 50, 40 which is a big spread in temperature.

Standard deviation in wages

The standard deviation of employee wages for a company is large. What does this mean about the employee wages in this company?

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The large standard deviation in wages implies large differences in wages, some get paid a lot more than others. Unfortunately, large standard deviation does not mean large wages. It points to inequality in wages.

Standard deviation in health care

Suppose the standard deviation of annual health care costs for 20-year olds is small, and the standard deviation of annual health care costs for 60-year olds is large. What does it mean about the health care costs for the 20-year olds? For the 60-year olds?

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Why is it important for health insurance company to know the standard deviations for the different groups? How do they use this information to assess risk categories?

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The smaller standard deviation implies the 20-year olds have similar health care needs. The greater standard deviation implies the 60-year olds have greater variations (not similar) in their health care costs and needs. In order to evaluate the cost of health care for the 60-year olds, the insurance company may need to break down into different risk categories such as gender, smoking, alcohol, and other health factors. The standard deviation in the risk category is smaller because they are more similar in health. Insurance company charges a fee according to the risk group.

Standard deviation in real estate

What does standard deviation in housing price mean when standard deviation is large? When it's small? Why does the real estate agents want to know the standard deviation of housing price? Do you think the housing price standard deviation in NYC is small or large?

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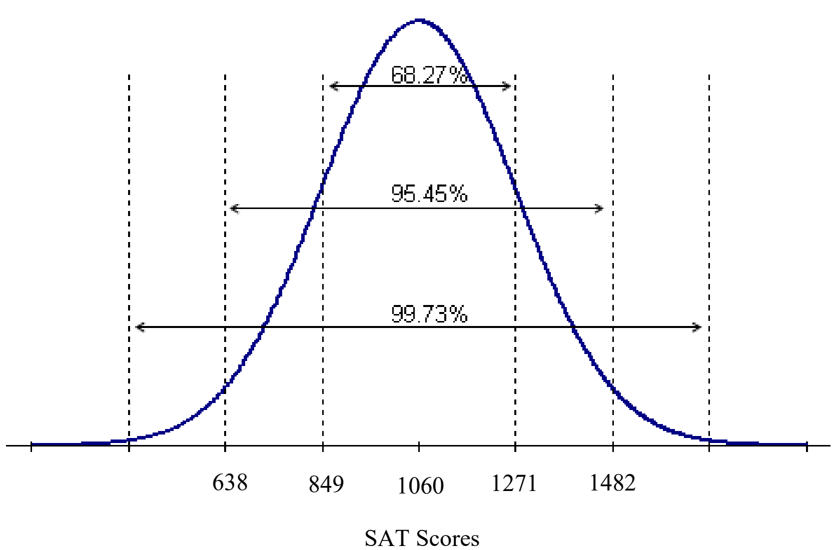
Real estate agents calculate the standard deviation of housing price in a particular area so they can advise their clients about the type of variation in prices to expect. In NYC, one expects the housing price standard deviation to be large reflecting its wide range of housing prices and their variability from year to year. A region with a small housing price standard deviation implies it is relatively stable in housing prices.

Standard deviation in SAT scores

In a normal distribution, ± 1 standard deviation corresponds to 68.27% of the population; ± 2 standard deviations correspond to 95.45% of the population; ± 3 standard deviations correspond to 99.73%, almost the entire population.

In standardized test such as SAT, the standard deviation is high by design to create variability between the scores. If the standard deviation is small, majority of the students would score around the mean, this wouldn't effectively differentiate the scores. For example, if SAT standard deviation is 50 points with a mean of 1050, then 95.45% of the students would score between ± 2 standard deviations which correspond to scores between 950 to 1150. If the standard deviation is 200 points, then 95.45% of the students would score between 650 and 1450, which is a more reasonable spread of the scores.

(The mean SAT scores for the class of 2021 is 1060. The standard deviation is 211.)



SECTION 8 SUPPLEMENTARY EXERCISES:

1. According to US Census, the top 10 most populous countries and their land areas are listed in the table below. Data retrieved from the US Census website: **https://www.census.gov/popclock/** and Worldometers: **https://www.worldometers.info/geography/largest-countries-in-the-world/**

Country	Population	Land area in square miles
China	1397897720	3,747,877
India	1339330514	1,269,345
United States	332475723	3,618,783
Indonesia	275122131	735,358
Pakistan	238181034	340,508
Nigeria	219463862	356,669
Brazil	213445417	3,287,955
Bangladesh	164098818	56,977
Russia	142320790	6,601,665
Mexico	130207371	758,449

a) Find the central tendency mean, median, and mode of the Population.
b) Find the central tendency mean, median, and mode of the Land Area.
c) Create a Country and Population graph.
d) Create a Country and Land Area graph.
e) Create a new column for Population Density. The formula for determining Population Density is Population divided by Land Area?
f) Find the central tendency mean, median, and mode of the Population Density.
g) Create a Country and Population Density graph.
h) Compare the three graphs. What are the similarities and differences between the three graphs?

2. The number of home runs the team hit in eight consecutive games were: 2, 3, 0, 3, 4, 1, 3, 0
Find mean, median, mode, and standard deviation

3. The quiz grades were: 8, 4, 10, 10, 6, 2, 9,
Find mean, median, mode, and standard deviation