Taylor polynomials of second order for functions of two real variables

Recall that if $u(t)$ is a function of one variable, then if $u'(c)$ is defined, that is if u is differentiable at c, then the first order Taylor polynomial of u at c is:

$$T_1(t) = u(c) + u'(c)(t-c)$$

which is exactly the tangent line to u at c. This is the unique polynomial of order one whose value and derivative agree with u at c, because

$$\begin{aligned} T_1(c) &= u(c) + u'(c)\cdot 0 = u(c) \\ \text{and} \quad T_1'(t) &= u'(c) \quad \Rightarrow T_1'(c) = u'(c) \end{aligned}$$

We obtain the second order Taylor polynomial in a similar way, matching the values of u, u', and u'' at c:

$$T_2(t) = u(c) + u'(c)(t-c) + \frac{1}{2}u''(c)(t-c)^2$$

Then,

$$\begin{aligned} T_2(c) &= u(c) + u'(c)\cdot 0 + \tfrac{1}{2}u''(c)\cdot 0^2 &\Rightarrow \quad T_2(c) &= u(c), \\ T_2'(t) &= u'(c) + u''(c)(t-c) &\Rightarrow \quad T_2'(c) &= u'(c), \\ \text{and} \quad T_2''(t) &= u''(c) &\Rightarrow \quad T_1''(c) &= u''(c) \end{aligned}$$

Notice that

$$T_2(t) = T_1(t) + \frac{1}{2}u''(c)(t-c)^2,$$

One way that we can look at $T_2(t)$ is as a correction to $T_1(t)$, that is $T_2(t)$ gives a closer approximation to $u(t)$ than $T_1(t)$ does when t is close enough to c. Taylor's inequality can be used to bound the error in the estimates of Taylor polynomials. In particular,[1].

$$|u(t) - T_1(t)| \leq K_1|t-c|^2 \quad \text{and} \quad |u(t) - T_2(t)| \leq K_2|t-c|^3$$

for $c-\ell \leq t \leq c+\ell$ where $|u''(t)| \leq 2K_1$ and $|u'''(t)| \leq 6K_2$ for all t in $[c-\ell, c+\ell]$.

Let's develop these ideas for functions of two variables. Take $U \subset \mathbb{R}^2$ and let $f: U \to \mathbb{R}$ be a function. We will also assume that all second order partial derivatives of f exist and are continuous on a disc centered at (a,b). Then, all partial derivatives of f are defined at (a,b), and

$$f_{xy}(a,b) = f_{yx}(a,b). \tag{1}$$

If we set

$$T_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \tag{2}$$

[1] See, for example, Theorem 6.7 in E. Herman and G. Strang, *Calculus*, Volume 2, OpenStax, Rice University, Houston, Texas 2017. Most single variable calculus textbooks give an estimate of the error for estimating a function with its nth order Taylor polynomial.

then it's easy to see that T_1 is equal to the linearization of f at (a, b):

$$\begin{aligned} L(x,y) &= f(a,b) + Df(a,b)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}\right) = f(a,b) + [f_x(a,b)\ f_y(a,b)]\begin{bmatrix} x-a \\ y-b \end{bmatrix} \\ &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= T_1(x,y) \end{aligned}$$

Now let's choose T_2 to be an adjustment of T_1 as defined in (2), so that T_2 has exactly the same value, the same first order partial derivatives, and the same second order partial derivatives as f at (a, b):

$$\begin{aligned} T_2(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \frac{1}{2}f_{yy}(a,b)(y-b)^2 + f_{xy}(a,b)(x-a)(y-b) \end{aligned} \tag{3}$$

Let's compare the values of T_2, and its first and second order partial derivatives at (a, b) to those of f and its derivatives.

$$\begin{aligned} T_2(a,b) &= T_1(a,b) + \frac{1}{2}f_{xx}(a,b)(a-a)^2 + \frac{1}{2}f_{yy}(a,b)(b-b)^2 + f_{xy}(a,b)(a-a)(b-b) \\ &= T_1(a,b) = f(a,b) \\ &\Rightarrow \boxed{T_2(a,b) = f(a,b)} \\ \frac{\partial}{\partial x}T_2(x,y) &= \frac{\partial}{\partial x}T_1(x,y) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b) \\ &= f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b) \\ &\Rightarrow \boxed{\tfrac{\partial}{\partial x}T_2(a,b) = f_x(a,b)} \\ \frac{\partial^2}{\partial x^2}T_2(x,y) &= 0 + f_{xx}(a,b) + 0 \\ &\Rightarrow \boxed{\tfrac{\partial}{\partial xx}T_2(a,b) = f_{xx}(a,b)} \\ \frac{\partial}{\partial y}T_2(x,y) &= \frac{\partial}{\partial y}T_1(x,y) + f_{yy}(a,b)(y-b) + f_{xy}(a,b)(x-a) \\ &= f_y(a,b) + f_{yy}(a,b)(y-b) + f_{xy}(a,b)(x-a) \\ &\Rightarrow \boxed{\tfrac{\partial}{\partial y}T_2(a,b) = f_y(a,b)} \\ \frac{\partial^2}{\partial y^2}T_2(x,y) &= 0 + f_{yy}(a,b) + 0 \\ &\Rightarrow \boxed{\tfrac{\partial}{\partial yy}T_2(a,b) = f_{yy}(a,b)} \\ \frac{\partial^2}{\partial x\partial y}T_2(x,y) &= 0 + 0 + f_{xy}(a,b) = f_{xy}(a,b) \\ &\Rightarrow \boxed{\tfrac{\partial}{\partial xy}T_2(a,b) = f_{xy}(a,b)} \end{aligned}$$

Therefore $T_2(a,b) = f(a,b)$, and all of their respective first and second order partial derivatives, are equal at (a,b).

Let's use (3) to find the second order Taylor polynomials for some examples.

Example 1

$$p(x,y) = x^2 + xy + y^2$$

Calculating:

$$p_x = 2x + y \qquad p_y = x + 2y \qquad p_{xx} = 2 \qquad p_{yy} = 2 \qquad p_{xy} = p_{yx} = 1$$

Evaluating at $(0,0)$ we can find the second order Taylor polynomial of p at $(0,0)$:

$$\begin{aligned} T_2(x,y) &= p(0,0) + p_x(0,0)x + p_y(0,0)y + \frac{1}{2}p_{xx}(0,0)x^2 + \frac{1}{2}p_{yy}(0,0)y^2 + p_{xy}(0,0)xy \\ &= 0 + 0 + 0 + \frac{1}{2}(2)x^2 + \frac{1}{2}(2)y^2 + (1)xy \\ &= x^2 + y^2 + xy \end{aligned}$$

Notice that this is the same as the given polynomial. Does this depend on the point of expansion? Let's find out by expanding at (a,b).

$$\begin{aligned} T_2(x,y) &= p(a,b) + p_x(a,b)x + p_y(a,b)y + \frac{1}{2}p_{xx}(a,b)x^2 + \frac{1}{2}p_{yy}(a,b)y^2 + p_{xy}(a,b)xy \\ &= a^2 + ab + b^2 + (2a+b)(x-a) + (a+2b)(y-b) \\ &\quad \frac{1}{2}(2)(x-a)^2 + \frac{1}{2}(2)(y-b)^2 + (1)(x-a)(y-b) \\ &= a^2 + ab + b^2 + 2ax + bx - 2a^2 - ab + ay + 2by - ab - 2b^2 \\ &\quad + x^2 - 2ax + a^2 + y^2 - 2by + b^2 + xy - bx - ay + ab \\ &= x^2 + y^2 + xy \end{aligned}$$

As is the case for functions of one variable a second order Taylor polynomial of a degree two polynomial is exactly equal to that polynomial no matter the point of expansion.

Example 2

$$f(x,y) = \sin^2(x) + \sin(xy) + \sin^2(y) \tag{4}$$

Calculating:

$$\begin{array}{ll} f_x = 2\sin(x)\cos(x) + y\cos(xy) & f_y = x\cos(xy) + 2\sin(y)\cos(y) \\ f_{xx} = 2\cos^2(x) - 2\sin^2(x) - y^2\sin(xy) & f_{yy} = -x^2\sin(xy) + 2\cos^2(y) - 2\sin^2(y) \\ \multicolumn{2}{c}{f_{xy} = f_{yx} = \cos(xy) - xy\sin(xy)} \end{array} \tag{5}$$

Evaluating at $(0,0)$ we can find the second order Taylor polynomial of f at $(0,0)$:

$$\begin{aligned} T_2(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + \frac{1}{2}f_{yy}(0,0)y^2 + f_{xy}(0,0)xy \\ &= 0 + 0 + 0 + \frac{1}{2}(2)x^2 + \frac{1}{2}(2)y^2 + (1)xy \\ &= x^2 + y^2 + xy \end{aligned}$$

It turns out that this function has the same Taylor polynomial at $(0, 0)$ as Example 1. Figure 1 is a graph of this example with its Taylor polynomial.

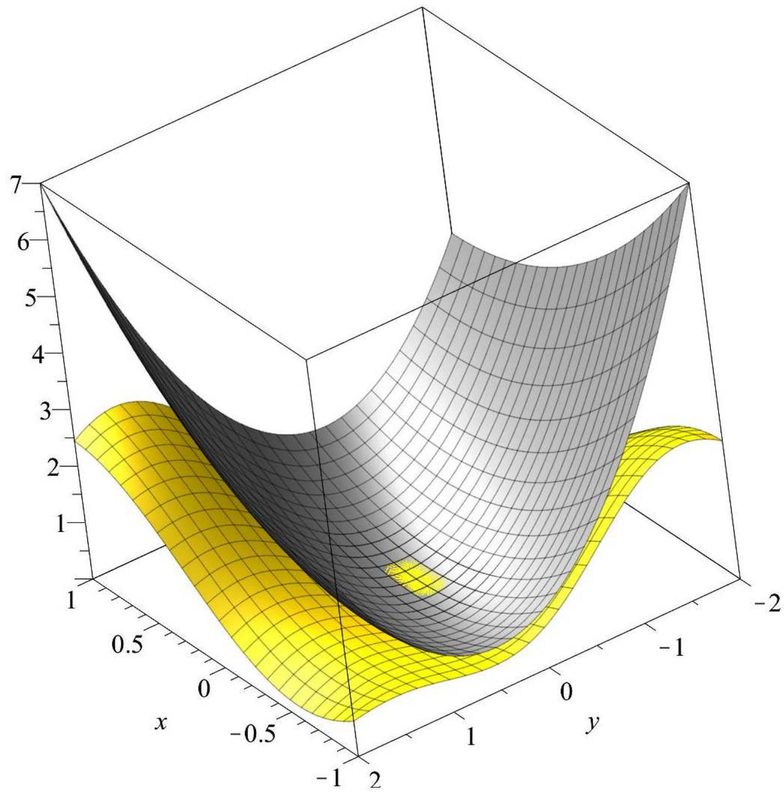

Figure 1: The graph of $z = f(x, y)$ in yellow and $z = T_2(x, y)$ in gray.[2]

Example 3 Let's repeat the calculation for the function used in Example 2, but at a different point: $(\frac{1}{2}, \pi)$. Using the calculations we did for Example 2, but evaluating instead at $(\frac{1}{2}, \pi)$:

$$\begin{aligned}
T_2(x, y) &= f(\tfrac{1}{2}, \pi) + f_x(\tfrac{1}{2}, \pi)(x - \tfrac{1}{2}) + f_y(\tfrac{1}{2}, \pi)(y - \pi) \\
&\quad + \frac{1}{2} f_{xx}(\tfrac{1}{2}, \pi)(x - \tfrac{1}{2})^2 + \frac{1}{2} f_{yy}(\tfrac{1}{2}, \pi)(y - \pi)^2 \\
&\quad + f_{xy}(\tfrac{1}{2}, \pi)(x - \tfrac{1}{2})(y - \pi) \\
&= \sin^2(\tfrac{1}{2}) + 1 + 2\sin(\tfrac{1}{2})\cos(\tfrac{1}{2})(x - \tfrac{1}{2}) + 0(y - \pi) \\
&\quad + \frac{1}{2}(2\cos^2(\tfrac{1}{2}) - 2\sin^2(\tfrac{1}{2}) - \pi^2)(x - \tfrac{1}{2})^2 + \frac{1}{2}(-\tfrac{1}{4} + 2)(y - \pi)^2 \\
&\quad + (-\tfrac{\pi}{2})(x - \tfrac{1}{2})(y - \pi) \\
&= \sin^2(\tfrac{1}{2}) + 1 + \sin(1)(x - \tfrac{1}{2}) + \left(\cos(1) - \tfrac{\pi^2}{2}\right)(x - \tfrac{1}{2})^2 \\
&\quad + \tfrac{7}{8}(y - \pi)^2 - \tfrac{\pi}{2}(x - \tfrac{1}{2})(y - \pi)
\end{aligned}$$

In the last step we used the identity called the double angle formulas for sine:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta),$$

[2]All figures were kindly provided by Prof. C. Koca.

and for cosine:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Note that unlike in Example 2, this polynomial has constant terms, so the Taylor polynomial changes depending on the point of expansion. This is a more typical case. The case of a degree two polynomial discussed as part of Example 1 is an exception to this rule.

Example 4

$$P(x, y) = (x - y)^3$$

Calculating:

$$P_x = 3(x-y)^2 \qquad P_y = -3(x-y)^2$$
$$P_{xx} = 6(x-y) \qquad P_{xy} = P_{yx} = -6(x-y) \qquad P_{yy} = 6(x-y)$$

Evaluating at $(0, 0)$ we can find the second order Taylor polynomial of p at $(0, 0)$:

$$\begin{aligned} T_2(x, y) &= P(0,0) + P_x(0,0)x + P_y(0,0)y \\ &\quad + \frac{1}{2}P_{xx}(0,0)x^2 + \frac{1}{2}P_{yy}(0,0)y^2 + P_{xy}(0,0)xy \\ &= 0 \end{aligned}$$

Notice that since all the terms are zero, $T_2 = T_1 = 0$, so the second order Taylor polynomial offers no correction to the first order Taylor polynomial. Neither is equal to the original function P except on the line $x = y$. Let's expand at $(-1, 0)$ to see a case where the first and second order Taylor polynomials are different.

$$\begin{aligned} T_2(x, y) &= P(-1,0) + P_x(-1,0)(x+1) + P_y(-1,0)y \\ &\quad + \frac{1}{2}P_{xx}(-1,0)(x+1)^2 + \frac{1}{2}P_{yy}(-1,0)y^2 + f_{xy}(-1,0)(x+1)y \\ &= -1 + 3(x+1) - 3y - 3(x+1)^2 - 3y^2 + 6(x+1)y \end{aligned}$$

We can get the first order Taylor polynomial by including just the terms up to order one:

$$T_1(x, y) = -1 + 3(x+1) - 3y$$

Example 5

$$q(x, y) = \frac{1}{1 + x - y} = (1 + x - y)^{-1}$$

Calculating:

$$q_x = -(1+x-y)^{-2} \qquad q_y = (1+x-y)^{-2}$$
$$q_{xx} = 2(1+x-y)^{-3} \qquad q_{yy} = 2(1+x-y)^{-3}$$
$$q_{xy} = -2(1+x-y)^{-3} = q_{yx} = -2(1+x-y)^{-3}$$

Evaluating at $(-1,1)$ we can find the second order Taylor polynomial of q at $(-1,1)$:

$$\begin{aligned} T_2(x,y) &= q(-1,1) + q_x(-1,1)(x+1) + q_y(-1,1)(y-1) \\ &\quad + \frac{1}{2}q_{xx}(-1,1)(x+1)^2 + \frac{1}{2}q_{yy}(-1,1)(y-1)^2 + q_{xy}(-1,1)(x+1)(y-1) \\ &= -1 - (x+1) + (y-1) - (x+1)^2 - (y-1)^2 + 2(x+1)(y-1) \end{aligned}$$

Now we will focus on the error in the estimates $f \approx T_j$ for $j = 1, 2$. Taylor's inequality for functions of one variable is something covered in Calculus II. Here we give only the special case for first and second order Taylor polynomials.[3] This theorem gives us an upper bound on the error in the approximations $f \approx T_1$ and $f \approx T_2$.

Theorem 1 (Taylor's Theorem in degree 1 and 2 for functions of 2 variables.)
If all second order partial derivatives of f are continuous on a disk of radius $r > 0$ centered at (a,b), and for some $M \geq 0$ all second order partial derivatives of f have values between M and $-M$ on that disk, then

$$|f(x,y) - T_1(x,y)| \leq 2M\|(x,y) - (a,b)\|^2 \tag{6}$$

where T_1 is the first order Taylor polynomial of f expanded at (a,b).

If all third order partial derivatives of f are continuous on a disk of radius $r > 0$ centered at (a,b), and for some $N \geq 0$ all third order partial derivatives of f have values between N and $-N$ on that disk, then

$$|f(x,y) - T_2(x,y)| \leq \frac{4}{3}N\|(x,y) - (a,b)\|^3 \tag{7}$$

where T_2 is the second order Taylor polynomial of f expanded at (a,b).

Let's apply Taylor's Theorem to estimate error in Taylor polynomial approximations by continuing to examine one of the examples we have already considered. We will use the methods that we studied to find absolute minima and absolute maxima to obtain values for M and N.

Example 6 Let's look at the error in the approximation by the first and second order Taylor polynomials we found in Example 1 centered at $(0,0)$. We have:

$$p(x,y) = x^2 + xy + y^2, \qquad T_1(x,y) = 0, \text{ and } T_2(x,y) = x^2 + y^2 + xy$$

To estimate the error in the approximation $p \approx T_1$ on the disk of radius $r > 0$ centered at $(0,0)$ using (6) we use the second derivatives of p found in Example 1:

$$p_{xx} = 2, \qquad p_{yy} = 2, \qquad p_{xy} = p_{yx} = 1$$

[3]The estimates given here are proved in J.E. Marsden and M.J. Hoffman *Elementary Classical Analysis*, 2nd Ed., W.H. Freeman, New York, 1993.

It's clear that these functions all take values between -2 and 2, so we can let $M = 2$ and we get

$$|p(x,y) - T_1(x,y)| \leq 2(2)\|(x,y) - (0,0)\|^2 = 4\|(x,y)\|^2$$

Next we'll examine the error in the approximation $p \approx T_2$, again on the disk of radius $r > 0$ centered at $(0,0)$. To do this we have to calculate the third order partial derivatives of p, which it turns out are all the same:

$$p_{xxx} = p_{yyy} = p_{xxy} = p_{yyx} = 0$$

It's easy to see that all the mixed partial derivatives are the same. Again finding the maximum value is simple, we can let $N = 0$. Applying (7) we get:

$$|p(x,y) - T_2(x,y)| \leq \frac{4}{3}(0)\|(x,y) - (0,0)\|^3 = 0$$

so there is no error and indeed we found that $p(x,y) = T_2(x,y)$ for all (x,y) no matter what the point of expansion is.

Example 7 Let's now consider the estimates $P \approx T_1$ and $P \approx T_2$ on the disk of radius $\frac{1}{\sqrt{2}}$ centered at $(-1,0)$, where

$$\begin{aligned} &P(x,y) = (x-y)^3, \qquad T_1(x,y) = -1 + 3(x+1) - 3y, \\ &T_2(x,y) = -1 + 3(x+1) - 3y - 3(x+1)^2 - 3y^2 + 6(x+1)y \end{aligned}$$

as in Example 4.

First, to use (6) we need to find the maximum of all the second order partial derivatives on the disk of radius $\frac{1}{\sqrt{2}}$ centered at $(-1,0)$. We calculated these in Example 4:

$$P_{xx} = -P_{xy} = -P_{yx} = P_{yy} = 6(x-y)$$

The function

$$m(x,y) = 6(x-y)$$

has gradient $\nabla m = (6,-6)$ which is never zero, so there are no critical points. This means that the maximum and minimum of m on the disk of radius $\frac{1}{\sqrt{2}}$ centered at $(-1,0)$ each occur on the boundary of that disk, in other words the circle whose points satisfy the equation

$$g(x,y) = (x+1)^2 + y^2 = \tfrac{1}{2}$$

We can use Lagrange multipliers to find the maximum and minimum under this constraint. There are two systems of equations to solve:

$$\left\{ \begin{array}{rcl} \nabla m(x,y) &=& \lambda \nabla g(x,y) \\ g(x,y) &=& \frac{1}{2} \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{rcl} \nabla g(x,y) &=& (0,0) \\ g(x,y) &=& \frac{1}{2} \end{array} \right.$$

Looking at the second system first,

$$\nabla g(x,y) = (2(x+1), 2y) = (0,0) \;\Rightarrow\; x = -1 \text{ and } y = 0 \text{ but } g(-1,0) = 0 \neq 1$$

so there are no possible maxima or minima from the second systems. Next we'll solve the first system of equations. Using $\nabla m = (6, -6)$,

$$\left\{ \begin{array}{rcc} 6 & = & 2\lambda(x+1) \\ -6 & = & 2\lambda y \end{array} \right. \quad \Rightarrow \quad -(x+1) = -\frac{6}{2\lambda} = y \tag{8}$$

Note that $\lambda = 0$ has no solution because $\pm 6 \neq 0$. If we substitute (8) into the constraint $g(x, y) = \frac{1}{2}$ we get

$$(x+1)^2 + (-(x+1))^2 = \tfrac{1}{2} \;\Rightarrow\; (x+1)^2 = \tfrac{1}{4} \;\Rightarrow\; x = -1 \pm \tfrac{1}{2}$$

Substituting these values of x into (8) gives two candidates for maximum/minimum points:

$$(-\tfrac{1}{2}, -\tfrac{1}{2}) \quad \text{and} \quad (-\tfrac{3}{2}, \tfrac{1}{2}) \tag{9}$$

Evaluating m at these two points gives:

$$\begin{array}{rcl} m(-\frac{1}{2}, -\frac{1}{2}) & = & 6(-\frac{1}{2} + \frac{1}{2}) = 0 \\ m(-\frac{3}{2}, \frac{1}{2}) & = & 6(-\frac{3}{2} - \frac{1}{2}) = -12 \end{array}$$

So m has a maximum of 0 and a minimum of -12 on the disk in question, which means we can take $M = 12$ and applying (6):

$$|P(x, y) - T_1(x, y)| \leq 24\|(x, y) - (-1, 0)\|^2 \tag{10}$$

Now we'll apply (7) to find the error in the estimate $P \approx T_2$. Calculating the third order partial derivatives of P we get:

$$P_{xxx} = -P_{xxy} = P_{yyx} = -P_{yyy} = 6$$

It's clear that these functions are all between -6 and 6, so we can take $N = 6$. Applying (7):

$$|P(x, y) - T_2(x, y)| \leq 8\|(x, y) - (-1, 0)\|^3 \tag{11}$$

If we compare (10) and (11) we can see that the error estimate for T_2 is less than a third of that of T_1.

Example 8 Here we continue with Example 5 where we expanded the function

$$q(x, y) = \frac{1}{1 + x - y} \;=\; (1 + x - y)^{-1}$$

at $(-1, 1)$ and found the second order Taylor polynomial to be:

$$T_2(x, y) \;=\; -1 - (x+1) + (y-1) - (x+1)^2 - (y-1)^2 + 2(x+1)(y-1)$$

and so the first order Taylor polynomial consists of the first order terms of T_2:

$$T_1(x, y) \;=\; -1 - (x+1) + (y-1)$$

Let's estimate the error in the approximation $q \approx T_1$ on the disk of radius $\frac{1}{\sqrt{32}}$ centered at $(-1, 1)$ using (6). To do this we must find the maximum and minimum values of all second order partial derivatives of q. These were found in Example 5 to be:

$$q_{xx} = q_{yy} = -q_{xy} = -q_{yx} = \frac{2}{(1 + x - y)^3}$$

Since all these derivatives are either the same or differ by a factor of -1, we only need to find the absolute maximum and absolute minimum of one function

$$m(x, y) = 2(1 + x - y)^{-3}$$

over the disk of radius $\frac{1}{\sqrt{32}}$ centered at $(-1, 1)$. To do this we seek all critical points inside the disk. Finding that there are none, we apply the Lagrange multiplier method on the boundary of the disk, which is given by all (x, y) such that

$$(x + 1)^2 + (y - 1)^2 = \tfrac{1}{32}$$

For the critical points of m we have $\nabla m = (0, 0)$, so

$$\begin{aligned} -6(1 + x - y)^{-4}(1) = 0 \quad &\text{and} \quad -6(1 + x - y)^{-4}(-1) = 0 \\ \Rightarrow \ \frac{-6}{(1 + x - y)^4} = 0 \quad &\text{and} \quad \frac{6}{(1 + x - y)^4} = 0 \end{aligned}$$

which has no solution, so m has no critical points inside the disk. Now we apply the Lagrange multiplier method to find the maximum and minimum of $m(x, y)$ with the constraint $g(x, y) = (x + 1)^2 + (y - 1)^2 = \frac{1}{32}$. So we solve the two systems:

$$\left\{ \begin{array}{rcl} \nabla m(x, y) & = & \lambda \nabla g(x, y) \\ g(x, y) & = & \frac{1}{32} \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{rcl} \nabla g(x, y) & = & (0, 0) \\ g(x, y) & = & \frac{1}{32} \end{array} \right.$$

The system of equations on the right has no solutions because

$$\nabla g(x, y) = (0, 0) \ \Rightarrow \ (2(x + 1), 2(y - 1)) = (0, 0) \ \Rightarrow \ x = -1, y = 1$$

This point does not satisfy the constraint because $0^2 + 0^2 = 0 \neq \frac{1}{32}$, so there are no solutions from the second system.

To solve the first system, we calculate $\nabla m = (-6(1+x-y)^{-4}, 6(1+x-y)^{-4})$, and so

$$\left\{ \begin{array}{rcl} -6(1 + x - y)^{-4} & = & 2\lambda(x + 1) \\ 6(1 + x - y)^{-4} & = & 2\lambda(y - 1) \end{array} \right. \quad \Rightarrow \quad -(x + 1) = \frac{6}{2\lambda(1 + x - y)^4} = y - 1 \qquad (12)$$

Note that $\lambda = 0$ has no solution because $\pm 6(1 + x - y)^{-4}$ is never zero. Plugging $-(x + 1) = y - 1$ into the constraint $g(x, y) = \frac{1}{32}$ gives:

$$(x + 1)^2 + (-(x + 1))^2 = \tfrac{1}{32} \quad \Rightarrow \quad (x + 1)^2 = \tfrac{1}{64} \quad \Rightarrow \quad x = -1 \pm \tfrac{1}{\sqrt{64}} = \tfrac{-8 \pm 1}{8}$$

From (12) we get that $y = -x$, so there are two points that are possible maxima/minima:

$$(-\tfrac{7}{8}, \tfrac{7}{8}) \quad \text{and} \quad (-\tfrac{9}{8}, \tfrac{9}{8}) \qquad (13)$$

Therefore, the absolute minimum and absolute maximum of m on the disk of radius $\frac{1}{\sqrt{32}}$ centered at $(-1,1)$ are:

$$\begin{aligned} m(-\tfrac{7}{8},\tfrac{7}{8}) &= 2\left(1-\tfrac{7}{8}-\tfrac{7}{8}\right)^{-3} &= 2\left(-\tfrac{3}{4}\right)^{-3} &= -\tfrac{2^7}{3^3} \\ m(-\tfrac{9}{8},\tfrac{9}{8}) &= 2\left(1-\tfrac{9}{8}-\tfrac{9}{8}\right)^{-3} &= 2\left(-\tfrac{5}{4}\right)^{-3} &= -\tfrac{2^7}{5^3} \end{aligned}$$

These are both negative, so the most negative, the minimum, is what we can take for $-M$, which means that $M = \frac{2^7}{3^3}$. Applying (6) we have that if (x,y) is in the disc of radius $\frac{1}{\sqrt{32}}$ centered at $(-1,1)$, then

$$|q(x,y) - T_1(x,y)| \leq \frac{2^8}{3^3}\|(x,y)-(-1,1)\|^2 \tag{14}$$

Now let's use (7) to estimate the error in the approximation $q \approx T_2$ on the same disk. First calculating the third order partial derivatives of q gives us:

$$q_{xxy} = q_{yyy} = -q_{xxx} = -q_{yyx} = \frac{6}{(1+x-y)^4}$$

It's easy to see that the order of the mixed partial derivatives doesn't change the result, so $q_{xxy} = q_{xyx} = q_{yxx}$, and $q_{yyx} = q_{yxy} = q_{xyy}$. As with the first order case we only have to find the extrema of a single function, which cuts down on our work considerably! (Note that we can't expect this to be the case for most problems.) So we are now looking for the absolute maximum and absolute minimum of the function

$$n(x,y) = 6(1+x-y)^{-4}$$

on that same disk. Using a quick calculation similar to the first order case, it's simple to see that there are no critical points of n in this disk. We are left to use the Lagrange multiplier with the same constraint as before:

$$g(x,y) = (x+1)^2 + (y-1)^2 = \tfrac{1}{8}.$$

Again there is no solution to $\nabla g = (0,0)$ that satisfies the constraint, so we are left with solving $\nabla n = \lambda \nabla g$ with $g(x,y) = \frac{1}{8}$. This gives the system of equations

$$\left\{ \begin{array}{rcl} -24(1+x-y)^{-5} &=& 2\lambda(x+1) \\ 24(1+x-y)^{-5} &=& 2\lambda(y-1) \end{array} \right. \quad \Rightarrow \quad -(x+1) = \frac{24}{2\lambda(1+x-y)^5} = y-1 \tag{15}$$

This is the same result as (12), and the constraint is the same, so the critical points are also the same, given by (13). Using similar simplifications as before we get that the absolute maximum and absolute minimum of n on the disk of radius $\frac{1}{\sqrt{32}}$ centered at $(-1,1)$ are:

$$\begin{aligned} n(-\tfrac{7}{8},\tfrac{7}{8}) &= 6\left(1-\tfrac{7}{8}-\tfrac{7}{8}\right)^{-4} &= 6\left(-\tfrac{3}{4}\right)^{-4} &= \tfrac{2^9}{3^3} \\ n(-\tfrac{9}{8},\tfrac{9}{8}) &= 6\left(1-\tfrac{9}{8}-\tfrac{9}{8}\right)^{-4} &= 6\left(-\tfrac{5}{4}\right)^{-4} &= \tfrac{3\cdot 2^9}{5^4} \end{aligned}$$

These are both positive and the first is larger, so we can take $N = \frac{2^9}{3^3}$. Applying (7) gives:

$$|q(x,y) - T_2(x,y)| \leq \frac{2^{11}}{3^4} \|(x,y) - (-1,1)\|^3 \tag{16}$$

Now since we know that $\|(x,y) - (-1,1)\| < \frac{1}{\sqrt{32}}$ for (x,y) in the disk of radius $\frac{1}{\sqrt{32}}$ centered at $(-1,1)$, equation (16) gives that,

$$|q(x,y) - T_2(x,y)| \leq \frac{2^9}{3^3}\|(x-y) - (-1,1)\|^2 \cdot \frac{1}{\sqrt{32}} = \frac{2^7}{3^3\sqrt{2}}\|(x-y) - (-1,1)\|^2.$$

Comparing this to (14) we see that the error estimate error in the approximation using the second order Taylor polynomial is less than half that of the first order Taylor polynomial on that whole disk. The advantage of T_2 over T_1 will only improve as (x,y) gets closer to $(-1,1)$.

Exercises

1. Find the second degree Taylor polynomial for the given function expanded at the given point.

   (a) $p(x,y) = 4(x+y)^2 + 3x - 2y + 1$ expanded at $(0,0)$.

   (b) $p(x,y) = 4(x+y)^2 + 3x - 2y + 1$ expanded at $(-1,1)$.

   (c) $q(x,y) = \dfrac{1}{1+x-y}$ expanded at $(0,2)$.

   (d) $f(x,y) = x^y$ expanded at $(1,0)$.

   (e) $f(x,y) = x^y$ expanded at $(e,2)$.

   (f) $g(r,\theta) = (r+\theta)\sin(\frac{\theta}{r})$ expanded at $(6,\pi)$.

   (g) $h(x,y) = \arctan\left(\frac{y}{x}\right)$ expanded at $(1,1)$.

   (h) $k(x,y) = (e^x + e^{-x})\cos(y)$ expanded at $(0,\pi)$.

2. For each of the functions in Exercise 1 use the Taylor polynomial you found to estimate that function at the given point (a,b). Also find the estimate using the first degree Taylor polynomial $T_1(a,b)$ a.k.a. the linearization. Calculate the exact value of the function at (a,b) and compare the error in $T_2(a,b)$ and $T_1(a,b)$.

   (a) Compare the error in $T_2(0.1,0.1)$, and $T_1(0.1,0.1)$, as estimates for $p(0.1,0.1)$, where $p(x,y) = 4(x+y)^2 + 3x - 2y + 1$, and the Taylor polynomials are expanded at $(0,0)$.

   (b) Compare the error in $T_2(0.1,0.1)$, and $T_1(0.1,0.1)$, as estimates for $p(0.1,0.1)$, where $p(x,y) = 4(x+y)^2 + 3x - 2y + 1$, and the Taylor polynomials are expanded at $(-1,1)$.

(c) Compare the error in $T_2(0.1, 1.8)$, and $T_1(0.1, 1.8)$, as estimates for $q(0.1, 1.8)$, where $q(x, y) = \dfrac{1}{1+x-y}$, and the Taylor polynomials are expanded at $(0, 2)$.

(d) Compare the error in $T_2(0.9, 0.1)$, and $T_1(0.9, 0.1)$, where $f(x, y) = x^y$, and the Taylor polynomials are expanded at $(1, 0)$.

(e) Compare the error in $T_2(3, 2.1)$, and $T_1(3, 2.1)$, where $f(x, y) = x^y$, and the Taylor polynomials are expanded at $(e, 2)$.

(f) Compare the error in $T_2(5.5, 3)$, and $T_1(5.5, 3)$, where $g(r, \theta) = (r + \theta)\sin(\frac{\theta}{r})$ is expanded at $(6, \pi)$.

(g) Compare the error in $T_2(1.2, 0.8)$, and $T_1(1.2, 0.8)$, where $h(x, y) = \arctan\left(\frac{y}{x}\right)$, is expanded at $(1, 1)$.

(h) Compare the error in $T_2(-0.1, 3)$, and $T_1(-0.1, 3)$, where

$$k(x, y) = \left(e^x + e^{-x}\right)\cos(y)$$

is expanded at $(0, \pi)$.

3. Consider the function,

$$p(x, y) = (2x + y + 1)^3$$

Find T_1 and T_2 centered at $(0, 0)$. Using Example 7 as a guide, compare the error in the estimates $p \approx T_1$ to $p \approx T_2$ on the disk of radius $\frac{1}{\sqrt{6}}$ centered at $(0, 0)$.

4. Using Example 8 as a guide, compare the error in the estimates $q \approx T_1$ to $q \approx T_2$ on the disk of radius $\frac{1}{\sqrt{50}}$ centered at $(0, 2)$. Here T_1 and T_2 are as in exercise 1(c).