College Algebra and Trigonometry

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Preface

This textbook is based on the course "College Algebra and Trigonometry" taught at New York City College of Technology, CUNY. It is designed to prepare students for the precalculus class at the next level.

The book is meant to be concise, while at the same time including all the material taught at the present time (as of Fall 2018) by mathematics department of NYCCT. Special attention is paid to present the material in a motivated and intuitively clear manner.

The book is divided into three parts and 25 sessions based on the total of 30 sessions in one semester, leaving remaining 5 sessions for 3 in-class exams, one review for final exam, and one final exam.

Each session ends with exercises. Most exercises are designed in pairs with consecutive odd and even numbers in such a way that exercises are similar in each pair. A recommended approach is for instructions to work out the examples in the lecture part of the session during class, then allow students to work on even exercises during class and assign the odd exercises for homework (or vice versa). The textbook also contains answers to almost all exercises.

Some sessions contain challenge problems. In sessions 8, 14 and 23, challenge problems are given in parametric form so they also can be used by instructors to generate additional specific exercises and problems for quizzes and exams.

I would like to thank my brother, Dr. Leonid Rozenblum, for numerous corrections and recommendations.

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Let's recall the well-known notation of multiplication. Everybody knows that $3 \times 4 = 12$. But what does exactly multiplication mean? Why the result is 12? We've got this result by **adding** number 3 to itself 4 times:

$$3 \times 4 = 3 + 3 + 3 + 3 = 12.$$

Multiplication means repetition with **addition**. It allows to write the summation of a number with itself in a short, compact form.

There are cases when we need repetition with **multiplication**. In other words, we want to **multiply** a number by itself several times. For example, consider the product $3 \times 3 \times 3 \times 3$. It would be a good idea to invent a special notation, similar to the notation for multiplication, that allows to write such a product in a short form, using number 3 only one time (which tells us the number we want to multiply by itself) and number 4 (which tells us how many time to multiply). We cannot use the notation 3×4 because it is already taken for multiplication to express repetition with summation. The following notation was invented to express repetition with multiplication: 3^4. This expression is called the **exponent**. So, by definition

$$3^4 = 3 \times 3 \times 3 \times 3.$$

Note. In some computer languages and calculators, to keep both numbers on one line, the notation 3^4 is used.

In similar way we can define exponent in general form.

Definition. For arbitrary number a and arbitrary positive integer n the exponent a^n is defined by the formula:

$$a^n = a \times a \times ... \times a \text{ (multiply } n \text{ times)}$$

Number a is called the **base**, and n is the **power** of exponent. We can say that we raise a to the power n. In particular, $a^1 = a$ (we "repeat" number a one time). Also, $1^n = 1$ for any n. For two special cases, when power $n = 2$ and $n = 3$, we also say that a^2 is a-square, and a^3 is a-cube. The reason for that is a^2 represents the area of a square, and a^3 represents the volume of a cube with sides a.

Note. Some textbooks define exponent as power n. We define exponent as a^n.

The notation of exponents is useful in many situations, in particular, when we work with very big numbers (for example, with distances between planets). Bellow we show a way in which exponents can also be used for very small numbers (such as, for example, distances inside molecules or atoms).

Let's consider examples, and study some properties of exponents.

Example 1.1. Some people believe that one kilobyte (KB) of computer memory is equal to 1000 bytes (B). However, 1 KB = 1024 B, not 1000 B. The reason is that the number 1024 is a power of 2 but the number 1000 is not. Express number 1024 in exponential form with the base of 2.

Solution. Let's divide 1024 by 2 several times: $1024 \div 2 = 512$, $512 \div 2 = 256$, and so on. We will get that 1024 is the product of 10 copies of 2: $1024 = 2 \times 2 \times ... \times 2$ (multiply 10 times). Therefore, $1024 = 2^{10}$.

Example 1.2. It is known that the distance from our Earth to the Sun is about 150,000,000 km (150 million kilometers). Represent this distance in a short form using exponents.

Solution. We can write this number in the form $150,000,000 = 1.5 \times 100,000,000$. Number 100,000,000 contains 8 zeros and can be written as $100,000,000 = 10^8$.

Therefore, $150,000,000 = 1.5 \times 10^8$.

Note. Representation of big numbers such as in the example 1.2 in exponential form with the base of 10 is widely used in science. This form is called scientific notation.

In general, we say that positive number n is in **scientific notation**, if it is written as product of two parts:
1) Number between 1 and 10.
2) Power of 10.

Example 1.3. Consider three numbers: 15.3×10^8, 0.15×10^6, and 2.73×10^4. Are these numbers in scientific notation?

Solution. It looks like all three numbers are in scientific notation. However, it is not true. The first number 15.3×10^8 is not in scientific notation, because its first part, number 15.3, is greater than 10, so it is outside the range from 1 to 10. The second number 0.15×10^6 is also not in scientific notation, because its first part $0.15 < 1$, so it is also not inside interval from 1 to 10. The third number 2.73×10^4 is in scientific notation: its first part 2.73 is between 1 and 10.

Now, how about very small numbers? Consider, for example, the diameter of DNA helix. It is known that this diameter is about 0.0000002 cm. Is it possible somehow to represent this number also in a short form using exponents? The answer is yes. We will solve this problem in example 1.5 below. To come up with the idea how to do this we need to learn more about exponents. Let's start with some basic properties.

Basic Properties of Exponents

We will not give proofs here since proofs are very simple and follow directly from the definition of exponents (if you wish you can try to proof yourself).

Product Rule. For any number a, and any positive integers n and m,

$$a^n \times a^m = a^{n+m}$$

Notice that all exponents in this formula have the same base a. This restriction is very important. If, for example, you need to multiply $3^4 \times 2^5$, there is no simple rule to represent the answer as exponent. Also notice how product rule works: to multiply exponents with the same base, we **add** powers. A common mistake here is to multiply powers instead of adding them.

Another rule is how to raise exponents into a power.

Power Rule. For any number a and any positive integers n and m,

$$\left(a^n\right)^m = a^{n \cdot m}$$

This time, contrary to product rule, we **multiply** powers.

Quotient Rule. For any nonzero number a, and any positive integers n and m, such that $n > m$,

$$\frac{a^n}{a^m} = a^{n-m}$$

So, to **divide** exponents, we **subtract** powers (we **do not** divide them).

Notice that in the above formula power of numerator is greater than power of denominator. But what if we need to divide exponents when the power of numerator is less than the power of denominator: $n < m$? One possible way is just to reduce this fraction by dividing numerator and denominator by a^n, and we get

$$\frac{a^n}{a^m} = \frac{1}{a^{m-n}}$$

Power of Product Rule. For any two numbers a and b, and any positive integer n,

$$\left(ab\right)^n = a^n b^n$$

Power of Quotient Rule. For any number a, any nonzero b, and any positive integer n,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

As you can see, according to power of product and quotient rules, we can raise a and b to the power n separately.

Exponents with Negative Integer Powers

Let's reconsider Quotient Rule when the power n of numerator is less than the power m of denominator: $\frac{a^n}{a^m} = \frac{1}{a^{m-n}}$, $n < m$. It would be a good idea to somehow write it in the same exponential form as for the case when $n > m$: $\frac{a^n}{a^m} = a^{n-m}$. In doing this, we come up to the exponents with negative powers! For example, we can write that

$$\frac{a^2}{a^6} = \frac{1}{a^4} = a^{2-6} = a^{-4}, \text{ or } a^{-4} = \frac{1}{a^4}.$$

You may say that negative exponent (exponent with negative power) does not make any sense. Indeed, by the initial definition of exponent, its power tells us how many times to multiply the base by itself. How can we multiply anything "negative number of times"? Of course, we cannot. However, there is a way to give a sense to the above formula with negative power.

The idea is to replace the initial definition (which does not make sense for negative powers) with the definition based on the above expression for a^{-4}.

Definition. For any nonzero number a, and for any positive integer n, we define a^{-n} as

$$a^{-n} = \frac{1}{a^n}$$

So, exponents with negative powers are **reciprocals** to exponents with positive powers. In particular,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

Now, let's try to invent the definition of a^0 (exponent with zero power).

We can use similar approach as for negative exponents, using Quotient Rule for $m = n$.

We will have $\frac{a^n}{a^n} = a^{n-n} = a^0$. Because $\frac{a^n}{a^n} = 1$, we get $a^0 = 1$. We come up with the

Definition. For any nonzero number a, $a^0 = 1$.

Example 1.4. Calculate
$$\text{a) } 10^0, \quad \text{b) } 10^{-1}, \quad \text{c) } 10^{-2}, \quad \text{d) } 10^{-n}.$$

Solution. By definition, we have

$$\text{a) } 10^0 = 1, \qquad \text{b) } 10^{-1} = \frac{1}{10} = 0.1, \qquad \text{c) } 10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01,$$

$$\text{d) } 10^{-n} = \frac{1}{10^n} = 0.0...01 \quad (n-1 \text{ zeros after decimal point}).$$

Example 1.5. The diameter of DNA helix is about 0.0000002 cm. Represent this number in exponential form.

Solution. $0.0000002 = 2 \times 0.0000001 = 2 \times 10^{-7}$.

This example shows that exponents with negative powers are useful for representation small numbers in a compact form. The above representation, as for big numbers, is also called scientific notation.

In conclusion of this session, consider several examples. It can be shown that all the above properties of exponents with positive powers are also true for negative powers. In all problems below, it is required to simplify given expression and write the answer using positive exponents (meaning positive powers) only.

Example 1.6. $\dfrac{a^n b^{-p}}{a^{-m} b^q}$.

Solution. Technically, we can get rid of negative exponent a^{-m} in denominator, and b^{-p} in numerator, by moving them into the opposite part of the fraction: move a^{-m} up to the numerator and move b^{-p} down to the denominator. Then apply product rule. We will get

$$\frac{a^n b^{-p}}{a^{-m} b^q} = \frac{a^m a^n}{b^p b^q} = \frac{a^{m+n}}{b^{p+q}} .$$

Example 1.7. $\left(ax^{-5} y^3\right)\left(bxy^{-1}\right)$.

Solution. Numbers a and b are not powers, they are coefficients, so we simply multiply them (not add). For exponents, we use product rule (note that x can be written as x^1):

$$\left(ax^{-5} y^3\right)\left(bxy^{-1}\right) = abx^{-5+1} y^{3-1} = abx^{-4} y^2 .$$

Now, to get rid of negative exponent x^{-4}, similar to Example 1.6, move x^{-4} down:

$$abx^{-4} y^2 = \frac{aby^2}{x^4} .$$

Example 1.8. $\left(\dfrac{45u^{-3} v^8}{18u^{-6} v^{-4}}\right)^{-2}$.

Solution. It is possible to simplify this expression in different ways. As a first step, let's get rid of negative power -2, by taking reciprocal of given fraction:

$$\left(\frac{45u^{-3} v^8}{18u^{-6} v^{-4}}\right)^{-2} = \left(\frac{18u^{-6} v^{-4}}{45u^{-3} v^8}\right)^2 .$$

Next, we simplify fraction inside parentheses by reducing coefficients 18 and 45 by 9, and moving both exponents u^{-6} and v^{-4} down. Then we use product rule:

$$\left(\frac{18u^{-6} v^{-4}}{45u^{-3} v^8}\right)^2 = \left(\frac{2}{5u^{-3} u^6 v^8 v^4}\right)^2 = \left(\frac{2}{5u^{-3+6} v^{8+4}}\right)^2 = \left(\frac{2}{5u^3 v^{12}}\right)^2 .$$

Finally, we use power of quotient and power rules:

$$\left(\frac{2}{5u^3 v^{12}}\right)^2 = \frac{2^2}{5^2 \left(u^3\right)^2 \left(v^{12}\right)^2} = \frac{4}{25u^{3\cdot2} v^{12\cdot2}} = \frac{4}{25u^6 v^{24}} .$$

Exercises 1

In exercises 1.1 – 1.4, write given numbers in scientific notation.

1.1. The Earth's circumference at the equator is approximately 25,000 mi.

1.2. Mount Everest (on the border of Nepal and China) is the highest place on Earth above sea level, at about 29,000 ft.

1.3. A tiny space inside a computer chip has been measured to be 0.000256 cm wide.

1.4. A tiny space inside a computer chip has been measured to be 0.000014 cm long.

In exercises 1.5 – 1.8, write given numbers as ordinary numbers.

1.5. The length of a bacterium is about 4×10^{-5} in.

1.6. The weight of a flea is about 8.75×10^{-2} g.

1.7. The diameter of the Moon is 3.475×10^{6} m.

1.8. The speed of sound in dry air at is 1.236×10^{3} km/h.

In Exercises 1.9 and 1.10, numbers are **not** written in scientific notation (why?). Write these numbers in scientific notation.

1.9. a) 34.7×10^{-4}

   b) 0.25×10^{3}

1.10 a) 43.8×10^{-6}

   b) 0.36×10^{5}

In Exercises 1.11 and 1.12, calculate and write the answer without exponents.

1.11. a) $2\cdot3^{0}$

   b) $\left(2\cdot3\right)^{0}$

   c) 4^{-2}

   d) -4^{-2}

   e) $\left(-4\right)^{-2}$

1.12. a) $3\cdot4^{0}$

   b) $\left(3\cdot4\right)^{0}$

   c) 5^{-2}

   d) -5^{-2}

   e) $\left(-5\right)^{-2}$

In Exercises 1.13 – 1.22, simplify and write the answer using **positive** powers only (assume that all letters represent positive numbers).

1.13. a) $a^{5}a^{-7}$

   b) $\left(a^{3}\right)^{-2}$

1.14. a) $c^{-6}c^{3}$

   b) $\left(d^{-4}\right)^{3}$

1.15. a) $\dfrac{p^{-4}}{p^8}$

b) $\dfrac{p^4}{p^{-8}}$

c) $\dfrac{p^{-4}}{p^{-8}}$

1.16. a) $\dfrac{r^9}{r^{-3}}$

b) $\dfrac{r^{-9}}{r^{-3}}$

c) $\dfrac{r^{-9}}{r^3}$

1.17. $\dfrac{x^{-a}y^b}{x^c y^{-d}}$

1.18. $\dfrac{p^w q^{-x}}{p^{-y} q^z}$

1.19. $\left(ma^4 b^{-2}\right)\left(na^{-5} b^3\right)$

1.20. $\left(xu^{-5} v^3\right)\left(yu^7 v^{-6}\right)$

1.21. $\left(-\dfrac{35 r^6 s^{-12}}{42 r^{-2} s^3}\right)^{-2}$

1.22. $\left(-\dfrac{12 x^{-15} y^9}{30 x^5 y^{-3}}\right)^{-3}$

Session 2

Rational Expressions and Complex Fractions

Recall that a **rational number** is a number that can be written as a fraction (having a **numerator** on the top and a **denominator** on the bottom). Usually, we write a fraction in the form $\dfrac{m}{n}$, where m and n are two integers (m is the numerator, and n is the denominator). We treat a fraction as a ratio of its numerator to denominator, so, we can write $\dfrac{m}{n} = m \div n$. We will always assume that denominator n is not equal to zero.

We consider here **rational expressions**. These are also fractions. However, their numerators and denominators are not necessary numbers only. They are expressions that are called **polynomials**. Non-formally speaking, a polynomial is an expression (or a function) that can be written in the form that contains a variable, say x, together with the operations of addition, subtraction, and multiplication of x by numbers and by itself. When x is multiplied by itself, it usually is written as exponent. Here are examples of polynomials:

$$\frac{1}{2}x - 3, \ 5x^3 - 3x + 2, \ 3x^2 - 7x + 4.$$

The last polynomial is called the **quadratic trinomial**. The expression $\dfrac{3x^2 + x}{x}$ is also a polynomial because after reducing by x it can be written as $3x + 1$. On the contrary, $3 + \dfrac{1}{x}$ is not a polynomial because it contains a variable x in denominator (so, division by variable) and can not be reduced to a polynomial.

Definition. A rational expression is a ratio of two polynomials (or a fraction whose numerator and denominator are polynomials). Here are several examples of rational expressions:

$$\frac{5x^2 - 3x + 2}{2x - 1}, \ \frac{1}{x}, \ -\frac{3}{x^2 - 1}, \ \frac{3x + 2}{x^3 - 5x^2}.$$

Below we consider examples on how to add and subtract rational expressions. Mostly, these operations can be done in a manner, similar to rational numbers (ordinary fractions). Where possible, we will point out the similarity between rational numbers and rational expressions.

Simplification of Rational Expressions

When adding or subtracting, we will also simplify (if possible) resulting expressions. Basic technique to simplify is to factor numerator and denominator, and reduce (cancel out) a common factor. Let's see some examples on simplification.

Example 2.1. Simplify $\dfrac{10x^2-15x}{20x}$.

Solution. Possible mistake here is to reduce x from $15x$ and $20x$ and not from $10x^2$, and as a result to get a wrong answer $(10x^2-15)/20$. As we mentioned above, the correct way is to factor numerator before reducing. We can factor $5x$ from the numerator and then reduce:

$$\frac{10x^2-15x}{20x}=\frac{5x(2x-3)}{20x}=\frac{2x-3}{4}.$$

Example 2.2. Simplify $\dfrac{16x^2+12x}{6x-10}$.

Solution. Let's factor both numerator and denominator and then reduce by 2:

$$\frac{16x^2+12x}{6x-10}=\frac{4x(4x+3)}{2(3x-5)}=\frac{2x(4x+3)}{3x-5}.$$

Example 2.3. Simplify $\dfrac{5x^2+30x+40}{3x^2-3x-18}$.

Solution. As in the previous problem, we start with factoring numerator and denominator. We can factor them in two steps. First, factor 5 from numerator and 3 from denominator:

$$\frac{5x^2+30x+40}{3x^2-3x-18}=\frac{5(x^2+6x+8)}{3(x^2-x-6)}.$$

Second step is to factor quadratic trinomials in parentheses. Latter, in session 8 on quadratic equations, we will consider factoring quadratic trinomials in more details. For now, if you have difficulties to factor, the following method can be used. We may assume that both quadratic trinomials inside parentheses have the same factor. In this case this factor exists in the difference of the two trinomials. The difference is

$$x^2+6x+8-(x^2-x-6)=x^2+6x+8-x^2+x+6=7x+14=7(x+2).$$

From here we can guess that $x+2$ is (probably) the common factor of numerator and denominator. It is easy to check that this is really the case:

$$x^2+6x+8x=(x+2)(x+4)\ \text{ and }\ x^2-x-6=(x+2)(x-3).$$

We can complete the factorization and then reduce $x+2$:

$$\frac{5(x^2+6x+8)}{3(x^2-x-6)}=\frac{5(x+2)(x+4)}{3(x+2)(x-3)}=\frac{5(x+4)}{3(x-3)}.$$

Adding and Subtraction of Rational Expressions

Example 2.4. Add $\dfrac{5x}{2x-1}+\dfrac{3}{2x-1}$.

Solution. Recall that it is very easy to add (or subtract) numerical fractions if they have the same denominator: just add (or subtract) numerators, and keep (do not add or subtract) their common denominator. For example, $\dfrac{2}{7}+\dfrac{3}{7}=\dfrac{5}{7}$, $\dfrac{5}{9}-\dfrac{7}{9}=-\dfrac{2}{9}$.

The same rule applies to the rational expressions:

$$\frac{5x}{2x-1}+\frac{3}{2x-1}=\frac{5x+3}{2x-1}.$$

Example 2.5. Subtract $\dfrac{3a+4}{8}-\dfrac{a-2}{6}$.

Solution. This time the denominators are different. Recall how we would subtract rational numbers (fractions), let's say $\dfrac{7}{8}-\dfrac{5}{6}$.

To subtract, we replace these fractions with equivalent ones having the same denominator, which is called **LCD** (Least Common Denominator). LCD is the smallest number that is divisible by both denominators. Technically, we can subtract fractions in three steps.

1) Find LCD. For given denominators 8 and 6, LCD = 24. We put LCD into the denominator of the resulting fraction.

2) Find **complements** of each denominator to LCD. A complement is the number such that if we multiply it by denominator, we get LCD. To find complements, just divide LCD by each denominator. For the denominator 8, the complement is 3 ($24\div 8=3$), and for the denominator 6, the complement is 4 ($24\div 6=4$).

3) Calculate the numerator of the resulting fraction: multiply numerator of each fraction by complement to its denominator and subtract the results. For given fractions 7/8 and 5/6, multiply numerator 7 by 3 (complement to denominator 8), and numerator 5 by 4 (complement to denominator 6):

$$\frac{7}{8}-\frac{5}{6}=\frac{7\cdot 3-5\cdot 4}{24}=\frac{21-20}{24}=\frac{1}{24}.$$

To subtract rational expressions, we do the same thing:

$$\frac{3a+4}{8}-\frac{a-2}{6}=\frac{(3a+4)3-(a-2)4}{24}=\frac{9a+12-4a+8}{24}=\frac{5a+20}{24}.$$

Now consider an example when denominators are also different and contain variables.

Example 2.6. Add $\dfrac{3}{10x} + \dfrac{4}{15y}$.

Solution. As before, we construct LCD first. Denominators contain both numbers and letters (variables). For numbers 10 and 15, the numerical part of LCD is 30. Letters x and y do not have common factors. Therefore, the letter part of LCD is their product xy. The entire LCD is the product of numerical and letter parts:

$$LCD = 30xy.$$

Next, we find complements of each denominator to LCD by dividing LCD by denominators.

For the denominator $10x$, the complement is $30xy/10x = 3y$.

For the denominator $15y$, the complement is $30xy/15y = 2x$.

Finally, we add given fractions:

$$\frac{3}{10x} + \frac{4}{15y} = \frac{3 \cdot 3y + 4 \cdot 2x}{30xy} = \frac{9y + 8x}{30xy}.$$

Example 2.7. Combine $\dfrac{5}{a^2} - \dfrac{11}{6a} + \dfrac{9}{14}$.

Solution. To construct LCD, similar to previous example, we construct separately its numerical and letter parts.

For numbers 6 and 14, the numerical part of LCD is 42.

For letters a^2 and a, the letter part of LCD is a^2 .

The entire LCD is the product of both parts: LCD = $42a^2$.

Next, we find complements for each denominator to LCD:

For a^2, the complement is $42a^2 / a^2 = 42$.

For $6a$, the complement is $42a^2 / 6a = 7a$.

For 14, the complement is $42a^2 / 14 = 3a^2$.

From here,

$$\frac{5}{a^2} - \frac{11}{6a} + \frac{9}{14} = \frac{5 \cdot 42 - 11 \cdot 7a + 9 \cdot 3a^2}{42a^2} = \frac{210 - 77a + 27a^2}{42a^2}.$$

Note. In general, if denominators contain exponents with the same base and different powers, put into LCD exponent with the **biggest** power.

Example 2.8. Subtract $\dfrac{4}{5x-3} - \dfrac{2}{3x-5}$.

Solution. Denominators $5x-3$ and $3x-5$ do not have common factors, therefore, LCD is simply their product:

$$LCD = (5x-3)(3x-5).$$

The denominators $5x-3$ and $3x-5$ are complement to each other, therefore

$$\frac{4}{5x-3}-\frac{2}{3x-5}=\frac{4(3x-5)-2(5x-3)}{(5x-3)(3x-5)}=\frac{12x-20-10x+6}{(5x-3)(3x-5)}=\frac{2x-14}{(5x-3)(3x-5)}.$$

Example 2.9. Subtract $\dfrac{y-5}{y-6}-\dfrac{y+5}{6-y}$.

Solution. Notice that denominators of these fractions "almost" the same. We can make them exactly the same by using the following connection between expressions $a-b$ and $b-a$:

$$a-b=-(b-a).$$

Therefore, $6-y=-(y-6)$ and

$$\frac{y-5}{y-6}-\frac{y+5}{6-y}=\frac{y-5}{y-6}-\frac{y+5}{-(y-6)}=\frac{y-5}{y-6}+\frac{y+5}{y-6}=\frac{2y}{y-6}.$$

Example 2.10. Add $\dfrac{7}{x-3}+4$.

Solution. We can treat the integer 4 as a fraction with the denominator 1: $4=\dfrac{4}{1}$.

From here, LCD of the denominators $x-3$ and 1 is $x-3$, and these denominators are complement to each other. Therefore,

$$\frac{7}{x-3}+4=\frac{7}{x-3}+\frac{4}{1}=\frac{7\cdot1+4(x-3)}{x-3}=\frac{7+4x-12}{x-3}=\frac{4x-5}{x-3}.$$

Example 2.11. Add $\dfrac{5x}{x^2-4}+\dfrac{3}{2x-4}$.

Solution. At the first glance, it looks like the denominators x^2-4 and $2x-4$ do not have common factors. However, they have. To see that, factor both. It is easy to factor the denominator $2x-4$ by factoring number 2: $2x-4=2(x-2)$.

This factorization gives a hint, how to factor the denominator x^2-4: it may contain the factor $x-2$. It is really so: $x^2-4=(x-2)(x+2)$. We can check that by opening parentheses on the right side. Now, compare the denominators in the factoring form:

$$2(x-2) \text{ and } (x-2)(x+2).$$

We see the common factor $x-2$. This is a part of LCD. Also, we put into LCD other factors 2 and $x+2$. Therefore, the entire $LCD=2(x-2)(x+2)$. Next, we find the complements for each denominator:

For $x^2-4=(x-2)(x+2)$, the complement is 2.

For $2x-4=2(x-2)$, the complement is $x+2$.

Finally, we add the fractions:

$$\frac{5x}{x^2-4}+\frac{3}{2x-4}=\frac{5x\cdot 2+3(x+2)}{2(x-2)(x+2)}=\frac{10x+3x+6}{2(x-2)(x+2)}=\frac{13x+6}{2(x-2)(x+2)}.$$

Note. To factor x^2-4, we can use the general formula

$$\boxed{a^2-b^2=(a-b)(a+b)}$$

We will use this formula many times further. Try to memorize it.

Example 2.12. Subtract $\dfrac{4}{x^2-2x-35}-\dfrac{3}{x^2+2x-15}$.

Solution. Again, as a first step, we factor each denominator. We can use the same idea as in example 2.3: if these denominators have a common factor, the same factor should be in the difference of the denominators. The difference is

$$x^2-2x-35-(x^2+2x-15)=x^2-2x-35-x^2-2x+15=-4x-20=-4(x+5).$$

We can guess that $x+5$ is the common factor for the denominators. It is easy to check that this is really the case:

$$x^2-2x-35=(x+5)(x-7) \text{ and } x^2+2x-15=(x+5)(x-3).$$

Now we construct LCD by multiplying all factors from both denominators (taking the common factor $x+5$ only one time):

$$\text{LCD}=(x+5)(x-3)(x-7).$$

Next, using LCD, we find complements for each denominator.

For $x^2-2x-35=(x+5)(x-7)$, the complement is $x-3$.

For $x^2+2x-15=(x+5)(x-3)$, the complement is $x-7$.

Finally, we subtract given fractions:

$$\frac{4}{x^2+2x-35}-\frac{3}{x^2-2x-15}=\frac{4(x-3)-3(x-7)}{(x+5)(x-3)(x-7)}$$

$$=\frac{4x-12-3x+21}{(x+5)(x-3)(x-7)}=\frac{x+9}{(x+5)(x-3)(x-7)}.$$

Example 2.13. Combine $\dfrac{3}{a^2-4a-12}+\dfrac{4}{a-6}-\dfrac{2}{a+2}$.

Solution. Looking at the denominators, our guess is that the first one is the product of two others:

$$a^2-4a-12=(a-6)(a+2).$$

By opening parentheses on the right, we can check that this is true. So,

$$\text{LCD} = (a - 6)(a + 2).$$

Because LCD coincides with the first denominator, we need to find complements only for the second and third denominators.

For $a - 6$, the complement is $a + 2$.

For $a + 2$, the complement is $a - 6$.

We have

$$\frac{3}{a^2 - 4a - 12} + \frac{4}{a - 6} - \frac{2}{a + 2} = \frac{3 + 4(a + 2) - 2(a - 6)}{(a - 6)(a + 2)}$$

$$= \frac{3 + 4a + 8 - 2a + 12}{(a - 6)(a + 2)} = \frac{2a + 23}{(a - 6)(a + 2)}.$$

Complex Fractions

Complex fractions are fractions that contain another fractions in their numerators and/or denominators. We consider methods how to simplify them.

Example 2.14. Simplify $\dfrac{\dfrac{7}{15ab^2}}{\dfrac{14}{27ab}}$.

Solution. We can represent this complex fraction in the form of division of its numerator by denominator: $\dfrac{7}{15ab^2} \div \dfrac{14}{27ab}$. Now we use the rule for division of fractions: multiply the first fraction by the reciprocal of the second: $\dfrac{7}{15ab^2} \cdot \dfrac{27ab}{14}$. After simplification (reducing) we get the final answer $\dfrac{9}{10b}$.

Note. Working with complex fractions, it is important not to confuse those that look similar, but in reality are different. We need to carefully identify the position of the "main" (longest) fraction line. This line shows the place where we divide numerator by denominator. For example, compare the following complex fractions (which have different "longest" lines):

$$\frac{\dfrac{a}{b}}{c} = \frac{a}{b} \div c = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}. \qquad\qquad \frac{a}{\dfrac{b}{c}} = a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{ac}{b}.$$

As you see, these fractions are different.

In the examples below, to simply complex fractions, we use two methods.

First method: find LCD for all inner fractions, and then multiply the numerator and denominator of the complex fraction by this LCD.

Second method: simplify separately numerator and denominator of a complex fraction, and then divide its numerator by denominator.

Example 2.15. Simplify $\dfrac{\dfrac{2}{3}-\dfrac{5}{6}}{\dfrac{4}{15}+\dfrac{7}{12}}$.

Solution.

First method. The denominators of four inner fractions are 3, 6, 15, and 12. Their LCD is 60. Multiply the numerator and denominator of the complex fraction by this LCD:

$$\frac{60\left(\dfrac{2}{3}-\dfrac{5}{6}\right)}{60\left(\dfrac{4}{15}+\dfrac{7}{12}\right)}=\frac{60\cdot\dfrac{2}{3}-60\cdot\dfrac{5}{6}}{60\cdot\dfrac{4}{15}+60\cdot\dfrac{7}{12}}=\frac{40-50}{16+35}=\frac{-10}{51}=-\frac{10}{51}.$$

Second method. Simplify the numerator and denominator separately:

$$\frac{2}{3}-\frac{5}{6}=\frac{2\cdot2-5}{6}=-\frac{1}{6}, \qquad \frac{4}{15}+\frac{7}{12}=\frac{4\cdot4+7\cdot5}{60}=\frac{51}{60}.$$

Divide numerator by denominator: $-\dfrac{1}{6}\div\dfrac{51}{60}=-\dfrac{1}{6}\cdot\dfrac{60}{51}=-\dfrac{10}{51}.$

Example 2.16. Simplify $\dfrac{\dfrac{3}{x^2y}+\dfrac{2}{xy}}{\dfrac{5}{xy^2}-\dfrac{4}{x^2}}$.

Solution.

First method. The denominators of four inner fractions are x^2y, xy, xy^2, and x^2. Their LCD is x^2y^2. Multiply numerator and denominator of complex fraction by this LCD:

$$\frac{x^2y^2\left(\dfrac{3}{x^2y}+\dfrac{2}{xy}\right)}{x^2y^2\left(\dfrac{5}{xy^2}-\dfrac{4}{x^2}\right)}=\frac{\dfrac{3x^2y^2}{x^2y}+\dfrac{2x^2y^2}{xy}}{\dfrac{5x^2y^2}{xy^2}-\dfrac{4x^2y^2}{x^2}}=\frac{3y+2xy}{5x-4y^2}.$$

Second method.

Simplify the numerator: $\dfrac{3}{x^2 y} + \dfrac{2}{xy} = \dfrac{3+2x}{x^2 y}$.

Simplify the denominator: $\dfrac{5}{xy^2} - \dfrac{4}{x^2} = \dfrac{5x-4y^2}{x^2 y^2}$.

Divide numerator by denominator:

$$\frac{3+2x}{x^2 y} \div \frac{5x-4y^2}{x^2 y^2} = \frac{3+2x}{x^2 y} \cdot \frac{x^2 y^2}{5x-4y^2} = \frac{(3+2x)y}{5x-4y^2} = \frac{3y+2xy}{5x-4y^2}.$$

Example 2.17. Simplify $\dfrac{6 - \dfrac{5}{n-2}}{7 + \dfrac{3}{n-2}}$.

Solution.

First method. The LCD of inner denominators is $n - 2$. Multiply top and bottom of the complex fraction by $n - 2$:

$$\frac{6 - \dfrac{5}{n-2}}{7 + \dfrac{3}{n-2}} = \frac{(n-2)\left(6 - \dfrac{5}{n-2}\right)}{(n-2)\left(7 + \dfrac{3}{n-2}\right)} = \frac{6(n-2)-5}{7(n-2)+3} = \frac{6n-12-5}{7n-14+3} = \frac{6n-17}{7n-11}.$$

Second method.

Simplify the numerator: $6 - \dfrac{5}{n-2} = \dfrac{6(n-2)-5}{n-2} = \dfrac{6n-17}{n-2}$.

Simplify the denominator: $7 + \dfrac{3}{n-2} = \dfrac{7(n-2)+3}{n-2} = \dfrac{7n-14+3}{n-2} = \dfrac{7n-11}{n-2}$.

Divide numerator by denominator:

$$\frac{6n-17}{n-2} \div \frac{7n-11}{n-2} = \frac{6n-17}{n-2} \cdot \frac{n-2}{7n-11} = \frac{6n-17}{7n-11}.$$

Exercises 2

In exercises 2.1 – 2.8, simplify given expression.

2.1. $\dfrac{18y^2 - 12y}{15y}$

2.2. $\dfrac{24z^2 - 16z}{14z}$

2.3. $\dfrac{15z^2 + 25z}{10z - 30}$

2.4. $\dfrac{12y^2 + 18y}{9y - 15}$

2.5. $\dfrac{4x^2 + 8x - 60}{6x^2 + 6x - 72}$

2.6. $\dfrac{6x^2 + 24x - 72}{8x^2 + 40x - 48}$

2.7. $\dfrac{2x^2 - 6x - 56}{7x^2 - 49x}$

2.8. $\dfrac{3x^2 - 3x - 36}{4x^2 - 16x}$

2.9. Add and simplify

$\dfrac{6x}{3x + 2} + \dfrac{4}{3x + 2}$

2.10. Add and simplify

$\dfrac{10x}{5x + 1} + \dfrac{2}{5x + 1}$

2.11. Subtract

$\dfrac{4m + 3}{12} - \dfrac{m - 3}{18}$

2.12. Subtract

$\dfrac{2n + 5}{10} - \dfrac{n - 4}{12}$

2.13. Add

$\dfrac{5}{8x} + \dfrac{7}{12y}$

2.14. Add

$\dfrac{7}{6x} + \dfrac{3}{8y}$

2.15. Combine

$\dfrac{4}{15x^2} - \dfrac{5}{3x} + \dfrac{7}{6}$

2.16. Combine

$\dfrac{5}{12x^2} - \dfrac{7}{6x} + \dfrac{3}{8}$

2.17. Subtract

$\dfrac{3}{4x - 7} - \dfrac{5}{7x - 4}$

2.18. Subtract

$\dfrac{6}{3x - 4} - \dfrac{7}{4x - 3}$

2.19. Subtract

$\dfrac{a - 2}{a - 4} - \dfrac{a + 2}{4 - a}$

2.20. Subtract

$\dfrac{b - 4}{b - 5} - \dfrac{b + 4}{5 - b}$

2.21. Add

$$\frac{6}{x-2}+3$$

2.22. Add

$$\frac{8}{x-4}+2$$

2.23. Add and simplify

$$\frac{x-2}{5x+25}+\frac{3x+1}{x^2-25}$$

2.24. Add and simplify

$$\frac{4x+2}{x^2-16}+\frac{x-3}{4x+16}$$

2.25. Subtract and simplify

$$\frac{3}{x^2+7x+10}-\frac{1}{x^2+5x+6}$$

2.26. Subtract and simplify

$$\frac{6}{x^2+4x-5}-\frac{5}{x^2+3x-4}$$

2.27. Combine and simplify

$$\frac{5}{b^2+2b-8}+\frac{3}{b+4}-\frac{3}{b-2}$$

2.28. Combine and simplify

$$\frac{7}{c^2+2c-15}-\frac{4}{c-3}+\frac{4}{c+5}$$

In exercises 2.29 – 2.38, simplify complex fractions.

2.29.

$$\frac{\dfrac{5}{12c^3d}}{\dfrac{15}{16cd^2}}$$

2.30.

$$\frac{\dfrac{18}{5m^6n}}{\dfrac{6}{25m^3n^2}}$$

2.31.

$$\frac{\dfrac{2}{5}-\dfrac{3}{4}}{\dfrac{7}{10}+\dfrac{1}{2}}$$

2.32.

$$\frac{\dfrac{5}{6}+\dfrac{7}{8}}{\dfrac{3}{4}-\dfrac{2}{3}}$$

2.33.

$$\frac{\dfrac{4}{xy^2}+\dfrac{3}{xy}}{\dfrac{6}{x^2y}-\dfrac{2}{y^2}}$$

2.34.

$$\frac{\dfrac{6}{xy}-\dfrac{5}{x^2y}}{\dfrac{4}{x^2}+\dfrac{3}{xy^2}}$$

2.35.

$$\dfrac{4-\dfrac{7}{k-3}}{6+\dfrac{5}{k-3}}$$

2.36.

$$\dfrac{5+\dfrac{8}{m-4}}{7-\dfrac{6}{m-4}}$$

2.37.

$$\dfrac{1+\dfrac{1}{x}-\dfrac{2}{x^2}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}$$

2.38.

$$\dfrac{1+\dfrac{2}{x}-\dfrac{3}{x^2}}{1+\dfrac{5}{x}+\dfrac{6}{x^2}}$$

Session 3

Rational Equations

In previous session we worked with rational **expressions**. In this session we will work with rational **equations**. These two types of mathematical objects may look similar, but they are different. Namely, they have different final goal. For expressions, we **modify** or simplify them. In other words, we change the appearance of expressions. For equations, we **solve** them. It means that we want to find numerical values of variables for which equations become true statements. Such values are called **solutions** or **roots** of equations.

It is easy to distinguish expression and equations: expressions do not contain equal sign, while equations do. For example, $2x + 1$ is an expression, and $2x + 1 = 5$ is an equation. In other words, expressions contain only one part, while equations contain two parts or sides: left side and right side, connected with the equal sign. We may also say that an equation is the equality of two expressions.

To some extent, the technique to operate with equations and expressions is similar: each side of an equation is an expression, and we can manage them as any expression: open (remove) parentheses, combine like terms and so on. However, there are some operations that can be done on equations only. Among these operations are moving terms from one side of equation to another (with changing the sign of terms), dividing both sides by the same expression or number. In particular, we can always write an equation in the form when its right side equals to zero (by moving all terms to the left).

In this session we consider **rational (fractional) equations**. Both sides of such equations are sums or differences of rational expressions. Here is an example: $\dfrac{x+5}{4} = \dfrac{3}{4} - \dfrac{x-7}{6}$.

Below we solve this equation by reducing it to an equation with no fractions. As with expressions, we will use LCD to do this. However, the main technical difference here is that in expression we must keep LCD (and write it in the denominator of the answer), while in an equation we can drop LCD.

The reason to drop denominator is this. If we keep it (as we do with expressions), then both sides of the equation become fractions with the same denominator (which is LCD). If two fractions are equal and have the same denominator, then their numerators are also equal, so we equate numerators and drop denominators.

Example 3.1. Solve the equation $\dfrac{x+5}{4} = \dfrac{3}{4} - \dfrac{x-7}{6}$.

Solution. The first step is the same as for expressions: find LCD. For the denominators 4 and 6, LCD = 12 (notice that the number 24 is a common denominator but not the least). The second step is also the same: find complements for each denominator to LCD. For the denominator 4, the complement is 3, and for the denominator 6, the complement is 2. The third step is again the same: multiply each numerator by corresponding complement. But now, contrary to expressions, **we may drop all denominators!** As a result, the original equations becomes the equation with no fractions:

$$3(x+5) = 3\cdot 3 - 2(x-7)\,.$$

Now it is easy to solve it:

$$3x + 3\cdot 5 = 3\cdot 3 - 2x + 2\cdot 7 \;\Rightarrow\; 3x + 15 = 9 - 2x + 14 \;\Rightarrow\; 3x + 15 = 23 - 2x\,.$$

From this point we collect all terms with the variable x on the left side, and all other terms (numbers) on the right side:

$$3x + 2x = 23 - 15 \;\Rightarrow\; 5x = 8 \;\Rightarrow\; x = \frac{8}{5}\,.$$

Note. When collecting terms on one side of the equation, we can freely move terms from one side to another and simultaneously change their sign to opposite. This is equivalent to adding or subtracting terms to/from both sides of the equation.

Next we consider equations in which denominators contain a variable. The technique here is the same. The only additional (and very important) thing is that we need to check that the final answer does not cause any denominator of the original equation to be zero. If this happens, we must reject such a solution.

Example 3.2. Solve the equation $\dfrac{5}{6x} + \dfrac{7}{6} = \dfrac{3}{14x}$.

Solution. For the denominators $6x$, 6 and $14x$, LCD $= 42x$. Next we find complements for each denominator to LCD:

> For $6x$, the complement is 7.
> For 6, the complement is $7x$.
> For $14x$, the complement is 3.

We multiply each numerator by corresponding complement and drop LCD. The equation becomes free of fractions:

$$5\cdot 7 + 7\cdot 7x = 3\cdot 3$$

Now we solve it:

$$35 + 49x = 9 \;\Rightarrow\; 49x = 9 - 35 \;\Rightarrow\; 49x = -26 \;\Rightarrow\; x = -\frac{26}{49}\,.$$

None of the denominators of the original equation is zero for this value of x, so this is the final answer.

Example 3.3. Solve the equation $\dfrac{3}{4} + \dfrac{5}{x-5} = \dfrac{x}{x-5}$.

Solution. For the denominators 4 and $x - 5$, LCD $= 4(x - 5)$.

Complements for denominators to LCD are:

> For 4, the complement is $x - 5$.
> For $x - 5$, the complement is 4.

The equation becomes

$$3(x-5) + 5\cdot 4 = 4x\,.$$

We solve it:
$$3x - 15 + 20 = 4x \implies 3x + 5 = 4x \implies 3x - 4x = -5,$$
$$-x = -5 \implies x = 5.$$

For $x = 5$, the denominator $x - 5$ becomes zero, so we reject solution $x = 5$. Because this is the only possible solution, the original equation does not have solutions at all.

Example 3.4. Solve the equation $\dfrac{3}{2} + \dfrac{4}{x^2 - 16} = \dfrac{3x}{2x - 8}$.

Solution.

1) To find LCD, factor the second and third denominators.
   Second denominator:
   $$x^2 - 16 = (x - 4)(x + 4).$$
   Third denominator:
   $$2x - 8 = 2(x - 4).$$
   LCD of all three denominators is $2(x - 4)(x + 4)$.

2) Find complement for each denominator to LCD:

   For 2, the complement is $(x - 4)(x + 4) = x^2 - 16$.

   For $x^2 - 16 = (x - 4)(x + 4)$, the complement is 2.

   For $2x - 8 = 2(x - 4)$, the complement is $x + 4$.

3) Multiply each numerator of the original equation by corresponding complement, and drop denominator. The equation becomes
   $$3(x^2 - 16) + 4 \cdot 2 = 3x(x + 4).$$

4) Solve the above equation:
   $$3x^2 - 3 \cdot 16 + 4 \cdot 2 = 3x^2 + 3x \cdot 4 \implies 3x^2 - 48 + 8 = 3x^2 + 12x,$$
   $$3x^2 - 3x^2 - 12x = 48 - 8 \implies -12x = 40 \implies x = -\frac{40}{12} = -\frac{10}{3}.$$

5) None of the denominators of the original equation is zero if $x = -\dfrac{10}{3}$, so this is the solution.

Example 3.5. Solve the equation $\dfrac{x}{x - 3} + \dfrac{1}{x + 2} = \dfrac{18 - x}{x^2 - x - 6}$.

Solution.

1) To find LCD, factor the third denominator. First denominator $x - 3$ and second $x + 2$, may give a hint on how to factor the third denominator and get LCD:
   $$\text{LCD} = x^2 - x - 6 = (x - 3)(x + 2).$$

2) Find complement for each denominator to LCD:

For $x - 3$, the complement is $x + 2$.

For $x + 2$, the complement is $x - 3$.

For $x^2 - x - 6$ the complement is 1.

3) Multiply each numerator of the original equation by corresponding complement, and drop denominator. The equation becomes

$$x(x+2) + 1 \cdot (x-3) = 18 - x.$$

4) Solve the above equation:

$$x^2 + 2x + x - 3 = 18 - x \implies x^2 + 3x - 3 = 18 - x,$$

$$x^2 + 3x - 3 - 18 + x = 0 \implies x^2 + 4x - 21 = 0.$$

The last equation is the quadratic equation. We can solve it by factoring:

$$(x+7)(x-3) = 0 \implies x = -7 \text{ and } x = 3.$$

Note. Later in session 8 we consider quadratic equations in more details.

5) Let's check solutions of the above quadratic equation (numbers –7 and 3) with the original equation. As we mentioned above, we need to check whether these numbers make any of denominators zero. Number –7 does not make, but 3 does, so we must reject number 3.

Final answer: original equation has only one solution $x = -7$.

Example 3.6. Solve the equation $\dfrac{3}{4n+20} - \dfrac{2}{n-3} = \dfrac{6}{n^2+2n-15}$.

Solution.

1) To find LCD, first we factor denominators that are factorable. It is possible to factor first and third denominators:

$$4n + 20 = 4(n+5) \text{ and } n^2 + 2n - 15 = (n+5)(n-3).$$

2) Now we construct LCD, by multiplying all of the above factors (taking common factor $n + 5$ only one time):

$$\text{LCD} = 4(n+5)(n-3).$$

3) Find complement for each denominator to LCD:

For $4n + 20 = 4(n+5)$, the complement is $n - 3$.

For $n - 3$, the complement is $4(n+5) = 4n + 20$.

For $n^2 + 2n - 15 = (n+5)(n-3)$, the complement is 4.

4) Multiply each numerator of the original equation by corresponding complement, and drop denominator. The equation becomes

$$3(n-3) - 2(4n+20) = 6 \cdot 4.$$

5) Solve the above equation:

$$3n - 9 - 8n - 40 = 24 \ \Rightarrow \ -5n - 49 = 24 \ \Rightarrow -5n = 24 + 49,$$

$$-5n = 73 \ \Rightarrow \ n = -\frac{73}{5}.$$

6) None of the denominators of original equation is zero for this value of n; so, it is the solution.

In conclusion of this session consider an equation that has the form of equality of two fractions. Such an equation is called the **proportion**. Of course, it can be solved in the same way as before, using LCD. Another way is to use the important property of proportion: **cross-multiplication rule**.
This rule means the following:

$$\text{if } \ \frac{a}{b} = \frac{c}{d} \ \text{ then } \ ad = bc.$$

In words: product along one diagonal (a times d) is equal to the product along another diagonal (b times c).

Example 3.7. Solve the equation $\dfrac{5}{2x - 4} = \dfrac{2}{3x + 5}$.

Solution. The equation is written in the form of proportion, and we can use cross-multiplication rule. We have

$$5(3x + 5) = 2(2x - 4),$$

$$15x + 25 = 4x - 8 \ \Rightarrow \ 15x - 4x = -8 - 25 \ \Rightarrow \ 11x = -33 \ \Rightarrow \ x = -3.$$

None of the denominators of the original equation is zero for this value of x, so the final answer is $x = -3$.

Exercises 3

In exercises 3.1 – 3.14, solve given equations.

3.1. $\dfrac{x-2}{8} = \dfrac{5}{8} - \dfrac{x+3}{6}$

3.2. $\dfrac{x+2}{6} = \dfrac{7}{6} - \dfrac{x-5}{9}$

3.3. $\dfrac{2}{4x} + \dfrac{3}{4} = \dfrac{5}{6x}$

3.4. $\dfrac{7}{8x} + \dfrac{5}{12} = \dfrac{3}{8x}$

3.5. $\dfrac{8}{x-4} - 3 = \dfrac{3}{x-4}$

3.6. $\dfrac{9}{x-6} - 4 = \dfrac{5}{x-6}$

3.7. $\dfrac{4}{5} + \dfrac{7}{x^2-9} = \dfrac{4x}{5x-15}$

3.8. $\dfrac{5}{7} + \dfrac{6}{x^2-4} = \dfrac{5x}{7x+14}$

3.9.
$$\dfrac{x}{x-4} + \dfrac{1}{x+5} = \dfrac{40-x}{x^2+x-20}$$

3.10.
$$\dfrac{x}{x-6} + \dfrac{1}{x-2} = \dfrac{30-x}{x^2-8x+12}$$

3.11.
$$\dfrac{9}{5m+15} - \dfrac{4}{m-2} = \dfrac{7}{m^2+m-6}$$

3.12.
$$\dfrac{8}{3m+12} - \dfrac{5}{m-3} = \dfrac{7}{m^2+m-12}$$

3.13. $\dfrac{2}{7x-9} = \dfrac{3}{4x+6}$

3.14. $\dfrac{4}{5x+3} = \dfrac{7}{8x-3}$

Challenge Problems

Hint: m grams of $a\%$ solution contains $\dfrac{ma}{100}$ grams of pure substance.

3.15. A solution of antifreeze contains 20% alcohol. How much pure alcohol must be added to 6 gallons of the solution to make a 40% solution?

3.16. How many gallons of a 15% sugar solution must be mixed with 6 gallons of a 40% sugar solution to make a 30% sugar solution?

3.17. A chemist mixed 4 liters of 18% acid solution with 8 liters of 45% acid solution. What percent of acid is in the mixture?

3.18. Nick mixed 9 oz of apple drink with 8 oz of 48% carrot drink. Find the percent of pure apple juice in apple drink if the mixture contained 30% fruit juice.

Session 4

Radicals and Fractional Exponents

Definition of Radicals

Suppose we want to construct a box in the shape of a cube having the volume of 8 cm^3. The problem is to find its size (i.e. the length of its edges). If we denote this size by x, then the volume is x^3. So, to find x, we need to solve the equation $x^3 = 8$. It is not difficult to see that $x = 2$ cm. Equations like $x^3 = 8$ may appear in different problems, and it would be a good idea to invent a special notation for their solutions. Let's consider more general equation $x^n = a$, in which a and n are given, and we need to find x. The following symbol was invented for the solution of this equation: $\sqrt[n]{}$. This symbol is called the **radical** or ***n*-th root** or **root of the *n*-th degree**. Using it, the solution of the equation $x^n = a$ can be written as $x = \sqrt[n]{a}$, so $\left(\sqrt[n]{a}\right)^n = a$. We may say that to find x, we take n^{th} root of a. Number n is called the **degree** or **order** of the root. For example, we can read $\sqrt[3]{8}$ as "root of the 3rd order of 8", or "3rd root of 8", or "cube root of 8" (based on the above example with the volume of a cube).

The case when degree $n = 2$ is of special interest as it appears most often. For example, assume that we want to construct a square with the area of 9, and we are interested in its side. If we denote this side by x, then the area is x^2. To find x, we need to solve the equation $x^2 = 9$. And the solution is $x = \sqrt[2]{9} = 3$. It was the agreement to drop number 2 in the radical $\sqrt[2]{}$, and we simply write $\sqrt{}$. So, $\sqrt{9} = 3$. We read the expression $\sqrt{9}$ as "radical 9" or "square root of 9" (based on the example with the area of a square).

Note. Formally speaking, the equation $x^2 = 9$ has two solutions: $x = 3$ and $x = -3$ since the square of both numbers is 9. However, the radical $\sqrt{}$ always means a **nonnegative** number, so $\sqrt{9} = 3$, not –3. Both solutions of the equation $x^2 = 9$ can be written as $x = \sqrt{9} = 3$ and $x = -\sqrt{9} = -3$. Similar, if a is nonnegative number, by $\sqrt[n]{a}$ we always mean a nonnegative number.

Now, let's give the formal definition of the nth root.

Definition. Let a be a nonnegative number, and n be a positive integer. Then the nth root of a, denoted as $\sqrt[n]{a}$, is a nonnegative solution of the equation $x^n = a$.

As we mentioned above, this definition says that $\left(\sqrt[n]{a}\right)^n = a$.

Note. We defined $\sqrt[n]{a}$ only for nonnegative a. But what if a is a negative? In this case, we can also define $\sqrt[n]{a}$ but only if degree n is an **odd** number. For example, $\sqrt[3]{-8} = -2$,

since $(-2)^3 = -8$. If n is **even** and a is **negative**, then n^{th} root $\sqrt[n]{a}$ **does not exist**. For example, $\sqrt{-9}$ does not exist because there is no (real) number x such that $x^2 = -9$.

Note. You may think that even for some positive numbers, n^{th} root does not exist. What would you say about $\sqrt{8}$? It seems that we cannot find a number x such that $x^2 = 8$. However, this is true only if we think about integers. Actually, number $\sqrt{8}$ exists, but it is neither an integer nor a fraction (rational number). The existence of such a number is clearly seeing if we want to find a side for a square with the area of 8. Numbers like $\sqrt{8}$ are called **irrational** numbers. If we want to find this number as a decimal, we can get only its approximation with certain number of digits after decimal point. For example, using a calculator, we can find that $\sqrt{8} \approx 2.8$ or $\sqrt{8} \approx 2.83$ and so on.

If square root of a number is integer, we call such number a **perfect square**. For example, 9 is a perfect square, but 8 is not. To get a list of all perfect squares, we can take a list of integers 0, 1, 2, 3, …., and square these numbers. We will get the list of perfect squares: 0, 1, 4, 9, …

Properties of Radicals

As with exponents, we can multiply and divide radicals of the same order very easily. We will assume in this session that a and b are any nonnegative numbers, and n is any positive integer.

Product Rule: $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

So, to multiply radicals of the same order, just combine them in one single radical.

Example 4.1. Simplify $\sqrt{2} \times \sqrt{18}$.

Solution. $\sqrt{2} \times \sqrt{18} = \sqrt{2 \times 18} = \sqrt{36} = 6$.

Of course, we can re-write the product rule from right to left: $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$. In this way we may split one radical into a product of two. We will see below that this property may help in simplification of more complicated radicals.

Quotient Rule: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

As for product rule, we can combine two radicals of the same order in one.

Example 4.2. Simplify $\dfrac{\sqrt{50}}{\sqrt{2}}$.

Solution. $\dfrac{\sqrt{50}}{\sqrt{2}} = \sqrt{\dfrac{50}{2}} = \sqrt{25} = 5$.

Below we will show how to combine radicals with different orders.

Power Rule: $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ for positive integer m.

In particular, $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$.

Example 4.3. Simplify $\left(\sqrt{2}\right)^4$.

Solution. $\left(\sqrt{2}\right)^4 = \sqrt{(2)^4} = \sqrt{16} = 4.$

Exponents with Fractional Powers

So far we multiply and divide radicals of the same order only. But what if we need to operate with different orders? For example, is it possible somehow to represent the product $\sqrt{2} \times \sqrt[3]{2}$ as one single radical? Here we develop the technique to do this. The idea is to set up connection between radicals and exponents. In this way, we could apply product, quotient and power rules of exponents to radicals.

Let's try to represent radical $\sqrt[n]{a}$ as exponent with the base a and some power m: $\sqrt[n]{a} = a^m$. If we raise this equation (meaning both sides of the equation) into n^{th} power, we will have

$$\left(\sqrt[n]{a}\right)^n = \left(a^m\right)^n = a^{m \cdot n} . \text{ But } \left(\sqrt[n]{a}\right)^n = a = a^1 . \text{ Therefore, } a^1 = a^{m \cdot n} .$$

From here we can equate powers: $1 = mn$, and $m = \dfrac{1}{n}$. Now , the expression $\sqrt[n]{a} = a^m$ can

be written as $\sqrt[n]{a} = a^{\frac{1}{n}}$. We've got the representation of nth root as an exponent with the

fractional power $\dfrac{1}{n}$! We use this representation as definition.

Definition. Let a be any nonnegative number, and n be any positive integer. Then

$$a^{\frac{1}{n}} = \sqrt[n]{a} .$$

For example, $a^{\frac{1}{2}} = \sqrt{a}$.

We can easily generalize this definition to exponents with arbitrary fractional power $\dfrac{m}{n}$.

To do this, just raise the equation $a^{\frac{1}{n}} = \sqrt[n]{a}$ into m^{th} power and use power rules for exponents and radicals:

$$\left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \Rightarrow a^{\frac{m}{n}} = \sqrt[n]{a^m} . \text{ We come up to the following}$$

Definition. Let a be any non-negative number, and $\dfrac{m}{n}$ be any positive fraction. Then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \ .$$

Note. Be careful to put m and n in the radical in correct places: numerator m goes into the power of a, and n goes into degree of radical.

Representation of radicals as exponents with fractional powers extends our ability to manipulate with radicals. Here is an example on how it may help to simplify radical expressions of different orders.

Example 4.4. Simplify the expression $\sqrt{2} \times \sqrt[3]{2}$ (combine into one radical).

Solution. $\sqrt{2} \times \sqrt[3]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{3+2}{6}} = 2^{\frac{5}{6}} = \sqrt[6]{2^5} = \sqrt[6]{32} \ .$

In session 1 on integer exponents, we considered exponents with negative integer powers and in this session we dealt with positive fractional exponents. Now we can combine both together using the corresponding formulas

$$a^{-k} = \frac{1}{a^k} \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m} \ .$$

From here we get connection between **negative fractional exponents** and **radicals**:

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}} \ .$$

Example 4.5. Simplify the expression $\dfrac{\sqrt[5]{3}}{\sqrt[10]{3^7}}$.

Solution. $\dfrac{\sqrt[5]{3}}{\sqrt[10]{3^7}} = \dfrac{3^{\frac{1}{5}}}{3^{\frac{7}{10}}} = 3^{\frac{1}{5}-\frac{7}{10}} = 3^{\frac{2-7}{10}} = 3^{-\frac{5}{10}} = 3^{-\frac{1}{2}} = \dfrac{1}{3^{\frac{1}{2}}} = \dfrac{1}{\sqrt{3}} \ .$

Simplification of Square Roots

Here we focus on square roots only, but similar technique can also be used for general exponents. The simplification means to leave a smallest possible expression inside the radicals.

1. Simplification of numerical expression $\sqrt{a}$ (a is a number).

The idea is to split number a as a product of two factors, one of which is a perfect square, and then use the product rule.

Example 4.6. Simplify $\sqrt{12}$.

Solution. We split 12 as 4×3. Then $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$.

Note. We can split 12 in different way: $12 = 2 \times 6$. However, this way will not lead to simplification since both factors, 2 and 6, are not perfect squares.

Example 4.7. Simplify $\sqrt{48}$.

Solution. It is possible to split number 48 as a product of two factors, one of which is a perfect square, in two ways: $48 = 4 \times 12$ and $48 = 16 \times 3$. If we use the first way, we need to continue splitting 12 as 4×3. So, even the first way works, it's better to go the second way. In general, try to split given number in such a way that a factor which is not a perfect square can not be split further to contain another perfect square. So, we choose the factorization $48 = 16 \times 3$. Using this, we get $\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$.

2. Simplification of $\sqrt{x^n}$ for even n.

We can represent square root as exponent with power $\dfrac{1}{2}$, and then use power rules:

$$\sqrt{x^n} = \left(x^n\right)^{\frac{1}{2}} = x^{\frac{n}{2}}.$$

We come up to the following method how to simplify square root from exponent with even power: **divide power by 2** and remove radical.

Example 4.8. Simplify $\sqrt{x^{16}}$.

Solution. According to the above rule, $\sqrt{x^{16}} = x^{\frac{16}{2}} = x^8$.

Note. Do not be mislead that in the above example power 16 is a perfect square, and you may think to take square root of 16. **Do not** take square root from the power, instead, **divide** power by two.

3. Simplification of $\sqrt{x^n}$ for odd n.

Any odd number can be written as $2m + 1$, where m is an integer, so we can write $\sqrt{x^n}$ as $\sqrt{x^{2m+1}}$. To simplify this radical, we represent x^{2m+1} as $x^{2m} \cdot x$ (separate even power and single x). Then

$$\sqrt{x^{2m+1}} = \sqrt{x^{2m} \cdot x} = \sqrt{x^{2m}} \cdot \sqrt{x} = x^m \sqrt{x}.$$

We come up to the following rule how to simplify square root from exponent with odd power: detach (separate) variable x from x^{2m+1}, leave it inside the radical, and take square root from x^{2m} which is x^m.

Example 4.9. Simplify $\sqrt{x^{25}}$.

Solution. Using the above rule, $\sqrt{x^{25}} = \sqrt{x^{24} \cdot x} = \sqrt{x^{24}} \cdot \sqrt{x} = x^{12} \sqrt{x}$.

As you can see, even after simplification, we still have radical. Similar to the above note for even power, **do not** take square root from 25.

In conclusion consider an example that combines all three types of radicals: numbers, and exponents with variables that have even and odd powers.

Example 4.10. Simplify the radical $4 \cdot \sqrt{150 x^{12} y^7}$.

Solution.

1st way. (Process each item separately). Inside the radical, we have three items: number 150, and two exponents: x^{12} and y^7 . We can process each of them separately:

$$\sqrt{150} = \sqrt{25 \times 6} = 5\sqrt{6},$$

$$\sqrt{x^{12}} = x^6,$$

$$\sqrt{y^7} = \sqrt{y^6 \cdot y} = y^3 \sqrt{y}.$$

From here, $4 \cdot \sqrt{150 x^{12} y^7} = 4 \cdot \sqrt{150} \cdot \sqrt{x^{12}} \cdot \sqrt{y^7} = 4 \cdot 5 \cdot \sqrt{6} \cdot x^6 \cdot y^3 \sqrt{y} = 20 x^6 y^3 \sqrt{6y}$.

2nd way. (Process the entire expression).

$$4 \cdot \sqrt{150 x^{12} y^7} = 4 \cdot \sqrt{25 \cdot 6 \cdot x^{12} \cdot y^6 \cdot y} = 4 \cdot 5 \cdot x^6 \cdot y^3 \sqrt{6 \cdot y} = 20 x^6 y^3 \sqrt{6y} .$$

Exercises 4

In exercises 4.1 and 4.2, evaluate given expressions without using calculator.

4.1. a) $\sqrt{2} \times \sqrt{8}$

 b) $\dfrac{\sqrt{27}}{\sqrt{3}}$

 c) $\left(\sqrt{5}\right)^4$

 d) $16^{\frac{1}{2}}$

 e) $8^{-\frac{1}{3}}$

 f) $25^{\frac{3}{2}}$

 g) $27^{-\frac{2}{3}}$

 h) $\sqrt[3]{4} \times \sqrt[6]{4}$

4.2. a) $\sqrt{3} \times \sqrt{27}$

 b) $\dfrac{\sqrt{80}}{\sqrt{5}}$

 c) $\left(\sqrt{6}\right)^4$

 d) $27^{\frac{1}{3}}$

 e) $36^{-\frac{1}{2}}$

 f) $8^{\frac{2}{3}}$

 g) $4^{-\frac{3}{2}}$

 h) $\sqrt[6]{81} \times \sqrt[12]{81}$

In exercises 4.3 and 4.4, combine into one radical and simplify.

4.3. $\dfrac{\sqrt[7]{a}}{\sqrt[14]{a^9}}$

4.4. $\dfrac{\sqrt[6]{b^5}}{\sqrt[3]{b}}$

In exercises 4.5 – 4.14, simplify given expressions.

4.5. $\sqrt{50}$

4.6. $\sqrt{54}$

4.7. $\sqrt{32}$

4.8. $\sqrt{72}$

4.9. a) $\sqrt{x^8}$

 b) $\sqrt{x^7}$

4.10. a) $\sqrt{x^6}$

 b) $\sqrt{x^5}$

4.11. a) $\sqrt{36y^{36}}$

 b) $\sqrt{9y^9}$

4.12. a) $\sqrt{49z^{49}}$

 b) $\sqrt{64z^{64}}$

4.13. $3 \cdot \sqrt{75x^{10}y^5z^{16}}$

4.14. $5 \cdot \sqrt{75x^{14}y^9z^{13}}$

Session 5

Multiplication, Addition and Subtraction Radicals

Multiplication Radicals

In previous session we considered product rule for radicals of the same degree n: $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$. This rule allows to combine a product of two (or more) radicals in one radical. Here we consider more examples for multiplication of radicals. We restrict ourselves to square roots only. Also we assume that all letters represent nonnegative numbers.

Let's recall that $\sqrt{a \cdot a} = a$. This simple formula allows to avoid tedious calculations in some cases: if you notice inside the radical a product of a number by itself, do not multiply, just take this number out of radical.

Example 5.1. Multiply and simplify $\sqrt{23} \cdot \sqrt{46}$.

Solution. One way is to directly multiply numbers inside radicals:

$$\sqrt{23} \cdot \sqrt{46} = \sqrt{23 \cdot 46} = \sqrt{1058}.$$

Now you need to simplify $\sqrt{1058}$. Even it is possible, this is not the best way: it may be not clear what to do with 1058. Notice, however, that $46 = 23 \cdot 2$, and it is much easier to process like this:

$$\sqrt{23} \cdot \sqrt{46} = \sqrt{23} \cdot \sqrt{23 \cdot 2} = \sqrt{23 \cdot 23 \cdot 2} = 23\sqrt{2}.$$

Example 5.2. Multiply and simplify $\left(5\sqrt{7x^5 y}\right)\left(3\sqrt{14x^3 y^4}\right)$

Solution. We multiply separately numbers outside the radicals (5 times 3) and expressions inside radicals (we also represent 14 as $7 \cdot 2$):

$$\left(5\sqrt{7x^5 y}\right)\left(3\sqrt{14x^3 y^4}\right) = (5 \cdot 3)\sqrt{\left(7x^5 y\right) \cdot \left(7 \cdot 2x^3 y^4\right)}$$

$$= 15\sqrt{7 \cdot 7 \cdot 2x^8 yy^4} = 15 \cdot 7x^4 y^2 \sqrt{2y} = 105x^4 y^2 \sqrt{2y}$$

Addition and Subtraction Radicals

Contrary to multiplication rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, there is no so simply rule to add or subtract radical: in general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. Here is an example:

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7, \text{ but } \sqrt{9+16} = \sqrt{25} = 5.$$

We can add or subtract radicals directly only if inside radicals we have exactly the same expressions. This procedure is similar to combining like terms.

Example 5.3. Add $5\sqrt{7} + 3\sqrt{7}$.

Solution. Similar to combining like terms: $5x + 3x = 8x$, we have $5\sqrt{7} + 3\sqrt{7} = 8\sqrt{7}$.

Example 5.4. Simplify the expression $x\sqrt{2y} + 3\sqrt{5z} - 4\sqrt{2y} - x\sqrt{5z}$.

Solution. This expression contains four terms. As we indicated above, we can combine only those terms that have the same expressions inside radicals. We can combine first term with third, and second with fourth:

$$x\sqrt{2y} + 3\sqrt{5z} - 4\sqrt{2y} - x\sqrt{5z} = (x-4)\sqrt{2y} + (3-x)\sqrt{5z} .$$

Note. As you can see, the final answer contains two radicals. We cannot combine them in one radical because they have different expressions inside, so it is not possible to process further.

There are cases when even the original expression contains different radicals, it is possible to combine them. It can be done by simplifying of individual radicals before combining.

Example 5.5. Simplify the expression $6\sqrt{8} + 5\sqrt{27} - 4\sqrt{32} - 2\sqrt{75}$.

Solution. There are four radicals here and all of them are different, so we cannot combine them initially. Let's simplify them first (we will process the entire expression):

$$6\sqrt{8} + 5\sqrt{27} - 4\sqrt{32} - 2\sqrt{75} = 6\sqrt{4\cdot 2} + 5\sqrt{9\cdot 3} - 4\sqrt{16\cdot 2} - 2\sqrt{25\cdot 3}$$
$$= 6\cdot 2\sqrt{2} + 5\cdot 3\sqrt{3} - 4\cdot 4\sqrt{2} - 2\cdot 5\sqrt{3} = 12\sqrt{2} + 15\sqrt{3} - 16\sqrt{2} - 10\sqrt{3}.$$

Now, the first and the third terms have the same $\sqrt{2}$, and the second and fourth – the same $\sqrt{3}$. Therefore, we can combine them:

$$12\sqrt{2} + 15\sqrt{3} - 16\sqrt{2} - 10\sqrt{3} = (12-16)\sqrt{2} + (15-10)\sqrt{3} = -4\sqrt{2} + 5\sqrt{3} .$$

It is not possible to combine further.

Mixed Problems

Example 5.6. Multiply and simplify $3\sqrt{5}\left(2\sqrt{15} - 4\sqrt{30}\right)$.

Solution. We can open parentheses using the usual distributive property:

$$3\sqrt{5}\left(2\sqrt{15} - 4\sqrt{30}\right) = 3\sqrt{5}\cdot 2\sqrt{15} - 3\sqrt{5}\cdot 4\sqrt{30} .$$

Next, we can process each term separately:

$$3\sqrt{5}\cdot 2\sqrt{15} = 3\cdot 2\cdot \sqrt{5}\cdot \sqrt{15} = 6\sqrt{5\cdot 15} = 6\sqrt{5\cdot 5\cdot 3} = 6\cdot 5\sqrt{3} = 30\sqrt{3} ,$$

$$3\sqrt{5}\cdot 4\sqrt{30} = 3\cdot 4\sqrt{5}\cdot\sqrt{30} = 12\sqrt{5\cdot 30} = 12\sqrt{5\cdot 5\cdot 6} = 12\cdot 5\sqrt{6} = 60\sqrt{6}\,.$$

Finally, we subtract the last expression from the previous and get the answer

$$3\sqrt{5}\left(2\sqrt{15} - 4\sqrt{30}\right) = 30\sqrt{3} - 60\sqrt{6}\,.$$

Example 5.7. Multiply and simplify $\left(3\sqrt{2} - 4\right)\left(5\sqrt{3} + \sqrt{6}\right)$.

Solution. As in Example 5.6, we start with distributive property:

$$\left(3\sqrt{2} - 4\right)\left(5\sqrt{3} + \sqrt{6}\right) = 3\sqrt{2}\cdot 5\sqrt{3} + 3\sqrt{2}\cdot\sqrt{6} - 4\cdot 5\sqrt{3} - 4\cdot\sqrt{6}\,.$$

Again, we can process each term separately (if you want, you may process them simultaneously):

$$3\sqrt{2}\cdot 5\sqrt{3} = 3\cdot 5\sqrt{2\cdot 3} = 15\sqrt{6}\,,$$
$$3\sqrt{2}\cdot\sqrt{6} = 3\sqrt{2\cdot 6} = 3\sqrt{2\cdot 2\cdot 3} = 3\cdot 2\sqrt{3} = 6\sqrt{3}\,.$$
$$4\cdot 5\sqrt{3} = 20\sqrt{3}\,.$$

From here,

$$\left(3\sqrt{2} - 4\right)\left(5\sqrt{3} + \sqrt{6}\right) = 15\sqrt{6} + 6\sqrt{3} - 20\sqrt{3} - 4\sqrt{6}$$
$$= = (15 - 4)\sqrt{6} + (6 - 20)\sqrt{3} = 11\sqrt{6} - 14\sqrt{3}\,.$$

Below, we consider an example that can be easily solved using the following simple but useful formula, that we already mentioned in session 2 about rational expressions (see note after example 2.11 from session 2):

$$(a - b)(a + b) = a^2 - b^2\,.$$

Example 5.8. Multiply and simplify $\left(2\sqrt{5} - 3\sqrt{7}\right)\left(2\sqrt{5} + 3\sqrt{7}\right)$.

Solution. We can solve this problem in the same way as in Example 5.7, using the distributive property. Notice, however, that in both pairs of parentheses we have difference and sum of the same expressions $2\sqrt{5}$ and $3\sqrt{7}$. Therefore, we can use the above formula with $a = 2\sqrt{5}$ and $b = 3\sqrt{7}$. We have

$$a^2 = \left(2\sqrt{5}\right)^2 = 2^2\cdot\left(\sqrt{5}\right)^2 = 4\cdot 5 = 20\,,$$
$$b^2 = \left(3\sqrt{7}\right)^2 = 3^2\cdot\left(\sqrt{7}\right)^2 = 9\cdot 7 = 63\,.$$

Now, subtract and get the answer

$$\left(2\sqrt{5} - 3\sqrt{7}\right)\left(2\sqrt{5} + 3\sqrt{7}\right) = 20 - 63 = -43\,.$$

In the following example we will solve a problem similar to Example 5.8 in more general form.

Example 5.9. Multiply and simplify $\left(m\sqrt{n} - x\sqrt{y}\right)\left(m\sqrt{n} + x\sqrt{y}\right)$.

Solution.

$$\left(m\sqrt{n} - x\sqrt{y}\right)\left(m\sqrt{n} + x\sqrt{y}\right) = \left(m\sqrt{n}\right)^2 - \left(x\sqrt{y}\right)^2$$
$$= m^2\left(\sqrt{n}\right)^2 - x^2\left(\sqrt{y}\right)^2 = m^2 n - x^2 y.$$

Note. The expressions $m\sqrt{n} - x\sqrt{y}$ and $m\sqrt{n} + x\sqrt{y}$ are called **conjugate** to each other. Example 5.9 shows that the product of conjugate expressions does not contain radicals. In the next session we will use this property to "rationalize" denominators.

Exercises 5

In exercises 5.1 – 5.6, multiply and simplify given expressions.

5.1. $\sqrt{2018}\cdot\sqrt{2018}$

5.2. $\sqrt{2019}\cdot\sqrt{2019}$

5.3. $\sqrt{47}\cdot\sqrt{94}$

5.4. $\sqrt{26}\cdot\sqrt{78}$

5.5. $4\sqrt{24a^3b^2}\cdot6\sqrt{72a^7b}$

5.6. $2\sqrt{21u^9v^6}\cdot7\sqrt{42u^3v}$

In exercises 5.7 and 5.8, add given expressions.

5.7. $3\sqrt{6}+8\sqrt{6}$

5.8. $7\sqrt{5}+2\sqrt{5}$

In exercises 5.9 – 5.12, simplify given expressions.

5.9.

$3m\sqrt{6n}-2\sqrt{7k}+4\sqrt{7k}+5m\sqrt{6n}$

5.10.

$7p\sqrt{3q}-2\sqrt{6r}+4\sqrt{6r}+5p\sqrt{3q}$

5.11.

$8\sqrt{24}+4\sqrt{20}-2\sqrt{54}-6\sqrt{45}$

5.12.

$6\sqrt{63}-7\sqrt{48}+5\sqrt{28}-2\sqrt{108}$

In exercises 5.13 – 5.22, multiply and simplify given expressions.

5.13. $5\sqrt{7}\left(4\sqrt{63}-3\sqrt{7}\right)$

5.14. $4\sqrt{3}\left(6\sqrt{3}-2\sqrt{12}\right)$

5.15. $\left(3\sqrt{6}-2\sqrt{3}\right)\left(3\sqrt{6}+2\sqrt{3}\right)$

5.16. $\left(4\sqrt{7}+3\sqrt{5}\right)\left(4\sqrt{7}-3\sqrt{5}\right)$

5.17. $\left(5\sqrt{5}+2\sqrt{2}\right)\left(5\sqrt{5}-2\sqrt{2}\right)$

5.18. $\left(6\sqrt{6}-3\sqrt{3}\right)\left(6\sqrt{6}+3\sqrt{3}\right)$

5.19. $\left(3\sqrt{7}+2\sqrt{5}\right)^2$

5.20. $\left(2\sqrt{6}+4\sqrt{3}\right)^2$

5.21. $\left(3\sqrt{7}-2\sqrt{5}\right)^2$

5.22. $\left(2\sqrt{6}-4\sqrt{3}\right)^2$

Session 6

Rationalizing the Denominators and Solving Radical Equations

Rationalizing the Denominators

Similar to product rule, we can use the following quotient rule to divide radicals (again, without loss of generality, we consider square roots only): $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$. As with product rule, this rule allows to replace two "umbrellas" (two radicals) for a and b, with one "umbrella" that covers both. For example, $\dfrac{\sqrt{10}}{\sqrt{6}} = \sqrt{\dfrac{10}{6}} = \sqrt{\dfrac{5}{3}}$. In some cases, it is desirable to modify expressions like this further to get rid of radical in the denominator. The procedure to do this is called **rationalization the denominator**.

The general idea to rationalize the denominator is to use the main property of fraction: if we multiply both sides of a fraction by the same nonnegative expression, the value of the fraction remains the same (even if the fraction will look differently). We consider here two types of fractions: one with a single term with the radical in denominator, and another with two terms (where at least one of them contains radical).

Fractions with a single radical term in denominator.

Such fractions have the following general form $\dfrac{\exp r}{m\sqrt{n}}$, where expr means some expression. To rationalize the denominator here, we multiply both numerator and denominator by radical $\sqrt{n}$ located in the denominator, using the property $\sqrt{n} \cdot \sqrt{n} = n$. Then we will get a fraction with no radical in the denominator (so, we rationalize the denominator):

$$\frac{\exp r}{m\sqrt{n}} = \frac{\exp r\sqrt{n}}{m\sqrt{n}\sqrt{n}} = \frac{\exp r\sqrt{n}}{mn}.$$

Example 6.1. Rationalize the denominator: $\sqrt{\dfrac{5}{3}}$.

Solution. We have $\sqrt{\dfrac{5}{3}} = \dfrac{\sqrt{5}}{\sqrt{3}}$. To continue (to get rid of $\sqrt{3}$ in the denominator), we multiply both sides by this $\sqrt{3}$:

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{15}}{3}.$$

Example 6.2. Rationalize the denominator: $\dfrac{6}{5\sqrt{8}}$.

Solution. Following the same method, we multiply top and bottom by $\sqrt{8}$ (it is not needed to multiply also by 5):

$$\frac{6}{5\sqrt{8}} = \frac{6\cdot\sqrt{8}}{5\sqrt{8}\cdot\sqrt{8}} = \frac{6\sqrt{8}}{5\cdot 8} = \frac{6\sqrt{4\cdot 2}}{5\cdot 8} = \frac{6\cdot 2\sqrt{2}}{5\cdot 8} = \frac{3\sqrt{2}}{10} .$$

Here we also simplified $\sqrt{8}$. If you compare initial fraction $\dfrac{6}{5\sqrt{8}}$ with the final answer $\dfrac{3\sqrt{2}}{10}$, you see that they look completely different. However they have exactly the same numerical values (you can check this using calculator).

Fractions with two terms in denominator.

Such fractions have the following general form

$$\frac{\exp r}{m\sqrt{n} - x\sqrt{y}} \quad \text{or} \quad \frac{\exp r}{m\sqrt{n} + x\sqrt{y}} ,$$

where expr, as before, means some expression. The denominators in these two fractions are **conjugate** to each other. As we already saw in example 5.9 from the previous session, their product is the rational expression (i.e. expression with no radicals):

$$\left(m\sqrt{n} - x\sqrt{y}\right)\left(m\sqrt{n} + x\sqrt{y}\right) = m^2 n - x^2 y$$

This property allows to rationalize denominators: multiply both sides of given fraction by expression which is conjugate to the denominator.

Note. If you try to multiply both sides of the above fractions only by one of the radicals, you will still have radicals in denominator. So, do not confuse two cases: single term in the denominator (when we multiply both sides of the fraction by radical in the denominator) and two terms (when we multiply both sides by expression conjugate to the entire denominator).

Example 6.3. Rationalize the denominator: $\dfrac{1}{2-\sqrt{3}}$.

Solution. Here we have two terms in the denominator: 2 and $\sqrt{3}$. Therefore, we multiply both sides of the fraction by the expression conjugate to denominator. This expression is $2+\sqrt{3}$.

$$\frac{1}{2-\sqrt{3}} = \frac{1\cdot\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)} = \frac{2+\sqrt{3}}{2^2 - \left(\sqrt{3}\right)^2} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3} .$$

Notice that we've came up to a pretty nice equality $\dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3}$.

Example 6.4. Rationalize the denominator: $\dfrac{\sqrt{x}-3}{3\sqrt{x}+2\sqrt{y}}$.

Solution. The expression conjugate to the denominator is $3\sqrt{x}-2\sqrt{y}$. We multiply both sides of the fraction by it:

$$\frac{\sqrt{x}-3}{3\sqrt{x}+2\sqrt{y}} = \frac{\left(\sqrt{x}-3\right)\left(3\sqrt{x}-2\sqrt{y}\right)}{\left(3\sqrt{x}+2\sqrt{y}\right)\left(3\sqrt{x}-2\sqrt{y}\right)} = \frac{\sqrt{x}\cdot3\sqrt{x}-2\sqrt{x}\cdot\sqrt{y}-3\cdot3\sqrt{x}+3\cdot2\sqrt{y}}{3^2\left(\sqrt{x}\right)^2-2^2\left(\sqrt{y}\right)^2}$$

$$= \frac{3x-2\sqrt{xy}-9\sqrt{x}+6\sqrt{y}}{9x-4y} .$$

Solving Radical Equations

We consider here equations that contain radicals. They can be transformed to equations with no radicals by making two simple steps: isolate radical (i.e. leave it along on one side of the equation), and then square both sides of the equation.

Note. When you square both sides, it is possible to get answers that are not roots of the original equation. Here is simple example: $x+1=2$. Obvious, this equation has only one solution $x=1$. Now, let's square both sides of this equation: $(x+1)^2 = 2^2$, or

$$x^2 + 2x+1 = 4 \Rightarrow x^2+2x-3=0 \Rightarrow (x-1)(x+3)=0 .$$

The last equation has two solutions: $x=1$ and $x=-3$. However, the value $x=-3$ is not a root of the original equation $x+1=2$. The conclusion from this note is this: check your final answer with the original equation.

Example 6.5. Solve the equation $\sqrt{2x-1}=7$.

Solution. Here radical is already isolated and the first step is not needed. We just square both sides:

$$\left(\sqrt{2x-1}\right)^2 = 7^2 \Rightarrow 2x-1=49 \Rightarrow 2x=49+1 \Rightarrow 2x=50 \Rightarrow x=25 .$$

It is easy to verify that number 25 is really a root of the original equation:

$\sqrt{2\cdot25-1} = \sqrt{49} = 7$. Final answer: $x=25$.

Example 6.6. Solve the equation $\sqrt{6x-5}+7=6$.

Solution. Here the radical is not isolated, and we isolate it by moving number 7 to the right side (this is the same as subtracting 7 from both sides):

$$\sqrt{6x-5} = 6-7 \text{ or } \sqrt{6x-5} = -1.$$

Now, radical is isolated and we square both sides:

$$\left(\sqrt{6x-5}\right)^2 = (-1)^2 \Rightarrow 6x-5 = 1 \Rightarrow 6x = 1+5 \Rightarrow 6x = 6 \Rightarrow x = 1.$$

So, it looks like the answer is $x = 1$. Let's check it with the original equation:

$$\sqrt{6\cdot 1 - 5} + 7 = \sqrt{1} + 7 = 1 + 7 = 8.$$

But right side of the original equation is 6, not 8, so $x = 1$ is not a solution and we reject it. The original equation doesn't have roots at all.

Note. It is easy to see at the very beginning without doing anything that original equation doesn't have roots. Indeed, square root on the left side is never negative, and by adding to it number 7, we cannot get number 6.

Example 6.7. Solve the equation $\sqrt{4x^2 + 14x + 3} - 2x - 3 = 0$.

Solution. Here again the radical is not isolated, and we isolate it by moving terms $2x$ and 3 to the right side:

$$\sqrt{4x^2 + 14x + 3} = 2x + 3.$$

Now square both sides:

$$\left(\sqrt{4x^2 + 14x + 3}\right)^2 = (2x+3)^2 \Rightarrow 4x^2 + 14x + 3 = 4x^2 + 12x + 9.$$

Reducing (cancel out) $4x^2$ from both sides, we get

$$14x + 3 = 12x + 9 \Rightarrow 2x = 6 \Rightarrow x = 3.$$

Finally, we check $x = 3$ with the original equation:

$$\sqrt{4\cdot 3^2 + 14\cdot 3 + 3} - 2\cdot 3 - 3 = \sqrt{36 + 42 + 3} - 6 - 3 = \sqrt{81} - 9 = 9 - 9 = 0.$$

So, $x = 3$ is a solution.

Further, in the session 8 on quadratic equations (example 8.7) we will consider more complicated example.

Exercises 6

In exercises 6.1 – 6.14, rationalize the denominator and simplify.

6.1. $\sqrt{\dfrac{3}{5}}$

6.2. $\sqrt{\dfrac{6}{7}}$

6.3. $\dfrac{2}{3\sqrt{20}}$

6.4. $\dfrac{3}{5\sqrt{27}}$

6.5. $\dfrac{4}{5\sqrt{3}}$

6.6. $\dfrac{2}{3\sqrt{5}}$

6.7. $\dfrac{4}{5+\sqrt{3}}$

6.8. $\dfrac{2}{3+\sqrt{5}}$

6.9. $\dfrac{1}{2-\sqrt{3}}$

6.10. $\dfrac{1}{5+\sqrt{6}}$

6.11. $\dfrac{1}{\sqrt{6}+\sqrt{3}}$

6.12. $\dfrac{1}{\sqrt{6}-\sqrt{5}}$

6.13.

$\dfrac{\sqrt{u}+2}{4\sqrt{u}-3\sqrt{v}}$

6.14.

$\dfrac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}+5\sqrt{b}}$

In exercises 6.15 – 6.24, solve given equation.

6.15. $\sqrt{3x+4}=5$

6.16. $\sqrt{4x-3}=7$

6.17. $\sqrt{5x-4}+2=8$

6.18. $\sqrt{7x+1}-4=2$

6.19. $\sqrt{6x-8}+5=3$

6.20. $\sqrt{8x+9}+7=2$

6.21.

$\sqrt{16x^2+70x-7}-4x-7=0$

61.22.

$\sqrt{25x^2-12x+28}-5x+2=0$

6.23.

$\sqrt{9x^2+7x-18}-4=3x$

6.24.

$\sqrt{36x^2-19x-25}+3=6x$

Session 7

Complex Numbers

When we solve linear equation $ax + b = 0$, $a \neq 0$, it always has a unique solution $x = -\dfrac{b}{a}$. In some problems, more complicated equations may appear.

Example 7.1. Suppose that you need to measure a piece of land in the shape of rectangle, having given area A and given perimeter P. What are the sides of this rectangle?

Solution (equation only). Let's denote the sides of the rectangle by letters x and y. Then $x \cdot y = A$ (area), and $2x + 2y = P$ (perimeter). We can solve the last equation for y:

$2y = P - 2x$, and $y = \dfrac{P - 2x}{2}$. If we substitute this expression for y into the first equation $x \cdot y = A$, we will get

$$x \cdot y = x \cdot \frac{P - 2x}{2} = A \implies x(P - 2x) = 2A \implies xP - 2x^2 = 2A \implies 2x^2 - Px + 2A = 0.$$

We come up with the equation that contains x^2. Such equations are called the **quadratic equations** (as opposite to linear equations: $ax + b = 0$). We already considered simple cases when quadratic equation can be solved by factoring. We will discuss non-factorable cases in the next session.

Contrary to linear equations, quadratic equations not always have (real) solutions. A simple example is the equation $x^2 + 1 = 0$. It can be written as $x^2 = -1$. Obvious, this equation does not have solutions because square of a real number cannot be negative.

However, it is possible to introduce a special symbol that can be treated as a solution of the equation $x^2 + 1 = 0$. Usually, this symbol is denoted by the letter i and is called the **imaginary unit** (that's why the letter i). Of course, i is not a real number. It has the property that $i^2 = -1$. Also, we can write $i = \sqrt{-1}$.

Note. You may be disappointed with such "definition" of number i. Indeed, it looks like we introduce an object that does not exist: the equation $x^2 + 1 = 0$ does not have real solutions, and we use letter i for non-existing solution. If you have such feelings, you are not along. For more than two hundred years similar feelings had many mathematicians. Only in 18th century the exact theory of so-called **complex numbers** was created which includes symbol i as well as other related to it "magic" numbers. In short, a complex number is the same as ordered pair of two real numbers.

Definition. Any complex number z has the form $z = a + bi$. Here a and b are two real numbers, and i is a symbol with the property $i \cdot i = i^2 = -1$. Actually, we are saying that $i \cdot i$ is equal to -1 **by definition.** Symbol i is called the **imaginary unit**, number a is the **real part**, and number b is the **imaginary part** of complex number z. The form $a + bi$ is called the **standard form** of complex number.

Using the symbol i, a square root of any negative number can be written as a complex number: if a is a non-negative number, then $\sqrt{-a} = \sqrt{a} \cdot \sqrt{-1} = \sqrt{a} \cdot i$, so

$$\boxed{\sqrt{-a} = \sqrt{a} \cdot i} \quad , \ a \geq 0 .$$

Example 7.2. Express $\sqrt{-48}$ in terms of i and simplify.

Solution. $\sqrt{-48} = \sqrt{48} \cdot i = \sqrt{16 \cdot 3} \cdot i = 4\sqrt{3} \cdot i$.

We can operate with complex numbers in the same way as with polynomials or rational expressions: we can make all arithmetic operations with them, open parentheses, combine like terms. The only specific property is that if we multiply i by i, the result is -1. In particular, two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are equal, if separately equal their real and imaginary parts: $a_1 = a_2$ and $b_1 = b_2$.

Example 7.3. Show that any real number can be considered as a complex number.

Solution. If a is a real number, then it can be written as a complex one: $a = a + 0i$.

So, we can treat the set of complex numbers as an extension of the set of real numbers. Let's consider arithmetic operations with complex numbers.

Example 7.4.

1) Add $(3 + 2i) + (1 + 5i)$.
2) Subtract $(6 + 3i) - (4 - 7i)$.
3) Multiply $(1 + i)(2 + 3i)$.
4) Multiply $(3 + 4i)(3 - 4i)$.

Solution. As we mentioned above, we can perform these operations as with ordinary algebraic expressions just keeping in mind that $i^2 = -1$.

1) $(3 + 2i) + (1 + 5i) = (3 + 1) + (2 + 5)i = 4 + 7i$.
2) $(6 + 3i) - (4 - 7i) = (6 - 4) + (3 - (-7))i = 2 + 10i$.
3) $(1+i)(2+3i) = 2 + 3i + 2i + 3i^2 = 2 + 3i + 2i + 3 \cdot (-1) = 2 + 3i + 2i - 3 = -1 + 5i$.
4) We can proceed the same way as in part 3) or use as a shortcut formula that we already used in sessions 2 (see note after example 2.11) and in session 5 (see example 5.8): $(a - b)(a + b) = a^2 - b^2$. Using this formula we have

$$\left(3 + 4i\right)\left(3 - 4i\right) = 3^2 - \left(4i\right)^2 = 9 - 16 \cdot (-1) = 9 + 16 = 25$$

Notice that the result of multiplication here is a real number. We will use similar result in general form below.

Now, consider division of complex numbers. At the first glance, the quotient of two complex numbers does not look as a complex number. For example, does the quotient

$\dfrac{5+4i}{3+2i}$ can be represented as one single complex number? The answer is yes. The method to do that is similar to that we used to rationalize the denominator of radical expressions with two terms in denominators: we multiplied both sides of a fraction by expression conjugate to the denominator. In similar way, consider the following pair of complex numbers z_1 and z_2:

$$z_1 = a+bi, \quad z_2 = a-bi.$$

These numbers have the same real parts, while imaginary parts are differed only by sign. Such complex numbers are called **complex conjugate** to each other. The important property is that if we multiply them, the result is real number:

$$z_1 \cdot z_2 = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2.$$

This property allows to represent the quotient $\dfrac{c+di}{a+bi}$ as a single complex number by multiplying both sides of this fraction by $a-bi$ which is conjugate to the denominator:

$$\frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{ac - bci + adi - bdi^2}{a^2 + b^2} = \frac{(ac+bd)+(ad-bc)i}{a^2 + b^2}$$

So,

$$\frac{c+di}{a+bi} = \frac{ac+bd}{a^2+b^2} + \frac{ad-bc}{a^2+b^2} i.$$

As you see, the result is written as the sum of two parts: real and imaginary. Therefore, this result is a complex number in standard form. We got a general formula for division of two complex numbers.

Note. The above formula looks rather complicated. Don't worry: you do not need to memorize it. Just keep in mind the **method** for division complex numbers: multiply top and bottom by the number conjugate to the denominator.

Example 7.5. Divide $(5 + 4i)$ by $(3 - 2i)$.

Solution.

$$\frac{5+4i}{3-2i} = \frac{(5+4i)(3+2i)}{(3-2i)(3+2i)} = \frac{15+10i+12i+8i^2}{9+4} = \frac{(15-8)+(10+12)i}{13} = \frac{7+22i}{13} = \frac{7}{13} + \frac{22}{13} i.$$

If denominator of a fraction contains only imaginary part of complex number (so, real part is equal to zero), to divide, simply multiply top and bottom by i.

Example 7.6. Divide 1 by i.

Solution. $\dfrac{1}{i} = \dfrac{1 \cdot i}{i \cdot i} = \dfrac{i}{-1} = -i$. This result can also be written as $i^{-1} = -i$.

In conclusion consider how to raise imaginary unit i to positive integer power.

Example 7.7. Calculate

    1) i^3 2) i^4 3) i^5 4) i^6

Solution.

1) $i^3 = i^2 \cdot i = (-1) \cdot i = -i$.

2) $i^4 = i^3 \cdot i = (-i) \cdot i = -i^2 = -(-1) = 1$.

3) $i^5 = i^4 \cdot i = 1 \cdot i = i$.

4) $i^6 = i^5 \cdot i = i \cdot i = -1$.

Note. Similar to part 2) of the above example, $i^n = 1$ if power n is divisible by 4. Indeed, we can write $n = 4k$ where k is an integer, and $i^n = i^{4k} = \left(i^4\right)^k = 1^k = 1$. Based on this property, we can calculate i^n for any positive integer n.

Example 7.8. Calculate

    1) i^{100} 2) i^{101} 3) i^{102} 4) i^{103}

Solution. Notice that 100 is divisible by 4, and $101 = 100 + 1$, $102 = 100 + 2$, $103 = 100 + 3$. Therefore,

$$i^{100} = 1, \ i^{101} = i^{100} \cdot i = i, \ i^{102} = i^{100} \cdot i^2 = -1, \ i^{103} = i^{100} \cdot i^3 = -i.$$

Exercises 7

In exercises 7.1 and 7.2, represent given expression in terms of i and simplify.

7.1. a) $\sqrt{-25}$

    b) $\sqrt{-32}$

7.2. a) $\sqrt{-16}$

    b) $\sqrt{-27}$

In exercises 7.3 and 7.4, add or subtract.

7.3. a) $(7+3i)+(2-5i)$

    b) $(4-6i)-(8+2i)$

7.4. a) $(9-4i)+(3-i)$

    b) $(6-3i)-(2-8i)$

In exercises 7.5 and 7.6, multiply complex numbers.

7.5. a) $3(2-4i)$

    b) $(6-3i)(2-8i)$

    c) $(5+3i)(6-5i)$

    d) $(3+2i)(3-2i)$

7.6. a) $4(3+5i)$

    b) $7i(6-3i)$

    c) $(4-6i)(7+i)$

    d) $(2-4i)(2+4i)$

In exercises 7.7 and 7.8, divide. Write the answer in standard form for comples numbers.

7.7. a) $(8+2i)\div 5$

    b) $3\div(2i)$

    c) $(9-6i)\div(-4i)$

    d) $(-4+3i)\div(5+6i)$

    e) $(3-2i)\div(4-5i)$

7.8 . a) $(-6+7i)\div 2$

    b) $2\div(-3i)$

    c) $(4+3i)\div(6i)$

    d) $(5-6i)\div(4-3i)$

    e) $(4+5i)\div(-3+2i)$

In exercises 7.9 and 7.10, calculate

7.9. a) i^{2018}

    b) i^{2019}

    c) i^{2020}

    d) i^{2021}

7.10. a) i^{2023}

    b) i^{2024}

    c) i^{2025}

    d) i^{2026}

Session 8

Quadratic Equations: Factoring and Square Forms

In previous session we mentioned about quadratic equations. These are polynomial equations in which the highest degree of a variable is 2. In general, equation is called the **quadratic**, if it can be written in the form

$$ax^2 + bx + c = 0, \ a \neq 0.$$

This form is called the **standard form**. Here a, b, and c are given constant numbers which are called the coefficients: a is the leading, b is the middle, and c is the last (or free) coefficient.

Note. If we omit the restriction $a \neq 0$, the above equation will not be necessary a quadratic: for $a = 0$, the equation becomes linear $bx + c = 0$. Therefore, we will always assume that the leading coefficient $a \neq 0$. Coefficients b and c may be any real numbers, including 0. Also notice that in the standard form the right side of the equation is equal to zero.

Of course, a quadratic equation may be given in different forms.

Example 8.1. Write the following equations as quadratic equations in standard form. Identify the coefficients a, b, and c.

1) $(2x - 1)(x + 5) = 0.$

2) $(3x + 2)^2 = 5.$

Solution. In both equations we just need to open parentheses, combine like terms, and bring all terms from the right side to the left (if needed).

1) $(2x - 1)(x + 5) = 2x^2 + 10x - x - 5 = 2x^2 + 9x - 5.$

   We've got the standard form

$$2x^2 + 9x - 5 = 0; \quad a = 2, \ b = 9, \ c = -5.$$

2) $(3x + 2)^2 = 5 \ \Rightarrow \ 9x^2 + 12x + 4 = 5 \ \Rightarrow \ 9x^2 + 12x + 4 - 5 = 0 \ \Rightarrow \ 9x^2 + 12x - 1 = 0.$

   We've got the standard form

$$9x^2 + 12x - 1 = 0; \ a = 9, \ b = 12, \ c = -1.$$

Notes.

1) In Example 8.1, 1), we call the equation to be written in the **factoring form**.
2) In Example 8.1, 2), we call the equation to be written in the **square form**.
3) In solving Example 8.1, 2), we used the following formula (square of the sum)

$$(a + b)^2 = a^2 + 2ab + b^2.$$

   Another useful formula is the square of the difference:

$$(a-b)^2 = a^2 - 2ab + b^2.$$

When solving a quadratic equation, it is not needed to always represent it in standard form. In some cases, it is even more preferable another forms like in Example 8.1: factoring form or square form. These are the forms in which the quadratic equation can be solved very easily. Let's consider both forms separately.

Factoring Form of the Quadratic Equation

Before, we already solved some quadratic equations by factoring. Here, we consider this method in more details.

In general, factoring form of quadratic equation looks like this

$$(mx + n)(px + q) = 0.$$

Method to solve this equation is based on the following simple observation: if the product of two values A and B is zero, i.e. $A \cdot B = 0$, then at least one of them is zero: $A = 0$ or $B = 0$. Therefore, the equation $(mx + n)(px + q) = 0$ can be split into two linear equations:

$$mx + n = 0 \text{ and } px + q = 0,$$

which can be easily solved.

Example 8.2. Solve the equation from Example 8.1, 1): $(2x - 1)(x + 5) = 0$.

Solution. Since this equation is written in factoring form, it can be immediately split into two equations: $2x - 1 = 0$ and $x + 5 = 0$. From the first equation, $x = \dfrac{1}{2}$, and from the second, $x = -5$. So, the original equation has two solutions:

$$x = \frac{1}{2} \text{ and } x = -5.$$

Many quadratic equations (but not all) can be easily solved by **factoring**. Using this method, we first represent given equation in factoring form, and then split it into two linear equations like in Example 8.2. The main part of this method is to factor the equation. Let's consider two cases of factoring: when leading coefficient $a = 1$ and $a \neq 1$.

Case to factor: leading coefficient $a = 1$.

In this case the standard form of the equation is

$$x^2 + bx + c = 0.$$

Such equation (when leading coefficient is 1) is called the **reduced** equation. To factor, we need to represent the left side as a product of two linear expressions (two pairs of parentheses): $(x + p)(x + q) = 0$. Let's open parentheses here:

$$x^2 + px + qx + pq = 0 \text{ or } x^2 + (p + q)x + pq = 0.$$

If we compare the last equation with the original $x^2 + bx + c = 0$, we conclude that $p + q = b$, and $p \cdot q = c$. This conclusion gives us an idea: to factor, we need to find two

numbers p and q such that their sum is the middle coefficient b and the product is the last coefficient c.

Technically, to factor we can start with the template (skeleton) for the equation:

$$(x + \underline{})(x + \underline{}) = 0.$$

Then to fill in blanks, consider possible ways to factor the last coefficient c, and select such factors that their sum is b. Replace blanks with these numbers.

Example 8.3. Solve the quadratic equation $x^2 + 5x + 6 = 0$ by factoring.

Solution. Start with the template $(x + \underline{})(x + \underline{}) = 0$. For last coefficient 6, there are two ways to factor: $6 = 2 \times 3$ and $6 = 1 \times 6$. We select 2 and 3 since their sum is the middle coefficient 5. Replacing blanks with these numbers, we get the factoring form

$$(x + 2)(x + 3) = 0.$$

Now split it into two equations: $x + 2 = 0$ and $x + 3 = 0$. Solve them and get two solutions:

$$x = -2 \text{ and } x = -3.$$

Note. Keep in mind that in factoring form, the right side of given equation must be zero. For example, the equation $(x - 1)(x + 2) = 4$ is **NOT** written in factoring form and cannot be split immediately into two liner equations.

Example 8.4. Solve the above equation $(x - 1)(x + 2) = 4$ by factoring.

Solution. To write this equation in factoring form, we first represent it in standard form by opening parentheses and combining like terms

$$x^2 + 2x - x - 2 = 4, \text{ or } x^2 + x - 6 = 0.$$

Now to factor, we write the template $(x + \underline{})(x + \underline{}) = 0$, and try to find two numbers such that the product is -6 and the sum is 1 (which is coefficient for x). By guessing and checking we find 3 and -2. Substitute these numbers for blanks and get the factoring form $(x + 3)(x - 2) = 0$. Solving the equations $x + 3 = 0$ and $x - 2 = 0$, we get two solutions:

$$x = -3 \text{ and } x = 2.$$

Example 8.5. Solve the quadratic equation $x^2 + 7x = 0$ by factoring.

Solution. Here coefficient $c = 0$. Such an equation is easy to factor just by taking x out of parentheses: $x(x + 7) = 0$. From here, $x = 0$ and $x + 7 = 0 \Rightarrow x = -7$. Final answer:

$$x = 0 \text{ and } x = -7.$$

Example 8.6. Solve the quadratic equation $3x^2 - 48 = 0$ by factoring.

Solution. Here the leading coefficient is not 1 (it is 3). Notice, however, that both coefficients divisible by 3, and we can factor out this 3: $3(x^2 - 16) = 0$. From here we conclude that expression inside parentheses must be zero, so we just drop factor 3 and get equation with leading coefficient 1: $x^2 - 16 = 0$. Another way to get this equation is just

to divide coefficients of the original equation by 3. Now to factor the left side of this equation, we use the formula for factoring the difference of two squares (we already used this formula several times, see, for example, note after example 2.11 from section 2): $a^2 - b^2 = (a-b)(a+b)$. Using this formula, we factor the last equation as $(x-4)(x+4) = 0$, then solve two equations $x-4 = 0$ and $x+4 = 0$, and get final answer: $x = 4$ and $x = -4$. Final answer can also be written as $x = \pm 4$, meaning that we combined both roots in one formula.

Some equations are not quadratic, but can be reduced for such. Let's solve the following radical equation, using the technique that we used in session 6 for solving radical equations: isolate radical and square both sides.

Example 8.7. Solve the equation $\sqrt{2x+11} - x = 4$.

Solution. Here the radical is not isolated, and we isolate it by moving x to the right side (or adding x to both sides): $\sqrt{2x+11} = 4 + x$. Now we square both sides:

$$\left(\sqrt{2x+11}\right)^2 = (4+x)^2 \Rightarrow 2x+11 = 16 + 8x + x^2.$$

We've got a quadratic equation. Let's write it in standard form $ax^2 + bx + c = 0$. For this, we first switch left and right sides: $16 + 8x + x^2 = 2x + 11$, and then move $2x + 11$ to the right:

$$16 + 8x + x^2 - 2x - 11 = 0 \text{ or } x^2 + 6x + 5 = 0.$$

The last equation can be solved by factoring: $(x+1)(x+5) = 0$. We've got two solutions of the quadratic equation: $x = -1$ and $x = -5$. Let's check them with the original equation.

$x = -1$: $\sqrt{2x+11} - x = \sqrt{2\cdot(-1)+11} - (-1) = \sqrt{9} + 1 = 3 + 1 = 4$. So, $x = -1$ is a solution.

$x = -5$: $\sqrt{2x+11} - x = \sqrt{2\cdot(-5)+11} - (-5) = \sqrt{1} + 5 = 1 + 5 = 6 \neq 4$. So, $x = -5$ is not a solution and we reject it.

Final answer: original equation has only one solution $x = -1$.

Note. As we see in the above example, and as we mentioned in session 6 on solving radical equations, it is important to check final answer with the original equation to exclude possible wrong solutions.

Case to factor: leading coefficient $a \neq 1$.

This is more complicated case for factoring of the equation $ax^2 + bx + c = 0$. We will show a method how to reduce it to the case $a = 1$. The method includes three steps.

1) Construct (temporary) a new reduced equation $x^2 + bx + ac = 0$. In words: take away coefficient a from x^2 and multiply it by c.

2) Factor this new equation: $(x + p)(x + q) = 0$.

3) Divide both numbers p and q by a: $\left(x+\dfrac{p}{a}\right)\left(x+\dfrac{q}{a}\right)=0$. This is the factoring form of the original equation.

Example 8.8. Solve the equation $6x^2+5x-4=0$ by factoring.

Solution. Let's use the above method.

1) Construct the reduced equation $x^2+5x-6\cdot4=0$ or $x^2+5x-24=0$.

2) To factor it, find two numbers such that the product is -24 and the sum is 5. These numbers are 8 and -3. The above equation is factorized: $(x+8)(x-3)=0$.

3) Divide both numbers 8 and -3 by 6 (the coefficient for x^2) and get the factoring form of the original equation:

$$\left(x+\frac{8}{6}\right)\left(x-\frac{3}{6}\right)=0 \text{ or } \left(x+\frac{4}{3}\right)\left(x-\frac{1}{2}\right)=0.$$

Now split this equation into two: $x+\dfrac{4}{3}=0$ and $x-\dfrac{1}{2}=0$. By solving, we get two solutions of the original equation: $x=-\dfrac{4}{3}$ and $x=\dfrac{1}{2}$.

Square Form of the Quadratic Equation

Not every quadratic equation can be solved by factoring (using rational numbers). We will consider here another form of quadratic equation that also can be solved very easily but not by factoring. This is the square form such as in Example 8.1, 2).

In general, the **square form** can be written as

$$(px+q)^2=r.$$

To solve this equation, just take square root from both sides. However, we need to be careful with the square root of number r. There are two numbers such that their squares are r: $\sqrt{r}$ and $-\sqrt{r}$. Both numbers must be taken into account. Therefore, after taking square root from both sides of the equation $(px+q)^2=r$, we split it into two linear equations: $px+q=\sqrt{r}$ and $px+q=-\sqrt{r}$. It is a very common to write both equations in just one using the symbol "$\pm$": $px+q=\pm\sqrt{r}$. From here, $px=-q\pm\sqrt{r}$, and $x=\dfrac{-q\pm\sqrt{r}}{p}$. This is the final answer for the solutions of the equation $(px+q)^2=r$.

Note. Keep in mind that the formula $x=\dfrac{-q\pm\sqrt{r}}{p}$ means combination of two formulas:

$$x=\frac{-q+\sqrt{r}}{p} \text{ and } x=\frac{-q-\sqrt{r}}{p}.$$

Example 8.9. Solve the quadratic equation from Example 8.1, 2): $(3x+2)^2 = 5$.

Solution. Take square root from both sides using the symbol "$\pm$": $3x+2 = \pm\sqrt{5}$. From

here, $3x = -2 \pm\sqrt{5}$ and $x = \dfrac{-2 \pm\sqrt{5}}{3}$.

Note. The final answer $x = \dfrac{-2 \pm\sqrt{5}}{3}$ may look strange. It represents two **exact** solutions

written in radical form: $x = \dfrac{-2 + \sqrt{5}}{3}$ and $x = \dfrac{-2 - \sqrt{5}}{3}$. If we want to get solutions as

usual decimal numbers, we can do this only approximately. Using a calculator to

approximate $\sqrt{5}$ as 2.236, we can get the following approximations for the solutions:

$$x = \frac{-2 + \sqrt{5}}{3} \approx \frac{-2 + 2.236}{3} \approx 0.079 \quad \text{and} \quad x = \frac{-2 - \sqrt{5}}{3} \approx \frac{-2 - 2.236}{3} = -1.412.$$

Example 8.9 shows also that not any quadratic equation may be solved by factoring in terms of rational numbers.

Example 8.10. Solve the quadratic equation $(4x-3)^2 + 7 = 0$.

Solution. Write the equation in square form: $(4x-3)^2 = -7$. Next, we want to take square root from both sides. Let's recall from the session 7 "Complex Numbers" that a square root of negative number can be written in terms of imaginary unit i: $\sqrt{-7} = \sqrt{7} \cdot i$. Therefore, taking square root from both sides of the equation $(4x-3)^2 = -7$, we get

$4x-3 = \pm\sqrt{7} \cdot i$. From here, $4x = 3 \pm\sqrt{7} \cdot i$, and $x = \dfrac{3 \pm\sqrt{7} \cdot i}{4}$. So, the final answer

represents two complex conjugate numbers:

$$x_1 = \frac{3 + \sqrt{7} \cdot i}{4} = \frac{3}{4} + \frac{\sqrt{7}}{4} i \text{ and } x_2 = \frac{3 - \sqrt{7} \cdot i}{4} = \frac{3}{4} - \frac{\sqrt{7}}{4} i.$$

Example 8.11. Solve the quadratic equation $x^2 + 5 = 0$.

Solution. Let's write it in square form $x^2 = -5$. From here, $x = \pm\sqrt{-5} = \pm\sqrt{5} \cdot i$.

Example 8.12. Solve the quadratic equation $(5x-2)^2 = 0$.

Solution. By taking square root from both sides, we get the only linear equation

$5x-2 = 0$. From here, $x = \dfrac{2}{5}$. So, given quadratic equation has only one (unique)

solution $x = \dfrac{2}{5}$.

In the next session we will discuss how to transform any quadratic equation from general form to square form.

Exercises 8

In exercises 8.1 and 8.2, write given equations as quadratic equations in standard form. Identify the coefficients a, b, and c.

8.1. a) $(3x+7)(x-2)=0$

  b) $(4x-3)^2 = 6$

8.2. a) $(4x-3)(2x+1)=0$

  b) $(5x+2)^2 = 3$

In exercises 8.3 and 8.4, solve given equations.

8.3. a) $(3x+7)(x-2)=0$

  b) $(6x-5)^2 = 0$

8.4. a) $(4x-3)(2x+1)=0$

  b) $(7x+4)^2 = 0$

In exercises 8.5 and 8.6, solve given equations by factoring.

8.5. a) $3x^2 +5x=0$

  b) $2x^2 -32=0$

  c) $x^2 +x-12=0$

  d) $(x+4)(x-3)=8$

8.6. a) $6x^2 -7x=0$

  b) $5x^2 -45=0$

  c) $x^2 -2x-15=0$

  d) $(x+3)(x-5)=9$

In exercises 8.7 and 8.8, solve given equations.

8.7. a) $\sqrt{3x+10} - x = 2$

  b) $\sqrt{3x+7} - x = 3$

8.8. a) $\sqrt{4x+13} - x = 4$

  b) $\sqrt{4x+21} - x = 4$

In exercises 8.9 and 8.10, solve given equations by factoring.

8.9. a) $8x^2 -2x-3=0$

  b) $5x^2 +12x=9$

8.10. a) $6x^2 +13x-5=0$

  b) $7x^2 +3x=4$

In exercises 8.11 and 8.12, solve given equations.

8.11 a) $(4x-3)^2 = 6$

  b) $x^2 +7=0$

  c) $(6x+4)^2 +3=0$

8.12 a) $(5x+2)^2 = 3$

  b) $x^2 +3=0$

  c) $(7x-5)^2 +6=0$

Challenge Problems

8.13. Consider the equation $\sqrt{ax+b^2} - x = b, \ a > 0, \ b > 0.$
Prove the following statements:

1) If $a < b$, then the equation has the only solution $x = 0$.

2) If $a \geq b$, then the equation has two solutions $x = 0$ and $x = a - 2b$.

8.14. Consider the equation $\sqrt{ax+b^2 - ar} - x = b - r, \ a > 0.$
Prove the following statements:

1) If $b < 0$, then the equation has only one solution $x = a + r - 2b$.

2) If $b > a$, then the equation has only one solution $x = r$.

3) If $0 \leq b \leq a$, then the equation has both the above solutions.

Hints: 1) Check that quadratic equation $ax + b^2 - ar = (x+b-r)^2$ has both solutions $x = r$ and $x = a + r - 2b$.

2) Use the property $\sqrt{u^2} = |u| = \begin{cases} u, & \text{if } u \geq 0, \\ -u, & \text{if } u < 0. \end{cases}$

Session 9

Completing the Square and Quadratic Formula

In previous session we saw that if a quadratic equation is written in factoring or square forms, it can be solved very easily. Also, we saw that not any quadratic equation can be written in factoring form (using rational numbers). However, it turns out that any quadratic equation can be written in the square form. In this way, we also get the quadratic formula, i.e. formula that allows to solve any quadratic equation by simply substituting its coefficients into the formula.

Completing the Square

The procedure for converting quadratic equation from the standard form $ax^2 + bx + c = 0$ into the square form $(px + q)^2 = r$ is called the **Completing the Square**. This procedure is based on the following two formulas that we already mentioned in the previous session – square of the sum and square of the difference:

$$(x + p)^2 = x^2 + 2px + p^2 \text{ (Square of the Sum).}$$

$$(x - p)^2 = x^2 - 2px + p^2 \text{ (Square of the Difference).}$$

Let's start with the special case: consider how to complete the square for the equation

$$x^2 + 2px + c = 0 \quad (a = 1,\ b = 2p).$$

We can re-write it like this (bring c to the right):

$$x^2 + 2px = -c.$$

Compare the left side of this equation with the square of the sum formula (written from right to left)

$$x^2 + 2px + p^2 = (x + p)^2.$$

We see that the left side of the equation $x^2 + 2px = -c$ is not complete to be the square $(x + p)^2$ because the term p^2 is missing. To compensate this deficit (to complete the square), we add p^2 to both sides of the equation $x^2 + 2px = -c$, and it becomes

$$x^2 + 2px + p^2 = -c + p^2, \text{ or } \left(x + p\right)^2 = -c + p^2.$$

We have completed the square and presented the equation in the square form.

Now consider the general case of the equation $ax^2 + bx + c = 0$ with arbitrary $a \neq 0$. We describe the procedure to complete the square step-by-step.

Procedure to Complete the Square

1) Divide both sides of the equation $ax^2 + bx + c = 0$ by the leading coefficient a. We get the reduced equation with the leading coefficient 1: $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$.

2) Bring the last term $\dfrac{c}{a}$ to the right: $x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$.

3) Divide middle coefficient $\dfrac{b}{a}$ by 2 and square it. We get $\left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2}$.

4) Add the above expression $\dfrac{b^2}{4a^2}$ to both sides of the equation $x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

5) Complete the square: $\left(x + \dfrac{b}{2a}\right)^2 = -\dfrac{c}{a} + \dfrac{b^2}{4a^2}$.

Note. The most important (and possible most complicated) part of this procedure is step 3): divide middle coefficient by 2 and square it.

Example 9.1. Solve the quadratic equation $x^2 + 6x + 5 = 0$ by completing the square.

Solution. Here the leading coefficient is 1, and step 1) is not needed. Bring the last term 5 to the right: $x^2 + 6x = -5$. Now, according to step 3), divide middle coefficient 6 by 2 and square it: $\left(\dfrac{6}{2}\right)^2 = 3^2 = 9$. Add this 9 to both sides of the equation: $x^2 + 6x + 3^2 = -5 + 9 = 4$. Complete the square: $(x+3)^2 = 4$.

We can finish with the solution by taking square root from both sides: $x + 3 = \pm\sqrt{4} = \pm 2$. From here, $x = -3 \pm 2$, and we get two solutions: $x = -3 + 2 = -1$, and $x = -3 - 2 = -5$. Final answer: $x = -1$ and $x = -5$.

Example 9.2. Solve the quadratic equation $x^2 + 3x - 18 = 0$ by completing the square.

Solution. Again, step 1) is not needed. Bring 18 to the right: $x^2 + 3x = 18$. According to step 3), divide 3 by 2 and square: $\left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$. Add $\dfrac{9}{4}$ to both sides of the equation:

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = 18 + \frac{9}{4} = \frac{81}{4}. \text{ Complete the square: } \left(x + \frac{3}{2}\right)^2 = \frac{81}{4}.$$

From here, $x + \dfrac{3}{2} = \pm\sqrt{\dfrac{81}{4}} = \pm\dfrac{9}{2}$, or $x = -\dfrac{3}{2} \pm \dfrac{9}{2}$. We get two solutions:

$$x = -\frac{3}{2} + \frac{9}{2} = \frac{6}{2} = 3 \text{ and } x = -\frac{3}{2} - \frac{9}{2} = -\frac{12}{2} = -6.$$

Final answer: $x = 3$ and $x = -6$.

Example 9.3. Solve quadratic equation $x^2 - 5x + 3 = 0$ by completing the square.

Solution. Bring 3 to the right: $x^2 - 5x = -3$. Make steps 3) and 4):

$$\left(-\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4} \quad \text{and} \quad x^2 - 5x + \left(\frac{5}{2}\right)^2 = -3 + \frac{25}{4}.$$

Complete the square $\left(x - \frac{5}{2}\right)^2 = \frac{13}{4}$. From here

$$x - \frac{5}{2} = \pm\sqrt{\frac{13}{4}} = \pm\frac{\sqrt{13}}{2}, \quad x = \frac{5}{2} \pm \frac{\sqrt{13}}{2} = \frac{5 \pm \sqrt{13}}{2}.$$

The answer is represented in radical form that combines two solutions

$$x = \frac{5 + \sqrt{13}}{2} \quad \text{and} \quad x = \frac{5 - \sqrt{13}}{2}$$

in one formula.

Example 9.4. Solve quadratic equation $3x^2 + 5x + 2 = 0$ by completing the square.

Solution. Here leading coefficient is not 1 (it is 3) and step 1) is needed.

1) Divide both sides by leading coefficient 3: $x^2 + \frac{5}{3}x + \frac{2}{3} = 0$.

2) Bring $\frac{2}{3}$ to the right: $x^2 + \frac{5}{3}x = -\frac{2}{3}$.

3) Divide middle coefficient $\frac{5}{3}$ by 2 and square it: $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$:

4) Add the last number to both sides:

$$x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = -\frac{2}{3} + \frac{25}{36}..$$

5) Complete the square: $\left(x + \frac{5}{6}\right)^2 = \frac{1}{36}$.

To get solutions, take square root from both sides: $x + \frac{5}{6} = \pm\sqrt{\frac{1}{36}} = \pm\frac{1}{6}$.

Finally, solve for x: $x = -\frac{5}{6} \pm \frac{1}{6}$. We come up with two solutions: $x = -\frac{5}{6} + \frac{1}{6} = -\frac{4}{6} = -\frac{2}{3}$

and $x = -\frac{5}{6} - \frac{1}{6} = -\frac{6}{6} = -1$. Final answer: $x = -\frac{2}{3}$ and $x = -1$.

Note. Probably, it would be easier to solve the equations in examples 9.1, 9.2, and 9.4 by factoring. However, we used the method of completing the square to demonstrate its

universal character: any quadratic equation can be solved by this method, contrary to the method of factoring that does not work in all cases as we can see from Example 9.3.

Quadratic Formula

To get this formula, we will use the result from step 5) above for completing the square. Starting with the equation $ax^2 + bx + c = 0$, $a \neq 0$, we already represented it in step 5) in the square form

$$\left(x + \frac{b}{2a} \right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

Let's combine fractions on the right side:

$$-\frac{c}{a} + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Now we can write

$$\left(x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take square roots from both sides:

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

To solve for x, bring the term $\dfrac{b}{2a}$ to the right:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Finally, we obtain the **quadratic formula**

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This formula gives the solution of quadratic equation written in standard form

$$ax^2 + bx + c = 0, \ a \neq 0.$$

Quadratic formula also allows to get an idea about possible solutions of quadratic equation. Let's analyze it. First of all, this formula represents two solutions x_1 and x_2:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ \text{ and } \ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

written in one, using "$\pm$" operation.

Also notice that the most complicated part of these formulas is the expression inside square root: $b^2 - 4ac$. This expression is important and it is given a special name.

Definition. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, and denoted by letter D: $D = b^2 - 4ac$.

Using discriminant, quadratic formula can be written in slightly more simple form

$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \text{or} \quad x_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{D}}{2a}.$$

Note. For the reduced quadratic equation $x^2 + bx + c = 0$ (when $a = 1$), $\sqrt{D}$, when $D > 0$, has the following geometric interpretation: it is the distance between roots x_1 and x_2 on number line: $\sqrt{D} = x_1 - x_2$.

As any real number, discriminant D may be positive, negative or zero. Let's see how the sign of discriminant affects the roots x_1 and x_2.

1) Discriminant is positive: $D > 0$. In this case, as you can see from the quadratic formula, the quadratic equation has two roots x_1 and x_2 which are real numbers and distinct. Roots x_1 and x_2 may be rational or irrational numbers.

2) Discriminant is zero: $D = 0$. In this case, roots x_1 and x_2 coincide and the equation has only one real root $x = -\dfrac{b}{2a}$.

3) Discriminant is negative: $D < 0$. In this case, equation has two complex roots x_1 and x_2 which are conjugate to each other.

As you can see, there are only three options regarding the nature of solutions of quadratic equation: it may have one real solution, two (distinct) real solutions, and two complex conjugate solutions. The sign of discriminant allows to distinguish these three cases.

Let's consider some examples of using quadratic formula. You can use either form: with discriminant or without it. We will use discriminant form

$$x = \frac{-b \pm \sqrt{D}}{2a}, \quad D = b^2 - 4ac.$$

Note. When using quadratic formula, make sure that the quadratic equation is written in the standard form $ax^2 + bx + c = 0$ (the right side must be zero) to be able to identify coefficients a, b, and c correctly.

Example 9.5. Solve quadratic equation $3x^2 + 6x = 2$ by quadratic formula.

Solution. The equation is not in standard form. To get standard form, bring 2 from right side to the left: $3x^2 + 6x - 2 = 0$. Now identify coefficients and calculate the discriminant:

$$a = 3, \ b = 6, \ c = -2, \ D = b^2 - 4ac = 6^2 - 4 \cdot 3 \cdot (-2) = 36 + 24 = 60.$$

By quadratic formula, we get two real solutions

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-6 \pm \sqrt{60}}{2 \cdot 3} = \frac{-6 \pm \sqrt{4 \cdot 15}}{6} = \frac{-6 \pm 2\sqrt{15}}{6} = \frac{-3 \pm \sqrt{15}}{3}.$$

Both of them are irrational numbers.

Example 9.6. Solve quadratic equation $4x^2 - 20x + 25 = 0$ by quadratic formula.

Solution. The equation is already in standard form. We have

$$a = 4, \ b = -20, \ c = 25, \ D = b^2 - 4ac = (-20)^2 - 4 \cdot 4 \cdot 25 = 400 - 400 = 0.$$

Discriminant is zero, and the equation has only one real solution

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-20) \pm \sqrt{0}}{2 \cdot 4} = \frac{20}{8} = \frac{5}{2}.$$

Example 9.7. Solve quadratic equation $6x^2 - 5x + 2 = 0$ by quadratic formula.

Solution. The equation is already in standard form. We have

$$a = 6, \ b = -5, \ c = 2, \ D = b^2 - 4ac = (-5)^2 - 4 \cdot 6 \cdot 2 = 25 - 48 = -23.$$

Discriminant is negative, and the equation has two complex conjugate solutions

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-5) \pm \sqrt{-23}}{2 \cdot 6} = \frac{5 \pm \sqrt{23} \cdot i}{12} = \frac{5}{12} \pm \frac{\sqrt{23}}{12} i.$$

Exercises 9

In exercises 9.1 and 9.2, solve given quadratic equation by **completing the square**.

9.1. a) $x^2 - 4x - 5 = 0$

    b) $x^2 + 5x + 6 = 0$

    c) $x^2 - 7x + 8 = 0$

    d) $2x^2 + 3x - 9 = 0$

9.2. a) $x^2 + 2x - 8 = 0$

    b) $x^2 - 9x + 14 = 0$

    c) $x^2 + 3x - 5 = 0$

    d) $3x^2 + 7x + 2 = 0$

In exercises 9.3 and 9.4, solve given quadratic equation by **quadratic formula**.

9.3. a) $5x^2 - 8x = 1$

    b) $16x^2 + 24x + 9 = 0$

    c) $4x^2 - 7x + 5 = 0$

    d) $-6x^2 + 17x - 12 = 0$

9.4. a) $7x^2 - 6x = 3$

    b) $9x^2 - 42x + 49 = 0$

    c) $6x^2 + 5x + 3 = 0$

    d) $-12x^2 + 28x - 15 = 0$

Session 10

Parabolas

Here we will relate the quadratic **equation** $ax^2 + bx + c = 0$ to quadratic **function** $y = ax^2 + bx + c$. The graph of this function (and the function itself) is called the **parabola**. Recall that for quadratic equation there are three cases of real solutions: it may have one solution, two solutions, or no solutions at all. As a graph, parabola allows to visualize all three cases as well as some other properties of quadratic function.

Let's start with the simplest (or basic) parabola $y = x^2$. Notice, first of all, that this function takes the same values for x and $-x$ since $(-x)^2 = x^2$. In general, if some function $y = f(x)$ has the property $f(-x) = f(x)$, then such a function is called the **even** function. Because both points $(x, f(x))$ and $(-x, f(x))$ are on the graph of the function $f(x)$, and they are symmetric to each other over the y-axis, the graph of any even function is also symmetric over the y-axis. In particular, this property holds for the parabola $y = x^2$. Therefore, if we draw this parabola only for positive x, then we can reflect this graph over the y-axis to get the entire picture.

Another simple property is that for positive x, the bigger x, the bigger x^2. We say that parabola $y = x^2$ increases (for positive x). However, this function is **not linear**. It means that its graph is not a straight line. Instead, the graph is a **curve**. To picture this curve, we can calculate several values of parabola for some values of x. The following table represents one of the possible calculations.

x	0	1	2	3
$y = x^2$	0	1	4	9
(x, y)	(0, 0)	(1, 1)	(2, 4)	(3, 9)

If we plot points (x, y) and connect them with a smooth curve, we will get the picture:
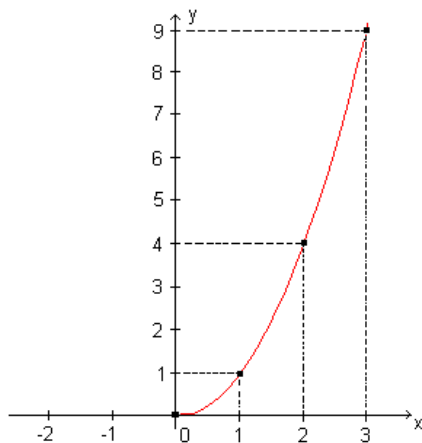
To get the entire parabola (to include negative x), we reflect this graph over the y-axis. Here is the final picture.

$$y = x^2$$
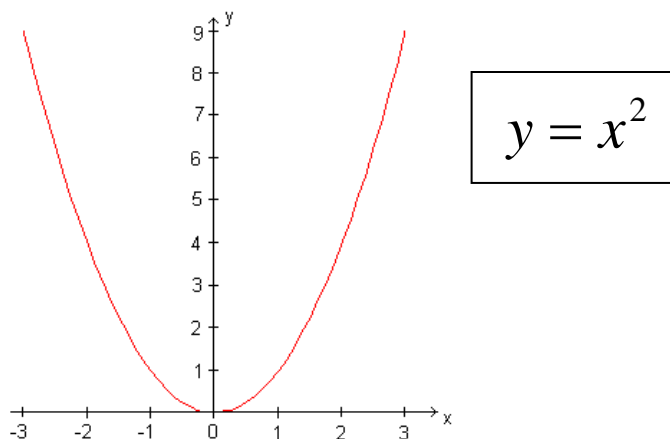
Let's observe this graph. Notice that for negative x parabola decreases (going down from left to right), and for positive x parabola increases. We say that this parabola opens up (or upward). Also, it has minimum (lowest) point (0, 0). This point is called the **vertex** of parabola.

Now consider the second basic parabola $y = -x^2$. We do not need any special analysis to graph this function. Just notice the relation between graphs of functions $y = f(x)$ and $y = -f(x)$. Because points $(x, f(x))$ and $(x, -f(x))$ are symmetric to each other with respect to x-axis, we can simply reflect the graph of $y = x^2$ over the x-axis and get the graph of $y = -x^2$. We say that parabola $y = -x^2$ opens down (downward). Here is its graph:

$$y = -x^2$$
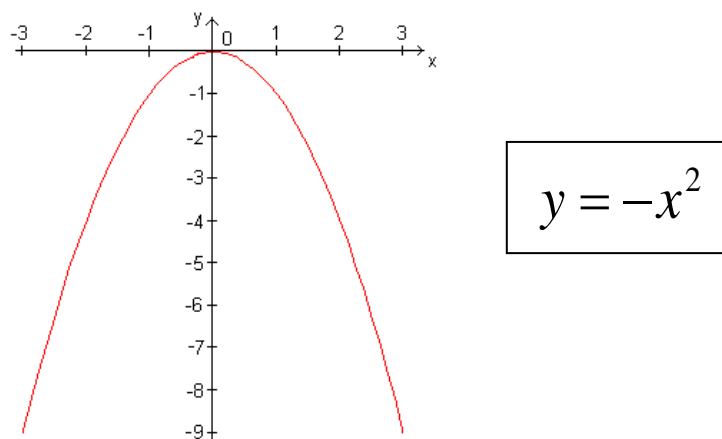
Now consider the general quadratic function $y = ax^2 + bx + c$. It turns out that the shape of its graph is similar to one of the above graphs of $y = x^2$ and $y = -x^2$ (depending on whether the leading coefficient a is positive or negative). To understand why, let's consider three types of transformations (deformations) of graphs of functions. Namely,

assume that we know the graph of $y = f(x)$, and consider how we can construct graphs of functions $y = af(x)$, $y = f(x) + k$, and $y = f(x + h)$.

1) Consider the graph of $y = af(x)$ for positive a. If $a > 1$, this graph is obtained from the graph of $y = f(x)$ by its shrinking towards y-axis (i.e. graph becomes like narrow apple tree), and if $a < 1$, by its stretching away from y-axis (i.e. graph becomes like wide apple tree). So, the shape of the graph of $af(x)$ resembles the graph of $f(x)$.

Graph of $y = -af(x)$ is obtained from the graph of $y = af(x)$ by reflection over the x-axis, so the shape of the graph of $y = -af(x)$ resembles the graph of $y = -f(x)$.

Back to parabola, we conclude that for positive a, the graph of $y = ax^2$ resembles the parabola $y = x^2$ and the graph of $y = -ax^2$ resembles the parabola $y = -x^2$. We need just to stretch or shrink the graphs of $y = x^2$ and $y = -x^2$ with respect to y-axis depending on whether $a < 1$ or $a > 1$. The vertices of parabolas remain at the origin $(0, 0)$. In the picture below, you can see graphs of $y = x^2$, $y = \frac{1}{2}x^2$ and $y = 2x^2$:
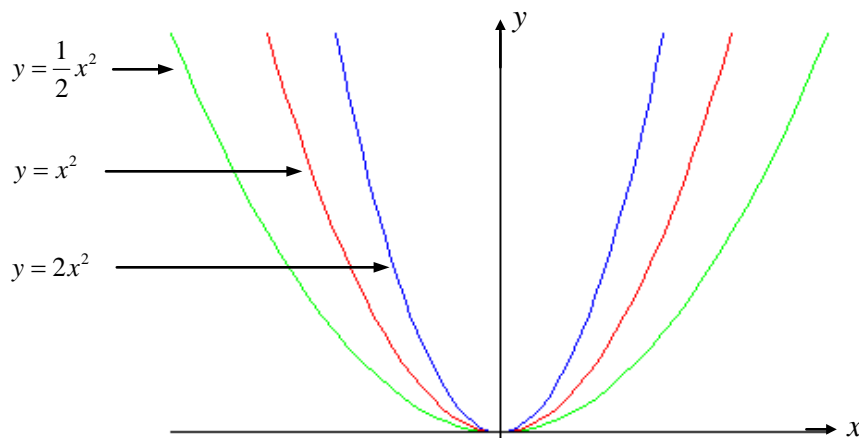
2) The graph of $y = f(x) + k$ is obtained from the graph $y = f(x)$ by its shifting along y-axis k units. If $k > 0$, the graph is shifted up, and if $k < 0$ – down. So, the transformation $f(x) \rightarrow f(x) + k$ does not change the shape of the graph of $f(x)$, it only changes the **position** of the graph.

3) Finally, consider the graph of $y = f(x + h)$. This graph is obtained from the graph of $y = f(x)$ by its horizontal shifting along x-axis by h units. It is important not to confuse the direction of shifting: to the left or to the right. It may seem that for positive h the graph is shifted to the right, and for negative h – to the left. However, this is wrong. The correct answer is just the opposite: if $h > 0$, graph is shifted to the left, and if $h < 0$, to the right. Here is the reason. For positive h, consider two points x_0 and $x_1 = x_0 - h$. Point x_1 lies on the left of x_0. At point x_1, function $f(x + h)$ takes the value $f(x_1 + h) = f(x_0 - h + h) = f(x_0)$, which is the same as the value of $f(x)$ at point x_0. Since $x_1 < x_0$, we have shift to the left. Similar reasoning is true when h is negative. We conclude that the shape of the parabola $y = (x + h)^2$ is exactly

the same as the shape of $y = x^2$, and only the location is different: if $h > 0$, $y = (x+h)^2$ is located h units to the left of $y = x^2$, and if $h < 0$, h units to the right. The same thing is true for functions $y = -(x+h)^2$ and $y = -x^2$. The vertex of both parabolas $y = (x+h)^2$ and $y = -(x+h)^2$ has the coordinates (–h, 0).

If we combine together the above three transformations, we see that if quadratic function is written in the **square form** $y = a(x+h)^2 + k$, then its graph resembles the basic parabolas $y = x^2$ or $y = -x^2$. In particular, the vertex of the parabola $y = a(x+h)^2 + k$ (i.e. its lowest or highest point) has the coordinates (– h, k). The graph of this parabola can be obtained from the graph of $y = ax^2$ by shifting in vertical direction by $|k|$ units (up or down) and in horizontal direction by $|h|$ units (left or right) as described above. The graph is symmetric over the vertical line that passes through its vertex (– h, k), so the line $x = -h$ is the line of symmetry of the parabola. The parabola opens up if $a > 0$, and opens down if $a < 0$.

To graph parabola, which is in the square form $y = a(x+h)^2 + k$, we can use the following steps.

1) Plot the vertex (– h, k).
2) Draw dotted vertical line through the vertex. This is the line of symmetry of the parabola.
3) Identify whether the parabola opens up or down by looking at the sign of the leading coefficient a. If $a > 0$, it opens up, if $a < 0$, it opens down.
4) Draw the parabola. To be more accurate, you may calculate several values of the parabola and plot corresponding points. In particular, you may put $x = 0$ to find y-intercept.

Example 10.1. Graph the parabola $y = 2(x+3)^2 + 4$.

Solution. Let's follow the above steps. We have $a = 2$, $h = 3$, $k = 4$.

1) Plot the vertex (– h, k) = (–3, 4):
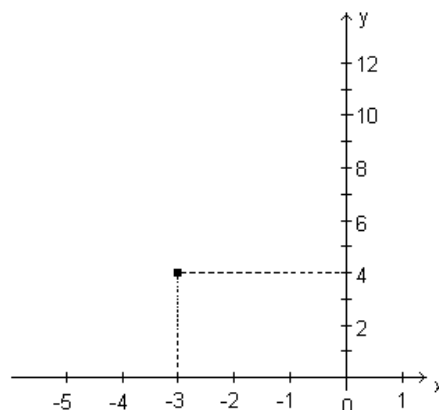
2) Draw dotted vertical line of symmetry through the vertex (–3, 4):
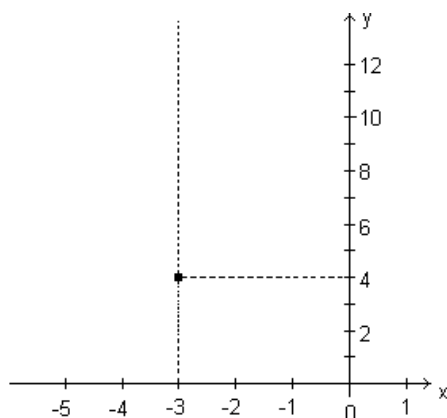
3) Identify how parabola opens (up or down) looking at the leading coefficient $a = 2$. It is positive, so parabola opens up.

4) To draw parabola more accurate, calculate several values:

$$y(-1) = 2(-1+3)^2 + 4 = 12$$, so the graph contains the point $(-1, 12)$.

$$y(-2) = 2(-2+3)^2 + 4 = 6$$, so the graph contains the point $(-2, 6)$.

The parabola looks like this
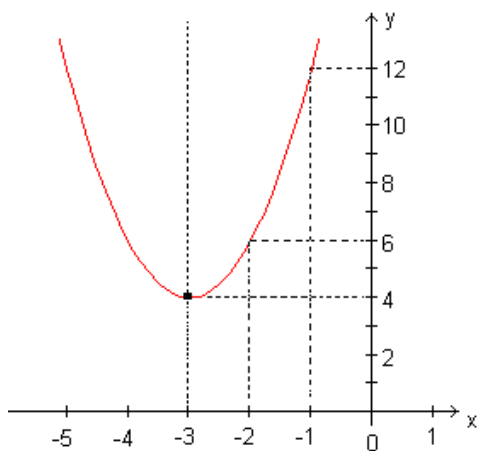
Note. It is not needed to always show dotted line for the line of symmetry. Final picture may look like this:
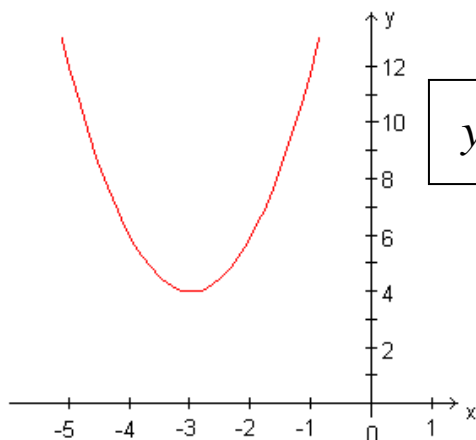
$$y = 2(x+3)^2 + 4$$

Notice that any quadratic function $y = ax^2 + bx + c$ can be represented in the square form $y = a(x+h)^2 + k$ (and, therefore, its graph has the same shape as above). Actually, we already did it in the previous session when we discussed the method of the completing the square for quadratic equation. Let's repeat this procedure one more time (with small modification). We can use these steps.

1) On the right side of $y = ax^2 + bx + c$, factor out the coefficient a from the first two terms:

$$y = a\left(x^2 + \frac{b}{a}x\right) + c.$$

2) Divide coefficient $\frac{b}{a}$ by 2 and square it: $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

3) Inside parentheses (in step 1), add and subtract the above expression:

$$x^2 + \frac{b}{a}x = x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}.$$

4) The first three terms on the right side of the above expression can be written as $\left(x + \frac{b}{2a}\right)^2$, therefore

$$x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}.$$

5) Write the expression for y as

$$y = a\left(x^2 + \frac{b}{a}x\right) + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

We've got the square form $y = a(x+h)^2 + k$ of the parabola $y = ax^2 + bx + c$:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

where

$$h = \frac{b}{2a}, \quad k = \frac{4ac - b^2}{4a}.$$

Actually, to draw the graph of the parabola $y = ax^2 + bx + c$, you do not need to go through the above steps every time. Just memorize the most important formula for **the x-coordinate of the vertex of parabola**, which we denote as x_v:

$$x_v = -\frac{b}{2a}$$

This is the first coordinate $-h$ for the vertex $(-h, k)$. The second coordinate k is the value $k = y_v = \frac{4ac - b^2}{4a}$. This formula is more complicated and you do not need to memorize it.

This coordinate can be calculated by substitution the value of x_v for x in the original parabola $y = ax^2 + bx + c$.

Here are possible steps to graph the general parabola $y = ax^2 + bx + c$.

1) Identify coefficients a, b, and c.

2) Calculate the x-coordinate x_v of the vertex of the parabola: $x_v = -\frac{b}{2a}$.

3) Calculate the y-coordinate y_v of the vertex by substituting x_v in the original equation.

4) Follow the above steps for graphing the parabola $y = a(x+h)^2 + k$, where $h = -x_v$, $k = y_v$.

Example 10.2. Graph the parabola $y = -2x^2 + 8x - 5$.

Solution.

1) Identify the coefficients a, b, and c: $a = -2$, $b = 8$, $c = -5$.

2) Calculate the x-coordinate of the vertex: $x_v = -\frac{b}{2a} = -\frac{8}{2 \cdot (-2)} = 2$.

3) Calculate the y-coordinate of the vertex by substitution $x_v = 2$ in the original equation: $y_v = -2 \cdot 2^2 + 8 \cdot 2 - 5 = 3$.

So, the vertex of the parabola has the coordinates $(2, 3)$.

4) Draw the parabola according to the steps described for the square form $y = a(x+h)^2 + k$. In particular, parabola opens down (because $a = -2 < 0$) and has a vertical line of symmetry that passes through the vertex $(2, 3)$. Also (for more accuracy), we can calculate the values $y(0) = -5$, and $y(1) = 1$. Here is the picture.
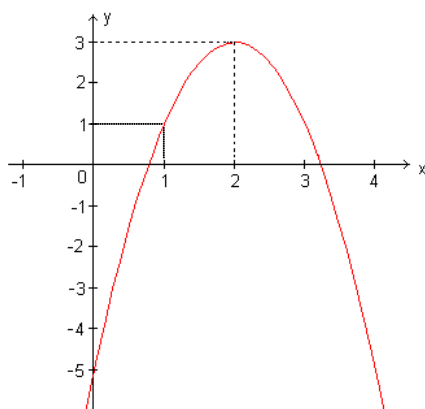
$$y = -2x^2 + 8x - 5$$

Note. To graph the parabola more accurate, you may want to find x- and y-intercepts. It is very easy to find y-intercept. Indeed, any point on y-axis has the first coordinate zero, so just put $x = 0$ into the function $y = ax^2 + bx + c$, and you will get that y-intercept is equal to c. To find x-intercepts, put $y = 0$ and solve the quadratic equation $ax^2 + bx + c = 0$.

Consider in more details how parabola shows possible cases about the number of real solutions (roots) of the equation $ax^2 + bx + c = 0$. In general, solving the equation $f(x) = 0$ means to find all values of x for which the function $y = f(x)$ takes the value of zero: $y = 0$. Geometrically, points $(x, 0)$ lie on the x-axis. Therefore, roots of the equation $f(x) = 0$ are x-coordinates of points of intersection of the graph of $y = f(x)$ with the x-axis. So, to solve the equation $f(x) = 0$ we just need to find all x-intercepts of the graph of the function $y = f(x)$.

In particular, to solve the quadratic equation $ax^2 + bx + c = 0$, we need to find all x-intercepts of the parabola $y = ax^2 + bx + c$. We consider the case $a > 0$ (parabola opens up). The case $a < 0$ is similar. Obvious there are only three possible positions of the parabola with respect to x-axis:

1) Vertex of the parabola is located below x-axis. In this case there are two x-intercepts, so there are two roots of the quadratic equation.

2) Parabola touches the x-axis at one point (at the vertex), so there is only one root.

3) Parabola is located above x-axis. In this case, no x-intercepts, so no roots.

Here are the corresponding pictures:
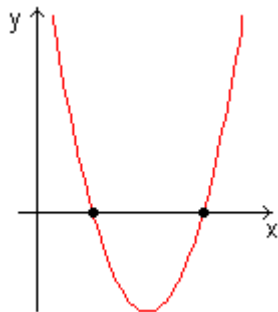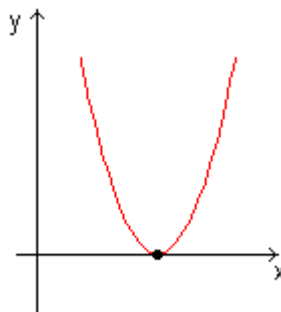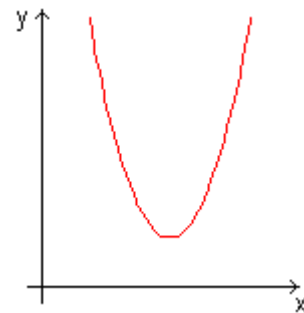
| Two roots | One root | No roots |

Exercises 10

In exercises 10.1 and 10.2,

    1) Identify coordinates of the vertex of given parabola.

    2) Determine whether the parabola opens up or down

10.1 a) $y = 3(x+4)^2 - 5$ 10.2 a) $y = -5(x-3)^2 + 7$

       b) $y = -4(x-2)^2 + 6$ b) $y = 6(x+7)^2 - 3$

       c) $y = 7(x-5)^2 - 4$ c) $y = -2(x+1)^2 + 8$

In exercises 10.3 and 10.4, the given graphs are shifted graphs of the parabola $y = 3x^2$.
Write the equations of the drawn parabolas

    1) In square form $y = a(x+h)^2 + k$.

    2) In general form $y = ax^2 + bx + c$.

10.3.
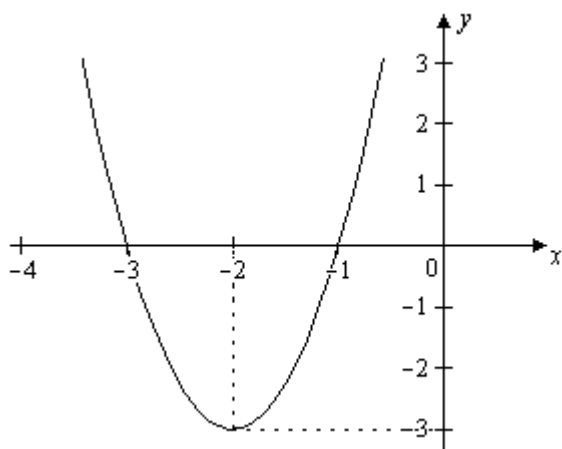
        a)   b)
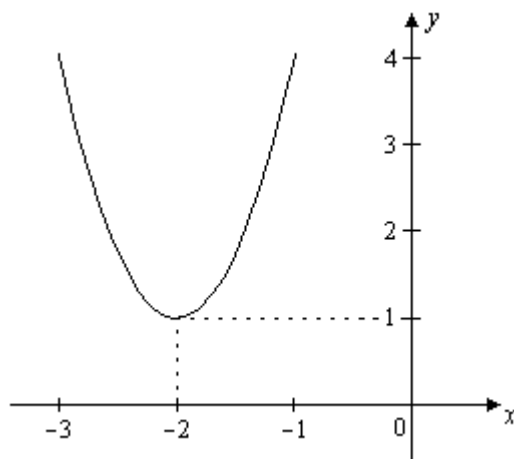
10.4.

a)
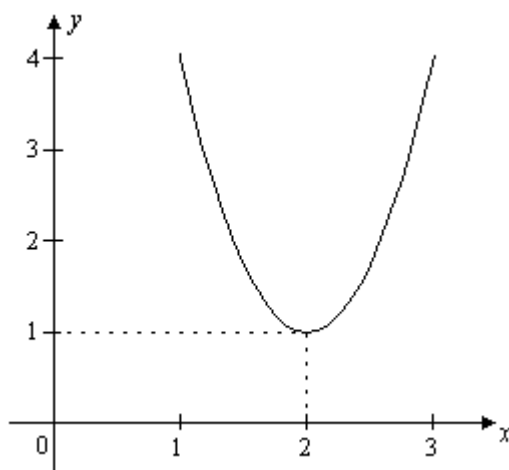
b)
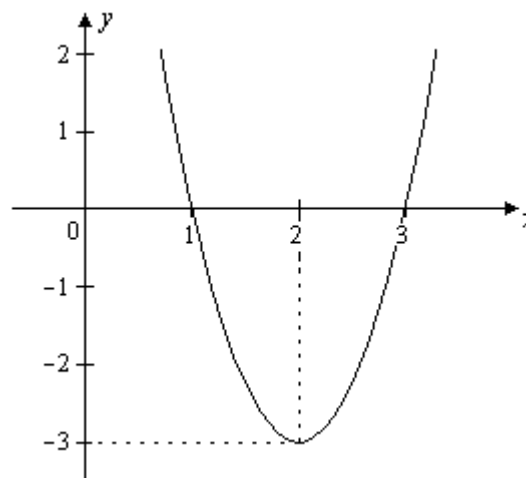
In exercises 10.5 and 10.6, the given graphs are shifted graphs of the parabola $y = -2x^2$.
Write the equations of the drawn parabolas

1) In square form $y = a(x+h)^2 + k$.

2) In general form $y = ax^2 + bx + c$.

10.5.

a)

b)

10.6.

a)

b)
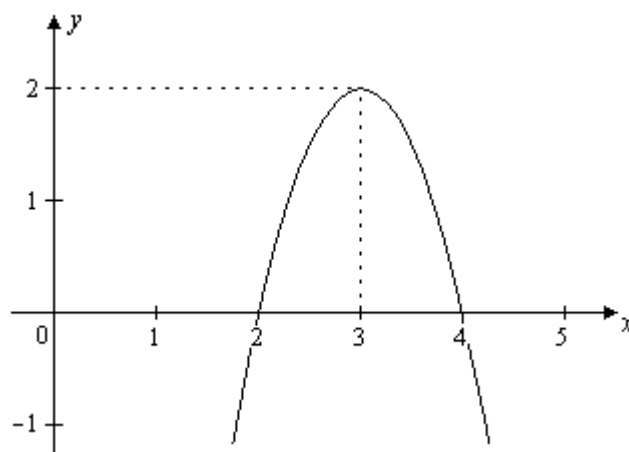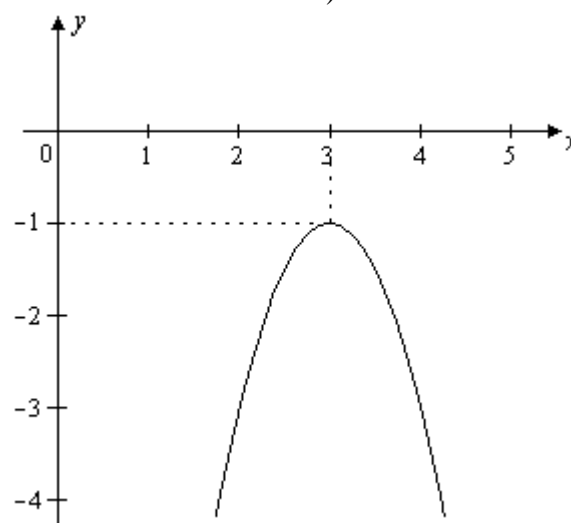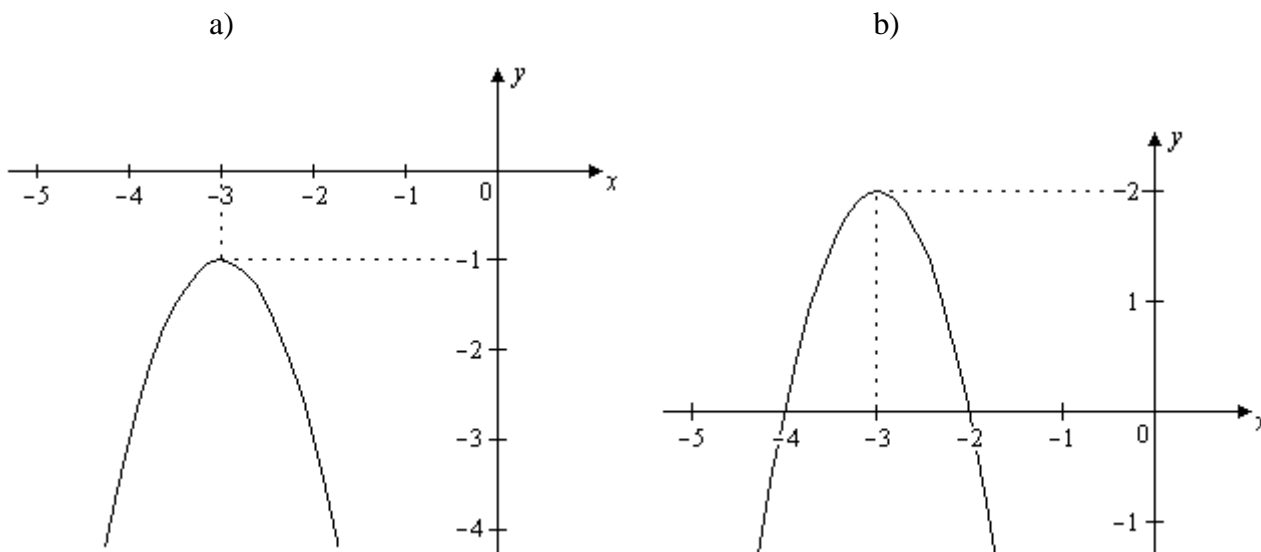
In exercises 10.7 and 10.8, for each of the quadratic functions

   1) Find the vertex of parabola.

   2) Find the y- intercept.

   3) Find the x- intercepts.

   4) Write the equation of the line of symmetry.

   5) Graph the parabola.

   6) Label the vertex and x- and y- intercepts with numbers or coordinates.

10.7. a) $y = x^2 - 2x - 3$

     b) $y = -x^2 - 4x + 5$

10.8. a) $y = -x^2 - 2x + 8$

     b) $y = x^2 - 6x + 5$

Session 11

Distance Formula, Midpoint Formula, and Circles

When we say that a point is given we mean that coordinates of the point are given. If A is a point in the plane and (x, y) are its coordinates in the system of coordinates, we will also denote this point as $A(x, y)$.

Distance Formula

Assume two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given. The problem is to find distance between them (i.e. to get a formula for this distance).

Recall that if we plot a point $C(x_0, y_0)$ in the system of coordinates, then x_0 is a horizontal coordinate (along x-axis), and y_0 is a vertical coordinate (along y-axis):
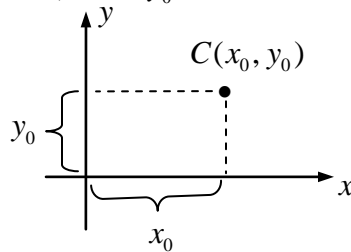
Now consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
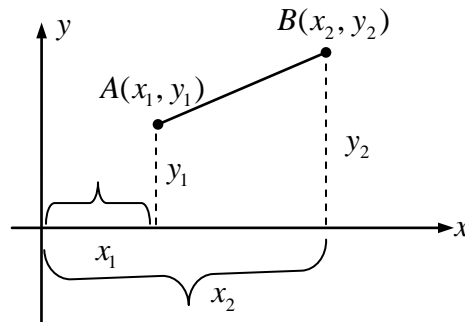
We denote $d(A, B)$ to be distance between points A and B (i.e. the length of the segment AB). To find this distance, we may draw a right triangle with the hypotenuse AB and horizontal and vertical legs:
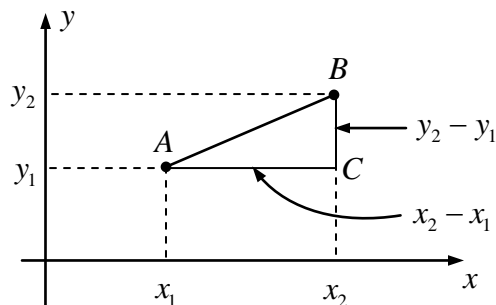
We can see that leg $AC = x_2 - x_1$ and leg $BC = y_2 - y_1$. By Pythagorean Theorem,

Session 11: Distance Formula, Midpoint Formula, and Circles

$$d^2(A,B) = AB^2 = AC^2 + BC^2 = \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2.$$

By taking square root from both sides, we get **Distance Formula**:

$$d(A,B) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

Here (x_1, y_1) and (x_2, y_2) are coordinates of the points A and B respectively.

Note. Distance formula will not change, if we switch (exchange) x_1 and x_2, and/or y_1 and y_2. The reason is that if a and b are two numbers, then $(a-b)^2 = (b-a)^2$. Also, the distance formula is valid for points located in any quadrants (not only in the 1st one).

Example 11.1. Calculate the distance between points $A(2,-3)$ and $B(-5,-7)$.

Solution. By distance formula we get:

$$d(A,B) = \sqrt{\left(2-(-5)\right)^2 + \left(-3-(-7)\right)^2} = \sqrt{\left(2+5\right)^2 + \left(-3+7\right)^2} = \sqrt{7^2+4^2} = \sqrt{49+16} = \sqrt{65}$$

Later, in Session 19 "Solving Oblique Triangles – Law of Cosines", in Example 19.4 we will justify the following method to check whether a triangle with given sides a, b, and c is acute, obtuse or right triangle:

Let c be the biggest side of the triangle. Calculate the value $E = a^2 + b^2 - c^2$.

If $E > 0$, the triangle is acute.

If $E < 0$, the triangle is obtuse.

If $E = 0$, the triangle is right.

Example 11.2. Let three vertices of triangle ABC be $A(2,-3)$, $B(-5,-7)$ and $C(-2, 6)$. Determine what kind of triangle it is: acute, obtuse, or right triangle. Also, determine which angle in the triangle ABC is the biggest angle, and which angle is the smallest one.

Solution. First, we calculate squares of all three sides of triangle ABC (there is no need to calculate the sides themselves because the above expression for E contains squares of sides only). Side AB we already calculated in Example 11.1, so $AB^2 = 65$. Using distance formula for sides AC and BC we have:

$$AC^2 = \left(2-(-2)\right)^2 + \left(-3-6\right)^2 = \left(2+2\right)^2 + \left(-9\right)^2 = 16+81 = 97,$$
$$BC^2 = \left(-5-(-2)\right)^2 + \left(-7-6\right)^2 = \left(-5+2\right)^2 + \left(-13\right)^2 = 9+169 = 178.$$

We see that side BC is the biggest one. Now we can construct the above expression for E:

$$E = AB^2 + AC^2 - BC^2 = 65+97-178 = -16.$$

Since E is negative, we conclude that triangle ABC is obtuse. In any triangle, the bigger side, the bigger opposite angle. In our case, the biggest side is BC, and the smallest side is AB. Therefore, the biggest angle is A, and the smallest one is C.

Midpoint Formula

Let A and B be two points. Midpoint formula gives coordinates of the point C located in the middle of the line segment AB. To get this formula, consider first the simplest case when points A and B lie on the number line (i.e. on horizontal x-axis):
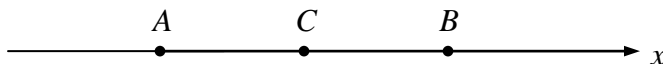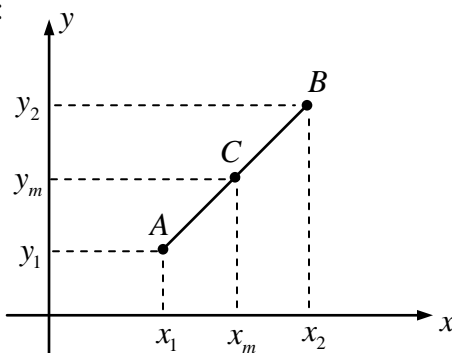

Here point C is the midpoint of the segment AB. Any point on number line is defined by its coordinate (a number). Let a, b, c be coordinates of points A, B, C correspondingly. Distance between points A and B (i.e. length of the segment AB) is $AB = b - a$. It is easy to show that coordinate c of the midpoint C is the average of coordinates a and b: $c = \dfrac{a+b}{2}$. Indeed, with this coordinate, distance AC is half of the distance AB:

$$AC = c - a = \frac{a+b}{2} - a = \frac{a+b-2a}{2} = \frac{b-a}{2} = \frac{AB}{2}.$$

Now, let points A, B and midpoint C lie in the plane and has coordinates (x_1, y_1), (x_2, y_2) and (x_m, y_m) respectively:

We can see that x_m is midpoint of the segment $[x_1, x_2]$ on x-axis, and y_m is midpoint of the segment $[y_1, y_2]$ on y-axis. Therefore, x_m is average of x_1 and x_2, and y_m is average of y_1 and y_2. We come up to the **Midpoint Formula**:

Coordinates (x_m, y_m) of the midpoint C of the line segment AB are averages of the corresponding coordinates of endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\boxed{x_m = \frac{x_1 + x_2}{2}, \ y_m = \frac{y_1 + y_2}{2}}$$

Example 11.3. Calculate the coordinates of the midpoint of the line segment with endpoints $(-3, 4)$ and $(-7, 10)$.

Solution. Let (x_m, y_m) be coordinates of the midpoint. By the midpoint formula

$$x_m = \frac{-3 + (-7)}{2} = \frac{-10}{2} = -5 \ \text{ and } \ y_m = \frac{4 + 10}{2} = \frac{14}{2} = 7 \ .$$

Answer: midpoint has coordinates $(-5, 7)$.

Example 11.4. Let $A(4, -7)$ be endpoint of a line segment, and $C(-6, 9)$ be its midpoint. Find the coordinates of another endpoint of the line segment.

Solution. Denote another endpoint as $B(x, y)$. We will use midpoint formula with given midpoint $C(-6, 9)$, so $x_m = -6$ and $y_m = 9$. We have

$$x_m = -6 = \frac{4 + x}{2} \implies -12 = 4 + x \implies x = -16,$$

$$y_m = 9 = \frac{-7 + y}{2} \implies 18 = -7 + y \implies y = 25.$$

Answer: endpoint B has coordinates $(-16, 25)$.

Circle

By definition, circle is a set of points in the plane equidistant (having the same distance) from a fixed point on this plane. This fixed point is called the **center** of the circle, and the distance from any point on the circle to the center is called the **radius**.

Equation of a circle can be easily derived directly from the distance formula. Let $C(a, b)$ be a center of a circle, and $A(x, y)$ be any point on the circle. If r is the radius of the circle, then, by definition, $d(A, C) = r$.

By distance formula, $d(A, C) = \sqrt{(x - a)^2 + (y - b)^2} = r$. Square both sides, and we get **Equation of the Circle**:

$$\boxed{(x - a)^2 + (y - b)^2 = r^2}$$

This equation is called the equation of circle in **standard form**. Here (a, b) are coordinates of the center of the circle, and r is its radius.

Example 11.5. Identify center and radius of the circle $(x - 3)^2 + (y + 5)^2 = 15$.

Solution. This equation in given in standard form, and we get answer immediately: center has coordinates $(3, -5)$, and radius is $\sqrt{15}$.

Note. Notice that in the above example, the second coordinate of the center is -5, not 5. This is because according to the equation of circle, we represent $y - b = y + 5$ as $y - (-5)$, so $b = -5$. Also, radius is equal to $\sqrt{15}$, but not 15, since number 15 is the square of the radius.

Equation of a circle may be given not in the standard form. In this case, to identify center and radius, represent the equation of circle in standard form first. Useful technique to do this is the completing the square. For review, you may take a look at Session 9 "Completing the Square and Quadratic Formula".

Example 11.6. Identify center and radius of the circle given by the equation

$$x^2 + y^2 + 8x - 10y + 32 = 0.$$

Solution. We reorganize terms and write the equation like this

$$\left(x^2 + 8x\right) + \left(y^2 - 10y\right) + 32 = 0.$$

Now complete the square for both x and y. According to procedure for completing the square that we described in Session 9, in each pair of parenthesis we divide coefficients for x and y by 2 and square them: $(8/2)^2 = 16$, $(-10/2)^2 = 25$. Then we add 16 and 25 to both sides of the equation:

$$\left(x^2 + 8x + 16\right) + \left(y^2 - 10y + 25\right) + 32 = 16 + 25 \implies (x+4)^2 + (y-5)^2 + 32 = 41,$$

$$(x+4)^2 + (y-5)^2 = 41 - 32 \implies (x+4)^2 + (y-5)^2 = 9.$$

We've got equation of circle in standard form. From here, coordinates of the center are $(-4, 5)$ and radius is 3.

Example 11.7. Graph the circle from example 11.6 and label four points on the circle.

Solution. In example 11.6, we calculated that the center is $(-4, 5)$ and radius is 3. We use this info to graph the circle by the following steps:

1) Plot center $(-4, 5)$.

2) From the center, draw dotted horizontal and vertical lines.

3) Along these lines, count 3 units (which is radius) starting from the center in all four directions: up, down, left and right. Mark four corresponding points as A, B, C, and D. These points are on the circle.

4) Draw the circle through the points A, B, C, and D.
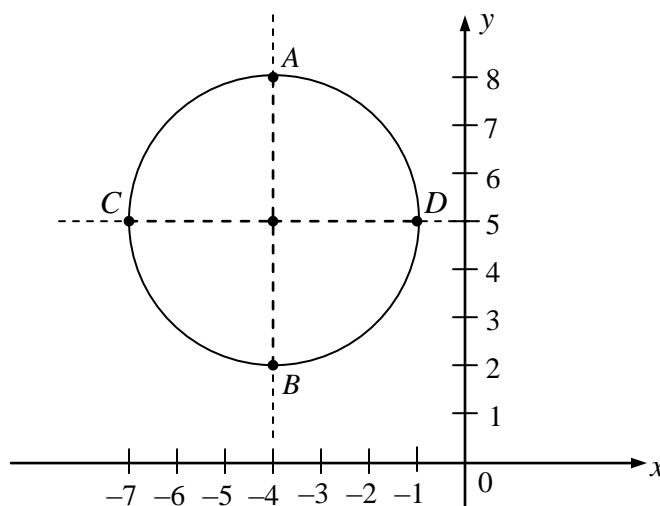
Points A, B, C, D has coordinates $A(-4, 8)$, $B(-4, 2)$, $C(-7, 5)$, $D(-1, 5)$.

Exercises 11

In exercises 11.1 and 11.2, calculate the distance between given points.

11.1. (5, 4) and (−1, 2)   11.2. (−2, 5) and (−3, 12)

In exercises 11.3 and 11.4, three vertices A, B and C of triangle ABC are given.
   1) Determine what kind of triangle it is: acute, obtuse, or right triangle.
   2) Determine which angle in the triangle ABC is the biggest angle, and which angle is the smallest one.
(You may want to review Example 11.2)

11.3. $A(-2, -3)$, $B(5, 2)$, $C(6, -5)$ 11.4. $A(-2, 4)$, $B(-7, -1)$, $C(4, -2)$

In exercises 11.5 and 11.6, calculate the coordinates of the midpoint of the line segment with given endpoints.

11.5. $(3, -4)$ and $(-5, -6)$ 11.6. $(4, -5)$ and $(2, 7)$

In exercises 11.7 and 11.8, C is midpoint of line segment AB. Coordinates of points A and C are given. Find the coordinates of point B.

11.7. $A(-9, 7)$, $C(-4, 5)$ 11.8. $A(5, 8)$, $C(-1, 3)$

In exercises 11.9 and 11.10, identify center and radius of the given circle.

11.9. a) $(x+2)^2 +(y+4)^2 = 36$ 11.10. a) $(x-6)^2 +(y-3)^2 = 32$

     b) $(x+5)^2 +(y-2)^2 = 20$ b) $(x-7)^2 +(y+8)^2 = 50$

In exercises 11.11 and 11.12, points A and B are given endpoints of the diameter of a circle.
Find the equation of the circle in standard form $(x-a)^2 +(y-b)^2 = r^2$.

11.11. $A(2, 8)$, $B(-8, -16)$ 11.12. $A(20, 25)$, $B(4, -5)$

In exercises 11.13 and 11.14, equations of circles are given. For each circle
   1) Find the coordinated of the center and radius.
   2) Graph the circle.
   3) Label four endpoints of vertical and horizontal diameters with coordinates.

11.13. a) $x^2 + y^2 +4x-2y-11=0$ 11.14. a) $x^2 + y^2 -8x+2y+8=0$

     b) $x^2 + y^2 -10x-8y+32=0$ b) $x^2 + y^2 +6x+4y-12=0$

Session 12

Systems of Three Linear Equations in Three Variables

The general form of such systems is this

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

Here x, y, and z are variables (unknown values). All other letters are given numbers. Numbers that are written next to variables (labeled with letters a, b and c) are called the **coefficients** of the system. The above system has 9 coefficients. A solution of the system is a triple (x, y, z) that satisfies the system (makes each equation a true statement after substitution numerical values of variables).

Note. Keep in mind that a triple (x, y, z) represents one solution, not three.

There are different methods of solving systems of linear equations with any number of equations and any number of variables. Here we consider the **elimination method**. This method suggests to eliminate one of the variables from two equations of the system using the third equation. After elimination of one variable, we get two equations with two other variables. We can solve this system using elimination method again. As a result, we will find the values of two unknowns. Finally, we substitute these values into one of the equations of the original system and solve it for the third unknown. To eliminate a variable, we multiply equations by appropriate numbers and then add them up. For this reason, this method is also called addition-elimination method.

Theoretically, there are three possibilities regarding the number of solutions of the linear system: it may have

1) One solution (so, one triple). We also say that the system has **unique** solution.

2) No solutions at all. Such system is called **inconsistent**.

3) Infinite many solutions. Such system is called **dependent**.

Let's consider corresponding examples.

Example 12.1. Solve the system

$$\begin{cases} 3x - y + 2z = -3 \\ 2x + 4y - 5z = 1 \\ -8x + 3y + 3z = 17 \end{cases}.$$

Solution. We have many options to eliminate variables. Actually, we can eliminate either one. Let's eliminate y from the second and third equations using the first equation (in which the coefficient for y is -1).

1) Consider together first and second equations:

$$\begin{cases} 3x - y + 2z = -3 \\ 2x + 4y - 5z = 1 \end{cases}$$

To eliminate y, we want coefficients for y in both equations to be equal by absolute values but have the opposite signs. In this case, if we add the equations, variable y will be cancelled (eliminated). To get this case, it's enough to multiply the first equation by 4.

Note. Multiplication the equation by a number means multiplication **all** terms of the equation by this number. We get

$$\begin{cases} 12x - 4y + 8z = -12 \\ 2x + 4y - 5z = 1 \end{cases}$$

Now we add these equations and y is eliminated:

$$12x + 2x + 8z - 5z = -12 + 1, \text{ or } 14x + 3z = -11.$$

2) To eliminate y from the third equation, consider first and third equations together:

$$3 \begin{cases} 3x - y + 2z = -3 \\ -8x + 3y + 3z = 17 \end{cases}$$

Number 3 outside the braces means that we intend to multiply the first equation by 3:

$$\begin{cases} 9x - 3y + 6z = -9 \\ -8x + 3y + 3z = 17 \end{cases}$$

Add these equations to eliminate y: $9x - 8x + 6z + 3z = -9 + 17$, or $x + 9z = 8$.

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} 14x + 3z = -11 \\ x + 9z = 8 \end{cases}.$$

4) Solve the above system (using the elimination method again):

$$-14 \begin{cases} 14x + 3z = -11 \\ x + 9z = 8 \end{cases} \Rightarrow \begin{cases} 14x + 3z = -11 \\ -14x - 126z = -112 \end{cases}$$

$$3z - 126z = -11 - 112 \Rightarrow -123z = -123 \Rightarrow z = 1.$$

5) At this point we found variable $z = 1$. Now we move back in the above steps. Substitute $z = 1$ into the second equation in step 3) and solve for x:

$$x + 9 \cdot 1 = 8 \Rightarrow x = -1.$$

6) Substitute the values $x = -1$ and $z = 1$ into the first equations of the original system, and solve it for y:

$$3 \cdot (-1) - y + 2 \cdot 1 = -3, \quad -3 - y + 2 = -3, \quad -y = -3 + 3 - 2, \quad -y = -2, \quad y = 2.$$

   Final answer: the system has one (**unique**) solution

$$x = -1, \ y = 2, \ z = 1, \text{ or as a triple } (-1, 2, 1).$$

Example 12.2. Solve the system

$$\begin{cases} x - 2y + 4z = 5 \\ 2x + 3y - z = 1 \\ 4x - y + 7z = 7 \end{cases}.$$

Solution. Let's eliminate z from the first and third equations using the second equation (which has coefficient -1 for z). Of course, you may eliminate any other variable.

1) Consider the first and second equations:

$$\begin{array}{c} \\ 4 \end{array} \begin{cases} x - 2y + 4z = 5 \\ 2x + 3y - z = 1 \end{cases} \Rightarrow \begin{cases} x - 2y + 4z = 5 \\ 8x + 12y - 4z = 4 \end{cases}$$

   Add the last equations to eliminate z: $x + 8x - 2y + 12y = 5 + 4, \quad 9x + 10y = 9.$

2) Consider the second and third equations:

$$\begin{array}{c} 7 \\ \end{array} \begin{cases} 2x + 3y - z = 1 \\ 4x - y + 7z = 7 \end{cases} \Rightarrow \begin{cases} 14x + 21y - 7z = 7 \\ 4x - y + 7z = 7 \end{cases}$$

   Add the last equations: $14x + 4x + 21y - y = 7 + 7, \quad 18x + 20y = 14, \quad 9x + 10y = 7.$

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} 9x + 10y = 9 \\ 9x + 10y = 7 \end{cases}$$

4) Solve the above system. Notice that the left sides of both equations are the same but the right sides are different. Therefore, this system does not have solutions, so the system is **inconsistent**.

   Final answer: the system does not have solutions, or, in other words, the solution set is empty set (the symbol for empty set is $\varnothing$).

Example 12.3. Solve the system

$$\begin{cases} 2x - 4y + 3z = 5 \\ 8x - 6y + 5z = 7 \\ x + 3y - 2z = -4 \end{cases}.$$

Solution. Let's eliminate x from the first and second equations using the third equation (which has coefficient 1 for x).

1) Consider the first and third equations:

$$\begin{matrix} & \begin{cases} 2x - 4y + 3z = 5 \\ x + 3y - 2z = -4 \end{cases} \\ -2 & \end{matrix} \Rightarrow \begin{cases} 2x - 4y + 3z = 5 \\ -2x - 6y + 4z = 8 \end{cases}$$

Add the last equations to eliminate x: $-4y - 6y + 3z + 4z = 5 + 8$, $-10y + 7z = 13$.

2) Consider the second and third equations:

$$\begin{matrix} & \begin{cases} 8x - 6y + 5z = 7 \\ x + 3y - 2z = -4 \end{cases} \\ -8 & \end{matrix} \Rightarrow \begin{cases} 8x - 6y + 5z = 7 \\ -8x - 24y + 16z = 32 \end{cases}$$

Add the last equations: $-6y - 24y + 5z + 16z = 7 + 32$, $-30y + 21z = 39$,
$-10y + 7z = 13$.

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} -10y + 7z = 13 \\ -10y + 7z = 13 \end{cases}.$$

4) Solve the above system. Notice that both equations coincide. So, actually, we have only one equation. In this case we cannot find the values of y and z uniquely. Indeed, we can assign any numerical value to one of the variables y or z, say to z. Then we can solve the above equation for y. Because there are infinite values of z to select, we get infinite number of pairs (y, z) which are solutions of the above equation. It means that the system has infinite many solutions. By substitution y and z in any of the original equations, we can find x. Finally, we will get infinite many triples (x, y, z). So, the system is **dependent**.

We come up to an interesting question how to describe the infinite set of all solutions of the system. Of course, we cannot create an infinite list of them. Instead, we can use the **parametric** form to describe the solution set. It means the following. Let's solve the above equation $-10y + 7z = 13$ for y in terms of z:

$$-10y = 13 - 7z, \quad y = -\frac{13}{10} + \frac{7}{10}z.$$

Here the variable z may take any values, and we call it the **free parameter**. Let's denote this parameter by the letter t: $z = t$. Then, $y = -\frac{13}{10} + \frac{7}{10}t$. Now, we can express the variables x in terms of the parameter t by substituting expressions for y and z into any equation of the original system. Let's substitute expressions for y and z into the thirds equation (in which coefficient for x is 1) and solve for x:

$$x + 3\left(-\frac{13}{10} + \frac{7}{10}t\right) - 2t = -4 \Rightarrow x - \frac{39}{10} + \frac{21}{10}t - 2t = -4$$

$$x = \frac{39}{10} - \frac{21}{10}t + 2t - 4 = \frac{39 - 21t + 20t - 40}{10} = \frac{-1-t}{10}$$

We can also write x as $x = -\frac{1}{10} - \frac{1}{10}t$. Now we have described all unknowns in the parametric form:

$$x = -\frac{1}{10} - \frac{1}{10}t, \quad y = -\frac{13}{10} - \frac{7}{10}t, \quad z = t.$$

Here t is a parameter that takes any numerical value.

Note. We can get the specific (particular) numerical solutions of the original system from the above parametric representation by assigning any specific number to the parameter t.

For example, if we put $t = 0$, we get the particular solution $x = -\frac{1}{10}, \quad y = -\frac{13}{10}, \quad z = 0$.

Exercises 12

In exercises 12.1 and 12.2, solve the system of equations.

If the system is inconsistent, state that.

If the system is dependent, state that.

12.1.

a) $\begin{cases} 2x - 3y + z = -10 \\ 5x + 2y - 3z = 1 \\ -3x + 4y + 5z = -5 \end{cases}$

b) $\begin{cases} 3x + 2y + z = 1 \\ 11x + 10y + 9z = 5 \\ x + 2y + 3z = 1 \end{cases}$

c) $\begin{cases} 5x + 3y - 8z = 6 \\ x + y - 2z = 1 \\ 3x + 2y - 5z = 4 \end{cases}$

12.2.

a) $\begin{cases} 4x + 2y - 3z = 10 \\ -x + 2y + 5z = -21 \\ 8x + 7y - 5z = 6 \end{cases}$

b) $\begin{cases} x - 3y + z = 4 \\ 3x + 2y + z = 3 \\ 6x + 4y + 2z = -1 \end{cases}$

c) $\begin{cases} 3x + 2y + z = 4 \\ x - y + 2z = 3 \\ 4x + y + 3z = 7 \end{cases}$

Challenge Problem

For the systems in exercises 12.1 and 12.2, that are dependent,

    1) Describe solutions in parametric form.
    2) Find a particular solution.

(Answers may vary).

Session 13

Determinants and Cramer's Rule

In previous session we considered solving systems of three linear equations by using elimination method. This method requires some specific operations upon the equations of given systems. Here we consider **formulas** that allow to calculate solutions explicitly by direct substitution of coefficients of equations into these formulas, rather than manipulating with equations. Such formulas are called the Cramer's rule named after Gabriel Cramer (1704 – 1752), a Swiss mathematician. Cramer's rule is not efficient for systems with many equations, and it is not used in practical calculations. However, it is easy to use for systems with two and three equations that we consider here. Cramer's rule is especially convenient if coefficients of systems are integers, but the solutions are fractions. Also, it has a theoretical importance.

Case of the system with two equations

Let's derive Cramer's rule for the system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

First, we solve this system by elimination method. Let's eliminate variable y by multiplying the first equation by e, the second equation by $-b$, and adding the resulting equations:

$$\begin{matrix} e \\ -b \end{matrix} \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} aex + bey = ce \\ -bdx - bey = -bf \end{cases}$$

Add the last equations, and solve for x: $aex - bdx = ce - bf \Rightarrow (ae - bd)x = ce - bf$,

$$x = \frac{ce - fb}{ae - bd}.$$

In similar way we can find y by eliminating x:

$$\begin{matrix} -d \\ a \end{matrix} \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} -adx - bdy = -cd \\ adx + aey = af \end{cases}$$

Add the last equations, and solve for y: $-bdy + aey = -cd + af \Rightarrow (ae - bd)y = af - cd$,

$$y = \frac{af - cd}{ae - bd}.$$

We come up to the following general formulas for the solutions of the system of two linear equations with two variables:

$$x = \frac{ce - fb}{ae - bd}, \quad y = \frac{af - cd}{ae - bd}.$$

Observe these formulas. Notice that the denominators of both fractions are the same, and structure of numerators looks similar to denominators. Cramer's rule represents these formulas in terms of a special number that is called the **determinant**. For the system of two equations, determinant is defined by four numbers, say k, l, m, and n. Here is the notation and the definition of the determinant:

$$\begin{vmatrix} k & l \\ m & n \end{vmatrix} = kn - ml.$$

We call it 2×2 determinant. As you can see, to calculate it, we take the product along the main diagonal (from left top corner to right bottom corner, so we multiply k by n) minus the product along the minor diagonal (from left bottom corner to right top corner, so we multiply m by l).

If you return back to the formulas for x and y, you may notice that their numerators and denominators can be written in terms of determinants. We come up to the following **Cramer's rule**. Solution of the system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

includes three steps:

1) Calculate the following determinant D which is called the determinant of the system:

$$D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd.$$

Notice that "free" coefficients c and f from the right side of the system are not used in the determinant D. It consists of the coefficients for x and y only.

2) Calculate another two determinants, D_x and D_y:

$$D_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf, \quad D_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd.$$

Notice that determinant D_x is obtained from D by replacing its first column with the column of "free" coefficients c and f. Similar, determinant D_y is obtained from D by replacing its second column with the column of "free" coefficients.

3) Calculate the solution of the system by the formulas

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}.$$

Note. As you see, the denominator in these fractions is the determinant D. Therefore, these formulas make sense only if $D \neq 0$. If $D = 0$, then the system does not have unique

solution. Instead, it either does not have solutions at all (the system is inconsistent), or it has infinite many solutions (the system is dependent). To detect which case we have, we should check D_x (or D_y) for zero. If $D_x \neq 0$, then there are no solutions. If $D_x = 0$, then the system has infinite number of solutions. (It can be shown that if $D = 0$, then both D_x and D_y are equal or not equal to zero simultaneously).

Example 13.1. Solve the following system using the Cramer's rule.

$$\begin{cases} 7x - 2y = 4 \\ 5x + 3y = 7 \end{cases}.$$

Solution.

1) Calculate the determinant D of the system:

$$D = \begin{vmatrix} 7 & -2 \\ 5 & 3 \end{vmatrix} = 7 \cdot 3 - 5 \cdot (-2) = 21 + 10 = 31.$$

2) Calculate the determinants D_x and D_y:

$$D_x = \begin{vmatrix} 4 & -2 \\ 7 & 3 \end{vmatrix} = 4 \cdot 3 - 7 \cdot (-2) = 12 + 14 = 26,$$

$$D_y = \begin{vmatrix} 7 & 4 \\ 5 & 7 \end{vmatrix} = 7 \cdot 7 - 5 \cdot 4 = 49 - 20 = 29.$$

3) Write the solution of the system:

$$x = \frac{D_x}{D} = \frac{26}{31}, \quad y = \frac{D_y}{D} = \frac{29}{31}.$$

Final answer: $x = \dfrac{26}{31}$, $y = \dfrac{29}{31}$, or, as a pair, $\left(\dfrac{26}{31}, \ \dfrac{29}{31} \right)$.

Case of the system with three equations

Consider the Cramer's rule for the system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

Similar to systems with two equations, the solutions of this system can also be represented in terms of determinants as ratios of determinants D_x, D_y, and D_z corresponding to variables x, y, and z, to the common determinant D of the system. Let's describe how to find these determinants.

We will not derive here corresponding formulas, and just provide the final result. The determinant D of the above system is denoted by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This is a 3×3 determinant constructing from the coefficients of the system. There are several methods to calculate it. We consider here only one method: direct calculation.

Direct calculation method. Here is the formula for determinant D:

$$D = a_1 b_2 c_3 + b_1 c_2 a_3 + a_2 b_3 c_1$$
$$- c_1 b_2 a_3 - b_1 a_2 c_3 - c_2 b_3 a_1.$$

This formula seems difficult to memorize. Notice that it contains six terms: three terms with the plus sign, and another three with the minus sign. Here is one of the possible ways to memorize the formula. Let's extend (double) the determinant D to the following table:

$$\begin{bmatrix} a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 & c_3 \end{bmatrix}.$$

To get three terms of the determinant with the plus sign, calculate products along the main diagonal a_1, b_2, c_3, and two parallel diagonals b_1, c_2, a_3 and c_1, a_2, b_3.

To get three terms with the minus sign, calculate products along the minor diagonal a_3, b_2, c_1, and two parallel diagonals b_3, c_2, a_1 and c_3, a_2, b_1.

Note. The last column of the above table is not used, so it is not necessary to write it.

Example 13.2. Calculate the following determinant by direct calculation

$$D = \begin{vmatrix} 5 & -6 & -2 \\ 3 & 2 & -4 \\ 2 & 0 & 3 \end{vmatrix}.$$

Solution. Construct the extended table (we dropped the last column)

$$\begin{bmatrix} 5 & -6 & -2 & 5 & -6 \\ 3 & 2 & -4 & 3 & 2 \\ 2 & 0 & 3 & 2 & 0 \end{bmatrix}$$

We have

$$D = 5 \cdot 2 \cdot 3 + (-6) \cdot (-4) \cdot 2 + (-2) \cdot 3 \cdot 0$$

$$- 2 \cdot 2 \cdot (-2) - 0 \cdot (-4) \cdot 5 - 3 \cdot 3 \cdot (-6) = 30 + 48 + 8 + 54 = 140.$$

Now, we are ready to describe the Cramer's rule for the system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

1) Calculate the determinant D of this system:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2) Calculate three other determinants, D_x, D_y and D_z that correspond to variables x, y, and z. These determinants are constructed by replacing corresponding columns of the determinant D with the column from the right side of the system:

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

3) Calculate the solution of the system by the formulas

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}.$$

Note. Similar to the case of the system with two variables, there is no solution or there is infinite number of solutions if $D = 0$.

Example 13.3. Solve the following system using the Cramer's rule.

$$\begin{cases} 5x - 6y - 2z = 7 \\ 3x + 2y - 4z = -8 \\ 2x + 3z = 5 \end{cases}$$

Solution.

1) The determinant D of the system is $D = \begin{vmatrix} 5 & -6 & -2 \\ 3 & 2 & -4 \\ 2 & 0 & 3 \end{vmatrix}.$

This is exactly the same determinant as in example 13.2, so $D = 140$.

2) Calculate the determinants D_x, D_y and D_z:

$$D_x = \begin{vmatrix} 7 & -6 & -2 \\ -8 & 2 & -4 \\ 5 & 0 & 3 \end{vmatrix}, \quad D_y = \begin{vmatrix} 5 & 7 & -2 \\ 3 & -8 & -4 \\ 2 & 5 & 3 \end{vmatrix}, \quad D_z = \begin{vmatrix} 5 & -6 & 7 \\ 3 & 2 & -8 \\ 2 & 0 & 5 \end{vmatrix}.$$

For determinate D_x, construct the extended table

$$\begin{bmatrix} 7 & -6 & -2 & 7 & -6 \\ -8 & 2 & -4 & -8 & 2 \\ 5 & 0 & 3 & 5 & 0 \end{bmatrix}.$$

$$D_x = 7 \cdot 2 \cdot 3 + (-6) \cdot (-4) \cdot 5 + (-2) \cdot (-8) \cdot 0$$
$$- 5 \cdot 2 \cdot (-2) - 0 \cdot (-4) \cdot 7 - 3 \cdot (-8) \cdot (-6) = 38$$

For determinate D_y, construct the extended table

$$\begin{bmatrix} 5 & 7 & -2 & 5 & 7 \\ 3 & -8 & -4 & 3 & -8 \\ 2 & 5 & 3 & 2 & 5 \end{bmatrix}.$$

$$D_y = 5 \cdot (-8) \cdot 3 + 7 \cdot (-4) \cdot 2 + (-2) \cdot 3 \cdot 5$$
$$- 2 \cdot (-8) \cdot (-2) - 5 \cdot (-4) \cdot 5 - 3 \cdot 3 \cdot 7 = -201$$

For determinate D_z, construct the extended table

$$\begin{bmatrix} 5 & -6 & 7 & 5 & -6 \\ 3 & 2 & -8 & 3 & 2 \\ 2 & 0 & 5 & 2 & 0 \end{bmatrix}.$$

$$D_z = 5 \cdot 2 \cdot 5 + (-6) \cdot (-8) \cdot 2 + 7 \cdot 3 \cdot 0$$
$$- 2 \cdot 2 \cdot 7 - 0 \cdot (-8) \cdot 5 - 5 \cdot 3 \cdot (-6) = 208$$

3) Calculate the solutions x, y, and z of the system

$$x = \frac{D_x}{D} = \frac{38}{140} = \frac{19}{70}, \quad y = \frac{D_y}{D} = -\frac{201}{140}, \quad z = \frac{D_z}{D} = \frac{208}{140} = \frac{52}{35}.$$

Final answer: $x = \dfrac{19}{70}$, $y = -\dfrac{201}{140}$, $z = \dfrac{52}{35}$, or, as a triple, $\left(\dfrac{19}{70}, \ -\dfrac{201}{140}, \ \dfrac{52}{35} \right)$.

Exercises 13

In exercises 13.1 and 13.2, solve systems of equations using Cramer's rule.

13.1. $\begin{cases} -3x + 5y = 4 \\ 7x - 6y = 8 \end{cases}$

13.2. $\begin{cases} 2x - 3y = 7 \\ 5x + 4y = 9 \end{cases}$

In exercises 13.3 and 13.4, calculate determinants.

13.3. $D = \begin{vmatrix} 3 & -2 & 1 \\ -5 & 4 & -3 \\ 2 & 1 & 7 \end{vmatrix}$

13.4. $D = \begin{vmatrix} 1 & 6 & -7 \\ 7 & -5 & 3 \\ 4 & -3 & 2 \end{vmatrix}$

In exercises 13.5 and 13.6, use the results of exercises 13.3 and 13.4 respectively to solve systems of equations using Cramer's rule.

13.5. $\begin{cases} 3x - 2y + z = 1 \\ -5x + 4y - 3z = 0 \\ 2x + y + 7z = 2 \end{cases}$

13.6. $\begin{cases} x + 6y - 7z = 2 \\ 7x - 5y + 3z = -1 \\ 4x - 3y + 2z = 0 \end{cases}$

Session 14

Nonlinear Systems of Equations in Two Variables

We consider here couple examples of systems of two equations with two variables in which one or both equations are not linear. We also consider geometrical interpretation of solutions.

Example 14.1. Solve the system of equations $\begin{cases} x + y = 6 \\ y = x^2 + 4x - 8 \end{cases}$

Solution. First of all, observe these equations. Notice that the first equation is linear, while the second is not (since it contains x^2). We can easily solve the first (linear) equation for x or y and substitute this expression into the second (nonlinear) equation. As a result, the second equation will contain only one variable. Such method is called the **substitution** method. For the first equation itself, it does not matter for which variable to solve: for x or for y. However, it is matter for the second equation. If we solve the first equation for x, then we need to substitute this expression for both x^2 and x in the second equation, which is a bit inconvenient (especially for x^2). It is more suitable to solve the first equation for y. In this case we substitute this expression into the second equation for y only, and we will avoid raising the expression into the second power.

Let's do this: solve the first equation for y: $y = 6 - x$. Substitute this expression into the second equation, and get the quadratic equation for x:

$$6 - x = x^2 + 4x - 8.$$

Solve this equation:

$$6 - x = x^2 + 4x - 8 \implies x^2 + 4x - 8 - 6 + x = 0 \implies x^2 + 5x - 14 = 0$$
$$\implies (x - 2)(x + 7) = 0 \implies x = 2 \text{ and } x = -7.$$

Now, using the expression $y = 6 - x$, we can find the corresponding values of y: if $x = 2$, then $y = 6 - 2 = 4$, and if $x = -7$, then $y = 6 - (-7) = 13$.
Final answer: the system has two solutions: $x = 2$, $y = 4$ and $x = -7$, $y = 13$. Or, as a solution set, $\{(2, 4), (-7, 13)\}$.

Note. We can interpret the above solutions geometrically using graphs of given equations. The graph of the first equation is a straight line, and the graph of the second equations is a parabola. Solutions of the system give the points of intersection of these graphs. According to the final answer, the straight line and the parabola intersect each other at two points with the coordinates (2, 4) and (–7, 13). You can draw the corresponding graphs and see it for yourself.

Example 14.2. Solve the system of equations $\begin{cases} y = x^2 - 3x + 14 \\ y = x^2 + 6x - 4 \end{cases}$.

Solution. In this example both equations are nonlinear and both are solved for y. In such a case we can easily eliminate y if we subtract these equations (in any order). Subtracting

the first equation from the second, we get

$$0 = 6x - 4 - (-3x + 14) \implies 0 = 6x - 4 + 3x - 14 \implies 0 = 9x - 18 \implies 9x = 18 \implies x = 2.$$

To find y, we can substitute $x = 2$ in either equation of the system. Substituting it into the first equation, we get

$$y = 2^2 - 3 \cdot 2 + 14 = 4 - 6 + 14 = 12.$$

Final answer: the system has one solution $x = 2$, $y = 12$, or as a pair (2, 12).

Note. As in example 14.1, we can interpret the above solution geometrically. The graphs of both equations of the system are parabolas. According to the final answer, these parabolas intersect each other only at one point (2, 12).

Example 14.3. Solve the system of equations $\begin{cases} 3x^2 + 2y^2 = 30 \\ 4x^2 + 3y^2 = 43 \end{cases}$.

Solution. Notice that in this system both variables, x and y, are in the second power only. We may temporary use new variables u and v: $u = x^2$, $v = y^2$. Then, in terms of u and v, we have linear system

$$\begin{cases} 3u + 2v = 30 \\ 4u + 3v = 43 \end{cases}.$$

Solve this system by elimination method:

$$\begin{matrix} -4 \\ 3 \end{matrix} \begin{cases} 3u + 2v = 30 \\ 4u + 3v = 43 \end{cases} \implies \begin{cases} -12u - 8v = -120 \\ 12u + 9v = 129 \end{cases}.$$

Add the last two equations to eliminate u and solve for v: $v = 9$. Substitute this value into the first equation of the system, and solve for u:

$$3u + 2 \cdot 9 = 30, \ 3u + 18 = 30, \ 3u = 12, \ u = 4.$$

So, $u = 4$ and $v = 9$. Now, we need to return from u and v to original variables x and y. We have $u = x^2 = 4$. From here, $x = \pm\sqrt{4} = \pm 2$. We got two values of x: 2 and -2.

Similar, $v = y^2 = 9 \implies y = \pm\sqrt{9} = \pm 3$. We have two values of y: 3 and -3.

From this point we need to be very careful to write final answer in correct way. Any solution of given system is a pair (x, y). Therefore, we need to combine each value of x with each value of y. As a result, original system has four solutions:

$$\{(2, 3), \ (2, -3), \ (-2, 3), \ (-2, -3)\}.$$

Note. It can be shown that graphs of the equations in given system are ellipses ("stretched" circles). The answer to the problem tells that these ellipses intersect each other at four above points.

Example 14.4. Solve the system of equations $\begin{cases} y = \sqrt{x} \\ x^2 + 2y^2 = 15 \end{cases}$.

Solution. Here both equations are non-linear. Square the first equation: $y^2 = x$. Substitute this expression into the second equation and get quadratic equation for x:

$$x^2 + 2x = 15 \Rightarrow x^2 + 2x - 15 = 0 \Rightarrow (x-3)(x+5) = 0.$$

From here we get two solutions of the quadratic equation: $x = 3$ and $x = -5$. Now we can use the first equation to find corresponding values of y.

For $x = 3$, $y = \sqrt{3}$. So, one solution is the pair $\left(3, \sqrt{3}\right)$. For $x = -5$, $y = \sqrt{-5}$. In this session, we consider only real numbers. Because $\sqrt{-5}$ is not a real number, we reject it. Finally answer: original system has only one solution $\left(3, \sqrt{3}\right)$.

Note. Geometrically, the graph of the first equation is the curve, which has the shape of the half of parabola that "lies on the side": it is going not along y-axis, but along x-axis, and the graph is located above x-axis. The second equation is the ellipse with the center in original. Both curves intersect each other at the point $\left(3, \sqrt{3}\right)$.

Example 14. 5. The area of a rectangular region is 96 square feet, and the perimeter is 40 feet. Find the dimensions of the region (i.e. find its length and width).

Solution. As for most word problems, we will solve it in two steps: set up equations and solve equations.

1) Let x represent the length of the region, and y represents its width. Then $xy = 96$ (area of the rectangle), and $2x + 2y = 40$ (perimeter of the rectangle). We come up to the system of equations:

$$\begin{cases} xy = 96 \\ 2x + 2y = 40 \end{cases}$$

2) We can reduce the second equation by 2 (divide all terms by 2):

$$\begin{cases} xy = 96 \\ x + y = 20 \end{cases}.$$

Here the second equation is a linear one, and we can easily solve it for x or y. Let's solve it for y: $y = 20 - x$. Substituting this expression in the first equation, we have $x(20 - x) = 96$, or $20x - x^2 = 96$. This is a quadratic equation that can be rewritten in a standard form $x^2 - 20x + 96 = 0$. We can solve it by factoring:

$(x - 8)(x - 12) = 0$. From here $x = 8$ or $x = 12$. Corresponding value of y we can get from the expression $y = 20 - x$. If $x = 8$, then $y = 20 - 8 = 12$. If $x = 12$, then $y = 20 - 12 = 8$.

Note. It looks like we have found two solutions: $x = 8$, $y = 12$, and $x = 12$, $y = 8$. This is true for the system of equations. However, for given rectangle these two solutions simply mean that one side of the rectangle is 8, and the other is 12. Assuming that the length is greater than the width, we come up to the unique solution.

Final answer: the length of the rectangle is 12 feet, and the width is 8 feet.

Exercises 14

In exercises 14.1 and 14.2, solve systems of equations (find real solutions).

14.1. a) $\begin{cases} 3x - y = 1 \\ x^2 + 2y = 5 \end{cases}$

14.2. a) $\begin{cases} 4x + y = 2 \\ x^2 + 3y = 19 \end{cases}$

b) $\begin{cases} x = y^2 + 5y - 2 \\ x = y^2 - 3y - 26 \end{cases}$

b) $\begin{cases} x = y^2 + 6y - 2 \\ x = y^2 + 3y + 16 \end{cases}$

c) $\begin{cases} \sqrt{x} + y = 0 \\ x^2 - 5y^2 = 36 \end{cases}$

c) $\begin{cases} \sqrt{x} - y = 0 \\ x^2 - 3y^2 = 4 \end{cases}$

d) $\begin{cases} x - 2y = 3 \\ x^2 - 3y^2 = -2 \end{cases}$

d) $\begin{cases} x + 3y = 2 \\ x^2 - 8y^2 = 4 \end{cases}$

e) $\begin{cases} 2x^2 + y^2 = 34 \\ 5x^2 + 3y^2 = 93 \end{cases}$

e) $\begin{cases} 5x^2 - 3y^2 = -28 \\ 3x^2 + 2y^2 = 44 \end{cases}$

In exercises 14.3 and 14.4, *A* represents the area of a rectangular region, and *P* represents its perimeter. Find the dimensions of the region (i.e. find its length and width).

14.3. $A = 12 \text{ m}^2$, $P = 14$ m.

14.4. $A = 30 \text{ yd}^2$, $P = 22$ yd .

Challenge Problems

14.5. Solve the system of equations $\begin{cases} ax - y = b \\ x^2 + cy = d(d + ac) - bc \end{cases}$.

Hint: Check that the quadratic equation $x^2 + c(ax - b) = d(d + ac) - bc$ has roots d and $-d - ac$.

14.6. Consider the system of equations $\begin{cases} \sqrt{x} - y = 0 \\ x^2 + (a - b)y^2 = ab \end{cases}$.

Prove the following statements:

1) If $a = 0$ and $b = 0$, the system has one solutions $(0,\ 0)$.

2) If $a < 0$ and $b > 0$, the system has two solutions $\left(-a,\ \sqrt{-a}\right)$ and $\left(b,\ \sqrt{b}\right)$.

3) If $a \le 0$ and $b < 0$, the system has one solutions $\left(-a,\ \sqrt{-a}\right)$.

4) If $a > 0$ and $b \ge 0$, the system has one solutions $\left(b,\ \sqrt{b}\right)$.

5) If $a > 0$ and $b < 0$, the system does not have solutions.

Part II

Trigonometric Functions

Session 15

Geometric and Trigonometric Angles

Historically, trigonometry studies the relationships between angles and sides of triangles. In Greek, the word "Trigonometry" literally means "Triangle-Measurement".

It is important to understand that in geometry and trigonometry we treat angles in different ways.

In geometry, an angle is simply a figure, created by two rays, coming from the same point. Also, we assign the measure to an angle as some positive number. A common measure is the **degree measure**. If you cut a round pizza-pie (theoretically) into 360 equal slices, the angle in one slice is of one degree: $1°$.

In trigonometry, we extend the meaning of an angle by assigning to it the "direction of rotation" and, as a result, the sign of its measure. That means that we assign to angles not only positive measure, but also negative. We can do this in the following way. Consider "geometric" angle

Terminal side
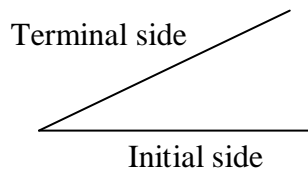
Initial side

Let's call one of its sides **initial side**, and the other – **terminal side**. Let's say, the horizontal side is the initial, and the slant side is the terminal.

We can treat this angle as a result of rotation of the terminal side when it starts from the position of initial side and then rotates to its current position. To rotate, we have two directions: clockwise and counterclockwise. We can mark these two directions of rotation by arrows:
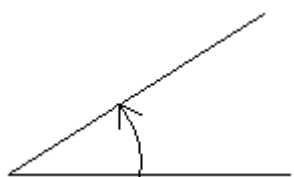
Counterclockwise rotation:
angle is positive.
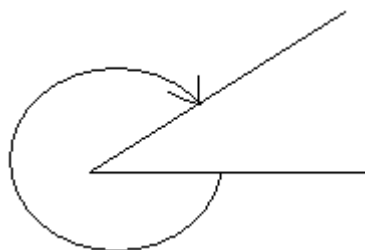
Clockwise rotation:
angle is negative.

It was an agreement to assign to an angle a **positive** measure if the direction of rotation is counterclockwise, and assign a **negative** measure if the direction of rotation is clockwise. On the left picture above, the angle is positive, and on the right – negative.
As you can see, taking one "geometric" angle (two rays, coming from the same point), we can consider two "trigonometric" angles: one positive and another negative depending

on direction in which we rotate the terminal side. Even more, we can assign to a given "geometric" angle infinite many "trigonometric" angles making multiple full rotations of terminal side in either direction. All such "trigonometric" angles have the same "geometric" angle and they are called **coterminal** angles. On two pictures above, the angles are coterminal.

Example 15.1. Consider the angle of $40°$:
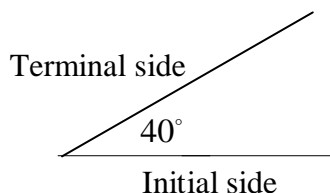
Describe all coterminal angles for this angle.

Solution. If we make one full rotation (rotation by $360°$) of the terminal side in either direction, the terminal side returns to its original position and we obtain coterminal angle (i.e. the same "geometric" angle). We get the same result, if instead of one, we make n full rotations (i.e. rotations by $360° \cdot n$). All such angles are coterminal to $40°$ angle and their values are described by the parametric formula $40° + 360° \cdot n$, where parameter n is any integer (positive, negative, or zero). We can write that $n = 0, \pm 1, \pm 2, \dots$. For positive n, we get positive values of the angle, and for negative n – negative values. For example, if we put $n = 1$ and $n = -1$, we get two specific coterminal angles: $40° + 360° = 400°$ and $40° - 360° = -320°$.

Two Special Right Triangles and Three Special Angles

In trigonometry, we often use the following two right triangles: one is a half of an equilateral triangle, and another is a half of a square:

$\triangle ABC$ is a half $\triangle ABC$ is a half
of an equilateral triangle   of a square
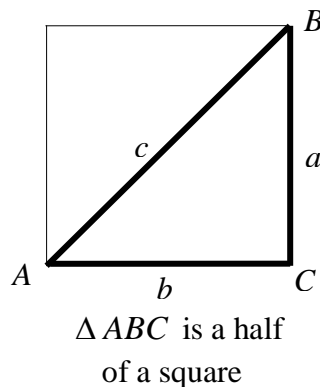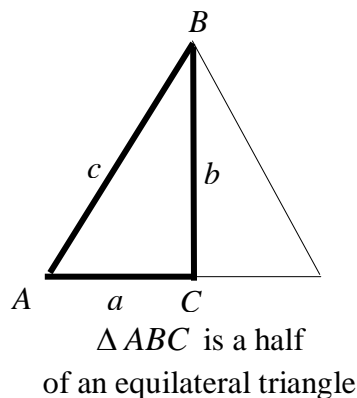
In the triangle ABC on the left picture, the acute angles are of $30°$ and $60°$. We will call such triangle a $30° - 60°$ triangle. In the triangle ABC on the right picture, both acute angles are of $45°$. We will call such triangle a $45° - 45°$ triangle. Both triangles:

$30° - 60°$ and $45° - 45°$, are called **special right triangles**, and angles $30°$, $45°$ and $60°$ are called **special angles**.

Let's consider special triangles in more details. For both, we will use the **Pythagorean Theorem** that states that for any right triangle with the hypotenuse c and legs a and b, the following equation is true:

$$a^2 + b^2 = c^2.$$

We will also use this theorem in the forms:

$$c = \sqrt{a^2 + b^2}, \ a = \sqrt{c^2 - b^2}, \ b = \sqrt{c^2 - a^2}.$$

$30° - 60°$ Triangle

Let's draw this triangle like this
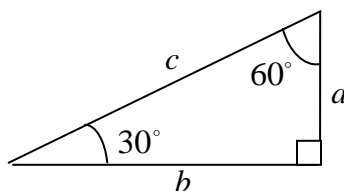
Recall that side c is the side of drawn above equilateral triangle, so side a is the half of side c: $a = \dfrac{c}{2}$ or $c = 2a$. Try to remember this fact:

> In any $30° - 60°$ triangle, the leg opposite to $30°$ is the half of the hypotenuse (or the hypotenuse is twice as this leg).

Example 15.2. Consider $30° - 60°$ triangle with legs a, b and hypotenuse c (see picture above). Solve the following problems.

1) $a = 7$. Find b and c.
2) $b = 5$. Find a and c.
3) $c = 10$. Find a and b.

Solution. In all problems, side a is opposite to $30°$ angle. Therefore, $c = 2a$.

1) $c = 2a = 2 \times 7 = 14$. By Pythagorean Theorem
$$b = \sqrt{c^2 - a^2} = \sqrt{14^2 - 7^2} = \sqrt{196 - 49} = \sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3}.$$

2) By Pythagorean Theorem $c^2 = a^2 + b^2$ and $c = 2a$. Therefore,
$$c^2 = (2a)^2 = a^2 + b^2 \Rightarrow 4a^2 = a^2 + b^2 \Rightarrow 3a^2 = b^2 \Rightarrow a^2 = \frac{b^2}{3} = \frac{25}{3}.$$
$$a = \sqrt{\frac{25}{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}, \ c = 2a = \frac{10\sqrt{3}}{3}.$$

3) Again, $c = 2a$, so $a = \dfrac{c}{2} = \dfrac{10}{2} = 5$. By Pythagorean Theorem

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}.$$

Let's describe the connection between sides of $30° - 60°$ triangle in general form. Let a be a side, opposite to $30°$. Then the hypotenuse $c = 2a$. Another side b which is opposite to $60°$, can be calculated by the Pythagorean Theorem:

$$b = \sqrt{c^2 - a^2} = \sqrt{(2a)^2 - a^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}.$$ We get the following picture
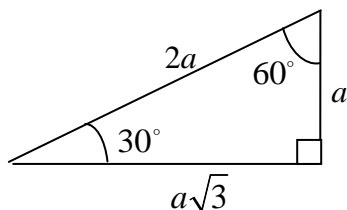
If you memorize this picture (or quickly get it), you can solve problems like in example 15.2 faster.

$45° - 45°$ Triangle

This triangle looks like this
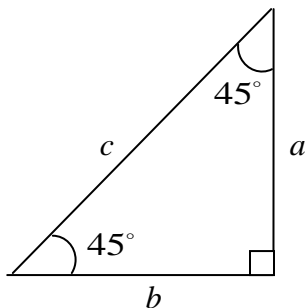
Both sides a and b are sides of the square above, therefore, they are equal: $a = b$. Try to remember this fact:

> In any $45° - 45°$ triangle both legs are equal.

Example 15.3. Consider $45° - 45°$ triangle with legs a, b and hypotenuse c (see picture above). Solve the following problems.

1) $a = 5$. Find b and c.

2) $b = 7$. Find a and c.

3) $c = 10$. Find a and b.

Solution. In all problems a and b are two legs, so they are equal: $a = b$. Therefore, problems 1) and 2) actually the same (just numbers are different).

1) $a = b = 5$. By Pythagorean Theorem

$$c = \sqrt{a^2 + b^2} = \sqrt{25 + 25} = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}.$$

2) $a = b = 7$, $c = \sqrt{a^2 + b^2} = \sqrt{7^2 + 7^2} = \sqrt{7^2 \cdot 2} = 7\sqrt{2}$.

3) By Pythagorean Theorem and using that $a = b$,

$$a^2 + b^2 = c^2,\ a^2 + a^2 = c^2,\ 2a^2 = 100,\ a^2 = 50,\ a = b = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}.$$

Similar to $30° - 60°$ triangle, let's describe the connection between sides of $45° - 45°$ triangle in general form. Let a be a side, opposite to one of the $45°$ angle. Then the other side b which is opposite to another $45°$ angle, is the same: $b = a$. The hypotenuse c can be calculated by the Pythagorean Theorem:

$c = \sqrt{a^2 + b^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$. We get the following picture
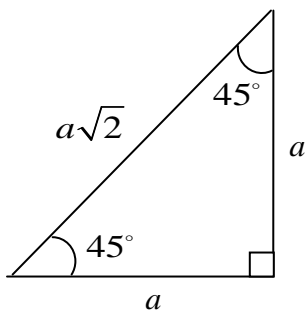
Exercises 15

In exercises 15.1 and 15.2, determine all coterminal angles for angle θ. Also, indicate two positive and two negative particular coterminal angles (answers may vary).

15.1.

    a) $\theta = 50°$.

    b) $\theta = -70°$.

15.2.

    a) $\theta = 27°$.

    b) $\theta = -35°$.

In exercises 15.3 and 15.4, $30° - 60°$ triangle is given. In it, A, B and C are angles, and a, b, and c are sides, which are opposite to corresponding angles. $\angle A = 30°$, $\angle C = 90°$. Solve the given problems.

15.3.

    a) $a = 6$. Find b and c.

    b) $b = 3$. Find a and c.

    c) $c = 8$. Find a and b.

15.4.

    a) $a = 8$. Find b and c.

    b) $b = 9$. Find a and c.

    c) $c = 4$. Find a and b.

In exercises 15.5 and 15.6, $45° - 45°$ triangle is given. In it, A, B and C are angles, and a, b, and c are sides, which are opposite to corresponding angles. $\angle A = 45°$, $\angle C = 90°$. Solve the given problems.

15.5.

    a) $a = 4$. Find b and c.

    b) $b = 8$. Find a and c.

    c) $c = 9$. Find a and b.

15.6.

    a) $a = 3$. Find b and c.

    b) $b = 6$. Find a and c.

    c) $c = 7$. Find a and b.

Challenge Problems

15.7. In $30° - 60°$ triangle, the hypotenuse is c. Express in terms of c two other sides.

15.8. In $30° - 60°$ triangle, the side opposite to $60°$ angle is b. Express in terms of b two other sides.

15.9. In $45° - 45°$ triangle, the hypotenuse is c. Express in terms of c two other sides.

Session 16

Trigonometric Functions for Acute Angles

Definition of six trigonometric functions

Consider the following "giraffe" problem:

"A giraffe's shadow is 8 meters. How tall is the giraffe if the sun is $28°$ to the horizon?"

Trigonometric functions that we introduce here, allow to solve this and many more problems that involve angles and sides of triangles. We will solve the above problem in the example 16.1 below.

To approach such problems, let's start with definition of trigonometric functions for acute angles.
Consider an acute angle θ:
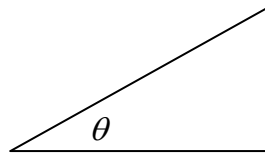
Trigonometric functions (in short trig functions) take this angle as its argument (as input) and assign some numerical values to it (output values). You will see shortly what exactly these values are.

Because angle θ is acute, we can always construct a right triangle with this angle:
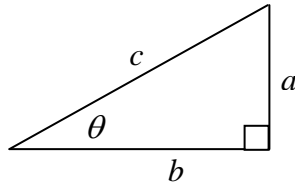
By proportionality properties of similar triangles, the ratios of sides of this triangle do not depend on the size of the triangle; instead, they depend on the value of angle θ only. In other words, if we take two right triangles with the same angle θ, but different sizes, then the ratios of corresponding sides remain the same. T**rigonometric functions** are exactly these **ratios**.

It is easy to see that there is a total of six possible ratios of the sides in a triangle. Here are all of them: a/c, b/c, a/b, b/a, c/b, c/a. So, there are exactly six trigonometric functions. Each of them has its own name and notation. The following table defines all six trig functions for angle θ.

Session 16: Trigonometric Functions for Acute Angles

Function Name	Function Notation	Definition
sine	$\sin \theta$	$\dfrac{a}{c}$
cosine	$\cos \theta$	$\dfrac{b}{c}$
tangent	$\tan \theta$	$\dfrac{a}{b}$
cotangent	$\cot \theta$	$\dfrac{b}{a}$
secant	$\sec \theta$	$\dfrac{c}{b}$
cosecant	$\csc \theta$	$\dfrac{c}{a}$

It may seem that it is difficult to memorize all of these functions. A simple advice (but, perhaps, not so simple to follow) is just to memorize them as you would the multiplication table.

From the above six trig functions, the first three are the most frequently used: sine, cosine, and tangent. They are called **basic** trig functions. The other three are reciprocals to basics: cotangent is reciprocal to tangent, secant is reciprocal to cosine, and cosecant is reciprocal to sine:

$$\cot \theta = \frac{1}{\tan \theta}, \ \sec \theta = \frac{1}{\cos \theta}, \ \csc \theta = \frac{1}{\sin \theta}.$$

Some people like the following mnemonic device **SohCahToa** to memorize the definition of basic trig functions. It works like this. In the above right triangle, we can treat legs a and b as opposite and adjacent to the angle θ:
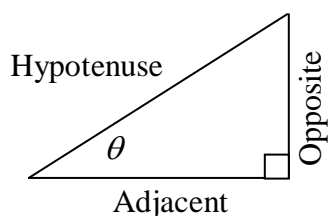
Now, the definition of sine, cosine, and tangent can be reformulated as

$$\sin \theta = \text{Opposite/Hypotenuse}$$
$$\cos \theta = \text{Adjacent/Hypotenuse}$$
$$\tan \theta = \text{Opposite/Adjacent}$$

The first three letters of the word SohCahToa mean: **S**ine is the ratio of **O**pposite leg to **H**ypotenuse, so we get **Soh**, and so on.

Trig Functions for Special Angles

In the previous session we have introduced three special angles $30°$, $45°$ and $60°$ as angles in special right triangles $30° - 60°$ and $45° - 45°$. Here we calculate basic trig functions sine, cosine and tangent for these angles.

Because trig functions do not depend on the size of a triangle, for calculations, we can choose any value for one of the sides. Let's select the value of 1 for the shortest leg of $30° - 60°$ triangle and for both legs of $45° - 45°$ triangle. Recall that in $30° - 60°$ triangle, hypotenuse is twice as the shortest leg (this leg is opposite to $30°$ angle), so the hypotenuse is 2. Then by Pythagorean Theorem the other leg is $\sqrt{2^2 - 1^2} = \sqrt{3}$. For $45° - 45°$ triangle, hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. We can draw the following two pictures
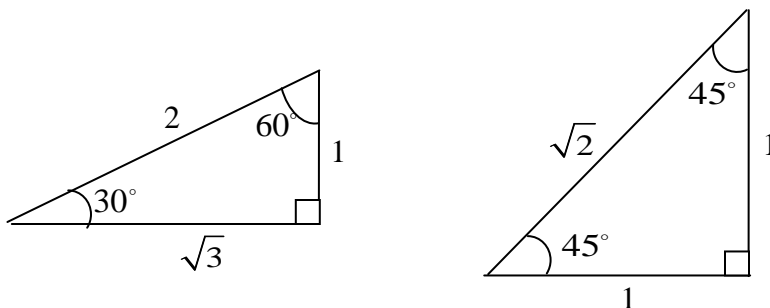
Now, we use the definition of basic trig functions.

$30°$ angle: opposite side is 1, adjacent side is $\sqrt{3}$, and hypotenuse is 2. Therefore,

$$\sin 30° = \frac{1}{2}, \; \cos 30° = \frac{\sqrt{3}}{2}, \; \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$60°$ angle: opposite side is $\sqrt{3}$, adjacent side is 1, and hypotenuse is 2. Therefore,

$$\sin 60° = \frac{\sqrt{3}}{2}, \; \cos 60° = \frac{1}{2}, \; \tan 60° = \sqrt{3}.$$

$45°$ angle: opposite side is 1, adjacent side is 1, and hypotenuse is $\sqrt{2}$. Therefore,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \; \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \; \tan 45° = 1.$$

We summarize these results in the following table

Angle θ	$30°$	$45°$	$60°$
$\sin \theta$	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$
$\cos \theta$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$
$\tan \theta$	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Working with arbitrary acute angles

Practical application of trigonometry essentially is based on the following general principal: it is easier to measure angles than distances. To measure angles, there is an optical device which is called the **clinometer**. Schematically, it looks like this:
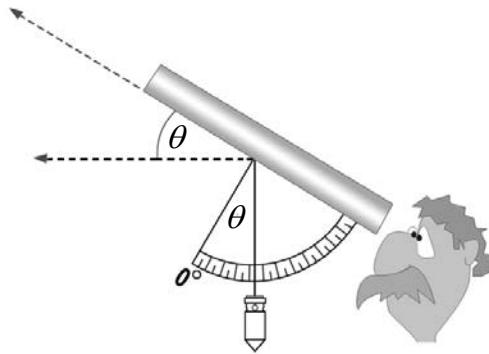

The angle from the horizontal line going up is called the angle of **elevation**. In the picture above, θ is the angle of elevation that can be measured by clinometer. In similar way, the angle from the horizontal line going down is called the angle of **depression:**
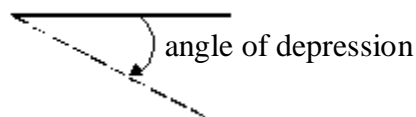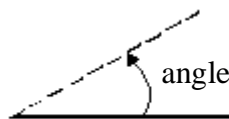
To find the values of basic trig functions for arbitrary angles, we can use buttons *sin*, *cos* and *tan* on scientific or graphing calculator.

Example 16.1. Let's solve the **"**giraffe**"** problem, stated at the beginning of this session. We can draw corresponding picture like this
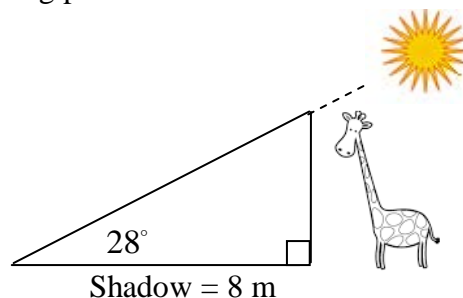
For the $28°$ angle, giraffe is the opposite side, and shadow – adjacent. A suitable trig function is tangent (ratio of the opposite side to adjacent). Let's denote giraffe's shadow by *s* and giraffe's height by *g*. We have $\tan 28° = \dfrac{g}{s}$. From here,

$$g = s\tan 28° = 8 \cdot 0.5317 = 4.25 \ \text{m.}$$

Example 16.2. Nick launched a kite on a 120-m thread. The angle of elevation of the thread is $37°$. At what altitude is the kite flying?

Solution. Here is the corresponding picture
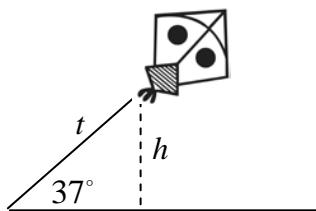
Let's denote the length of the thread by t. This is the hypotenuse and $t = 120$. The problem is to find height h which is opposite to the $37°$ angle. A suitable trig function is sine (ratio of the opposite side to hypotenuse). We have $\sin 37° = \dfrac{h}{t}$. From here,

$$h = t \sin 37° = 120 \cdot 0.6018 = 72.22 \text{ m.}$$

Example 16.3. A ladder is leaning against the wall such that the angle of depression of the top of the ladder is $56°$. What is the length of the ladder if the distance from its lower end to the wall is 2 m?
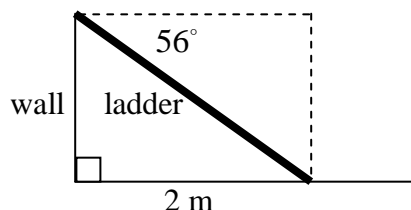
Solution. In the right triangle formed by the ladder and dotted lines, the ladder is the hypotenuse, and the top (horizontal) dotted line is the side adjacent to $56°$ angle. Let's denote the length of this side by d and the ladder's length by l. We have $d = 2$ m. The problem is to find l. A suitable trig function is cosine (ratio of the adjacent side to hypotenuse). We have $\cos 56° = \dfrac{d}{l}$. From here,

$$l = \frac{d}{\cos 56°} = \frac{2}{0.559} = 3.58 \text{ m.}$$

Trig functions allow also to find angles in right triangles when info about sides is known. To solve such problems, first identify, similar to previous examples, which trig function relates to given problem and find the value of this function. Then, to find the angle you can use buttons $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ on calculator. These buttons calculate the values of so-called **inverse** trigonometric functions. These functions restore angles from the values of corresponding trig functions. We will say more about inverse trig functions in sessions 18 and 19.

Session 16: Trigonometric Functions for Acute Angles

Example 16.4. A ship is 160 m away from the center of a horizontal barrier that measures 200 m from end to end. What is the minimum angle that the ship must be turned to avoid hitting the barrier?
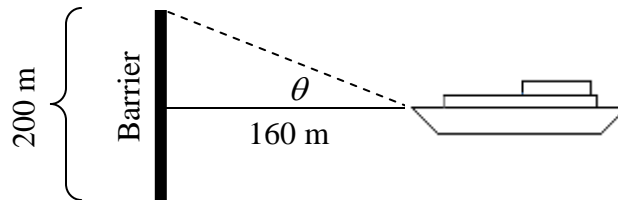
Solution. The problem is to find angle θ. Let's denote half of the length of barrier as a. We have $a = 200/2 = 100$ m. This is the side of right triangle on the picture and it is opposite to angle θ. Another side is the distance from the ship to the barrier. This side is adjacent to angle θ. We denote it as d. It is given that $d = 160$ m. Appropriate trig function here is tangent (ratio of the opposite side to adjacent).

$$\tan\theta = \frac{a}{d} = \frac{100}{160} = 0.625 \implies \theta = \tan^{-1}(0.625) = 32°.$$

Example 16.5. An airplane is flying at an altitude of 2.5 miles and is preparing for landing. It is 8.6 miles from the runway. Find the angle of depression that the airplane must make to land safely.
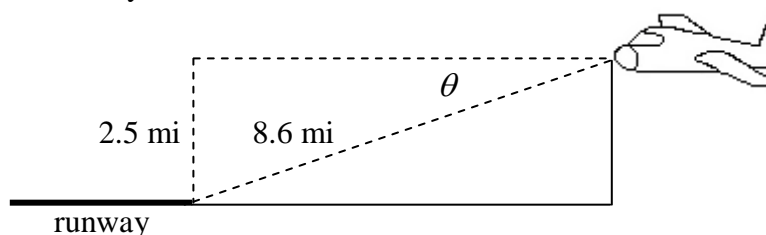
Solution. The problem is to find angle θ. Let's denote distance from the airplane to runway as d, and altitude as h. It is given that $d = 8.6$ mi and $h = 2.5$ mi. In the drawn right triangle (with dotted lines), d is hypotenuse, and h is opposite side for angle θ. Appropriate trig function here is sine (ratio of the opposite side to hypotenuse).

$$\sin\theta = \frac{h}{d} = \frac{2.5}{8.6} = 0.29 \implies \theta = \sin^{-1}(0.29) = 17°.$$

Example 16.6. Lillian wants to shingle her roof. The roofer asked her for the angle of elevation of the roof to make sure he can climb the roof safely. Help Lillian to calculate the angle according to the following picture:
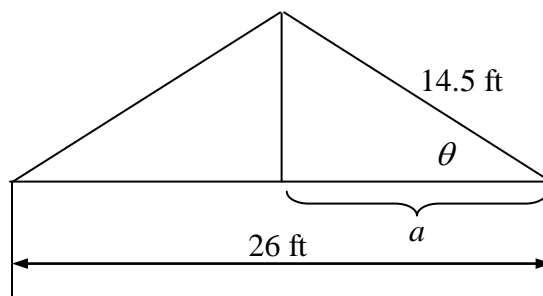
Solution. The problem is to find angle θ. Let's denote the marked horizontal line segment (half of the width of the house) as a, and slant line segment (the width of the roof) as s. We have $a = 26/2 = 13$ ft and $s = 14.5$ ft. In the triangle on the right side of the picture, s is hypotenuse and a is the side adjacent to angle θ. Appropriate trig function here is cosine (ratio of the adjacent side to hypotenuse).

$$\cos \theta = \frac{a}{s} = \frac{13}{14.5} = 0.9 \implies \theta = \cos^{-1}(0.9) = 26°.$$

Exercises 16

Round all answers to the nearest tenth.

16.1. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is $70°$. If the building is 80 ft tall, how far away is the fountain?

16.2. Allison is looking at the top of a tall building. Her eyes are 5 feet above the ground. The angle of elevation is $75°$ and she is 15 feet from the building. How tall is the building?

16.3. A 20 foot ladder rests against a wall. The ladder makes a $55°$ angle with the ground. How far from the wall is the base of the ladder?

16.4. Nick needs to reach a top window of the house using a ladder. He wants to put a ladder 1.5 meters from the wall. At this point, he measured that the angle of elevation to the window is $53°$. How long does the ladder have to be?

16.5. Ben is flying a kite and realizes that 260 feet of string are out. The angle of elevation of the kite is $40°$. How high is kite above the ground?

16.6. Lillian is swimming in the sea and notices a coral reef at the sea bottom. The angle of depression is $37°$ and the depth of the sea here is 50 feet. How far is she from the reef?

16.7. Over 6000 feet (horizontal), a road rises 330 feet (vertical). What is the angle of elevation?

16.8. Suppose a tree 15 m in height casts a shadow of length 27 m. What is the angle of elevation from the end of the shadow to the top of the tree?

16.9. A boat is sailing and spots a big shell 18 feet below the water. A diver jumps from the boat and swims 25 feet to reach the shell. What is the angle of depression from the boat to the shell?

16.10. A ladder leans against a wall. The foot of the ladder is 5.4 feet from the wall. The ladder is 15 feet long. Find the angle the ladder makes with the wall.

16.11. A vertical pole stands on the ground and has a support wire that runs from its top to the ground. The support is 50 feet long and anchored 22 feet from the base of the pole. Find the angle of elevation from the anchor point to the top of the pole.

16.12. Eli is putting up an antenna at the flat roof of a house. At its top, he attached a 50 ft guy wire and anchored it on the roof. Antenna is 30 ft long. What angle does the guy wire form with the antenna?

Challenge Problems

16.13. At a river shore (right near the water) a tree is standing. On the opposite side of the river, across the tree, Margaret is standing at the distance of 10 m from the water. She wants to determine the width of the river. She found that the angle of elevation to the top of the tree is $32°$. Then Margaret walked right to the water and found that the angle of elevation now became $43°$. What is the width of the river?

16.14. Esther and Nick stand at points E and N, 2 m apart in a dark room with a large mirror. Esther stands 2 m from the mirror, and Nick stands 1 m. At what angle EBA should Esther shine a flashlight on mirror so that the reflected light directly strikes Nick?

Note. According to the law of reflection, $\angle EBA = \angle NBC$.
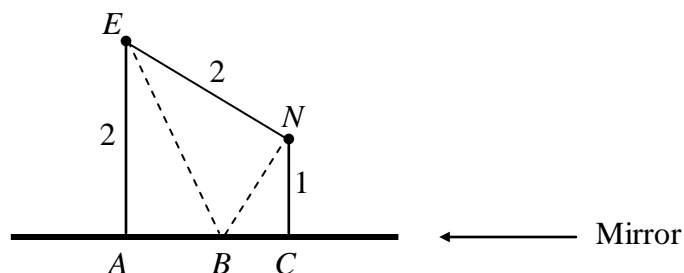
Session 17

# Trigonometric Functions for Arbitrary Angles

Unit Circle

In previous session we defined trig functions for acute angles: we constructed right triangle with given angle, and defined trig functions as ratios of sides in this triangle. This approach cannot be used for angles that are not acute like obtuse or negative angles: there are no right triangles with such angles.

Nevertheless, it is possible to define trig functions for arbitrary angles. To do this we will use a special tool that allows to reformulate definition of trig function of acute angles in such a way that a new definition can be used for arbitrary angles. This tool is called the **unit circle in the system of coordinates**.

This is just a circle with the radius of 1 and the center at the origin:
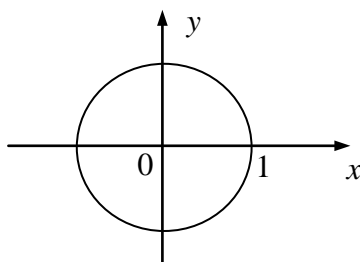
In this figure, we will draw angles in **standard position**. It means that their vertices are in the origin, and the initial sides goes along the positive part of x-axis. Here is an example of such angle θ in the 1st quadrant (i.e. acute angle):
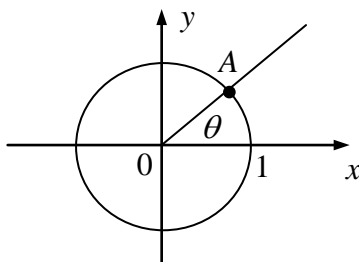
Angle θ is uniquely defined (up to coterminal angles) by the point A on the circle at which terminal side intersects the circle. We will call point A **corresponding** to angle θ.

Let (a, b) be coordinates of the point A (we also use the notation $A(a, b)$ for point A):
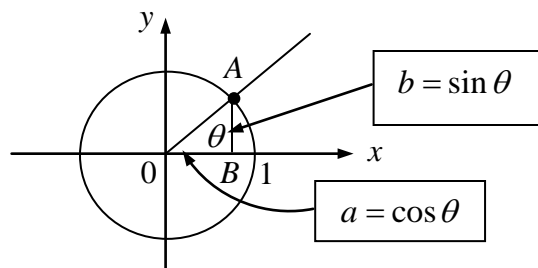
Notice that $0A = 1$ (radius of the unit circle). Then from the right triangle $0AB$, we have

$$\sin\theta = \frac{AB}{0A} = \frac{b}{1} = b, \ \cos\theta = \frac{0B}{0A} = \frac{a}{1} = a.$$

We see that for acute angles, **sine and cosine are coordinates** of the corresponding points on the unit circle: sine is the second coordinate (y-coordinate), and cosine is the first coordinate (x-coordinate). We've got the reformulation (i.e. a new definition) of sine and cosine for acute angles: they are coordinates of points on unit circle. We can use this reformulation as a general definition for arbitrary angles.

Definition. Let θ be an arbitrary angle in standard position, and $A(a, b)$ be the corresponding point on unit circle. Then, by definition,

$$\sin\theta = b, \ \cos\theta = a, \ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

Note. To memorize which of the trig functions – sine or cosine, is the first coordinate and which one is the second, you may use the alphabetical order of the first letters in the words **s**ine and **c**osine (c is before s, so **c**osine is the first coordinate, and **s**ine is the second).
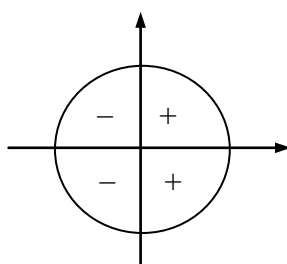
Other three trig functions can be defined as reciprocals to the basics:

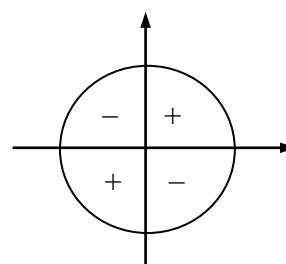$$\cot\theta = \frac{1}{\tan\theta}, \ \sec\theta = \frac{1}{\cos\theta}, \ \csc\theta = \frac{1}{\sin\theta}.$$

Because sine and cosine are coordinates, trig functions may take both positive and negative values depending on the quadrant in which angle θ lies. The following figures show the signs of basic trig functions.
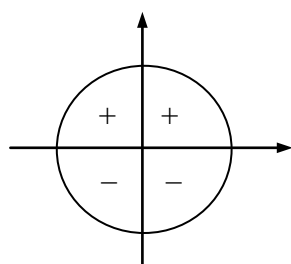
Signs of sine   Signs of cosine   Signs of tangent

Session 17: Trigonometric Functions for Arbitrary Angles

Note. The following phrase may help to memorize which of these functions is positive in each quadrant: "**A**ll **S**tudents **T**ake **C**alculus". This phrase hints that in the first quadrant all three are positive, in the second – only sine, in the third – only tangent, and in the fourth – only cosine.

Example 17.1. Calculate basic trig functions for quadrant angles of $0°$, $90°$, $180°$, $270°$, and $360°$.

Solution.

1) For $0°$ and $360°$ angles the corresponding point on unit circle has coordinates (1, 0). Therefore,

$$\sin 0° = \sin 360° = 0, \ \ \cos 0° = \cos 360° = 1, \ \ \tan 0° = \tan 360° = 0.$$

2) For $90°$ angle the corresponding point has coordinates $(0, 1)$. Therefore, $\sin 90° = 1$, $\cos 90° = 0$. By definition, $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$. Because $\cos 90° = 0$, $\tan 90°$ is undefined (we can not divide by zero).

3) For $180°$ angle the corresponding point has coordinates $(-1, 0)$. Therefore, $\sin 180° = 0$, $\cos 180° = -1$, $\tan 180° = 0$.

4) For $270°$ angle the corresponding point has coordinates $(0, -1)$. Therefore, $\sin 270° = -1$, $\cos 270° = 0$, $\tan 270°$ is undefined.

We summarize the results of example 17.1 in the following table

Angle θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin \theta$	0	1	0	−1	0
$\cos \theta$	1	0	−1	0	1
$\tan \theta$	0	undefined	0	undefined	0

We see that maximal and minimal values of sine and cosine of the quadrant angles are 1 and – 1 respectively. For all other angles sine and cosine are between –1 and 1. In general, for any angle θ

$$|\sin \theta| \le 1, \ \ |\cos \theta| \le 1.$$

There is no restriction for tangent.

Reduction Formulas (The "Head Rule")

In example 17.1 we calculated sine, cosine and tangent for quadrant angles $0°$, $90°$, $180°$, $270°$, and $360°$. Here we describe the way how to simplify sine and cosine of angles when we add (or subtract) angle θ to (from) quadrant angles. In other words we will simplify the following expressions:

$$\sin\left(90° \pm \theta\right), \ \cos\left(180° \pm \theta\right), \ \sin\left(270° \pm \theta\right), \ \sin\left(360° \pm \theta\right).$$

Formulas to simplify these expressions are called **reduction** formulas. For example, it is not difficult to get that $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$ (sine of an angle and cosine of complement angle are equal). Another example is $\cos(180° + \theta) = -\cos\theta$. We can analyze each of such expressions separately, and get all reduction formulas (there is total eight of them). Instead, we suggest a simple rule to get such formulas. We call this rule the **Head Rule**.

Head Rule works like this. To get the reduction formula assume that angle θ is acute. We need to answer two questions:

1) Should we put minus sign on the right side of the formula?

2) Should we change sine to cosine and/or vice versa?

To answer the first question, determine the quadrant in which angle under consideration lies. Based on the quadrant, determine the sign of trig function (as described above).

To answer the second question, move your head along the axis on which the quadrant angle lies. In doing this you automatically get answer "yes" or "no".

Example 17.2. Get reduction formulas for

$$\sin(90° + \theta), \quad \cos(180° + \theta), \quad \sin(270° + \theta).$$

Solution.

For $\sin(90° + \theta)$:

1) Angle $90° + \theta$ lies in 2nd quadrant. Here sine is positive, so minus sign is not needed.

2) Move your head along vertical axis (where $90°$ angle is located) and you get the answer "yes", so change sine to cosine. Final answer: $\sin(90° + \theta) = \cos\theta$.

For $\cos(180° + \theta)$:

1) Angle $180° + \theta$ lies in 3rd quadrant. Here cosine is negative, so minus sign is needed.

2) Move your head along horizontal axis (where $180°$ angle is located) and you get the answer "no", so do not change cosine to sine. Final answer: $\cos(180° + \theta) = -\cos\theta$.

For $\sin(270° + \theta)$:

1) Angle $270° + \theta$ lies in 4th quadrant. Here sine is negative, so minus sign is needed.

2) Move your head along vertical axis (where $270°$ angle is located) and you get the answer "yes", so change sine to cosine. Final answer: $\sin(270° + \theta) = -\cos\theta$.

Special cases of reduction formulas (when quadrant angle is $0°$) are

$$\sin(-\theta) = -\sin\theta \text{ (odd property of sine)}$$
$$\cos(-\theta) = \cos\theta \quad \text{(even property of cosine)}$$

Reference Angle

This is a useful tool to reduce calculation of trig functions of arbitrary angles to acute angles.

Definition. Let θ be an arbitrary angle in standard position. Angle θ_r is called the **reference angle** to θ, if it satisfies three conditions:

1) Terminal side of θ_r coincides with the terminal side of θ.
2) Initial side of θ_r is horizontal (it coincides with either the positive or negative parts of the x-axis).
3) Angle θ_r is acute angle.

Let's see how reference angle θ_r looks like depending on the quadrant in which original angle θ is located.
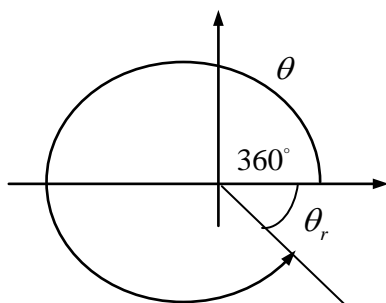
1) Angle θ is located in the first quadrant. Then θ_r coincides with θ: $\theta_r = \theta$.

2) Angle θ is located in the second quadrant. Then $\theta_r = 180° - \theta$:
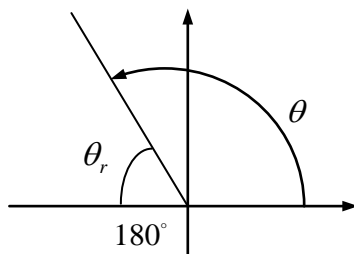
3) Angle θ is located in the third quadrant. Then $\theta_r = \theta - 180°$:
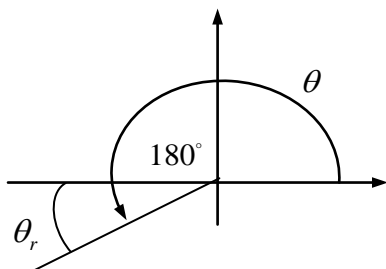
4) Angle θ is located in the fourth quadrant. Then $\theta_r = 360° - \theta$:

Reference angle is useful because up to sign, the values of trig functions of θ coincide with the value of the same trig function for the reference angle θ_r and θ_r is always acute angle. You can check this using reduction formulas described above.

Main Property of Reference Angle

> The absolute value of any trig function of any angle is equal to the value of the same trig function of the reference angle.

Hence, to calculate the value of a trig function, it is enough to find the sign of the function and calculate the value of trig function of the reference angle.

Example 17.3. Calculate $\cos 120°$.

Solution. Angle $120°$ is located in the 2$^{\text{nd}}$ quadrant, so $\cos 120° < 0$. This is the case 2) in the pictures above. Reference angle $\theta_r = 180° - 120° = 60°$. We have $\cos 60° = \dfrac{1}{2}$. Therefore,

$$\cos 120° = -\frac{1}{2}.$$

Example 17.4. Calculate $\sin 225°$.

Solution. Angle $225°$ is located in the 3$^{\text{rd}}$ quadrant, so $\sin 225° < 0$. This is the case 3) above. Reference angle $\theta_r = 225° - 180° = 45°$. We have $\cos 45° = \dfrac{\sqrt{2}}{2}$. Therefore,

$$\sin 225° = -\frac{\sqrt{2}}{2}.$$

Example 17.5. Calculate $\tan 330°$.
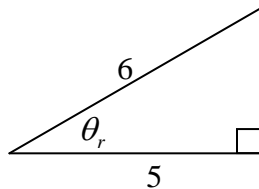
Solution. Angle $330°$ is located in the 4$^{\text{th}}$ quadrant, so $\tan 330° < 0$. This is the case 4) above. Reference angle $\theta_r = 360° - 330° = 30°$. We have $\tan 30° = \dfrac{\sqrt{3}}{3}$. Therefore,

$$\tan 330° = -\frac{\sqrt{3}}{3}.$$

Example 17.6. Find the values of other five trig functions, if $\cos \theta = -\dfrac{5}{6}$ and $\tan \theta > 0$.

Solution. For reference angle θ_r, $\cos \theta_r = \dfrac{5}{6}$. Let's draw a right triangle, using definition of $\cos \theta_r$ as ratio of adjacent side to hypotenuse:

By the Pythagorean theorem, vertical leg of this triangle is $\sqrt{6^2 - 5^2} = \sqrt{11}$. From here,

$\sin\theta_r = \dfrac{\sqrt{11}}{6}$ and $\tan\theta_r = \dfrac{\sqrt{11}}{5}$. Since $\cos\theta < 0$ and $\tan\theta > 0$, angle θ lies in the 3$^{\text{rd}}$

quadrant. Therefore, $\sin\theta = -\dfrac{\sqrt{11}}{6}$ and $\tan\theta = \dfrac{\sqrt{11}}{5}$. Other three trig functions are:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{6}{5}, \quad \csc\theta = \frac{1}{\sin\theta} = -\frac{6}{\sqrt{11}} = -\frac{6\sqrt{11}}{11}.$$

It is possible to define trig function using a circle with arbitrary radius r (not only unit circle with $r = 1$). Namely, sine, cosine and tangent of any angle θ (in a standard position), which has point $A(a, b)$ on its terminal side are:

$$\sin\theta = \frac{b}{r}, \quad \cos\theta = \frac{a}{r}, \quad \tan\theta = \frac{b}{a}, \quad r = \sqrt{a^2 + b^2}.$$

Note. In the above formulas, radius r is the distance from point $A(a, b)$ to the origin.

Example 17.7. Find the value of the six trig functions of the angle θ if point $(2, -3)$ lies on the terminal side of angle θ, and θ is in standard position.

Solution. We have $a = 2$, $b = -3$. Using the above formulas,

$$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-3)^2} = \sqrt{13},$$

$$\sin\theta = \frac{b}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}, \quad \cos\theta = \frac{a}{r} = \frac{2\sqrt{13}}{13}, \quad \tan\theta = \frac{b}{a} = \frac{-3}{2} = -\frac{3}{2}.$$

Other three trig function are

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{\sqrt{13}}{3}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{13}}{2}, \quad \cot\theta = \frac{1}{\tan\theta} = -\frac{2}{3}.$$

Exercises 17

In exercises 17.1 and 17.2, get reduction formulas for given expressions.

17.1.

  a) $\sin\left(180° + \theta\right)$

  b) $\cos\left(90° - \theta\right)$

17.2.

  a) $\cos\left(180° - \theta\right)$

  b) $\sin\left(270° - \theta\right)$

In exercises 17.3 and 17.4, find reference angles to given angles.

17.3.

  a) $130°$

  b) $320°$

  c) $250°$

  d) $85°$

17.4.

  a) $200°$

  b) $10°$

  c) $310°$

  d) $100°$

In exercises 17.5 and 17.6, reference angle of angle θ and quadrant in which angle θ is located are given. Find angle θ in the interval from $0°$ to $360°$.

17.5.

  a) $40°$, quadrant III

  b) $70°$, quadrant II

  c) $50°$, quadrant IV

  d) $20°$, quadrant I

17.6.

  a) $40°$, quadrant IV

  b) $70°$, quadrant I

  c) $50°$, quadrant II

  d) $20°$, quadrant III

In exercises 17.7 and 17.8, calculate the values of trig functions without using a calculator.

17.7.

  a) $\sin 210°$

  b) $\cos 300°$

  c) $\tan 135°$

17.8.

  a) $\sin 315°$

  b) $\cos 150°$

  c) $\tan 240°$

In exercises 17.9 and 17.10, use given information to identify the quadrant in which angle θ is located and find the values of five remaining trig functions.

17.9.

a) $\sin \theta = -\dfrac{2}{3}$ and $\tan \theta < 0$

b) $\cos \theta = -\dfrac{2}{5}$ and $\sin \theta > 0$

c) $\tan \theta = \dfrac{3}{5}$ and $\cos \theta < 0$

17.10.

a) $\sin \theta = \dfrac{4}{7}$ and $\cos \theta < 0$

b) $\cos \theta = -\dfrac{5}{8}$ and $\tan \theta > 0$

c) $\tan \theta = -\dfrac{7}{4}$ and $\sin \theta < 0$

In exercises 17.11 and 17.12, coordinates of a point are given. Find the values of six trig functions of an angle in standard position for which the terminal side passes through this point.

17.11. $(-1, -2)$

17.12. $(-3, 5)$

Session 18

Solving Oblique Triangles – Law of Sines

Oblique triangles – triangles that are not necessary right triangles. We are going to "solve" them. It means to find its basic elements – sides and angles, given some of them. First of all, let's see what elements must be given. Obvious, if only angles are given and no sides, this info is not enough to determine sides since triangles with the same angles are similar and may have different sizes. So, at least one side must be given. We consider all possible cases when one, two or all three sides are given as well as some number of angles. More precisely, the following four cases are possible in solving triangles:

1) One side and two angles are given.

2) Two sides and an angle opposite to one of them are given.

3) Two sides and angle between them are given.

4) Three sides are given.

Main tools to solve these problems are two important theorems: Law of Sines and Law of Cosines. Here we consider Law of Sines and the first two problems.

Law of Sines

It is clear that in any triangle, the bigger side, the bigger opposite angle. However, sides are **not** proportional to opposite angles. For example, in right triangle $30° - 60°$, if side opposite to $30°$ is a, then side opposite to $60°$ is $\sqrt{3}a$, which is not $2a$. Law of Sines says that in any triangle sides are proportional to the **sines** of opposite angles. In other words, the ratio of any side to the sine of the opposite angle remains the same for all three sides in a given triangle.

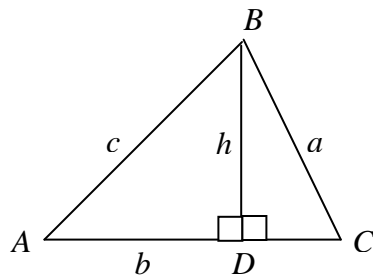More formally, the following theorem is true.

Theorem (Law of Sines). Consider triangle *ABC*:
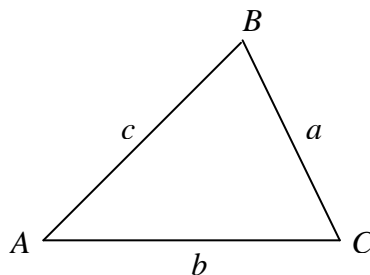
Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof. For simplicity, we consider only acute triangle (proof for obtuse triangle is slightly different, but similar). Let's draw height h to the side b:

Height h breaks triangle ABC into two right triangles: ABD and BCD. Let's consider sines of angles A and C:

From triangle ABD, $\sin A = \dfrac{h}{c}$. Solve for h: $h = c \sin A$.

From triangle BCD, $\sin C = \dfrac{h}{a}$. Solve for h: $h = a \sin C$.

Equate the above two expressions for h: $c \sin A = a \sin C$. Divide both sides of this equation by $\sin A \cdot \sin C$ and get

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Similar ratio is true for the side b and angle B. The proof is completed.

Law of Sines works perfectly good for solving triangles for the case 1) above when a side and two angles of a triangle are given. In this case triangle is defined uniquely (we assume that the sum of given angles is less than $180°$). With no problem we can find the third angle by subtracting two given angles from $180°$, and then use Law of Sines to find two other sides.

Example 18.1. Solve a triangle, if $a = 14$, $B = 40°$, and $C = 75°$.

Solution. We need to find angle A, and sides b and c.

1) $A = 180° - B - C = 180° - 40° - 75° = 65°$.

2) By Law of Sines, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. From here, using calculator, we get

$$b = \frac{a \sin B}{\sin A} = \frac{14 \cdot \sin 40°}{\sin 65°} = 9.9.$$

3) Again by Law of Sines, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$. From here

$$c = \frac{a \sin C}{\sin A} = \frac{14 \cdot \sin 75°}{\sin 65°} = 14.9.$$

Final answer: $A = 65°$, $b = 9.9$, $c = 14.9$.

## Using Law of Sines – Ambiguous Case

We consider how to solve a triangle for the case 2) above when two sides and an angle opposite to one of them are given. In this case a triangle is not always defined uniquely and we may face some difficulties to solve it. This is the ambiguous case. We will assume that the following data are given: sides a and b, and angle A opposite to side a.

Case: angle A is obtuse

This is a simple case since only two options are possible: triangle does not exist or triangle is unique. To understand why, let's draw angle A and mark side b on its slant side:
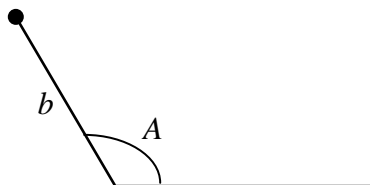
To get a triangle, we need to draw side a from the top point to meet with the horizontal side of angle A. Obvious, if side a is too short, it will not meet the horizontal side, and triangle does not exist:
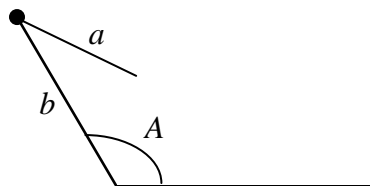
For triangle to exist, side a must be greater than b. Then triangle is defined uniquely. We come up to the following

Proposition 18.1. Let two sides a and b, and **obtuse** angle A opposite to side a are given. Then

  1) If $a \le b$, triangle does not exist.

  2) If $a > b$, triangle exists and unique.

Note. Conclusion in part 1) is also clear by the following reason: if $a \le b$, then $A \le B$. Angle A is obtuse, so B also must be obtuse. But triangle cannot have two obtuse angles.

Example 18.2. Solve a triangle, if $a = 18$, $b = 14$, and $A = 130°$.

Solution. We need to find angles B and C, and side c. Using Law of Sines, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. From here

$$\sin B = \frac{b \sin A}{a} = \frac{14 \cdot \sin 130^\circ}{18} = 0.596.$$

Notice, that at this point we calculated **sine** of angle B, but not angle B itself. To restore angle B from its sine, we can use the button $\sin^{-1}$ on calculator similar to what we did in session 16 for right triangles. This button corresponds to inverse sine. We have
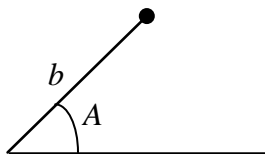
$$B = \sin^{-1}(0.596) = 37^\circ.$$

Now it is easy to find angle C: $\ C = 180^\circ - A - B = 180^\circ - 130^\circ - 37^\circ = 13^\circ$.
To find side c, we can use Law of Sines again:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{18 \sin 13^\circ}{\sin 130^\circ} = 5.3.$$

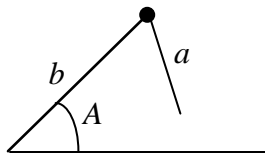Final answer: $\ B = 37^\circ, \ C = 13^\circ, \ c = 5.3$.

Case: angle A is acute

Similar to obtuse angle, let's draw angle A and mark side b on its slant side:
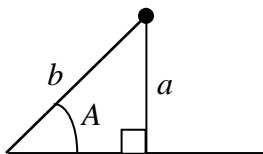

To create a triangle, we draw side a from the top point. Four cases are possible here:

1) Side a is too short to meet with the horizontal side:
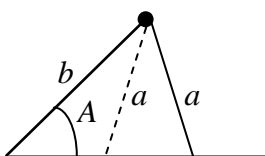

   Triangle does not exist.

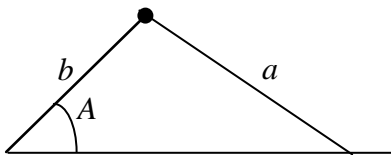2) Side a touches horizontal side exactly in one point:

   We have right triangle which is unique.

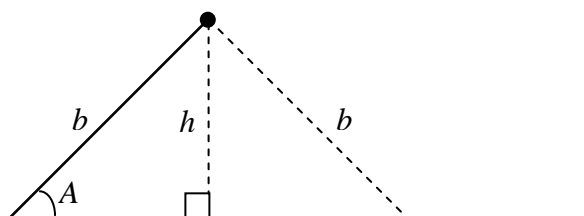3) Side a intersects horizontal side in two points:

We have two triangles with sides a, b and angle A.

4) Side a is long enough, and to create a triangle, side a intersects horizontal side only in one point:

   The triangle is unique. The top angle may be acute or obtuse.

How can we distinguish the above four cases using the values of sides a, b and angle A? Take a look at this picture

In your mind, draw side a from the top point. You can see that if $a < h$, side a is too short and triangle does not exist. If $a = h$, we can draw only one right triangle. If a is between h and b: $h < a < b$, we can draw side a on both sides (left and right) of the height h, and we have two triangles. Finally, if $a \geq b$, we can draw only one triangle. Notice that

$$\frac{h}{b} = \sin A, \text{ so } h = b \sin A.$$

We come up to the following

Proposition 18.2. Let two sides a and b, and **acute** angle A opposite to side a are given.

1) If $a \geq b$, triangle is unique. This triangle may be acute or obtuse.

2) If $a < b$, denote $h = b \sin A$.

   a) If $a < h$, triangle does not exist.

   b) If $a = h$, triangle is unique. It is a right triangle.

   c) If $a > h$, there are two triangles. Both of them may be obtuse, or one is acute and the other is obtuse (see Exercise 18.17).

Practical way to use Proposition 18.2 is to directly apply Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ and

solve this equation for $\sin B$: $\sin B = \dfrac{b \sin A}{a}$. Three cases are possible when calculating $\sin B$ by the above formula:

1) $\sin B > 1$. Because $\sin B$ cannot be greater than 1, triangle does not exist.

2) $\sin B = 1$. Then $B = \sin^{-1}(1) = 90°$ .The triangle is unique. It is a right triangle.

3) $\sin B < 1$. Let $\sin B = s$, then $B = \sin^{-1}(s)$. Angle B (as inverse sine of positive value) is always positive and acute. So, one triangle already exists. To understand whether another triangle exists, notice that there is one more angle B' such that $\sin B' = \sin B$. This angle is supplement to angle B : $B' = 180° - B$. Angle B' is obtuse. Should we accept it as a second solution or reject it? Just compare a and b :

    a) If $a \geq b$, then by Proposition 18.2 the triangle is unique, so second triangle does not exist.

    b) If $a < b$, the second triangle exists having the obtuse angle
        $B' = 180° - B$.

Note. Another way to see whether second triangle exists, is to calculate supplemental angle $B' = 180° - B$ in any case (regardless on which side is bigger: *a* or *b*). Then, if $B' + A < 180°$, accept B' , and if $B' + A \geq 180°$, reject it.

Example 18.3. Let *b* = 20 and $A = 30°$. Determine the number of triangles that satisfy the given conditions. If triangle exists, solve it.

    1) $a = 5$

    2) $a = 10$

    3) $a = 16$

    4) $a = 25$

Solution. Using Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, we have $\sin B = \dfrac{b \sin A}{a}$. From calculator (or just notice that $30°$ is a special angle), $\sin A = \sin 30° = 0.5$, and expression for $\sin B$ becomes $\sin B = \dfrac{20 \cdot 0.5}{a}$, so $\sin B = \dfrac{10}{a}$

1) If *a* = 5, then $\sin B = \dfrac{10}{5} = 2$. Because sine cannot be greater than 1, triangle does not exist.

2) If *a* = 10, then $\sin B = \dfrac{10}{10} = 1$ and $B = \sin^{-1}(1) = 90°$. This is a right triangle. To solve it, calculate angle *C* and side *c*.

    $C = 90° - A = 90° - 30° = 60°$. Side *c* can be found by Pythagorean Theorem (notice that *b* is hypotenuse, and *a* and *c* are legs):

    $c = \sqrt{b^2 - a^2} = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}$.

    Final answer: $B = 90°$, $C = 60°$, $c = 10\sqrt{3}$.

3) If $a = 16$, then $\sin B = \dfrac{10}{16} = 0.625$ and $B = \sin^{-1}(0.625) = 39°$. Another angle B', such that $\sin B' = \sin B$ is an obtuse angle and is supplement to angle B: $B' = 180° - B = 180° - 39° = 141°$. We accept it because

$$B' + A = 141° + 30° = 171° < 180°.$$

Another reason to accept B' is that $b > a$. So, we have two triangles. Let's solve them. It remains to find angle C and side c.

a) Triangle with angle $B = 39°$. We have $C = 180° - A - B = 180° - 30° - 39° = 111°$.

By Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{16 \sin 111°}{\sin 30°} = 20.87.$$

b) Triangle with angle $B = 141°$ (we use letter B instead of B').We have

$C = 180° - A - B = 180° - 30° - 141° = 9°$.
By Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{16 \sin 9°}{\sin 30°} = 5.01.$$

Final answer: There are two triangles:

$$B = 39°, \; C = 111°, \; c = 20.87.$$

$$B = 141°, \; C = 9°, \; c = 5.01.$$

4) If $a = 25$, then $\sin B = \dfrac{10}{25} = 0.4$ and $B = \sin^{-1}(0.4) = 24°$. Another angle B', such that $\sin B' = \sin B$ is supplement to B and is obtuse angle:

$$B' = 180° - B = 180° - 24° = 156°.$$

We reject it because

$$B' + A = 156° + 30° = 186° > 180°.$$

Another reason to reject B' is that $b < a$ and angle B' cannot be obtuse. So, we have only one triangle with angle $B = 24°$. To solve it, it remains to find angle C and side c.

$$C = 180° - A - B = 180° - 30° - 24° = 126°.$$

By Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{25 \sin 126°}{\sin 30°} = 40.45 \quad .$$

Final answer: $B = 24°, \; C = 126°, \; c = 40.45$.

Exercises 18

Round answers (where applicable) to the nearest tenth. For a triangle *ABC*, use the following notation: *A*, *B* and *C* are angles, and *a*, *b*, and *c* are sides opposite to the corresponding angles.

18.1. Three friends, Alice, Bob and Carol, are camping in their own tents on a flat meadow in the woodland. Alice and Carol are 12 m apart. The angle going from Alice to Bob and Carol is $50°$, and angle going from Carol to Alice and Bob is $110°$. How far apart are Alice and Bob, and Bob and Carol?

18.2. A post is standing on the ground and is supported by two wires (one on each side going in opposite direction). The ends of the wires on the ground are 7 ft apart. The angle of elevation of one of the wires is $80°$, and of the other wire is $65°$. Find the length of each wire.

18.3. Back to problem 18.1. On the next trip, the three friends set up tents such that Alice and Carol are 12 m apart, and Alice and Bob are 19 m apart. The angle going from Carol to Alice and Bob is $110°$. How far apart are Bob and Carol?

18.4. On another trip, the three friends set up tents such that Alice and Carol are 19 m apart, and Bob and Carol are 25 m apart. The angle going from Alice to Bob and Carol is $50°$. How far apart are Alice and Bob?

18.5. Nina intends to purchase a parcel of land in the shape of a triangle (say, *ABC*). She hired an assessor to measure the parcel. Nina told the assessor that it would be enough to provide her with minimum information such that she could calculate the remaining measurements (angles and sides) herself. The assessor submitted the following report: $a = 370$ ft, $A = 91°$, $b = 400$ ft,. Recently Nina took a course of Trigonometry, and she decided to fire this assessor. Why?

18.6. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $B = 115°$. Nina decided to fire this assessor as well. Why?

18.7. Nina then hired a third assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $B = 65°$, $C = 40°$. Nina again decided to fire this assessor. Why?

18.8. Nina hired yet another assessor. This assessor submitted the following report: $a = 370$ ft, $b = 500$ ft, $c = 120$ ft. Nina decided to fire this assessor too. Why?

Challenge Problems

18.9. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $b = 400$ ft. Nina decided to fire this assessor. Why?

18.10. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $b = 350$ ft, $C = 54°$. Nina decided to fire this assessor. Why?

18.11. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 61°$, $b = 400$ ft. This time Nina decided not to fire the assessor but to request additional information to be able to calculate remaining angles and sides. What additional info (non-numeric) is needed?

18.12. Two radar stations are located 15 miles apart. They detect an aircraft between them. The angle of elevation measured by the first station is $20°$, and the angle of elevation measured by the second station is $40°$. Find the altitude of the aircraft.

18.13. Back to previous problem, solve it in general form. Let d be the distance between radar stations, and let A and B be the angles of elevations to the aircraft from the stations. As before, the aircraft is between radar stations. Find the formula that expresses altitude of the aircraft in terms of A, B and d. In this case, it is possible to input this formula into a computer program to calculate the altitude instantly and track it during the aircraft flight between radar stations.

18.14. Back to problem 18.12. This time radar stations detect an aircraft on the right side of both stations. The distance between stations is 15 miles, and the angles of elevations to the aircraft are $20°$ and $70°$. Find the altitude of the aircraft.

18.15. Back to previous problem, solve it in general form. Let d be the distance between the radar stations, and let A and B be the angles of elevations to the aircraft. As before, the aircraft is on the same side (left or right) of both radar stations. Find the formula that expresses altitude of the aircraft in terms of A, B and d.

18.16. A 2.5 m flagpole is NOT standing up straight on the ground. It is supported by two wires (one on each side going in opposite direction), each 3 m long. Both wires make a $55°$ angle with the ground. How far apart is each wire from the flagpole?

18.17. Let two sides a and b, and angle A opposite to side a in a triangle ABC are given. Prove the following statements:

    a) If $0 < b \tan A < a < b$, there are two obtuse triangles.

    b) If $b \sin A < a < \min(b, \ b \tan A)$, there are two triangles: one is acute and the other is obtuse.

    Hint: Use the picture located above Proposition 18.2, and consider two cases for angle A: $A < 45°$ and $A > 45°$.

Session 19

Solving Oblique Triangles – Law of Cosines

In previous session, using Law of Sines, we considered two problems on solving triangles from the total of four: when one side and two angles are given, and when two sides and angle opposite to one of them are given.
Here we consider the remaining two problems:

1) Two sides and angle between them are given.

2) Three sides are given.

For both problems, triangle is unique and we do not have an ambiguous case. Method to solve these problems is based on another important law in trigonometry: Law of Cosines.

Note. Formally speaking, in problem 2) triangle does not exist, if one of the sides is greater or equal to the sum of two other sides. We will assume that this case will not happen.

Law of Cosines

This law can be treated as generalization of the Pythagorean Theorem from right triangles to oblique ones.
Consider the triangle
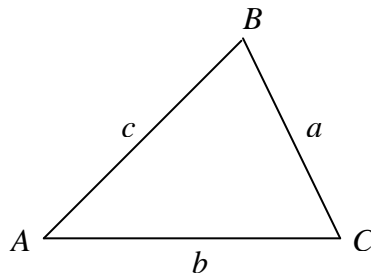
If angle C is not right angle, we cannot conclude that $c^2 = a^2 + b^2$, so Pythagorean Theorem is not true here. Instead, the following result is valid.

Theorem (Law of Cosines). For any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note. Consider the special case when $C = 90°$ (case of a right triangle). Then $\cos C = \cos 90° = 0$ and the above formula becomes $c^2 = a^2 + b^2$ which is exactly the Pythagorean Theorem. Therefore, the Law of Cosines can be considered as a generalization of the Pythagorean Theorem to oblique triangles.

Proof of the Law of Cosines.

Similar to the proof of Law of Sines, we consider only case of acute triangles. Let's draw the height h to the side b:
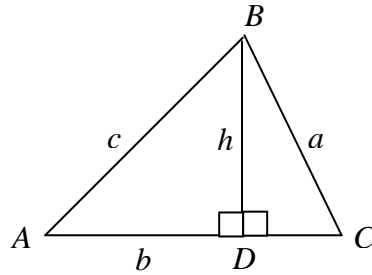
Height h breaks the triangle ABC into two right triangles: ABD and BCD. Let's write down the Pythagorean Theorem for each of them:

For triangle ABD: $c^2 = AD^2 + h^2$.

For triangle BCD: $a^2 = DC^2 + h^2$.

Now subtract the second equation from the first one to eliminate h^2:

$$c^2 - a^2 = AD^2 - DC^2 = (AD + DC)(AD - DC).$$

Notice that $AD + DC = b$. From here $AD = b - DC$ and

$$AD - DC = (b - DC) - DC = b - 2DC.$$

Formula for $c^2 - a^2$ becomes

$$c^2 - a^2 = b(b - 2DC) = b^2 - 2bDC$$

or

$$c^2 = a^2 + b^2 - 2bDC.$$

Now write down the definition of $\cos C$ from the triangle BCD: $\cos C = \dfrac{DC}{a}$. From here $DC = a\cos C$. Substitute this expression into the above formula for c^2:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The theorem is proved.

Note. In this theorem, we have expressed side c through sides a, b and the angle C that is between them. Because all three sides play the same role, no one has any privilege against the others. Therefore, we can write similar expressions for the sides a and b:

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ and } b^2 = a^2 + c^2 - 2ac\cos B.$$

Law of Cosines allows to express cosine of any angle through three sides. To do this, just solve the above equations for cosines:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

As we mentioned, using Law of Cosines we can solve triangles for the cases 1) and 2) indicated above. Also we will use the property $A + B + C = 180°$.

Case 1. Two sides and angle between them are given.

Example 19.1. Solve a triangle, if $a = 50$, $b = 15$, and $C = 55°$.

Solution. We need to find side c, and angles A and B.

    1) By Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C = 50^2 + 15^2 - 2\cdot 50\cdot 15\cdot\cos 55°.$$

    Using calculator, $\cos 55° = 0.5736$ and

$$c^2 = 2500 + 225 - 1500\cdot 0.5736 = 1864.6.$$

$$c = \sqrt{1864.6} = 43.2.$$

    2) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{15^2 + 1864.6 - 50^2}{2\cdot 15\cdot 43.2} = -0.3167.$

    Using calculator, $A = \cos^{-1}(-0.3167) = 108.5°$.

    3) $B = 180° - A - C = 180° - 108.5° - 55° = 16.5°.$

Final answer: $c = 43.2$, $A = 108.5°$, $B = 16.5°$.

Note. In solving problems for Case 1, it is possible in step 2) to use Law of Sines instead of Law of Cosines. However, you need to be very careful when using button $\sin^{-1}$ on calculator. This button always gives **only acute angle**, but the actual angle may be obtuse. To avoid possible mistake, we recommend, when using Law of Sines for calculation of angle, **do not** start with the angle opposite to the biggest side, because this angle may be obtuse. Always start with another angle which is definitely acute.

See, what may happen if you do not follow this advice. Let's return to Example 19.1, and try to use Law of Sines in step 2) to find angle A, which is opposite to the largest side $a = 50$:

We have $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$. From here

$$\sin A = \frac{a\sin C}{c} = \frac{50\sin 55°}{43.2} = \frac{50\cdot 0.8192}{43.2} = 0.9481,$$

and, using calculator, $\sin^{-1}(0.9481) = 71.5°$.

So, it looks like $A = 71.5°$. However, this answer is wrong. You can check it by calculating angle $B = 180° - A - C = 180° - 71.5° - 55° = 53.5°$ and using Law of Sines:

$$\frac{a}{\sin A} = \frac{50}{\sin 71.5°} = 52.7 \quad \text{but} \quad \frac{b}{\sin B} = \frac{15}{\sin 53.5°} = 18.7$$

Correct answer is the supplement obtuse angle $108.5° = 180° - 71.5°$.

When using Law of Cosines, not always you start with c^2. You need to start with the side for which opposite angle is given. The following example demonstrates it.

Example 19.2. Solve a triangle, if $b = 12$, $c = 15$, and $A = 25°$.

Solution. We need to find a, B and C.

1) Because angle A is given, we start with its opposite side a:

$$a^2 = b^2 + c^2 - 2bc \cos A = 12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cdot \cos 25°.$$

Using calculator, $\cos 25° = 0.9063$ and $a^2 = 144 + 225 - 360 \cdot 0.9063 = 42.73$.
$a = \sqrt{42.73} = 6.5$.

2) Let's find angle B using Law of Sines. It's safe to do because the opposite side b is not the largest one (see Note above). We have

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin B = \frac{b \sin A}{a} = \frac{12 \sin 25°}{6.5} = 0.78 \Rightarrow B = \sin^{-1}(0.78) = 51°.$$

3) $C = 180° - A - B = 180° - 25° - 51° = 104°$.

Final answer: $a = 6.5$, $B = 51°$, $C = 104°$.

Case 2. Three sides are given.

We only need to find three angles. Using Law of Cosines, we can start with any side. We recommend to start with the biggest side and find the opposite angle. In doing this, we guarantee that the other two angles are acute, and to find them we can use either Law of Cosines again or Law of Sines (without making mistake indicated in the Note above).

Here are our general recommendations:

> 1) When using Law of Sines, start with the smallest side.
> 2) When using Law of Cosines, start with the biggest side.

Example 19.3. Solve a triangle, if $a = 12$, $b = 20$, $c = 17$.

Solution. We need to find angles A, B and C.

1) According to the above recommendation, we use Law of Cosines starting with the largest side $b = 20$.

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

To find angle B, you can directly substitute given sides into this formula, or first solve it for $\cos B$. Let's solve for $\cos B$ first

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{12^2 + 17^2 - 20^2}{2 \cdot 12 \cdot 17} = 0.081, \ B = \cos^{-1}(0.081) = 85°.$$

2) To find angle A, let's use Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \ \sin A = \frac{a \sin B}{b} = \frac{12 \sin 85°}{20} = 0.598, \ A = \sin^{-1}(0.598) = 37°.$$

3) $C = 180° - A - B = 180° - 37° - 85° = 58°$.

Final answer: $A = 37°$, $B = 85°$, $C = 58°$.

Example 19.4. Justify the following method to check whether a triangle with given sides a, b, and c is an acute, an obtuse or a right triangle:

Let c be the biggest side of the triangle. Calculate the value $E = a^2 + b^2 - c^2$.

1) If $E > 0$, the triangle is acute.
2) If $E < 0$, the triangle is obtuse.
3) If $E = 0$, the triangle is right.

Solution. Using Law of Cosines, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{E}{2ab}$.

1) If $E > 0$, then $\cos C > 0$ and $C < 90°$. Since c is a biggest side, C is a biggest angle. Therefore, two other angels are also less than $90°$ and the triangle is acute.

2) If $E < 0$, then $\cos C < 0$ and $C > 90°$. The triangle is obtuse.

3) If $E = 0$, $\cos C = 0$ and $C = 90°$. The triangle is right.

Specific example of using this method was given in session 11, example 11.2.

Exercises 19

Round answers (where applicable) to the nearest tenth.

19.1. Eli and Ben came to the forest to pick some wild flowers. They started from the same point and each walked in a straight line at an angle of $40°$ relative to each other. Every minute they call out to each other to avoid being lost. A sound in this forest can be heard from up to 60 m away. After 10 minutes, Eli has walked 80 m and Ben has walked 70 m. Can they hear each other at that time?

19.2. Lillian wants to measure the distance between two trees that are on opposite sides of a small pond. She started at one of the trees and walked 240 ft in a straight line along the pond. Then she turned at $115°$ toward the second tree and walked another 310 ft until she reached the second tree. What is the distance between the trees?

19.3. Three friends, Alice, Bob and Carol, are camping in their own tents on a flat meadow in the woodland. Alice and Carol are 10 m apart, and Alice and Bob are 15 m apart. The angle going from Alice to Bob and Carol is $50°$. What is the angle going from Bob to Alice and Carol?

19.4. Back to previous problem. On the next trip, the three friends set up tents such that Alice and Bob are 15 m apart, Bob and Carol are 20 m apart, and the angle going from Bob to Alice and Carol is $110°$. Find the angle going from Carol to Alice and Bob.

19.5. A small airplane is 60 miles from the airport and is going down for landing. However its navigation device malfunctions and incorrectly shows the distance to be 65 miles. The dispatcher noticed this mistake and figured out that if airplane continues on its current course, it will end up 16 miles from airport. By how many degrees should dispatcher adjust the airplane heading?

19.6. An aircraft is making a flight to airport A. At some point a pilot receives information that due to bad weather, airport A is closed, and he needs to fly to airport B. At that moment, the aircraft is 520 mi apart from airport A, and is 650 mi apart from airport B. The distance between airports A and B is 570 mi. By what angle should the pilot change the course of the aircraft to fly to airport B?

Challenge Problem

19.7. After hurricane Sandy, a small tree was leaning. To keep it from falling, it was nailed by a 7-foot strap into the ground 5 feet from the base of the tree. The strap was attached to the tree 4 feet along the tree from the ground. By what angle from the vertical position was the tree leaning?

Session 20

Radian Measure of Angles

Most people familiar with the degree measure of angles. We already mentioned in session 15 that if we cut a round pizza pie (theoretically, of course) into 360 slices, the angle in one slice is of one degree (and this is a very tiny piece, so almost nothing to eat). But why the number 360 is used for the degree measure of angles?

This number was introduced by astronomers in ancient Babylon (at least 3000 B.C.). No one knows for sure why they settled for this number. At those times, it was already known that the yearly cycle consists of 365 and 1/4 days, even though astronomers didn't know yet that the earth revolves around the sun. It is reasonable to guess that they just rounded 365 and 1/4 to 360 because the number 360 has many more divisors. In other words, the number 360 can be divided into whole parts much better than 365. From this point of view, we could treat one degree angle as one day related to entire year. In any case, it's clear that angle measure based on the number 360 is artificial. It looks similar to the decimal system which is also an artificial one since it was introduced only because we have 10 fingers on our hands. In math, and especially in computer science, it is used more convenient systems like binary or octal which have as bases powers of two. These systems could be considered as natural ones.

And how about measurement of angles? Does some kind of natural measure of angles exist? The answer is "yes". This measure is called the **radian measure**.

To define the radian measure, consider an angle as a central angle in a circle. It means that we draw a circle and put the vertex of the angle in its center:
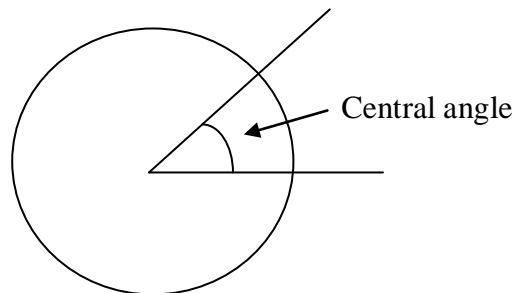

Of course, we can draw infinite many such circles. One of them is a **unit circle** (its radius is equal to 1). Using it, the **radian measure** (denoted as θ) of the central angle is the **length s of the corresponding arc** (arc between two radii):


$$\theta = s$$
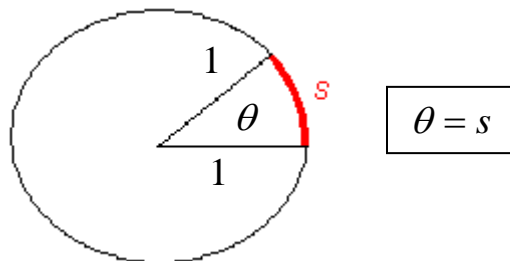
For any other circle (with arbitrary radius), by the proportionality, the ratio of the arc to the radius equals to the above arc of the unit circle. We come up to the following definition for arbitrary circle.

Definition of Radians. Consider an angle as a central angle: we draw a circle with the center in its vertex. Let the radius and the corresponding arc of the circle be r and s accordingly. Then the radian measure θ of the angle is defined as the ratio of s to r:

$$\theta = \frac{s}{r}$$
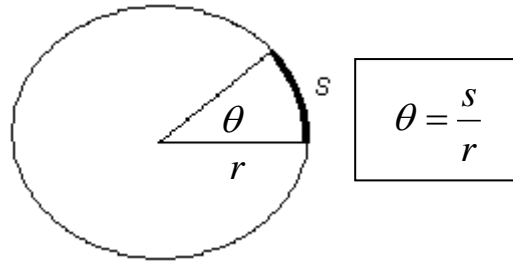
From here, $s = \theta \cdot r$. We may say that the radian measure of a central angle is the number of radii that can fit in the corresponding arc; hence the term "radian".

In particular, a central angle is of **one radian measure**, if the length of the corresponding arc is equal to the radius: $s = r$.
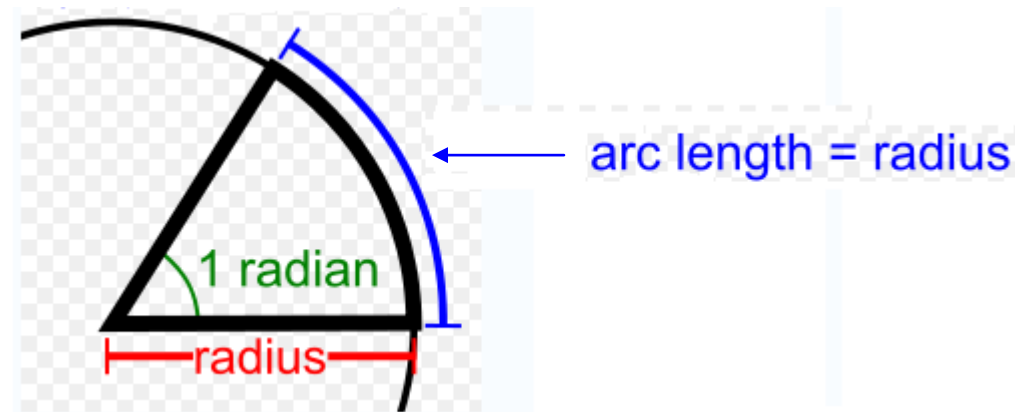
We may also say that a one-radian angle is an angle in a "curvilinear" equilateral triangle (sector) in which two sides are radii, the third side is an arc, and all three sides are equal.

From this point of view, it is easy to estimate the value of one radian. As we know, in a "normal" equilateral triangle all three angles are of $60°$. In "curvilinear" equilateral triangle, the central angle should be a bit less than $60°$ because the opposite side is an arc (a curve). Below in example 20.1 we will calculate that 1 radian $\approx 57.3°$. As we see, it is much better to cut our pizza pie by radians. In this case at least 6 people ($360/57.3 \approx 6$) will have something to eat.

At the first glance the radian measure may look a bit more complicated than the degree measure. However it is more useful in some problems in mathematics and science.

To understand the benefit of radian measure, let's re-write the above formula $\theta = \dfrac{s}{r}$ as $s = \theta \cdot r$. As you see, using the radian measure, the connection between arc, angle and radius is very simple. For any other measure of angles (for example, for degrees) this connection is more complicated and has the form $s = k \cdot \theta \cdot r$, where k is some numerical coefficient (we will show in example 20.4 below that for the degree measure, $k \approx 0.017$). Radian measure is different from all others by the simplest value $k = 1$. The main idea of the radian measure is to relate linear (length of the arc) and angular measurements in the simplest possible way. That's why many mathematical and technical calculations are simpler when using radians.

The idea of measuring angles by the length of the arc is credited to Roger Cotes in the early 1700s, an English mathematician who worked closely with Isaac Newton. But the term radian was first introduced only in the late 1800s by James Thomson, Ireland.

Let's set up connection between the radians and degrees. Consider the angle of $360°$. This angle corresponds to a full rotation around a circle. If we consider it as a central angle, the corresponding arc s is the entire circumference. Recall the formula for the circumference of a circle: $s = 2\pi \cdot r$. Compare this formula with the above $s = \theta \cdot r$. By equating both, we get $\theta \cdot r = 2\pi \cdot r$. From here, $\theta = 2\pi$. We see that angle $360°$ corresponds to 2π radians. This connection allows to express any degree measure in radians and vice versa. In particular, $180°$ corresponds to π radians. For any angle, let's denote its degree measure as $\theta°$, and the radian measure as θ_r. It is easy to set up connection between $\theta°$ and θ_r, if we use the proportion: $180°$ relates to π as $\theta°$ relates to θ_r

$$\frac{180°}{\pi} = \frac{\theta°}{\theta_r}$$

Let's call this proportion the **main proportion**.

Using cross-multiplication, we get $180° \cdot \theta_r = \pi \cdot \theta°$. From here we can express $\theta°$ through θ_r and vice versa:

$$\theta° = \frac{180°}{\pi} \cdot \theta_r, \qquad \theta_r = \frac{\pi}{180} \cdot \theta°.$$

Note. You do not need to memorize these formulas. Just remember that $180°$ corresponds to π radians:

$$180° = \pi_{rad}$$

and then use the main proportion.

Example 20.1. Express the angle of 1 radian in degrees.

Solution. The main proportion takes the form

$$\frac{180°}{\pi} = \frac{\theta°}{1_r}$$

By cross-multiplication, $\theta° \cdot \pi = 180°$. From here, $\theta° = \frac{180°}{\pi} \approx \frac{180°}{3.14} \approx 57.3°$.

So, 1 radian $\approx 57.3°$.

Note. If angle in radians is given in terms of π, there is no need to use proportion to convert this angle into degrees: simply replace π with 180. In this way we can say immediately that $\frac{\pi}{2}$ is $90°$, $\frac{3\pi}{2}$ is $270°$, 2π is $360°$ and so on.

Example 20.2. Express the angle of $\frac{5\pi}{12}$ radians in degrees.

Solution. Replace π with 180 and you are done: $\frac{5\pi}{12} = \frac{5 \cdot 180°}{12} = 75°$.

Example 20.3. Express the angle of $1°$ in radians.

Solution. The main proportion takes the form

$$\frac{180°}{\pi} = \frac{1°}{\theta_r}$$

By cross-multiplication, $180 \cdot \theta_r = \pi$. From here, $\theta_r = \frac{\pi}{180} \approx \frac{3.14}{180} \approx 0.017$.

So, $1° \approx 0.017$ radians.

Example 20.4. Express the arc length of a central angle through the radius of the circle and the degree measure of the angle.

Solution. Let s, r, and $\theta°$ be the arc length, radius, and degree measure of the central angle accordingly. Also, denote by θ_r the radian measure of the angle. As we mentioned above, $s = \theta_r \cdot r$, and $\theta_r = \frac{\pi}{180} \cdot \theta°$. From here, $s = \frac{\pi}{180} \cdot \theta° \cdot r$. Using calculation $\frac{\pi}{180} \approx 0.017$, we can write the approximate formula $s \approx 0.017 \cdot \theta° \cdot r$.

Session 20: Radian Measure of Angles

Note. Let's recall again that for radian measure, connection between arc length s, radius r, and central angle θ_r is the simplest:

$$s = \theta_r \cdot r$$

For any other measure this relation is more complicated, for example for degrees, $s \approx 0.017 \cdot \theta° \cdot r$.

Using the main proportion, we can calculate the radian measure of special angles $30°$, $45°$ and $60°$, as well as of quadrant angles $0°$, $90°$ $180°$ $270°$ $360°$. The following table summarizes the calculations.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π

In conclusion, let's mark quadrant angles in degrees and radians on the unit circle. Compare left and right figures.
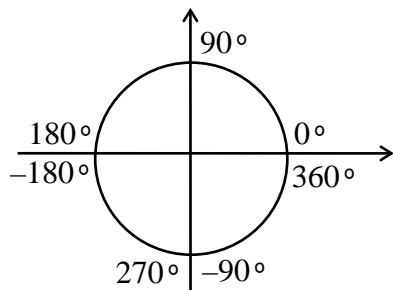
<u>Angles in degrees</u>   **<u>Angles in radians</u>**
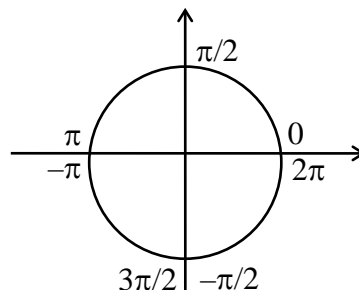
Exercises 20

In exercises 20.1 and 20.2, convert given angles from radians to degrees. Round the answers to the nearest tenth.

20.1. a) 1.3

   b) -0.6

20.2. a) 2.4

   b) -0.8

In exercises 20.3 and 20.4, convert given angles from radians to degrees.

20.3. a) $\dfrac{2\pi}{9}$

   b) $-\dfrac{3\pi}{10}$

20.4. a) $\dfrac{4\pi}{15}$

   b) $-\dfrac{6\pi}{5}$

In exercises 20.5 and 20.6, convert given angles from degrees to radians. Round the answers to the nearest hundreds.

20.5. a) $140°$

   b) $-85°$

20.6. a) $78°$

   b) $-237°$

In exercises 20.7 and 20.8, convert given angles from degrees to radians. Write the answers in terms of π. Do not round the answers.

20.7. a) $120°$

   b) $-150°$

20.8. a) $330°$

   b) $-225°$

In exercises 20.9 and 20.10, angle θ is given in radians. Without using a calculator, find the exact values of sine cosine and tangent of angle θ.

20.9. a) $\theta = \dfrac{4\pi}{3}$

   b) $\theta = -\dfrac{\pi}{4}$

   c) $\theta = \dfrac{5\pi}{6}$

20.10. a) $\theta = -\dfrac{\pi}{6}$

   b) $\theta = \dfrac{2\pi}{3}$

   c) $\theta = -\dfrac{3\pi}{4}$

In exercises 20.11 and 20.12, r is the radius of a circle, and θ is the degree measure of a central angle. Find the length of the arc bounded by angle θ. Round your answer to the nearest tenth.

20.11. $r = 1.3$ cm, $\theta = 80°$

20.12. $r = 2.5$ ft, $\theta = 115°$

Challenge Problems

20.13. The angles of a triangle are in the ratio of 3:4:5. Express these angles in radians.

20.14. The diameter of a Ferris wheel is 16 m. The spokes connecting two consecutive cabs to the center of the wheel make an angle of $\dfrac{\pi}{8}$. How many cabs are on the wheel? What is the length of the arc between two consecutive cabs?

20.15. Nick is running around a circular track of radius 30-meters. Esther is standing at the center and observing him. She found that she turned 6 radians in one minute. What was the Nick's speed?

20.16. A fly sat on the top of the second hand of a large clock and rode 48 cm until Lillian swatted it away. How long was the fly riding if the length of the second hand is 1.5 m?

Session 21

Graphs and Simplest Equations for Basic Trigonometric Functions

We consider here three basic trig functions: sine, cosine and tangent. For them, we will construct graphs, and solve three simplest (or basic) equations: $\sin x = a$, $\cos x = a$, and $\tan x = a$. We call these equations basic because solution of many more complicated equations can be reduced to them. We will use radian measure.

Function $y = \sin(x)$

Let's recall definition of sine for arbitrary angle: we draw an angle in standard position in system of coordinates with the unit circle, and consider point of interception of the terminal side of the angle with unit circle. Sine is the second (vertical) coordinate of this point.

To draw graph of sine, we will move along unit circle starting with the right-most position and observe how vertical coordinates of points on the unit circle change from quadrant to quadrant.

Obvious that in the first quadrant vertical coordinate (i.e. sine) increases from zero to one:
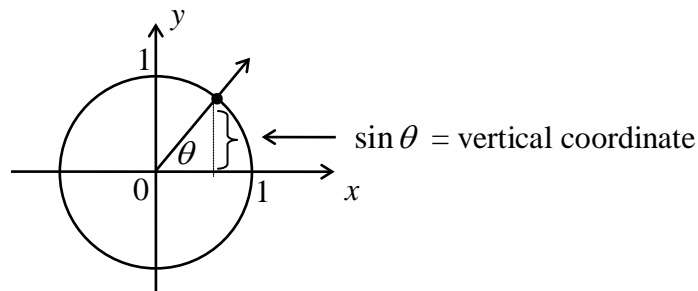
To graph sine, we will use another system of coordinates in which we mark angle on horizontal axis (we will use letter x instead of θ), and mark $\sin x$ on vertical y-axis. If you pick up several values of angles x in the first quadrant (i.e. from 0 to $\pi/2$), calculate $\sin x$, and plot points in the system of coordinates, you will see that sine increases not along a strait line. Instead, it increases along the curve:
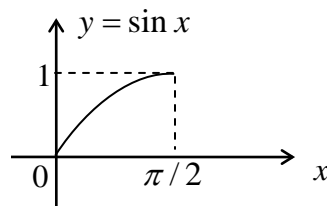
Graph of sine in the first quadrant

In similar way, in the second quadrant (from $\pi/2$ to π), sine decreases from 1 to 0:
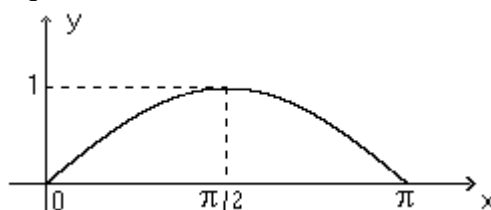
Graph of sine in the first and second quadrants

Continue moving along unit circle, we see that in third quadrant (from π to $3\pi/2$) sine decreases from 0 to –1, and in fourth quadrant (from $3\pi/2$ to 2π) sine increases from –1 to 0. At this point, we get the graph of sine at one full cycle (we also say, on one period interval):
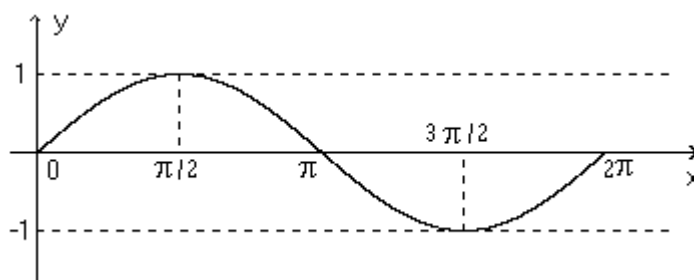
Graph of sine on one period interval $[0,\ 2\pi]$

If we continue moving around unit circle in either direction (positive or negative), we extend graph of sine to the entire number line, i.e. for all the values of x from $-\infty$ to $+\infty$:
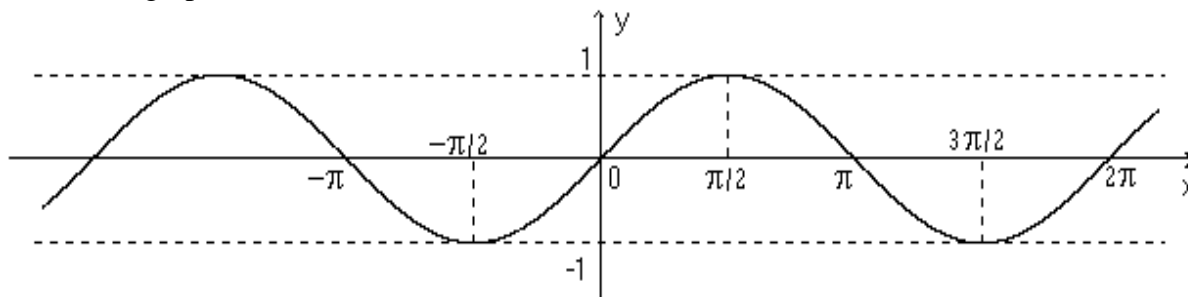
Entire graph of function $y = \sin x$

You can see that domain of sine is interval $(-\infty, +\infty)$ and range is $[-1, 1]$. Sine is periodical function with the period 2π. It means that sine repeats itself on each interval of the length 2π. More formally, for any x

$$\sin(x + 2\pi) = \sin(x) \quad \text{(Periodic property of sine)}$$

Also, graph is symmetric with respect to origin. Algebraically, it means that

$$\sin(-x) = -\sin(x) \quad \text{(Odd property of sine)}$$

Solving Simplest Equation $\sin(x) = a$ on One Period Interval $[0,\, 2\pi)$

Notice that the right point 2π is not included in the interval. The reason is that this point corresponds to the angle of 0, which is already taken for the left point.

In this interval, equation $\sin(x) = a$ may have zero, one or two solutions depending on the value of a. More precisely, the following statement is true.

Proposition 21.1. Consider the equation $\sin(x) = a$ in the interval $[0,\, 2\pi)$. Then

1) If $|a| > 1$, the equation does not have solutions.

2) If $|a| = 1$, the equation has one solution.

3) If $|a| < 1$, the equation has two solutions.

It is easy to check all three statements using geometric interpretation of the equation $\sin(x) = a$: its solutions are x-coordinates of points of intersection of the horizontal line $y = a$ with the graph of sine. By drawing this horizontal line, we can see three different locations of it, depending on three different values of number a.

1) $|a| > 1$. This inequality is equivalent to $a > 1$ or $a < -1$. Horizontal line $y = a$ is located above or below the graph of sine, so no points of intersection, and no solutions.

2) $|a| = 1$. This equality is equivalent to $a = 1$ or $a = -1$. In both cases line $y = a$ touches the graph of sine only in one point in the interval $[0,\, 2\pi)$:

For equation $\sin(x) = 1$, the only solution is $x = \pi / 2$.

For equation $\sin(x) = -1$, the only solution is $x = 3\pi / 2$.

3) $|a| < 1$. This inequality is equivalent to $-1 < a < 1$. Line $y = a$ is located between lines $y = -1$ and $y = 1$ and intersects the graph of sine exactly into two points in the interval $[0,\, 2\pi)$. In particular, if $a = 0$, equation $\sin(x) = 0$ has two solutions:

$$x = 0 \quad \text{and} \quad x = \pi.$$

To find roots of the equation $\sin(x) = a$ when $a \neq 0$, we can use reference angle. (For review of reference angles, see session 17). Here are possible steps to solve the equation $\sin(x) = a$.

a) Calculate the reference angle x_r by the formula

$$x_r = \sin^{-1}\left(|a|\right).$$

In other words, ignore the sign of number a and calculate the acute angle (root) of the equation $\sin(x) = |a|$. To find x_r, you may use button $\sin^{-1}$ on calculator.

b) Based on the sign of number a, identify quadrants in which angle x is located:

If $a > 0$, angle x is located in the 1st and 2nd quadrants.

If $a < 0$, angle x is located the 3rd and 4th quadrants.

c) Using quadrants and reference angle, find two solutions x_1 and x_2 of the equation $\sin(x) = a$.

If $a > 0$, then $x_1 = x_r$ and $x_2 = \pi - x_r$:
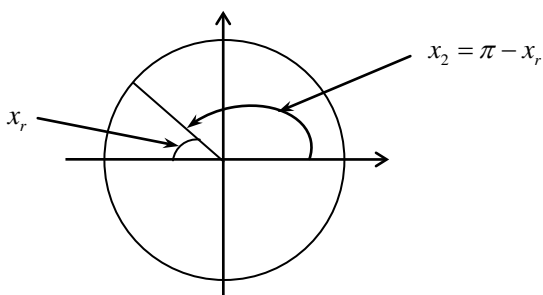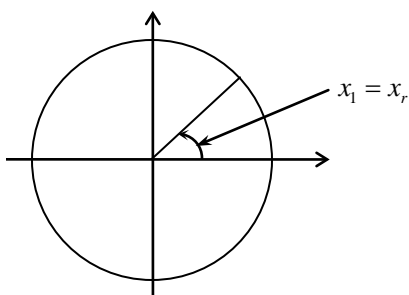
If $a < 0$, then $x_1 = \pi + x_r$ and $x_2 = 2\pi - x_r$:
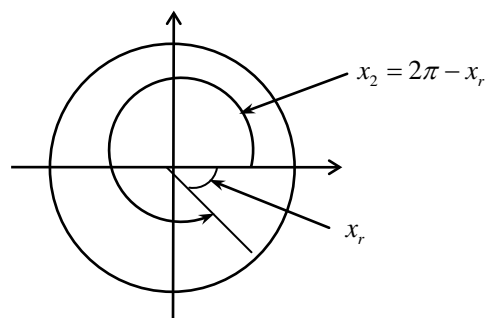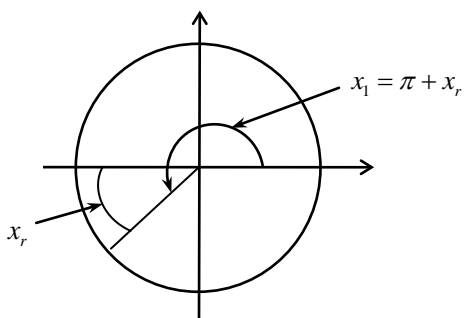
Example 21.1. Solve the equation $2\sin(x) + 4 = 5$ in the interval $[0, 2\pi)$.

Solution. It is easy to reduce this equation to the basic one by solving for $\sin(x)$: $\sin(x) = 1/2$. Here $a = 1/2 > 0$. Reference angle is $x_r = \sin^{-1}(1/2)$. We can find x_r with calculator or using special value $1/2$: $x_r = \sin^{-1}(1/2) = 30° = \pi/6$. Given equation has two roots. One of them, x_1, is in the 1st quadrant and coincides with the reference angle: $x_1 = x_r = \pi/6$. Second root x_2 is in the 2nd quadrant: $x_2 = \pi - x_r = \pi - \pi/6 = 5\pi/6$.

Example 21.2. Solve the equation $-2\sin(x) = \sqrt{2}$ in the interval $[0, 2\pi)$.

Solution. Solving for $\sin(x)$, we get basic equation $\sin(x) = -\dfrac{\sqrt{2}}{2}$. Here:

$a = -\dfrac{\sqrt{2}}{2} < 0$. Reference angle is $x_r = \sin^{-1}\left(\left|-\dfrac{\sqrt{2}}{2}\right|\right) = \sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$. With calculator, or

using special value $\dfrac{\sqrt{2}}{2}$, we can find that $x_r = \sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45° = \dfrac{\pi}{4}$. The equation has

two roots. One of them, x_1, is in 3rd quadrant: $x_1 = \pi + x_r = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$. Second root x_2

is in 4th quadrant: $x_2 = 2\pi - x_r = 2\pi - \dfrac{\pi}{4} = \dfrac{7\pi}{4}$.

Function $y = \cos(x)$

We can proceed here similar to function sine. Let's do this in brief form hoping that the reader can restore details yourself. By definition, cosine is first (horizontal) coordinate of a point on unit circle that corresponds to given angle:

$\cos\theta$ = horizontal coordinate
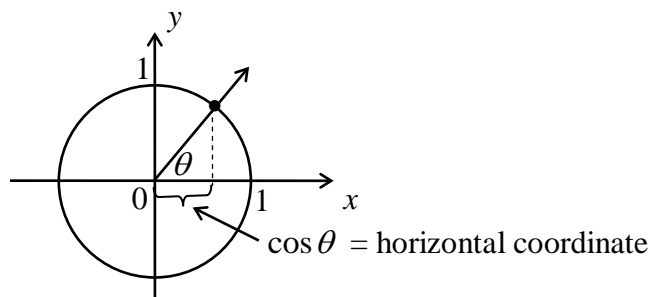
Moving around the unit circle from quadrant to quadrant, we can construct the graph of cosine by observing how horizontal coordinate is changing. For example, in the first quadrant when angle runs from 0 to $\pi/2$, cosine decreases from 1 to 0:
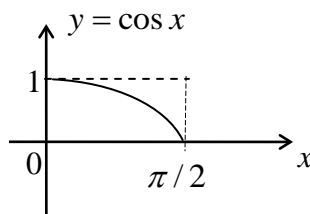
In second quadrant cosine continue to decrease from 0 to –1, in third quadrant it increases from –1 to 0, and, finally, in fourth quadrant increases from 0 to 1. Here is the graph of cosine at one full cycle (on one period interval) from 0 to 2π :
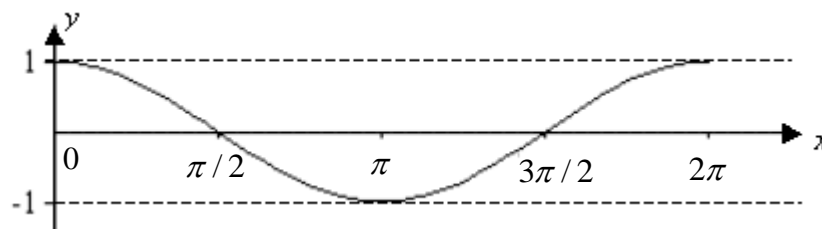
If we extend graph to the entire x-axis, we get complete graph of cosine:
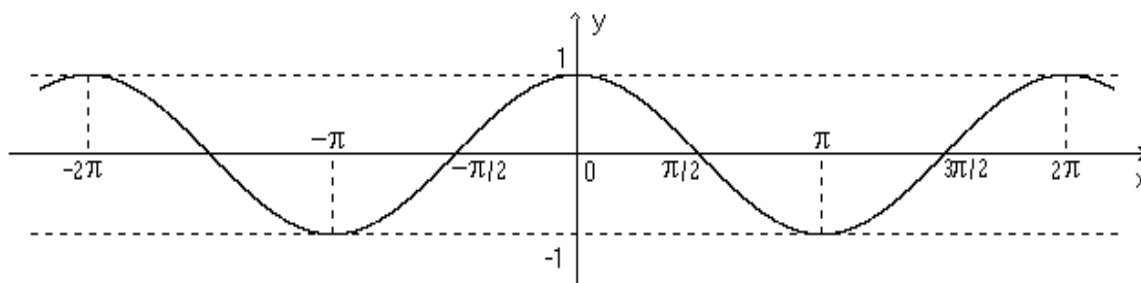
Entire graph of function $y = \cos x$

As for sine, the domain of cosine is $(-\infty, +\infty)$, range is $[-1, 1]$, and cosine is periodical function with the same period 2π. Graph of cosine is symmetric with respect to y-axis:

$$\cos(-x) = \cos(x) \text{ (Even property of cosine).}$$

Note. It is also possible to get the graph of cosine by shifting the graph of sine to the left by $\pi/2$ using the reduction formula $\cos(x) = \sin(x + \pi/2)$.

Solving Simplest Equation $\cos(x) = a$ on One Period Interval $[0, 2\pi)$

Number of solutions for this equation is exactly the same as for sine.

Proposition 21.2. Consider the equation $\cos(x) = a$ on the interval $[0, 2\pi)$. Then

1) If $|a| > 1$, the equation does not have solutions.

2) If $|a| = 1$, the equation has one solution in the interval $[0, 2\pi)$:

   For equation $\cos(x) = 1$, the only solution is $x = 0$.

   For equation $\cos(x) = -1$, the only solution is $x = \pi$.

3) If $|a| < 1$, the equation has two roots. In particular, if $a = 0$, roots of the equation $\cos(x) = 0$ are $x = \pi/2$ and $x = 3\pi/2$. If $a \neq 0$, both roots can be found using reference angle $x_r = \cos^{-1}(|a|)$ in the same way as we did above for the equation $\sin(x) = a$. Also, the following formulas can be used that are true any values of a, such that $|a| < 1$:

$$x = \cos^{-1}(a) \text{ and } x = 2\pi - \cos^{-1}(a).$$

Example 21.3. Solve the equation $2\sqrt{3}\cos(x) - 1 = 2$ in the interval $[0, 2\pi)$.

Solution. Solving this equation for $\cos(x)$, we have $\cos(x) = \dfrac{3}{2\sqrt{3}}$ or $\cos(x) = \dfrac{\sqrt{3}}{2}$.

This equation has two solutions: $x = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = 30° = \dfrac{\pi}{6}$, and $x = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$.

Example 21.4. Solve the equation $2\cos(x) + 4 = 3$ in the interval $[0,\, 2\pi)$.

Solution. Solving this equation for $\cos(x)$, we have $\cos(x) = -1/2$. This equation has two solutions: $x = \cos^{-1}\left(-\dfrac{1}{2}\right) = 120° = \dfrac{2\pi}{3}$, and $x = 2\pi - \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$.

Function $y = \tan(x)$

On unit circle in the system of coordinates, we can interpret tangent like this. On the right side of unit circle, draw vertical line and extend terminal side of the angle to meet with that line. Then tangent is the vertical coordinate of the point of interception. Here are pictures of tangent when angle is located in each of the quadrants:
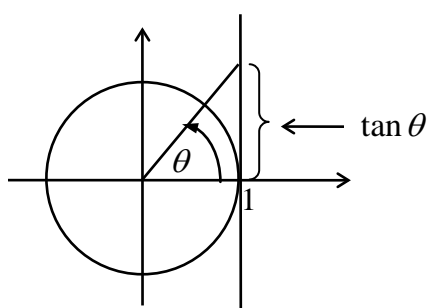
Angle is in 1st quadrant
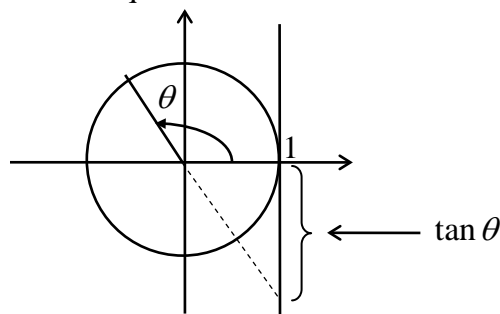
Angle is in 2nd quadrant
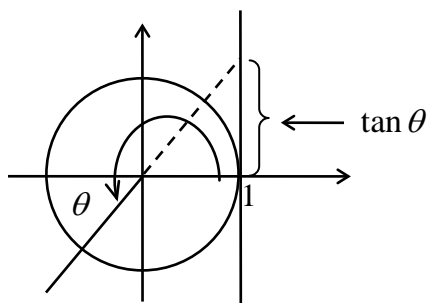
Angle is in 3rd quadrant
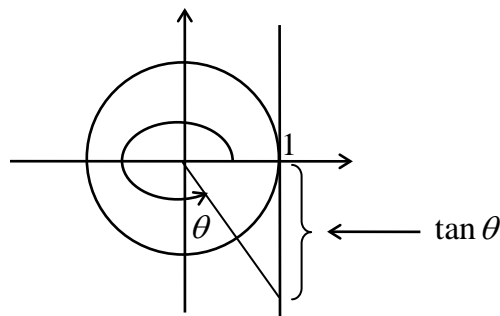
Angle is in 4th quadrant

We will draw graph of tangent in the way similar to as we did for sine and cosine. Moving along unit circle in the first quadrant, notice that tangent increases from zero to infinity, and its graph in the 1st quadrant is this
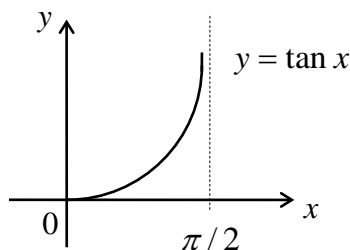
Line $x = \pi / 2$ becomes vertical asymptote. Continue moving in 2nd quadrant, we get the picture
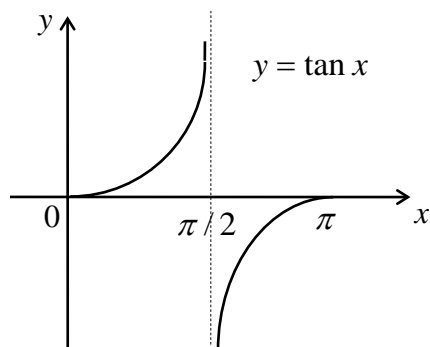
Moving in 3rd and 4th quadrants, we get graph of tangent on interval $[0, 2\pi)$:
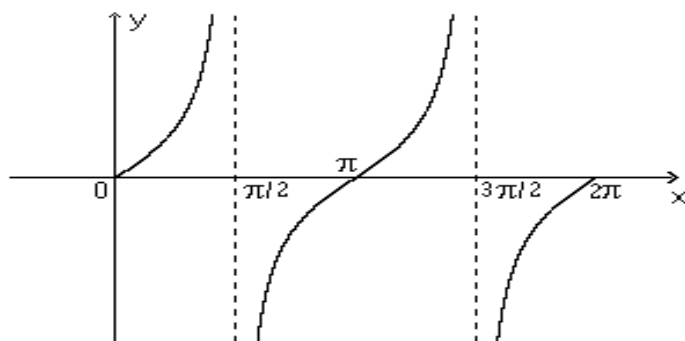
Graph of tangent on $[0, 2\pi]$ interval

Continue moving around unit circle in both directions, we can draw complete graph of tangent:
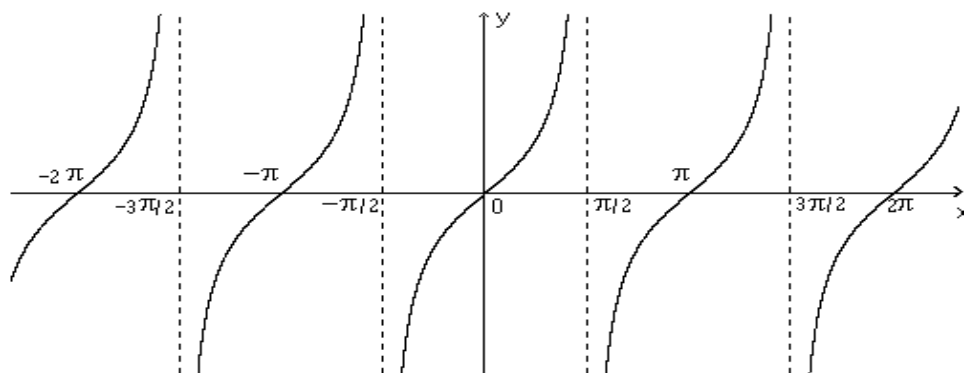
Entire graph of function $y = \tan x$

We see that graph consists of infinite number of branches, and it has infinite number of vertical asymptotes. The graph is symmetric with respect to origin, so tangent is odd

function: $\tan(-x) = -\tan(x)$. It repeats itself on π-length interval, so tangent has period π: $\tan(x+\pi) = \tan(x)$.

Solving Simplest Equation $\tan(x) = a$ on Interval $[0, 2\pi)$

Any horizontal line $y = a$ intersects the graph of tangent on $[0, 2\pi)$ interval always in two points, so the equation $\tan(x) = a$ always has exactly two roots for any a. The roots can be found in the same way as we described above for the equation $\sin(x) = a$, using reference angle.

Proposition 21.3. For any a, equation $\tan(x) = a$ has two roots in the interval $[0, 2\pi)$. If $a = 0$, the roots are $x_1 = 0$, $x_2 = \pi$. If $a \neq 0$, to find roots, you may first calculate the reference angle
$$x_r = \tan^{-1}(|a|).$$
It can be calculated with calculator, using the button $\tan^{-1}$. Then

1) If $a > 0$, roots are in the 1st and the 3rd quadrants and are
$$x_1 = x_r, \quad x_2 = \pi + x_r.$$

2) If $a < 0$, roots are in the 2nd and the 4th quadrants and are
$$x_1 = \pi - x_r, \quad x_2 = 2\pi - x_r.$$

Note. You don't need to memorize these formulas. Just draw unit circle and mark corresponding angles as we did above for the equation $\sin(x) = a$.

Example 21.5. Solve the equation $3\tan(x) - 2\sqrt{3} = \sqrt{3}$ in the interval $[0, 2\pi)$.

Solution. Solving the equation for $\tan(x)$ we get basic equation $\tan(x) = \sqrt{3}$. The reference angle is $x_r = \tan^{-1}(\sqrt{3}) = 60° = \pi/3$. Roots of given equation are located in the 1st and the 3rd quadrants and are
$$x_1 = x_r = \pi/3, \quad x_2 = \pi + x_r = \pi + \pi/3 = 4\pi/3.$$

Example 21.6. Solve the equation $4\tan(x) + 5 = 1$ in the interval $[0, 2\pi)$.

Solution. Solving the equation for $\tan(x)$ we get basic equation $\tan(x) = -1$. The reference angle is $x_r = \tan^{-1}(|-1|) = \tan^{-1}(1) = 45° = \pi/4$. Roots of given equation are located in the 2nd and the 4th quadrants and are
$$x_1 = \pi - x_r = \pi - \pi/4 = 3\pi/4,$$
$$x_2 = 2\pi - x_r = 2\pi - \pi/4 = 7\pi/4.$$

Exercises 21

In all exercises, solve given equations in the interval $[0, 2\pi)$. Use radian measure. Do not round the answers. Write answers in terms of π.

21.1. a) $2\sqrt{2}\sin(x)-1=1$

      b) $2\sqrt{3}\sin(x)+4=1$

21.2. a) $2\sqrt{3}\sin(x)-1=2$

      b) $2\sin(x)+3=2$

21.3. a) $2\sqrt{2}\cos(x)+3=5$

      b) $2\sqrt{3}\cos(x)+5=2$

21.4. a) $2\cos(x)+5=6$

      b) $\sqrt{2}\cos(x)+4=3$

21.5. a) $\sqrt{3}\tan(x)+2=3$

      b) $\sqrt{3}\tan(x)+6=3$

21.6. a) $3\tan(x)-1=2$

      b) $\sqrt{3}\tan(x)+6=5$

21.7. $2\sqrt{3}\sin(x)-3=1$

21.8. $3\sqrt{2}\cos(x)-4=2$

Session 22

Trigonometric Identities and None-Simplest Equations

Trigonometric Identities

Let's compare two statements:

$$\sin x + \cos x = 1 \text{ and } \sin^2 x + \cos^2 x = 1.$$

They look pretty much similar. However, they are completely different. The first one is **equation** and the second one is **identity**.

As we already know, equation is a statement that is true only for some specific values of variable, and the basic problem for equation is to **solve** it, which means to find these specific values. Such values are called solutions or roots of the equation. For example, the values $x = 0$ and $x = \pi/2$ are roots of the equation $\sin x + \cos x = 1$. It is easy to check by substitution these values into equation. We will have $\sin 0 + \cos 0 = 0 + 1 = 1$ and $\sin(\pi/2) + \cos(\pi/2) = 1 + 0 = 1$, so the equation becomes true statement. If we pick, for example, $x = \pi$, it is not a root, because by substitution π for x, the equation does not become a true statement: $\sin(\pi) + \cos(\pi) = 0 - 1 = -1 \neq 1$. Below in example 22.7 we show that roots 0 and $\pi/2$ are the only roots of the equation $\sin x + \cos x = 1$ in the interval $[0, 2\pi)$.

On the contrary, the second statement $\sin^2 x + \cos^2 x = 1$ is true for any value of x, no exceptions. It is easy to check if we use definitions of sine and cosine as vertical and horizontal coordinates of points on unit circle:
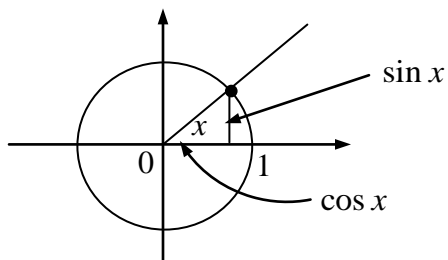
These coordinates (i.e. $\sin x$ and $\cos x$) together with the radius (which is equal to 1) form a right triangle if a point $(\sin x, \cos x)$ lies in the 1st quadrant. By Pythagorean Theorem, $\sin^2 x + \cos^2 x = 1$. We can check that this statement is true for all quadrants. Statements like this are called identities. Here is exact definition.

Definition. Statement $f(x) = g(x)$, where f and g are two functions, is called the **identity**, if this statement is true for all values of variable x from the common domain of functions f and g.

The basic problem for identity is to **prove** it, not to solve. In general, it is not possible to give exact recipe how to prove an identity. Common guideline is to try modify one or both sides f and g of the identity to get the same expression. Below we consider several examples.

Let's start with identities that we call **basic**. They can be used to prove more complicated identities. Four of them are simply expressions of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ through $\sin x$ and $\cos x$:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

Another identity we already proved above:

$$\sin^2 x + \cos^2 x = 1$$

We call this the **main** identity (it is also called Pythagorean identity). It allows to express $\sin^2 x$ through $\cos^2 x$ and vice versa:

$$\sin^2 x = 1 - \cos^2 x \quad \text{and} \quad \cos^2 x = 1 - \sin^2 x.$$

From the main identity, we can derive two more identities that connect $\tan x$ with $\sec x$, and $\cot x$ with $\csc x$: just divide both sides of the main identity by $\cos^2 x$ and $\sin^2 x$:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{and} \quad \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

From here, we get

$$\tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

There are a lot of trig identities that can be derived from basics. Let's consider some examples.

Example 22.1. Prove the identity $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \sec^2 x \cdot \csc^2 x$.

Solution. We will modify the left side to get the right side:

$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = \csc^2 x \cdot \sec^2 x$$

Example 22.2. Prove the identity $\sin^2 x - \cos^2 x = \sin^4 x - \cos^4 x$.

Solution. This time we modify the right side using formula $a^2 - b^2 = (a - b) \cdot (a + b)$:

$$\sin^4 x - \cos^4 x = \left(\sin^2 x - \cos^2 x\right)\cdot\left(\sin^2 x + \cos^2 x\right) = \sin^2 x - \cos^2 x.$$

Example 22.3. Prove the identity $\dfrac{1+\sin x}{1-\sin x} = \left(\tan x + \sec x\right)^2$.

Solution. Here both sides look rather complicated and we modify both of them.

To modify left side, we multiply numerator and denominator by $1+\sin x$. Then using the identity $1-\sin^2 x = \cos^2 x$ in the denominator, we get

$$\frac{1+\sin x}{1-\sin x} = \frac{\left(1+\sin x\right)\left(1+\sin x\right)}{\left(1-\sin x\right)\left(1+\sin x\right)} = \frac{\left(1+\sin x\right)^2}{1-\sin^2 x} = \frac{1+2\sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x + 2\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} + \tan^2 x$$

$$= \sec^2 x + 2\tan x \cdot \sec x + \tan^2 x.$$

Now let's modify right side of the original identity:

$$\left(\tan x + \sec x\right)^2 = \tan^2 x + 2\tan x \cdot \sec x + \sec^2 x$$

We've got the same expression as for left side. Identity is proved.

None-Simplest Trigonometric Equations

In previous session we solved simplest trig equations $\sin x = a$, $\cos x = a$ and $\tan x = a$. Now we consider slightly more complicated equations that can be reduced to simplest. To solve some of the equations, we will use basic identities. All equations we will solve in radians and in the interval $[0,\,2\pi)$.

Example 22.4. Solve the equation $8\sin^2 x - 14\sin x - 15 = 0$.

Solution. This equation can be treated as a quadratic equation with respect to $\sin x$. Using the notation $\sin x = u$, the equation becomes quadratic with respect to variable u: $8u^2 - 14u - 15 = 0$. It can be solved by factoring: $\left(4u - 3\right)\left(2u + 5\right) = 0$. From here $u = \dfrac{3}{4} = 0.75$ and $u = -\dfrac{5}{2} = -2.5$. Replacing u with $\sin x$, we get two simplest trig equations: $\sin x = 0.75$ and $\sin x = -2.5$. We can solve them in the same way as we did in the previous session.

1) Equation $\sin x = 0.75$ has two solutions in the interval $[0,\,2\pi)$ (rounded to 3 decimal places): $x = \sin^{-1}(0.75) = 0.848$ and $x = \pi - \sin^{-1}(0.75) = 2.294$.

2) Equation $\sin x = -2.5$ does not have solutions because $\sin x$ cannot be less than -1.

Final answer: there are two solutions $x = 0.848$ and $x = 2.294$.

Example 22.5. Solve the equation $8\sin^2 x - 2\cos x - 5 = 0$.

 Solution. We may re-write this equation in terms of $\cos x$ using the identity $\sin^2 x = 1 - \cos^2 x$. By substitution this expression into the original equation, we get

$$8\left(1 - \cos^2 x\right) + 2\cos x - 5 = 0 \;\Rightarrow\; 8 - 8\cos^2 x - 2\cos x - 5 = 0 \;\Rightarrow$$
$$-8\cos^2 x - 2\cos x + 3 = 0 \;\Rightarrow\; 8\cos^2 x + 2\cos x - 3 = 0.$$

Similar to example 22.4, we can treat this equation as quadratic with respect to $\cos x$. Letting $u = \cos x$, we have the equation $8u^2 + 2u - 3 = 0$. It can be solved by factoring $(2u - 1)\cdot(4u + 3) = 0$. From here $u = \dfrac{1}{2} = 0.5$ and $u = -\dfrac{3}{4} = -0.75$. Replacing u with $\cos x$, we get two simplest trig equation $\cos x = 0.5$ and $\cos x = -075$. Let's solve them.

1) Equation $\cos x = 0.5$ has two solutions (which are special angles)

$$x = \cos^{-1}(0.5) = 60° = \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

2) Equation $\cos x = -075$ also has two solutions (which can be find using calculator):

$$x = \cos^{-1}(-0.75) = 2.42 \text{ and } x = 2\pi - \cos^{-1}(-0.75) = 3.86.$$

Final answer: there are four roots: $x = \dfrac{\pi}{3}, \; x = \dfrac{5\pi}{3}, \; x = 2.42, \; x = 3.86$.

Note. First two roots are exact solutions, while the last two are approximations to two decimal places.

Example 22.6. Solve the equation $2\sin x \cdot \tan x + \sqrt{3}\tan x = 0$.

Solution. We can factor out $\tan x$: $\tan x\left(2\sin x + \sqrt{3}\right) = 0$. Now equation can be split into two: $\tan x = 0$ and $2\sin x + \sqrt{3} = 0$. The first one is simplest, and the second one can be written as simplest: $\sin x = -\sqrt{3}/2$. Let's solve them. Both of them have two solutions.

For $\tan x = 0$, $x = 0$ and $x = \pi$.

For $\sin x = -\sqrt{3}/2$ calculate $\sin^{-1} x\left(-\sqrt{3}/2\right) = -60° = -\pi/3$. From here,

$x = 2\pi - \pi/3 = 5\pi/3$ and $x = \pi + \pi/3 = 4\pi/3$.

Final answer: there are four solutions: $x = 0, \; x = \pi, \; x = 4\pi/3, \; x = 5\pi/3$.

Example 22.7. Solve the equation $\sin x + \cos x = 1$.

Solution. Let's square both sides:

$$\left(\sin x + \cos x\right)^2 = 1^2 \Rightarrow \sin^2 x + 2\sin x \cdot \cos x + \cos^2 x = 1.$$

Using the main identity $\sin^2 x + \cos^2 x = 1$, we can simplify the above equation:

$$1 + 2\sin x \cdot \cos x = 1 \Rightarrow 2\sin x \cdot \cos x = 0 \Rightarrow \sin x \cdot \cos x = 0.$$

This equation can be broken into two simplest equations: $\sin x = 0$ and $\cos x = 0$.

Equation $\sin x = 0$ has two solutions $x = 0$ and $x = \pi$. Equation $\cos x = 0$ also has two

solutions $x = \dfrac{\pi}{2}$ and $x = 2\pi - \dfrac{\pi}{2} = \dfrac{3\pi}{2}$. So, it looks like the original equation has four

solutions:

$$x = 0, \ x = \pi, \ x = \frac{\pi}{2}, \ x = \frac{3\pi}{2}.$$

However, this is not true. We need to be very careful, and check these values with the original equation. The reason is that when we square both sides of an equation, we may get additional roots that are not really roots of the original equation. We already saw that in session 6, example 6.6, and in session 8, example 8.7. Let's check the above four values.

Values $x = 0$, $x = \dfrac{\pi}{2}$ and $x = \pi$ we already checked at the beginning of this session: 0

and $\dfrac{\pi}{2}$ are roots, but $x = \pi$ is not. Let's check $x = \dfrac{3\pi}{2}$:

$\sin\left(\dfrac{3\pi}{2}\right) + \cos\left(\dfrac{3\pi}{2}\right) = -1 + 0 = -1 \neq 1$. So, $x = \dfrac{3\pi}{2}$ is not the root, and we reject it.

Final answer: there are only two roots $x = 0$ and $x = \dfrac{\pi}{2}$.

Exercises 22

In exercises 22.1 – 22.6, prove given identities

22.1. $\csc x - \sin x = \dfrac{\cos x}{\tan x}$

22.2. $\sec x - \cos x = \dfrac{\sin x}{\cot x}$

22.3. $\dfrac{\cot x}{1 + \csc x} - \dfrac{\cot x}{1 - \csc x} = 2\sec x$

22.4. $\dfrac{\tan x}{1 + \sec x} - \dfrac{\tan x}{1 - \sec x} = 2\csc x$

22.5. $\dfrac{\cos^4 x - \sin^4 x}{\cos^2 x} = 2 - \sec^2 x$

22.6. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x} = 2 - \csc^2 x$

In exercises 22.7 – 22.14, solve given equations in the interval $[0,\ 2\pi)$. Use radian measure. If it is possible to find exact solutions, write them in terms of π. If it is not possible, round the answers to the nearest thousandth.

22.7. $2\sin^2(x) - 3\sin(x) = 2$

22.8. $2\cos^2(x) + 7\cos(x) = -3$

22.9. $3\sin^2(x) - \cos(x) - 1 = 0$

22.10. $5\cos^2(x) + \sin(x) - 1 = 0$

22.11. $3\sin(x) \cdot \tan(x) = \sqrt{3}\sin(x)$

22.12. $\cos(x) \cdot \tan(x) = \sqrt{3}\cos(x)$

22.13. $\sin(x) - \cos(x) = 1$

22.14. $\cos(x) - \sin(x) = 1$

Part III

Exponential and Logarithmic Functions

Session 23

Logarithms

Consider the following problem. Suppose we have three numbers x, y and z, that are connected by the equation $y^x = z$. How to solve this equation for x and for y?

It is easy to solve for y: raise both sides of the equation to the power $\dfrac{1}{x}$, and get

$$\left(y^x\right)^{\frac{1}{x}} = y^{x \cdot \frac{1}{x}} = y^{\frac{x}{x}} = y^1 = y = z^{\frac{1}{x}}. \text{ So, } y = z^{\frac{1}{x}}.$$

It is important to understand that even we obtained the formula $y = z^{\frac{1}{x}}$, this formula, in general, does not give us a direct way (a finite sequence of arithmetic operations) to get the exact answer. Actually, the expression $z^{\frac{1}{x}}$ provides only the **notation** of a specific operation on x and z, and the question how to perform this operation is another story (which is out of the scope of this session).

Similar situation occurs when we want to solve the equation $y^x = z$ for x. In other words, we want to express power x in terms of the base y and number z. Of course, for some specific values of y and z, it is easy to do.

Example 23.1. Solve the equation $2^x = 8$.

Solution. This equation can be solved directly. Indeed, we can represent number 8 as an exponent with the base 2: $8 = 2^3$. Then the equation takes the form $2^x = 2^3$. From here we immediately conclude that $x = 3$.

However, in general, we cannot solve the equation $y^x = z$ for x so easily. Consider, for example, equation $2^x = 6$. Because $2^2 = 4$ and $2^3 = 8$, we can just estimate that x should be somewhere between 2 and 3. But where? We cannot indicate the exact value. At the end of this session we will be able to get an approximation. We will see in Example 23.8 that up to three decimals, $x \approx 2.585$.

In general, we may think of the solution x of the equation $y^x = z$ as a result of some specific operation that we perform on y and z. In other words, we consider x as some function of two variables y and z. Since we have function, we need a notation for it. You may invent your own notation. For example, using the abbreviation "sol" for solution, we can write $x = \text{sol}(y, z)$. In mathematics the following notation is used: $x = \log_y z$. We read this as "logarithm (or, in short, log) of number z with the base y". So, the solution of the equation $y^x = z$ with respect to x is $x = \log_y z$.

In the definition below, we simply change letter y to b and z to c.

Definition of Logarithm. Let b be a positive number not equal to 1, and c be any positive number. Then $x = \log_b c$ is the solution of the equation $b^x = c$. In other words, logarithm is a **power** to which we raise base b to get number c.

Note. You may be wondering why the base $b \neq 1$. Well, let's $b = 1$, so we consider the equation $1^x = c$. If we raise 1 to any power, the result is still 1, so we have $c = 1$, and any number x satisfies the equation $1^x = 1$. Therefore, in this case the solution $x = \log_1 1$ does not make sense: it can be any number. Another restriction is $b > 0$. This is to avoid problems with complex (not real) numbers. For example, $(-1)^{1/2} = \sqrt{-1}$ is not a real number, so we exclude negative base b. Also, we put the restriction on number c: $c > 0$. This is because $c = b^x$, and positive b raised to any power is positive, so for non-positive c logarithm does not exist.

In practice, often number 10 is used as the base of logarithms. Such logarithms are called **common** ones. Usually, for simplicity we drop the base 10 in the notation of common logs. So,

$$\log c = \log_{10} c.$$

Working with logs, it is often convenient to convert them into exponents. If we denote given logarithm by x (i.e. $\log_b c = x$), we can re-write it (by definition) as $b^x = c$. To be more comfortable with this technique, you may memorize the following **"Circular Rule"** for conversion: in $\log_b c = x$, take base b, raise it to power x, and you get c:

$$\boxed{\log_b c = x \quad \Leftrightarrow \quad b^x = c}$$

This rule says that two statements: $x = \log_b c$ and $c = b^x$ are equivalent.

Logarithms were invented by Scottish mathematician John Napier in early 1600, and the notation log was introduced by German mathematician Gottfried Leibniz in 1675.

In some cases, it is easier to operate with exponents rather than with logarithms. We will see this in the following example.

Example 23.2. Calculate or simplify

a) $\log_2 8$

b) $\log 100$

c) $\log 0.0001$

d) $\log_b 1$

e) $\log_b b$

f) $\log_b b^n$

g) $b^{\log_b c}$

Solution.

a) Problem to calculate $\log_2 8$ is, actually, the same as example 23.1, just written in different form. Indeed, if $x = \log_2 8$, then, by circular rule, $2^x = 8$. From example 23.1, we have $x = 3$, so $\log_2 8 = 3$.

b) Let $x = \log 100 = \log_{10} 100$. By the circular rule, $10^x = 100 = 10^2$, so $x = \log 100 = 2$.

c) Let $x = \log 0.0001$. Then $10^x = 0.0001 = 10^{-4}$, so $x = \log 0.0001 = -4$.

d) Let $x = \log_b 1$. Then $b^x = 1 = b^0$. Therefore, $x = \log_b 1 = 0$.

e) Let $x = \log_b b$. Then $b^x = b = b^1$. Therefore, $x = \log_b b = 1$.

f) Let $x = \log_b b^n$. Then $b^x = b^n$. Therefore, $x = \log_b b^n = n$.

g) At the first glance, the expression $b^{\log_b c}$ looks rather complicated. However, if you look at it carefully, you will realize that it is actually a very simple. Indeed, if we denote $\log_b c = x$, then $b^x = c$, so, $b^{\log_b c} = b^x = c$.

Note. Try to memorize answers of problems 23.2, d) and e):

$$\text{For any base } b, \quad \log_b 1 = 0 \text{ and } \log_b b = 1.$$

 Example 23.3. Prove that $\log_{\frac{1}{b}} c = -\log_b c$.

Solution. Let's use letters x and y for the above logs: $x = \log_{\frac{1}{b}} c$, $y = \log_b c$. Then

$$\left(\frac{1}{b}\right)^x = \frac{1}{b^x} = b^{-x} = c, \ b^y = c. \quad \text{From here,} \quad b^{-x} = b^y. \quad \text{So,} \quad y = -x. \quad \text{Therefore,}$$

$\log_{\frac{1}{b}} c = -\log_b c$.

Basic properties of logarithms

Multiplication Rule: $\log_b (x \cdot y) = \log_b x + \log_b y$.

In words: logarithm of product is equal to the sum of logarithms.

The proof of this statement can be done in a manner similar to examples 23.2 and 23.3. Denote each of three logs by letters: $A = \log_b (x \cdot y)$, $B = \log_b x$, and $C = \log_b y$. Next, use the circular rule to convert them into exponents: $b^A = x \cdot y$, $b^B = x$, and $b^C = y$. Now, multiply the second and third equations: $b^B \cdot b^C = x \cdot y$, or $b^{B+C} = x \cdot y$. Compare this equation with $b^A = x \cdot y$. From here, $b^A = b^{B+C}$, hence $A = B + C$, or $\log_b (x \cdot y) = \log_b x + \log_b y$.

Note. Before the era of calculators, there was a widely used device, so-called logarithmic ruler, or slide ruler, that allows to multiply numbers based on the Multiplication Rule for logarithms. Schematically speaking, this device works like this. It contains two rules that allow to convert numbers into logs and vice versa. To multiply two numbers, the device converts them to logs and adds them up. According to the Multiplication Rule this sum is the log of product. Then device converts this log of product back to the product of given numbers. So, the device makes (physically) summation, but mathematically, we get multiplication.

Example 23.4. Solve the equation $\log_2(x-1) + \log_2(x-3) = 3$.

Solution. Using the Multiplication Rule, we can combine both logs in one: $\log_2(x-1)(x-3) = 3$. From here, $(x-1)(x-3) = 2^3 = 8$. This is a quadratic equation that can be written in standard form $x^2 - 4x - 5 = 0$. We can solve it by factoring: $(x - 5)(x + 1) = 0$, and we get two solutions: $x = 5$ and $x = -1$. Let's check these solutions with the original equation. Let $x = 5$. Then

$$\log_2(5-1) + \log_2(5-3) = \log_2 4 + \log_2 2 = 2 + 1 = 3,$$

so everything is OK with this. Now, let $x = -1$. Then we come up with logs of negative numbers: $\log_2(-2)$ and $\log_2(-4)$. Such logs do not make sense. Therefore, we must reject the value $x = -1$. Final answer: the equation has the only solution $x = 5$.

Notes.

1) Example 23.4 shows that we need to be very careful when come up with the final answer: we must check the answer with the original equation.

2) Keep in mind that $\log_a x + \log_b y \neq \log_b(x \cdot y)$ if $a \neq b$. Multiplication Rule is applicable only to logs with the same base.

Quotient (or Division) Rule: $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$.

In words: logarithm of quotient is equal to the difference of logarithms.

The proof is similar to that given for multiplication rule. The proof of this statement as well as others statements given below, are left as exercises.

Note. The above two rules: multiplication and division rules for logs, can be considered as inverse rules for multiplication and division of exponents. If we multiply exponents, we add their powers (powers are logs), if we add logs, we multiply their numbers. If we divide exponents, we subtract their powers, if we subtract logs, we divide their numbers. In next session, we will discuss more about this "inverse connection".

Example 23.5. Solve the equation $\log(x+8) - \log(4x-7) = 1$.

Solution. Using the Division Rule, we can represent the left part as logarithm of quotient, and the equation takes the form $\log\dfrac{x+8}{4x-7} = 1$. From here $\dfrac{x+8}{4x-7} = 10^1 = 10$. Solving this equation, we get $x = 2$. Let's check this answer with the original equation:

$\log(2+8) - \log(4 \times 2 - 7) = \log 10 - \log 1 = 1 - 0 = 1$. So, $x = 2$ is the solution.

Power Rule: $\log_b x^n = n \cdot \log_b x$.

In words: logarithm of an exponent is equal to its power times logarithm of its base (base of exponent, not base of log).

Example 23.6. Calculate without using a calculator:

$$3 \cdot \log_6 3 - \log_6 75 + 2 \cdot \log_6 10$$

Solution. Here we can use all three rules listed above. We can modify first and third terms like this:

$$3 \cdot \log_6 3 = \log_6 3^3 = \log_6 27$$

$$2 \cdot \log_6 10 = \log_6 10^2 = \log_6 100$$

From here

$$3 \cdot \log_6 3 - \log_6 75 + 2 \cdot \log_6 10 = \log_6 27 - \log_6 75 + \log_6 100$$

$$= \log_6 \frac{27 \cdot 100}{75} = \log_6 36 = \log_6 6^2 = 2 \cdot \log_6 6 = 2.$$

Most scientific calculators allow to calculate logarithms only with specific bases: base 10 (common logs), and base e (this is a special number that we will discuss later in session 25). Logs with the base e are denoted with the symbol **ln** and are called the **natural logarithms**, so $\ln x = \log_e x$. To calculate logs with other bases, we need the way to convert logs from one base to another. The following rule allows to do this.

Change-of-Base Rule:

$$\log_b x = \frac{\log_d x}{\log_d b}.$$

According to this rule, if we need to calculate logarithm with the base b, but our ability is restricted with the base d only, we can make conversion from base b to d, and then perform the calculations. If we put $x = d$, we get special case of the Change-of-Base Rule:

$$\log_b d = \frac{1}{\log_d b}.$$

To prove Change-of-Base Rule, denote $y = \log_b x$ and convert it into exponential form $b^y = x$. Now apply log with the base d to both sides and use power rule:

$$\log_d b^y = \log_d x \implies y \log_d b = \log_d x \implies y = \frac{\log_d x}{\log_d b}.$$

Example 23.7. Assume that you have a calculator that allows to calculate common logs (logs with the base of 10). Calculate $\log_3 5$.

Solution. Using Change-of-Base Rule and calculator, we have

$$\log_3 5 = \frac{\log 5}{\log 3} \approx \frac{0.699}{0.477} \approx 1.465.$$

Often, logarithms are useful in solving exponential equations in which powers of exponents are unknown.

Example 23.8. Solve the equation $2^x = 6$ that we discussed at the beginning of this session. Approximate the solution to the nearest thousandth.

Solution. Let's take the logarithm of both sides of this equation to the base 10 (common log): $\log 2^x = \log 6$. Using the Power Rule, $x \cdot \log 2 = \log 6$, and $x = \dfrac{\log 6}{\log 2}$. This is "exact answer". Using calculator, we can get numerical approximation: $\dfrac{\log 6}{\log 2} \approx \dfrac{0.7781}{0.3010} \approx 2.585$.

So, $x \approx 2.585$.

Example 23.9. Solve the equation $3^{2x-1} = 5$. Approximate the solution to the nearest hundredth.

Solution. As in Example 23.8, we take log from both sides: $\log 3^{2x-1} = \log 5$. Using the Power Rule, we have $(2x-1)\log 3 = \log 5$. From here,

$$2x - 1 = \frac{\log 5}{\log 3}, \quad 2x = \frac{\log 5}{\log 3} + 1, \quad x = \frac{1}{2}\left(\frac{\log 5}{\log 3} + 1\right) \approx \frac{1}{2}\left(\frac{0.699}{0.477} + 1\right) \approx 1.23.$$

So, $x \approx 1.23$.

Note. Method of taking a logarithm of both side of a given equation, that we used in Examples 23.8 and 23.9, is used very often for equations that contain exponents. Theoretically, we can take logarithm with any base. We used common logs (base is 10) to be able to use calculator. We could also use natural logs (logs with the base e).

Exercises 23

In exercises 23.1 and 23.2, solve given equation.

23.1. $3^x = 27$

23.2. $5^x = 625$

In exercises 23.3 and 23.4, calculate without using a calculator.

23.3. a) $\log_3 81$

    b) $\log 1000$

    c) $\log 0.01$

    d) $\log_{\frac{1}{2}} 64$

23.4. a) $\log_4 16$

    b) $\log 10000$

    c) $\log 0.001$

    d) $\log_{\frac{1}{3}} 27$

In exercises 23.5 and 23.6, solve given equations.

23.5.

  a) $\log_3(x-4) + \log_3(x+2) = 3$

  b) $\log_5(7x-3) - \log_5(x-3) = 2$

23.6.

  a) $\log_4(x+4) + \log_4(x-2) = 2$

  b) $\log_2(15x+3) - \log_2(x+3) = 3$

In exercises 23.7 and 23.8, calculate without using a calculator.

23.7. $\log_8 32 - 2\cdot\log_8 6 + \log_8 9$

23.8. $3\cdot\log 20 + 2\cdot\log 3 - \log 72$

In exercises 23.9 and 23.10, assume that calculator allows to calculating only common logs ((logs with the base of 10). Calculate given expressions. Round your answers to the nearest thousands.

23.9. $\log_7 8$

23.10. $\log_8 7$

In exercises 23.11 and 23.12, solve given equations. Round your answers to the nearest thousands.

23.11. a) $6^x = 9$

    b) $7^{3x+2} = 4$

23.12. a) $5^x = 8$

    b) $4^{5x-3} = 7$

Challenge Problem

23.13. Consider the equation

$$\log_c(x+a) + \log_c(x+b) = p+q, \quad \text{where } b = a - c^p + c^q.$$

Prove that the only solution of this equation is $x = c^p - a$.

Session 24

Exponential and Logarithmic Functions

We already studied some functions: quadratic functions (parabolas) and trigonometric functions. In this session, we will study exponents and logarithms from the point of view of functions. For these functions, we will denote by letter **a** any positive number, not equal to 1, and call it the base of a function.

Exponential Functions

We can treat the exponent a^x as a function of x: if we pick any number x, the exponent will produce the value $y = a^x$. We can also write $f(x) = a^x$. The domain of this function (possible values of x) is the set of all real numbers (since any number x can be taken as a power, so no exceptions), but the range (possible values of y) is the set of only positive numbers (the value of a^x cannot be negative number or zero).

We are interested in the **behavior** of this function. It means that we want to know what happens with the value y when x takes some specific values, increases to positive infinity, or decreases to negative infinity. One of the ways for this is to visualize the functions, in other words, construct its **graph**. One point on the graph is easy to observe: if $x = 0$ then $y = a^0 = 1$. So, for any base a, the graph of the exponential function $y = a^x$ passes through the point (0, 1) which is located on the y-axis. It turns out that the shape of the graph of the function a^x depends whether base a is greater or less than 1.

1. Case $a > 1$. In this case, the bigger x, the bigger y. We say that function $y = a^x$ **increases**, and increases very fast. For example, if we take $a = 2$, we can construct the following tables of values of function $y = 2^x$ for non-negative and negative values of x:

x	0	1	2	3	4	5
$y = 2^x$	1	2	4	8	16	32

x	-1	-2	-3	-4	-5
$y = 2^x$	$2^{-1} = \dfrac{1}{2}$	$2^{-2} = \dfrac{1}{4}$	$2^{-3} = \dfrac{1}{8}$	$2^{-4} = \dfrac{1}{16}$	$2^{-5} = \dfrac{1}{32}$

Based on these two tables we can draw the graph of the function $y = 2^x$:
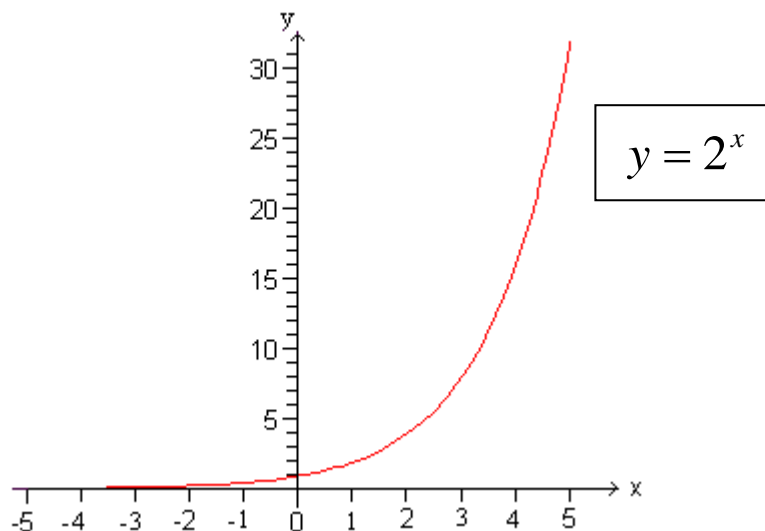
Notice that when x goes to positive infinity (moving to the right), y also goes to infinity (moving up), and when x goes to negative infinity (moving to the left), y approaches to zero (approaches to x-axis and never touches it). We say that x-axis is the horizontal **asymptote** of the function $y = a^x$.

2. Case $0 < a < 1$. In this case, the bigger x, the smaller y. We say that function $y = a^x$ **decreases**. Let's take as an example $a = \dfrac{1}{2}$. We can construct the graph of the function $y = \left(\dfrac{1}{2}\right)^x$ in a similar way as we did above for the function $y = 2^x$ by creating the table of its values. However, we can get this graph almost immediately, if we notice that $\left(\dfrac{1}{2}\right)^x = \dfrac{1}{2^x} = 2^{-x}$. So, we actually need to graph the function $y = 2^{-x}$.

Let's consider in general the relationship between the graphs of functions $f(x)$ and $f(-x)$. Points (x, y) and $(-x, y)$ are symmetric to each other with respect to y-axis. Therefore, graphs of $f(x)$ and $f(-x)$ are also symmetric to each other with respect to y-axis. It means that if we already drew the graph of $f(x)$, then, to get the graph of $f(-x)$, we can just reflect the graph of $f(x)$ with respect to y-axis.

We can apply the above reasoning to the function $y = \left(\dfrac{1}{2}\right)^x = 2^{-x}$ and reflect the graph of $y = 2^x$ with respect to y-axis. Her is the resulting picture
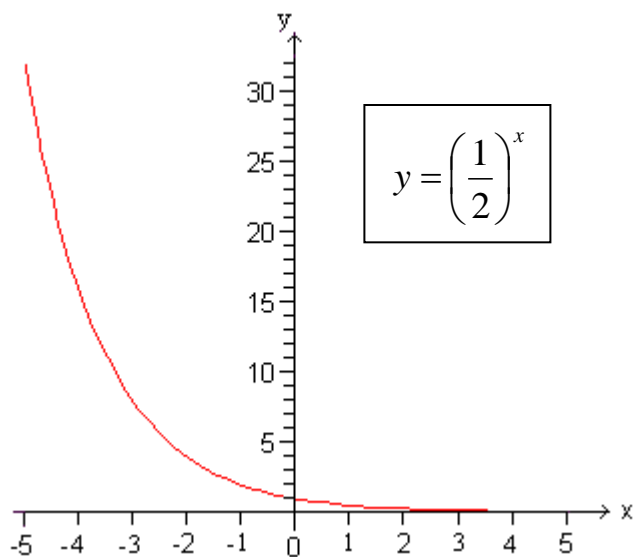
$$y = \left(\frac{1}{2}\right)^x$$

As you can see, when x goes to negative infinity, y increases to positive infinity, and when x goes to positive infinity, y approaches to zero (but doesn't equal to it), so x-axis is still horizontal asymptote of the function $y = \left(\frac{1}{2}\right)^x$.

Logarithmic Functions

Similar to exponents, we can treat the logarithm $\log_a x$ (with fixed base a) as a function of x: $y = \log_a x$. Its domain is the set of all positive numbers, and range – the set of all real numbers.

Note. Notice that the domain of $\log_a x$ is the range of a^x, and the range of $\log_a x$ is the domain of a^x. This is not a coincidence: we will see shortly that this is related to the concept of inverse functions.

All logarithmic functions (for all bases a) have the same value of zero at $x = 1$:

$$\log_a 1 = 0.$$

So, the graphs of all logs pass through the same point $(1, 0)$ on x-axis. Similar to exponential functions, the shape of the graph of the function $y = \log_a x$ depends whether a is greater or less than 1.

1. Case $a > 1$. In this case, as for an exponential function, the log function increases, but this time increases very slowly. Let's take as an example $a = 2$. Here are tables of values of the function $y = \log_2 x$ for values of x greater and less than 1.

x	1	2	4	8	16	32
$y = \log_2 x$	0	1	2	3	4	5

x	$\dfrac{1}{2}$	$\dfrac{1}{4}$	$\dfrac{1}{8}$	$\dfrac{1}{16}$	$\dfrac{1}{32}$
$y = \log_2 x$	-1	-2	-3	-4	-5

The graph of the function $y = \log_2 x$ is this
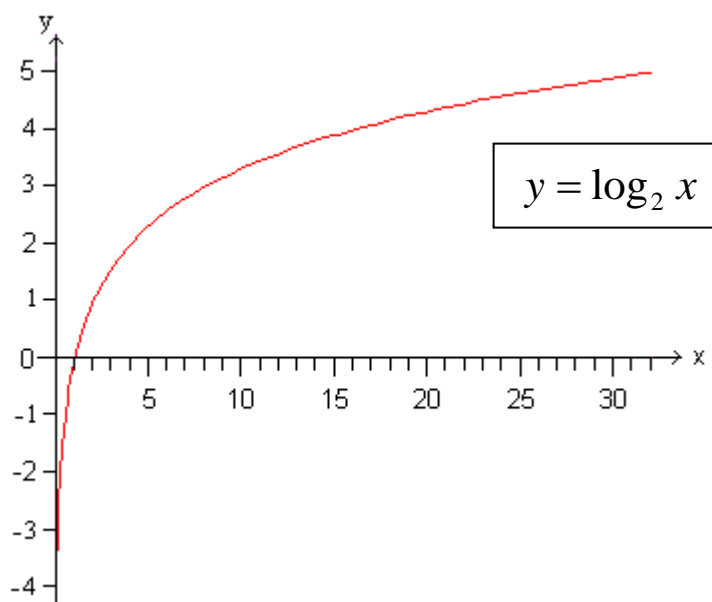
Here y-axis is the vertical asymptote: y goes to negative infinity when x approaches to zero, but never touches y-axis.

2. Case $0 < a < 1$. In this case, like for an exponential function, the function $y = \log_a x$ decreases, but now decreases very slowly. Let's take the example of $a = \dfrac{1}{2}$. To draw the graph of $y = \log_{\frac{1}{2}} x$ without creating a table of values, we can use the result of example 23.3 from the previous session: $\log_{\frac{1}{a}} x = -\log_a x$. In our particular case,

$$\log_{\frac{1}{2}} x = -\log_2 x.$$

Let's consider in general the connection between graphs of functions $y = f(x)$ and $y = -f(x)$. Points (x, y) and $(x, -y)$ are symmetric to each other with respect to x-axis. Therefore, the graphs of $f(x)$ and $-f(x)$ are also symmetric to each other with respect to x-axis. So, to get the graph of $-f(x)$, we can just reflect the graph of $f(x)$ with respect to x-axis.

Applying this reasoning to the function $y = \log_{\frac{1}{2}} x = -\log_2 x$, we will get the picture
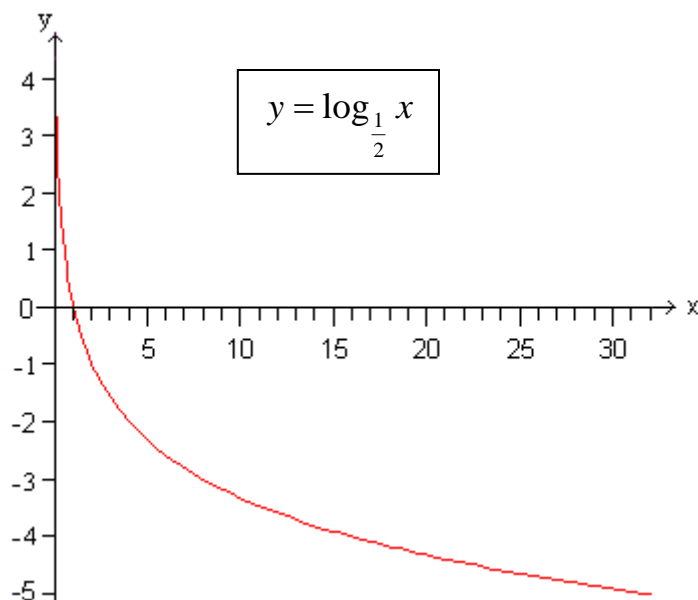
Here y-axis remains the vertical asymptote.

Relation between Exponential and Logarithmic Functions

If you compare the tables of values of functions $y = 2^x$ and $y = \log_2 x$, shown above, you may notice that variables x and y exchange their values. This is not a coincidence. The thing is that functions $y = a^x$ and $y = \log_a x$ are **inverse** to each other. Let's consider this concept in general form.

Let $y = f(x)$ be a function. As we already mentioned, we can treat variable x as input that goes into function f, then f operates on x and produces the output y. Schematically, we can represent function f by the diagram
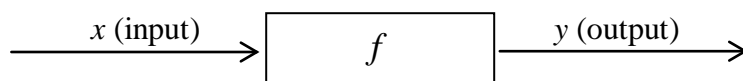
A function g is called the **inverse** to f, if it does the job opposite to f: it passes y back to x. In other words, the input of the inverse function is y, and the output is x. Usually, we denote the function inverse to f by f^{-1}. Schematically, we can represent the inverse function f^{-1} by the diagram
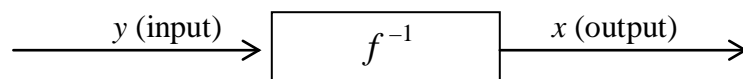
Note. The notation f^{-1} for inverse function may create confusion with the notation $\dfrac{1}{f}$ for the reciprocal function. Keep in mind that these are completely different functions.

To find the function inverse to f, we can solve the equation $y = f(x)$ for x, and then exchange x and y: replace x with y, and y with x.

Example 24.1. Find the inverse function to $y = x^2$, where $x \geq 0$.

Solution. If we solve the equation $y = x^2$ for non-negative x, we get $x = \sqrt{y}$. Now, just exchange x and y. The inverse function is $y = \sqrt{x}$.

Example 24.2. Find the inverse function to $y = \log_a x$.

Solution. As in example 24.1, solve the equation $y = \log_a x$ for x. Using the "Circular Rule" described in session 23, we have $x = a^y$. Now, exchange x and y, and get the inverse function $y = a^x$.

As you see, logarithmic and exponential functions are inverse to each other.

Let's return to a general case of the function f, and see how the graphs of f and f^{-1} are related. If $(x, f(x))$ is a point on the graph of f, then the point $((f(x), x)$ will be on the graph of f^{-1}. The points with coordinates (a, b) and (b, a) are symmetric to each other with respect to the line $y = x$ which is the bisector of the first and third quadrants. (To see that, you may consider some examples, like points $(3, 4)$ and $(4, 3)$, or try to prove this statement in general form). Therefore, the graphs of the function f and its inverse f^{-1} are symmetric to each other with respect to the line $y = x$.

Let's draw together the graphs of the functions $y = 2^x$ and $y = \log_2 x$ which are inverse to each other:
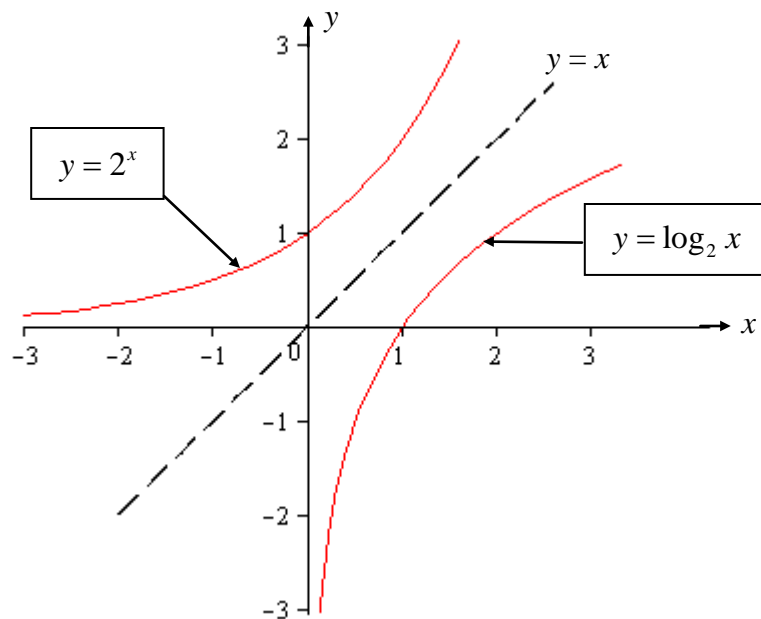
Exercises 24

In exercises 24.1 – 24.8, graphs of given functions f and g are drawn. One of the graphs is labeled as A and another as B. Identify functions f and g with the graphs. Explain your answers.

24.1. $f(x) = 3^x$, $g(x) = 4^x$.
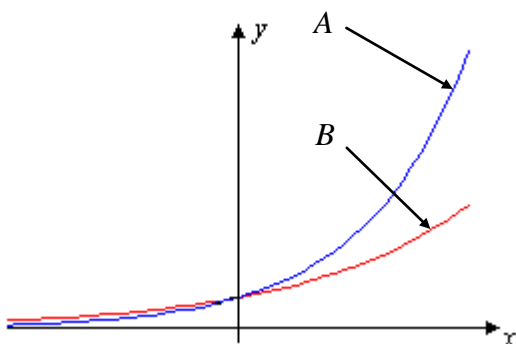
24.2. $f(x) = 4^x$, $g(x) = 5^x$.
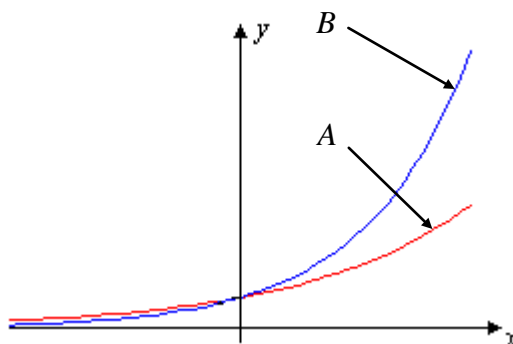
24.3. $f(x) = \left(\dfrac{1}{3}\right)^x$, $g(x) = \left(\dfrac{1}{4}\right)^x$.
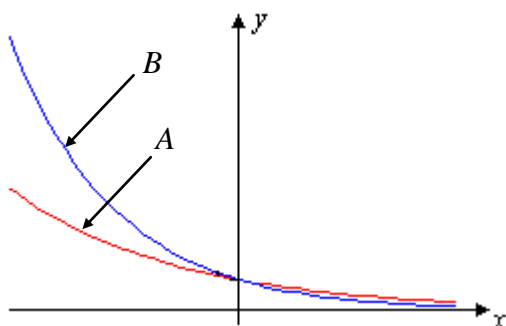
24.4. $f(x) = \left(\dfrac{1}{4}\right)^x$, $g(x) = \left(\dfrac{1}{5}\right)^x$.
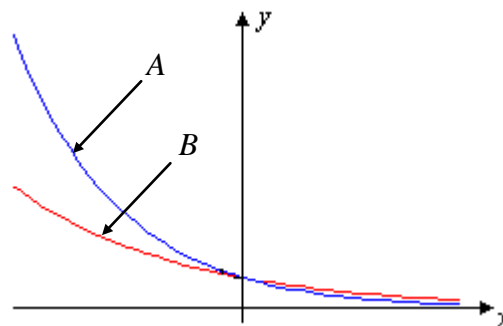
24.5. $f(x) = \log_3(x)$, $g(x) = \log_4(x)$.
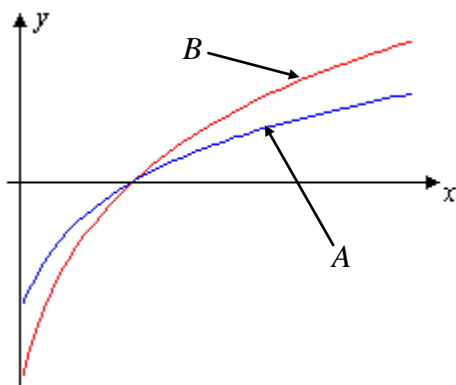
24.6. $f(x) = \log_4(x)$, $g(x) = \log_5(x)$.
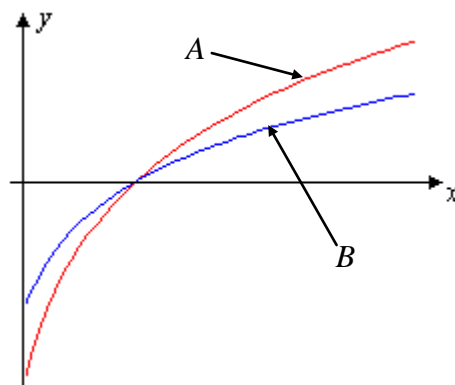
24.7. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{4}}(x)$.
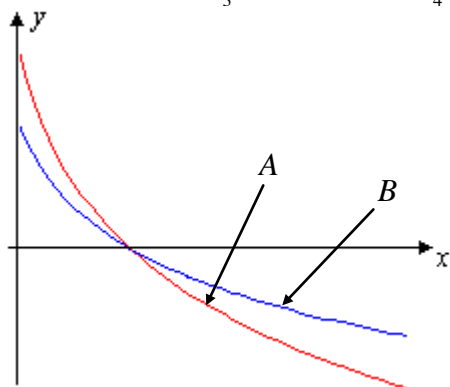
24.8. $f(x) = \log_{\frac{1}{4}}(x)$, $g(x) = \log_{\frac{1}{5}}(x)$.
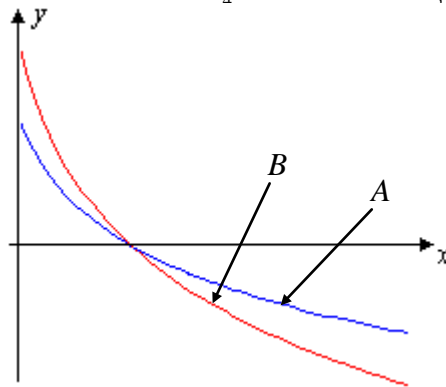
In exercises 24.9 and 24.10, find the inverse function to function $f(x)$.

24.9. a) $f(x) = 3^x$

b) $f(x) = \log_{\frac{2}{3}}(x)$

24.10. a) $f(x) = \left(\frac{3}{4}\right)^x$

b) $f(x) = \log_5(x)$

Session 25

Compound Interest and Number *e*

If you deposit money into a bank, bank pays you interest for usage of your money. If you borrow money from the bank, you pay interest to the bank. The interest may be **simple** or **compound**. To explain these, let's start with some terminology and notations.

P – Principal or Initial Value. This is amount of money that you deposit into a bank or borrow from the bank.

T – Time. This is a period of time that money is used (by you or by bank). In calculations, it is usually counted in years. If the time period is a few months, it is given by fraction or decimal. For example, *T* = 0.5 means 6 months.

R – Rate. This is interest rate used to pay for usage of money. Usually it is given as percentage per year. In calculations it is used as a decimal. For example, if interest rate is 1.7%, in calculations it is used as 0.017.

I –   Interest. This is amount of money that you (or bank) earn for usage of money for *T* years. (Do not confuse interest and rate: interest is dollar amount while rate is percentage).

A - Amount or Future Value. This is amount of money that you will have after *T* years. Obviously the future value is the sum of two parts: Initial Value and Interest. So, *A* = *P* + *I*.

Simple Interest

This type of interest is usually used when you keep money for a short period of time, like several months. This interest is really simple to calculate. If you deposit *P* dollars for one year with the rate *R* (taken as a decimal), then the interest *I* (this is what you earn), will be *I* = *PR*. If you keep money for *T* year, the total interest you earn will be *I* =*PRT*. This is the formula for simple interest. As you see, it's really simple. We can also calculate the future value *A*: *A* = *P* + *I* = *P* + *PRT* = *P*(1 + *RT*). So, basic formulas for simple interest are

$$I = PTR, \ A = P(1+RT)$$

Note. When using the above formulas, keep in mind that rate R must be taken as a decimal (not as percent), and time T should be in the same units of time as rate R (usually in years).

Example 25.1. Suppose you deposit $800 for 3 months into a bank that pays 5% simple interest. Calculate interest that bank will pay you and future value (amount that you withdraw) after 3 months.

Solution. We have

$$P = 800, \ R = 5\% = 0.05, \ T = 3\,\text{months} = \frac{3}{12}\,\text{years} = 0.25 \,.$$

Using the above formulas for *I* and *A*, we get

$$I = PRT = 800 \times 0.05 \times 0.25 = 10 \text{ (dollars)}, \quad A = P + I = 800 + 10 = 810 \text{ (dollars)}.$$

Compound Interest

If you keep money in a bank for a long period of time (for example on CD – Certificate of Deposit, for several years), it is not fair to calculate interest using the above formula for simple interest. Indeed, if the principal is P, and the bank rate is R, then the amount A_1 after the first year, according to the formula for future value with $T = 1$, is $A_1 = P(1 + R)$. Assume that after the first year you do not withdraw your money. Then for the second year it would be not fair to take as a principal the original value P. Instead, it is reasonable to take as a new principal the value A_1 (which is, of course, greater than P). In other words, for the second year rate R should be applied not only to the initial deposit P, but also to the interest $I = PR$ that you earned for the first year. Notice that according to the formula $A_1 = P(1 + R)$, the amount at the end of a year is equal to the amount at the beginning of the year times $(1 + R)$. Therefore, at the end of the second year the amount, denoted as A_2, should be

$$A_2 = A_1(1 + R) = P(1 + R)(1 + R) = P(1 + R)^2.$$

If we continue the same reasoning, then after T years you will accumulate the amount of $A_T = P(1 + R)^T$ dollars. Formula

$$A = P(1 + R)^T$$

allows to calculate the future value after T years, if the initial principal is P, and the bank interest is R.

Even so the above formula for future value is fairer than corresponding formula for simple interest, it still is not enough fair. Here is the reasoning. After some (even short) period of time, let's say after a half an year, your principal will be increased by the earned interest for that period. However, according to the above formula, the original principal remain the same for the entire year, and only at the beginning of the next year bank makes re-calculation, and replaces the original principal with a new value. It would be better (for customers), if such recalculations would be done more often. Many banks do that. They introduce a parameter which is called the **compounded period**. This is a period of time after which bank makes recalculation of the principal: bank takes the principal, adds earned interest, and uses this sum as a new principal. Usually, bank compounds (recalculates) semiannually (every half of a year), quarterly (every three months), monthly, and even daily. Therefore, the above formula for future value should be modified by including a new parameter N – number of compounded periods per year. For example, if investment is compounded semiannually, then $N = 2$, if quarterly, then $N = 4$, if monthly, then $N = 12$ and so on.

Let's modify the above formula for future value if investment compounded monthly, i.e. $N = 12$. Since the rate R is for the entire year, interest for one month will be $I = \dfrac{PR}{12}$.

After 1st month, future value is $A = P + I = P + \dfrac{PR}{12} = P\left(1 + \dfrac{R}{12}\right)$. So, in order to get future value for any month, we should take future value for the previous month and multiply it by expression $1 + \dfrac{R}{12}$. Therefore after 2nd month, future value

$A = P\left(1 + \dfrac{R}{12}\right)\left(1 + \dfrac{R}{12}\right) = P\left(1 + \dfrac{R}{12}\right)^2$. And, at the end of the year T, future value

$A = P\left(1 + \dfrac{R}{12}\right)^{12T}$. In similar way, for any compound period N, we can get the **general compound interest formula**:

$$A = P\left(1 + \dfrac{R}{N}\right)^{TN}$$

.

Here $\dfrac{R}{N}$ is the rate for one compound period, and TN is the total number of compounded periods for T years. For example, if rate R = 2.4%, deposit is compounded quarterly ($N = 4$), and number of years $T = 5$, then $\dfrac{R}{N} = \dfrac{0.024}{4} = 0.006$ and $TN = 5 \cdot 4 = 20$.

Interest I on this deposit is the difference between future value A and the original principal P:

$$I = A - P = P\left(1 + \dfrac{R}{N}\right)^{TN} - P = P\left[\left(1 + \dfrac{R}{N}\right)^{TN} - 1\right].$$

Example 25.2. Suppose you deposit $300 for 8 years at 3% compounded quarterly. Find the future value and earned interest.

Solution. We have: $P = 300$, $R = 3\% = 0.03$, $N = 4$, $T = 8$. Substitute these values into the compound interest formula and calculate future value A:

$$A = 300 \cdot \left(1 + \dfrac{0.03}{4}\right)^{8 \cdot 4} = 300 \cdot (1.0075)^{32} \approx 300 \cdot 1.2701 = 381.03.$$

So, the future value is $381.03. Interest I is the difference:

$$I = A - P = 381.03 - 300 = 81.03.$$

Therefore, you will earn $81.03 for 8 years.

The compound interest formula can be used to find rate R, or time T required to accumulate desirable amount in future.

Let's solve the problem to find time T in general form. Dividing both parts of the compound interest formula by P, we have

$$\frac{A}{P} = \left(1 + \frac{R}{N}\right)^{TN}.$$

Now, take log from both sides: $\log\dfrac{A}{P} = \log\left(1 + \dfrac{R}{N}\right)^{TN} = TN \cdot \log\left(1 + \dfrac{R}{N}\right).$

From here,

$$T = \frac{\log\dfrac{A}{P}}{N \cdot \log\left(1 + \dfrac{R}{N}\right)}.$$

Example 25.3. Suppose you deposit some amount of money at 6% compounded monthly. In how many years your deposit will be doubled?

Solution. According to the problem, the future value A is twice as principal P: $A = 2P$. Also, $R = 6\% = 0.06$, $N = 12$. By the above formula,

$$T = \frac{\log\dfrac{2P}{P}}{N\log\left(1 + \dfrac{R}{N}\right)} = \frac{\log 2}{12 \cdot \log\left(1 + \dfrac{0.06}{12}\right)} = \frac{\log 2}{12 \cdot \log 1.005} \approx \frac{0.3010}{12 \cdot 0.0022} \approx 11.5.$$

So, your deposit will be doubled in about 11.5 years.

When you make a decision in what bank to deposit your money, or what credit card to use to make only minimum payments, you need to take into consideration not only the rate, but also the compounded period. To make a true comparison of different rates, we can compare the interest that is accumulated on one dollar for one year. This value is called the **effective rate** or APY (Annual Percentage Yield). To get a formula for APY, we substitute into the above formula for interest I the values $P = 1$ and $T = 1$. We will have

$$APY = \left(1 + \frac{R}{N}\right)^{N} - 1.$$

 Usually, APY is presented as percentage.

Example 25.4. Suppose you have a choice of using two credit cards on which you want to make minimum payments only. On the 1st card, you will pay 18% interest compounded monthly, and on the 2nd card – 17. 9% compounded daily. Which deal is better for you?

Solution. On the first glance, it looks like the 2nd card is better (you pay smaller rate). However, let's compare APYs for these two cards.

1st card: $APY = \left(1 + \dfrac{0.18}{12}\right)^{12} - 1 \approx 0.1956$, or APY = 19.56%.

2nd card: $APY = \left(1 + \dfrac{0.179}{360}\right)^{360} - 1 \approx 0.1960$, or APY = 19.60%.

As you see, even the rate on the 1st credit card is higher, you would prefer this card because its APY is smaller, and in long run you will pay less interest.

Continuous compound interest. Number _e_

We saw that when you deposit money into a bank, it is more profitable for you, if bank uses compound interest formula instead of the simple one. Also, the shorter the compound period, the greater your profit. We mentioned the cases of compounding semiannually, quarterly, monthly, and daily. But why we need to be restricted only with these periods? Is it possible for the compound period to be one hour, one minute, or even one second? The answer is yes. In this way we come up to a formula which is called **continuous compound interest**.

To get this formula, let's modify the compound interest formula:

$$A = P\left(1 + \frac{R}{N}\right)^{TN} = P\left[\left(1 + \frac{R}{N}\right)^{\frac{N}{R}}\right]^{TR}.$$

If we denote $e_N = \left(1 + \dfrac{R}{N}\right)^{\frac{N}{R}}$, then $A = P(e_N)^{TR}$.

Let's see what happens with your income when compound period becomes shorter and shorter. In this case, recalculation of the interest happens more often, and, as a result, the future value (i.e. your money) becomes bigger. You may think that the future value will grow endless and, eventually, may become huge. However (unfortunately for you) this is not true. Your income has a limit. If N increases to infinity (becomes bigger and bigger), then the value $\dfrac{N}{R}$ also increases to infinity. However, this is not the case for e_N (and so for future value). This value also increases but not to infinity. Let's calculate e_N for some values of $\dfrac{N}{R}$.

$\dfrac{N}{R}$	1	2	3	4	5	10	100	1000
e_N	2	2.25	2.37	2.44	2.49	2.59	2.70	2.717

It can be shown that if N continues to increase, the value of e_N cannot be greater than 3. Actually e_N becomes closer and closer to a certain constant number. This number is

denoted by the letter e and is approximately equals to 2.718. In mathematics, this number is called Euler's number or the **base of natural logarithms**. Most scientific calculators have a button to calculate the number e with even better precision.

Let's return to the formula $e_N = \left(1 + \dfrac{R}{N}\right)^{\frac{N}{R}}$ and denote power $\dfrac{N}{R} = n$. Then, by taking

reciprocal, $\dfrac{R}{N} = \dfrac{1}{n}$ and $e_N = \left(1 + \dfrac{1}{n}\right)^n$. If N is big, number n is also big and $e_N \approx e$. We may say that

$$e \approx \left(1 + \frac{1}{n}\right)^n \approx 2.718, \text{ if number } n \text{ is big.}$$

If we replace e_N in the formula $A = P(e_N)^{TR}$ with the number e, we will get the **continuous compound interest formula**

$$\boxed{A = Pe^{TR}}$$

This formula gives the greatest possible future value compared to compound interest formula with any finite number N of compound periods per year.

Example 25.5. Suppose you made an investment of $300 for 8 years at 3% compounded continuously. Find the future value and earned interest.

Solution. We have: $P = 300$, $R = 3\% = 0.03$, $T = 8$. Substitute these values into the above formula and calculate future value A:

$$A = Pe^{TR} = 300 \cdot e^{8 \cdot 0.03} = 300 \cdot e^{0.24} = 300 \cdot 1.27125 = 381.37$$

So, the future value is $381.37. Interest I is the difference:

$$I = A - P = 381.37 - 300 = 81.37.$$

Compare this result with the result of example 25.2. The difference in interest is small (34 cents) but for large investments the difference becomes significant.

Exercises 25

Round all answers to the nearest cent.

25.1. Suppose you deposit $700 for 5 months into a bank that pays 2.5% simple interest. Calculate interest that bank will pay you and future value (amount that you withdraw) after 5 months.

25.2. Suppose you deposit $600 for 7 months into a bank that pays 4% simple interest. Calculate interest that bank will pay you and future value (amount that you withdraw) after 7 months.

25.3. Suppose you deposit $1,200 for 5 years at 5% compounded monthly. Find the future value and earned interest.

25.4. Suppose you deposit $2,400 for 6 years at 2% compounded semiannually. Find the future value and earned interest.

25.5. Suppose you made an investment of $1,200 for 5 years at 5% compounded continuously. Find the future value and earned interest. Compare with the results of example 25.3.

25.6. Suppose you made an investment of $2,400 for 6 years at 2% compounded continuously. Find the future value and earned interest. Compare with the results of example 25.4.

Challenge Problem

25.7. Suppose you want to invest $5,000 for the next 10 years. You are considering two banks for your investment. Bank *A* offers a rate of 7 % compounded semiannually. Bank *B* offers a rate of 6.9 % compounded daily. Which bank would you choose? Is it possible to solve this problem without information about investment amount and time?

Answers to Exercises

Session 1

1.1. 2.5×10^4 **1.2.** 2.9×10^4 **1.3.** 2.56×10^{-4} **1.4.** 1.4×10^{-5}

1.5. 0.00004 **1.6.** 0.0875 **1.7.** $3,475,000$ **1.8.** $1,236$

1.9. **a)** 3.47×10^{-3} **1.10. a)** 4.38×10^{-5} **1.11. a)** 2 **1.12. a)** 3

   b) 2.5×10^2 **b)** 3.6×10^4 **b)** 1 **b)** 1

   c) $1/16$ **c)** $1/25$

   d) $-1/16$ **d)** $-1/25$

   e) $1/16$ **e)** $1/25$

1.13. a) $1/a^2$ **1.14. a)** $1/c^3$ **1.15. a)** $1/p^{12}$ **1.16. a)** r^{12}

   b) $1/a^6$ **b)** $1/d^{12}$ **b)** p^{12} **b)** $1/r^6$

   c) p^4 **c)** $1/r^{12}$

1.17. y^{b+d} / x^{a+c} **1.18.** p^{w+y} / q^{x+z} **1.19.** mnb/a **1.20.** xyu^2 / v^3

1.21. $\left(36s^{30}\right)/\left(25r^{16}\right)$ **1.22.** $\left(-125x^{60}\right)/\left(8y^{36}\right)$

Session 2

2.1. $\dfrac{6y-4}{5}$ **2.2.** $\dfrac{12z-8}{7}$ **2.3.** $\dfrac{z(3z+5)}{2(z-3)}$ **2.4.** $\dfrac{2y(2y+3)}{3y-5}$

2.5. $\dfrac{2(x+5)}{3(x+4)}$ **2.6.** $\dfrac{3(x-2)}{4(x-1)}$ **2.7.** $\dfrac{2(x+4)}{7x}$ **2.8.** $\dfrac{3(x+3)}{4x}$

2.9. 2 **2.10.** 2 **2.11.** $\dfrac{10m+15}{36}$ **2.12.** $\dfrac{7n+50}{60}$

2.13. $\dfrac{14x+15y}{24xy}$ **2.14.** $\dfrac{9x+28y}{24xy}$ **2.15.** $\dfrac{35x^2-50x+8}{30x^2}$ **2.16.** $\dfrac{9x^2-28x+10}{24x^2}$

2.17. $\dfrac{x+23}{(4x-7)(7x-4)}$ **2.18.** $\dfrac{3x+10}{(3x-4)(4x-3)}$ **2.19.** $\dfrac{2a}{a-4}$ **2.20.** $\dfrac{2b}{b-5}$

Session 2 (*continued*)

2.21. $\dfrac{3x}{x-2}$

2.22. $\dfrac{2x}{x-4}$

2.23. $\dfrac{x+3}{5(x-5)}$

2.24. $\dfrac{x+5}{4(x-4)}$

2.25. $\dfrac{2}{(x+3)(x+5)}$

2.26. $\dfrac{1}{(x+4)(x+5)}$

2.27. $-\dfrac{13}{(b+4)(b-2)}$

2.28. $-\dfrac{25}{(c-3)(c+5)}$

2.29. $\dfrac{4d}{9c^2}$

2.30. $\dfrac{15n}{m^3}$

2.31. $-\dfrac{7}{24}$

2.32. $\dfrac{41}{2}$

2.33. $\dfrac{4x+3xy}{6y-2x^2}$

2.34. $\dfrac{6xy-5y}{4y^2+3x}$

2.35. $\dfrac{4k-19}{6k-13}$

2.36. $\dfrac{5m-12}{7m-34}$

2.37. $\dfrac{x-1}{x+4}$

2.38. $\dfrac{x-1}{x+2}$

Session 3

3.1. $x=9/7$

3.2. $x=5$

3.3. $x=4/9$

3.4. $x=-6/5$

3.5. $x=17/3$

3.6. $x=7$

3.7. $x=-1/12$

3.8. $x=-11/5$

3.9. $x=-11$

3.10. $x=-6$

3.11. $m=-113/11$

3.12. $m=-15$

3.13. $x=3$

3.14. $x=-11$

3.15. 2 gallons

3.16. 4 gallons

3.17. 36%

3.18. 14%

Session 4

4.1. **a)** 4 **b)** 3 **c)** 25 **d)** 4 **e)** 1/2 **f)** 125 **g)** 1/9 **h)** 2

4.2. **a)** 9 **b)** 4 **c)** 36 **d)** 3 **e)** 1/6 **f)** 4 **g)** 1/8 **h)** 3

4.3. $\dfrac{1}{\sqrt{a}}$

4.4. $\sqrt{b}$

4.5. $5\sqrt{2}$

4.6. $3\sqrt{6}$

4.7. $4\sqrt{2}$

4.8. $6\sqrt{2}$

4.9. **a)** x^4

   b) $x^3\sqrt{x}$

4.10. **a)** x^3

   b) $x^2\sqrt{x}$

4.11. **a)** $6y^{18}$

   b) $3y^4\sqrt{y}$

4.12. **a)** $7z^{24}\sqrt{z}$

   b) $8z^{32}$

4.13. $15x^5y^2z^8\sqrt{3y}$

4.14. $25x^7y^4z^6\sqrt{3yz}$

Session 5

5.1. 2018

5.2. 2019

5.3. $47\sqrt{2}$

5.4. $26\sqrt{3}$

5.5. $576a^5b\sqrt{3b}$

5.6. $294u^6v^3\sqrt{2v}$

5.7. $11\sqrt{6}$

5.8. $9\sqrt{5}$

5.9. $8m\sqrt{6n}+2\sqrt{7k}$

5.10. $12p\sqrt{3q}+2\sqrt{6r}$

5.11. $10\sqrt{6}-10\sqrt{5}$

5.12. $28\sqrt{7}-40\sqrt{3}$

5.13. 315

5.14. 24

5.15. 42

5.16. 67

5.17. 117

5.18. 189

5.19. $83+12\sqrt{35}$

5.20. $72+48\sqrt{2}$

5.21. $83-12\sqrt{35}$

5.22. $72-48\sqrt{2}$

Session 6

6.1. $\dfrac{\sqrt{15}}{5}$

6.2. $\dfrac{\sqrt{42}}{7}$

6.3. $\dfrac{\sqrt{5}}{15}$

6.4. $\dfrac{\sqrt{3}}{15}$

6.5. $\dfrac{4\sqrt{3}}{15}$

6.6. $\dfrac{2\sqrt{5}}{15}$

6.7. $\dfrac{5-\sqrt{3}}{4}$

6.8. $\dfrac{3-\sqrt{5}}{2}$

6.9. $2+\sqrt{3}$

6.10. $\dfrac{5-\sqrt{6}}{19}$

6.11. $\dfrac{\sqrt{6}-\sqrt{3}}{3}$

6.12. $\sqrt{6}+\sqrt{5}$

6.13. $\dfrac{4u+3\sqrt{uv}+8\sqrt{u}+6\sqrt{v}}{16u-9v}$

6.14. $\dfrac{2a-3\sqrt{ab}-5b}{4a-25b}$

6.15. $x=7$

6.16. $x=13$

6.17. $x=8$

6.18. $x=5$

6.19. No solutions

6.20. No solutions

6.21. $x=4$

6.22. No solutions

6.23. No solutions

6.24. $x=2$

Session 7

7.1. a) $5i$

7.2. a) $4i$

7.3. a) $9-2i$

7.4. a) $12-5i$

b) $4\sqrt{2}i$

b) $3\sqrt{3}i$

b) $-4-8i$

b) $4+5i$

7.5. a) $6-12i$ **b)** $-12-54i$ **c)** $45-7i$ **d)** 13

7.6. a) $12+20i$ **b)** $21+42i$ **c)** $34-38i$ **d)** 20

Session 7 (*continued*)

7.7. a) $\dfrac{8}{5}+\dfrac{2}{5}i$ **b)** $-\dfrac{3}{2}i$ **c)** $\dfrac{3}{2}+\dfrac{9}{4}i$ **d)** $-\dfrac{2}{61}+\dfrac{39}{61}i$ **e)** $\dfrac{22}{41}+\dfrac{7}{41}i$

7.8. a) $-3+\dfrac{7}{2}i$ **b)** $\dfrac{2}{3}i$ **c)** $\dfrac{1}{2}-\dfrac{2}{3}i$ **d)** $\dfrac{38}{25}-\dfrac{9}{25}i$ **e)** $-\dfrac{2}{13}-\dfrac{23}{13}i$

7.9. a) -1 **b)** $-i$ **c)** 1 **d)** i

7.10. a) $-i$ **b)** 1 **c)** i **d)** -1

Session 8

8.1. a) $3x^2+x-14=0;$ $a=3,\ \ b=1,\ \ \ c=-14$

    b) $16x^2-24x+3=0;$ $a=16,\ \ b=-24,\ \ c=3$

8.2. a) $8x^2-2x-3=0;$ $a=8,\ \ \ b=-2,\ \ c=-3$

    b) $25x^2+20x+1=0;$ $a=25,\ \ b=20,\ \ \ c=1$

8.3. a) $\left\{-\dfrac{7}{3},\,2\right\}$ **8.4. a)** $\left\{\dfrac{3}{4},\,-\dfrac{1}{2}\right\}$ **8.5. a)** $\left\{0,\,-\dfrac{5}{3}\right\}$ **8.6. a)** $\left\{0,\,\dfrac{7}{6}\right\}$

    b) $\left\{\dfrac{5}{6}\right\}$ **b)** $\left\{-\dfrac{4}{7}\right\}$ **b)** $\{\pm 4\}$ **b)** $\{\pm 3\}$

                                            c) $\{-4,\,3\}$ **c)** $\{-3,\,5\}$

                                              d) $\{4,\,-5\}$ **d)** $\{6,\,-4\}$

8.7. a) $\{2\}$ **8.8. a)** $\{-3,\,-1\}$ **8.9. a)** $\left\{\dfrac{3}{4},\,-\dfrac{1}{2}\right\}$ **8.10. a)** $\left\{\dfrac{1}{3},\,-\dfrac{5}{2}\right\}$

    b) $\{-2,\,-1\}$ **b)** $\{1\}$ **b)** $\left\{\dfrac{3}{5},\,-3\right\}$ **b)** $\left\{\dfrac{4}{7},\,-1\right\}$

8.11. a) $\left\{\dfrac{3\pm\sqrt{6}}{4}\right\}$ **b)** $\{\pm\sqrt{7}i\}$ **c)** $\left\{-\dfrac{2}{3}\pm\dfrac{\sqrt{3}}{6}i\right\}$

8.12. a) $\left\{\dfrac{-2\pm\sqrt{3}}{5}\right\}$ **b)** $\{\pm\sqrt{3}i\}$ **c)** $\left\{\dfrac{5}{7}\pm\dfrac{\sqrt{6}}{7}i\right\}$

Session 9

9.1. a) $\{-1,\,5\}$ **b)** $\{-3,\,-2\}$ **c)** $\left\{\dfrac{7\pm\sqrt{17}}{2}\right\}$ **d)** $\left\{-3,\,\dfrac{3}{2}\right\}$

9.2. a) $\{-4,\,2\}$ **b)** $\{2,\,7\}$ **c)** $\left\{\dfrac{-3\pm\sqrt{29}}{2}\right\}$ **d)** $\left\{-2,\,-\dfrac{1}{3}\right\}$

9.3. a) $\left\{\dfrac{4\pm\sqrt{21}}{5}\right\}$ **b)** $\left\{-\dfrac{3}{4}\right\}$ **c)** $\left\{\dfrac{7}{8}\pm\dfrac{\sqrt{31}}{8}i\right\}$ **d)** $\left\{\dfrac{4}{3},\,\dfrac{3}{2}\right\}$

9.4. a) $\left\{\dfrac{3\pm\sqrt{30}}{7}\right\}$ **b)** $\left\{\dfrac{7}{3}\right\}$ **c)** $\left\{-\dfrac{5}{12}\pm\dfrac{\sqrt{47}}{12}i\right\}$ **d)** $\left\{\dfrac{5}{6},\,\dfrac{3}{2}\right\}$

Session 10

10.1. a) $(-4,\,-5)$, Up **b)** $(2,\,6)$, Down **c)** $(5,\,-4)$, Up

10.2. a) $(3,\,7)$, Down **b)** $(-7,\,-3)$, Up **c)** $(-1,\,8)$, Down

10.3. a) square form: $y = 3(x+2)^2 - 3$, standard form: $y = 3x^2 + 12x + 9$

 b) square form: $y = 3(x+2)^2 + 1$, standard form: $y = 3x^2 + 12x + 13$

10.4. a) square form: $y = 3(x-2)^2 + 1$, standard form: $y = 3x^2 - 12x + 13$

 b) square form: $y = 3(x-2)^2 - 3$, standard form: $y = 3x^2 - 12x + 9$

10.5. a) square form: $y = -2(x-3)^2 + 2$, standard form: $y = -2x^2 + 12x - 16$

 b) square form: $y = -2(x-3)^2 - 1$, standard form: $y = -2x^2 + 12x - 19$

10.6. a) square form: $y = -2(x+3)^2 - 1$, standard form: $y = -2x^2 - 12x - 19$

 b) square form: $y = -2(x+3)^2 + 2$, standard form: $y = -2x^2 - 12x - 16$

10.7. a) vertex: $(1,\,-4)$, x-intercepts: $(-1,\,0)$ and $(3,\,0)$, y-intercepts: $(0,\,-3)$

 b) vertex: $(-2,\,9)$, x-intercepts: $(-5,\,0)$ and $(1,\,0)$, y-intercepts: $(0,\,5)$

10.8. a) vertex: $(-1,\,9)$, x-intercepts: $(-4,\,0)$ and $(2,\,0)$, y-intercepts: $(0,\,8)$

 b) vertex: $(3,\,-4)$, x-intercepts: $(1,\,0)$ and $(5,\,0)$, y-intercepts: $(0,\,5)$

Session 11

11.1. $2\sqrt{10}$ **11.2.** $5\sqrt{2}$

11.3. Acute triangle. Biggest angle C, smallest angle A.

11.4. Right triangle. Biggest angle A, smallest angle C.

11.5. $(-1, -5)$ **11.6.** $(3, 1)$ **11.7.** $(1, 3)$ **11.8.** $(-7, -2)$

11.9. a) Center $(-2, -4)$, radius $= 6$ **b)** Center $(-5, 2)$, radius $= 2\sqrt{5}$

11.10. a) Center $(6, 3)$, radius $= 4\sqrt{2}$ **b)** Center $(7, -8)$, radius $= 5\sqrt{2}$

11.11. $(x+3)^2 + (y+4)^2 = 13^2$ **11.12.** $(x-12)^2 + (y-10)^2 = 17^2$

11.13. a) Center $(-2, 1)$, radius $= 4$, points: $(-2, 5)$, $(-2, -3)$, $(-6, 1)$, $(2, 1)$

  b) Center $(5, 4)$, radius $= 3$, points: $(5, 1)$, $(5, 7)$, $(2, 4)$, $(8, 4)$

11.14. a) Center $(4, -1)$, radius $= 3$, points: $(4, 2)$, $(4, -4)$, $(1, -1)$, $(7, -1)$

  b) Center $(-3, -2)$, radius $= 5$, points: $(-3, -7)$, $(-3, 3)$, $(-8, -2)$, $(2, -2)$

Session 12

12.1. a) System has unique solution $(-2, 1, -3)$.

  b) System is dependent. Possible parametric form: $x = t$, $y = \dfrac{1}{2} - 2t$, $z = t$.

   Particular solutions for $t = 0$: $\left(0, \dfrac{1}{2}, 0\right)$.

  c) System is inconsistent: no solutions.

12.2. a) System has unique solution $(3, -4, -2)$.

  b) System is inconsistent: no solutions.

  c) System is dependent. Possible parametric form: $x = 1 - t$, $y = t$, $z = 1 + t$.

   Particular solutions for $t = 0$: $(1, 0, 1)$.

Session 13

13.1. $\left(\dfrac{64}{17}, \dfrac{52}{17}\right)$ **13.2.** $\left(\dfrac{55}{23}, -\dfrac{17}{23}\right)$ **13.3.** $D = 22$ **13.4.** $D = -6$

13.5. $\left(\dfrac{35}{22}, \dfrac{37}{22}, -\dfrac{9}{22}\right)$ **13.6.** $\left(\dfrac{11}{6}, \dfrac{17}{3}, \dfrac{29}{6}\right)$

Session 14

14.1. **a)** $\{(-7, -22), (1, 2)\}$

  b) $\{(-8, -3)\}$

  c) $\{(9, -3)\}$

  d) $\{(-19, -11), (1, -1)\}$

  e) $\{(3, 4), (-3, 4), (3, -4), (-3, -4)\}$

14.3. Length = 4 m, width = 3 m

14.2. **a)** $\{(-1, 6), (13, -50)\}$

  b) $\{(70, 6)\}$

  c) $\{(4, 2)\}$

  d) $\{(-34, 12), (2, 0)\}$

  e) $\{(2, 4), (-2, 4), (2, -4), (-2, -4)\}$

14.4. Length = 6 yd, width = 5 yd

Session 15

15.1. **a)** $50° + 360°n, \ n = 0, \pm 1, \pm 2, ...$
    $410°, \ 770°, \ -310°, \ -670°$

  b) $-70° + 360°n, \ n = 0, \pm 1, \pm 2, ...$
    $290°, \ 650°, \ -430°, \ -790°$

15.3. **a)** $b = 6\sqrt{3}, \ c = 12$

  b) $a = \sqrt{3}, \ c = 2\sqrt{3}$

  c) $a = 4, \ b = 4\sqrt{3}$

15.5. **a)** $b = 4, \ c = 4\sqrt{2}$

  b) $a = 8, \ c = 8\sqrt{2}$

  c) $a = b = \dfrac{9\sqrt{2}}{2}$

15.2. **a)** $27° + 360°n, \ n = 0, \pm 1, \pm 2, ...;$
    $387°, \ 747°, \ -333°, \ -693°$

  b) $-35° + 360°n, \ n = 0, \pm 1, \pm 2, ...$
    $325°, \ 685°, \ -395°, \ -755°$

15.4. **a)** $b = 8\sqrt{3}, \ c = 16$

  b) $a = 3\sqrt{3}, \ c = 6\sqrt{3}$

  c) $a = 2, \ b = 2\sqrt{3}$

15.6. **a)** $b = 3, \ c = 3\sqrt{2}$

  b) $a = 6, \ c = 6\sqrt{2}$

  c) $a = b = \dfrac{7\sqrt{2}}{2}$

15.7. Side opposite to $30°$ is $\dfrac{c}{2}$, side opposite to $60°$ is $\dfrac{c\sqrt{3}}{2}$

15.8. Side opposite to $30°$ is $\dfrac{b\sqrt{3}}{3}$, hypotenuse is $\dfrac{2b\sqrt{3}}{3}$

15.9. Both sides are $\dfrac{c\sqrt{2}}{2}$

Session 16

16.1. 29.1 ft   **16.2.** 61 ft   **16.3.** 11.5 ft   **16.4.** 2.5 m   **16.5.** 167.1 ft   **16.6.** 83.1 ft

16.7. $3.2°$ **16.8.** $29.1°$ **16.9.** $46.1°$ **16.10.** $21.1°$ **16.11.** $63.9°$ **16.12.** $53.1°$

16.13. 20.3 m   **16.14.** $60°$

Session 17

17.1. a) $-\sin\theta$ **b)** $\sin\theta$

17.2. a) $-\cos\theta$ **b)** $-\cos\theta$

17.3. a) $50°$ **b)** $40°$ **c)** $70°$ **d)** $85°$

17.4. a) $20°$ **b)** $10°$ **c)** $50°$ **d)** $80°$

17.5. a) $220°$ **b)** $110°$ **c)** $310°$ **d)** $20°$

17.6. a) $320°$ **b)** $70°$ **c)** $130°$ **d)** $200°$

17.7. a) $-\dfrac{1}{2}$ **b)** $\dfrac{1}{2}$ **c)** -1

17.8. a) $-\dfrac{\sqrt{2}}{2}$ **b)** $-\dfrac{\sqrt{3}}{2}$ **c)** $\sqrt{3}$

17.9.

a) IV, $\cos\theta = \dfrac{\sqrt{5}}{3}$, $\tan\theta = -\dfrac{2\sqrt{5}}{5}$, $\csc\theta = -\dfrac{3}{2}$, $\sec\theta = \dfrac{3\sqrt{5}}{5}$, $\cot\theta = -\dfrac{\sqrt{5}}{2}$

b) II, $\sin\theta = \dfrac{\sqrt{21}}{5}$, $\tan\theta = -\dfrac{\sqrt{21}}{2}$, $\csc\theta = \dfrac{5\sqrt{21}}{21}$, $\sec\theta = -\dfrac{5}{2}$, $\cot\theta = -\dfrac{2\sqrt{21}}{21}$

c) III, $\sin\theta = -\dfrac{3\sqrt{34}}{34}$, $\cos\theta = -\dfrac{5\sqrt{34}}{34}$, $\csc\theta = -\dfrac{\sqrt{34}}{3}$, $\sec\theta = -\dfrac{\sqrt{34}}{5}$, $\cot\theta = \dfrac{5}{3}$

17.10.

a) II, $\cos\theta = -\dfrac{\sqrt{33}}{7}$, $\tan\theta = -\dfrac{4\sqrt{33}}{33}$, $\csc\theta = \dfrac{7}{4}$, $\sec\theta = -\dfrac{7\sqrt{33}}{33}$, $\cot\theta = -\dfrac{\sqrt{33}}{4}$

b) III, $\sin\theta = -\dfrac{\sqrt{39}}{8}$, $\tan\theta = \dfrac{\sqrt{39}}{5}$, $\csc\theta = -\dfrac{8\sqrt{39}}{39}$, $\sec\theta = -\dfrac{8}{5}$, $\cot\theta = \dfrac{5\sqrt{39}}{39}$

c) IV, $\sin\theta = -\dfrac{7\sqrt{65}}{65}$, $\cos\theta = \dfrac{4\sqrt{65}}{65}$, $\csc\theta = -\dfrac{\sqrt{65}}{7}$, $\sec\theta = \dfrac{\sqrt{65}}{4}$, $\cot\theta = -\dfrac{4}{7}$

Session 17 (*continued*)

17.11.

$$\sin\theta = -\frac{2\sqrt{5}}{5}, \ \cos\theta = -\frac{\sqrt{5}}{5}, \ \tan\theta = 2, \ \csc\theta = -\frac{\sqrt{5}}{2}, \ \sec\theta = -\sqrt{5}, \ \cot\theta = \frac{1}{2}$$

17.12.

$$\sin\theta = \frac{5\sqrt{34}}{34}, \ \cos\theta = -\frac{3\sqrt{34}}{34}, \ \tan\theta = -\frac{5}{3}, \ \csc\theta = \frac{\sqrt{34}}{5}, \ \sec\theta = -\frac{\sqrt{34}}{3}, \ \cot\theta = -\frac{3}{5}$$

Session 18

18.1. $AB = 33$ m, $BC = 26.9$ m.

18.2. 11.1 ft and 12.0 ft.

18.3. 11.2 m

18.4. 32.5 m

18.12. 3.8 mi

18.13. $\dfrac{d \sin A \sin B}{\sin(A+B)}$

18.14. 6.3 mi

18.15. $\dfrac{d \sin A \sin B}{\sin|A-B|}$

18.16. 1.3 m and 2.2 m

Session 19

19.1. Yes

19.2. 301.4 ft

19.3. 41.8°

19.4. 29.3°

19.5. 14°

19.6. 57°

19.7. 11.5°

Session 20

20.1. a) 74.5°
   b) −34.4°

20.2. a) 137.5°
   b) −45.8°

20.3. a) 40°
   b) −54°

20.4. a) 48°
   b) −216°

20.5. a) 2.44
   b) − 1.48

20.6. a) 1.36
   b) − 4.14

20.7. a) $\dfrac{2\pi}{3}$
   b) $-\dfrac{5\pi}{6}$

20.8. a) $\dfrac{11\pi}{6}$
   b) $-\dfrac{5\pi}{4}$

Session 20 (*continued*)

20.9. **a)** $\sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$, $\cos\left(\dfrac{4\pi}{3}\right) = -\dfrac{1}{2}$, $\tan\left(\dfrac{4\pi}{3}\right) = \sqrt{3}$

b) $\sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$, $\cos\left(-\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$, $\tan\left(-\dfrac{\pi}{4}\right) = -1$

c) $\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2}$, $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$, $\tan\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$

20.10. **a)** $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$, $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$, $\tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$

b) $\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$, $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$

c) $\sin\left(-\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$, $\cos\left(-\dfrac{3\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$, $\tan\left(-\dfrac{3\pi}{4}\right) = 1$

20.11. 1.8 cm

20.12. 5 ft

20.13. $\dfrac{\pi}{4}$, $\dfrac{\pi}{3}$, $\dfrac{5\pi}{12}$

20.14. 16 cabs, arc $\approx$ 3.14 m

20.15. 180 m/min

20.16. 3 sec

Session 21

21.1. **a)** $\dfrac{\pi}{4}$, $\dfrac{3\pi}{4}$

b) $\dfrac{4\pi}{3}$, $\dfrac{5\pi}{3}$

21.2. **a)** $\dfrac{\pi}{3}$, $\dfrac{2\pi}{3}$

b) $\dfrac{7\pi}{6}$, $\dfrac{11\pi}{6}$

21.3. **a)** $\dfrac{\pi}{4}$, $\dfrac{7\pi}{4}$

b) $\dfrac{5\pi}{6}$, $\dfrac{7\pi}{6}$

21.4. **a)** $\dfrac{\pi}{3}$, $\dfrac{5\pi}{3}$

b) $\dfrac{3\pi}{4}$, $\dfrac{5\pi}{4}$

21.5. **a)** $\dfrac{\pi}{6}$, $\dfrac{7\pi}{6}$

b) $\dfrac{2\pi}{3}$, $\dfrac{5\pi}{3}$

21.6. **a)** $\dfrac{\pi}{4}$, $\dfrac{5\pi}{4}$

b) $\dfrac{5\pi}{6}$, $\dfrac{11\pi}{6}$

21.7. No solutions

21.8. No solutions

Session 22

22.7. $\dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$ **22.8.** $\dfrac{2\pi}{3},\ \dfrac{4\pi}{3}$ **22.9.** $\pi,\ 0.841,\ 5.442$

22.10. $\dfrac{\pi}{2},\ 4.069,\ 5.356$ **22.11.** $0,\ \pi,\ \dfrac{\pi}{6},\ \dfrac{7\pi}{6}$ **22.12.** $\dfrac{\pi}{3},\ \dfrac{4\pi}{3}$

22.13. $\dfrac{\pi}{2},\ \pi$ **22.14.** $0,\ \dfrac{3\pi}{2}$

Session 23

23.1. 3 **23.2.** 4 **23.3. a)** 4 **b)** 3 **c)** -2 **d)** -6

23.4. a) 2 **b)** 4 **c)** -3 **d)** -3 **23.5. a)** 7 **b)** 4

23.6. a) 4 **b)** 3 **23.7.** 1 **23.8.** 3 **23.9.** 1.069 **23.10.** 0.936

23.11. a) 1.226 **b)** -0.429 **23.12. a)** 1.292 **b)** 0.881

Session 24

24.1. $f \leftrightarrow B,\ g \leftrightarrow A$ **24.2.** $f \leftrightarrow A,\ g \leftrightarrow B$ **24.3.** $f \leftrightarrow A,\ g \leftrightarrow B$

24.4. $f \leftrightarrow B,\ g \leftrightarrow A$ **24.5.** $f \leftrightarrow B,\ g \leftrightarrow A$ **24.6.** $f \leftrightarrow A,\ g \leftrightarrow B$

24.7. $f \leftrightarrow A,\ g \leftrightarrow B$ **24.8.** $f \leftrightarrow B,\ g \leftrightarrow A$

24.9. a) $\log_3(x)$ **b)** $\left(\dfrac{2}{3}\right)^x$ **24.10. a)** $\log_{\frac{3}{4}}(x)$ **b)** 5^x

Session 25

25.1. $I = \$7.29,\ A = \$\,707.29$ **25.2.** $I = \$14.00,\ A = \$\,614.00$

25.3. $A = \$1{,}540.03,\ I = \$\,340.03$ **25.4.** $A = \$2{,}704.38,\ I = \$\,304.38$

25.5. $A = \$1{,}540.83,\ I = \$\,340.83$ **25.6.** $A = \$2{,}705.99,\ I = \$\,305.99$

25.7. Bank B. Yes, information about investment amount and time is not needed.