System of two equations with two unknowns

November 28, 2018

1 Introduction

We know how to solve a linear equation with one unknown, for example,

$$3x - 2 = 9 \implies 3x = 11 \implies x = \frac{11}{3}.$$

There is always one solution (provided it can't be rewritten to look like $0 \cdot x = 1$) and we can write it down.

In this chapter, we will look at solving equations with two variables $ax + by = c$, where a, b, and c are fixed real numbers.

We have already seen that the solutions to the equation

$$3x - 2y = 6$$

are ordered pairs (a, b) so that when you plug the first number in for x and the second in for y, we get a true statement. In this example, $(2, 0)$ is a solution, but $(2, 4)$ is not. We have seen in the section on lines, that there are infinitly many solutions, so writing them all down in a list is impossible, so we draw a picture of them on the cartesian plane. The solutions form a line. This is a cartoon of all solutions in that the graph gives only approximate information and moreover is only finite so the actual solutions necessarily go out of range of any such graph.

We can ask about solutions to a system of equations, like this example:

$$\begin{cases} 3x - 2y = 6 \\ 2x - y = 5 \end{cases} .$$

Definition 1.1. An ordered pair (a, b) is a solution to a system of equations if (a, b) is a solution to each of the equations in the system.

In the above example, $(2,0)$ is not a solution to the system since it is not a solution to the second equation $(2 \cdot 2 - 0 \neq 5)$. $(2,1)$ is not a solution because it is not a solution to the first equation.

Try 1.1. Check to see if the following are solutions to the system of equations:

1. $(-3,-6)$

2. $(1,1)$

3. $(0,-5)$

4. $(4,3)$

Challenge 1.2. Draw a picture of the solution set of each equation on a coordinate plane. Can you identify the solution set for the system on your picture (remember that a solution for the system is simultaniously a solution for each equation comprising the system)? Based on your answer, can you give different scenarios of a system of two equations with two unknowns where the system has zero, one, two, or infinity of solutions?

2 solving by substitution

It can be difficult to guess a solution and sometimes difficult to read off of a graph (especially when the solution isn't on a grid point of the graph).

Fortunately, we have a couple of methods for solving such a system algebraically.

We will start with simple examples of the first method: substitution.

Example 2.1. Solve

$$\begin{cases} 2x - 3y = 3 \\ x = 3 - y \end{cases}.$$

We solve this by noting that if there is a solution, then it satisfies the second equation so that the solution looks like $(3-y, y)$ for some particular value or values for y which we now aim to find. Since this must also satisfy the first equation, we will *substitute* this into the first equation to find

$$2 \cdot (3-y) - 3y = 3.$$

This is now a linear equation in y which we can solve. Distributing and collecting like terms we find that $3 - 4y = 3$ and so $y = 0$. So $(3,0)$ is the only solution to this system of equations.

Try 2.1. Solve the following by substitution:

$$\begin{cases} 4x + 2y = 8 \\ y = x - 2 \end{cases} .$$

Ususally, the equation isn't written so nicely and we have to first solve one equation for one of the variables that we will then substitute. "Which variable from which equation?", you may ask. Any variable from any equation will do the job, however you may end up having some time consuming computations which plenty of opportunities for errors if you don't choose wisely. The goal is to do the simplest thing.

Example 2.2. Solve by substitution:

$$\begin{cases} 3x - 2y = 6 \\ 2x - y = 5 \end{cases} .$$

First, we look over the problem to see what the easiest variable to solve for will be (this solving will hopefully not involve a division). If we solve the second equation for y we will not be dividing (by comparison, solve for x in the second equation). In fact, we see that $y = 2x - 5$. So our solution must look like $(x, 2x - 5)$. Substituting this into the first equation gives

$$3x - 2(2x - 5) = 6.$$

This is a linear equation in x which we can solve. We see that

$$3x - 2(2x - 5) = 6 \implies -x + 10 = 6 \implies x = 4.$$

So that our solution is $(4, 2 \cdot 4 - 5)$ or, $(4, 3)$ (as we saw earlier on in this chapter).

Try 2.2.

$$\begin{cases} x - 2y = -3 \\ 2x - 5y = -8 \end{cases} .$$

Challenge 2.3. Solve the following by substitution

$$\begin{cases} 3x - 2y = 5 \\ 2x - 3y = -5 \end{cases} .$$

3 Solve by elimination

Solving the challenge problem above creates some challenges! What are they?

It turns out that there is another way to solve the system that will avoid most of the difficulties. Let's practice this new method on the example from the last section:

Example 3.1. Solve by elimination:

$$\begin{cases} 3x - 2y = 6 \\ 2x - y = 5 \end{cases}.$$

I will multiply both sides of the second equation by -2 to get the equivalent system

$$\begin{cases} 3x - 2y = 6 \\ -4x + 2y = -10 \end{cases}.$$

Then the sum of the left-hand-sides and the sum of the right-hand-sides are equal. We find

$$-x = -4.$$

What happened? The y got *eliminated* from the sum. Can you see why multiplying by -2 accomplished this result?

So, we found that $x = 4$ and we can substitute this into either equation to find that $y = 3$, as we have seen before.

Let us, however, practice this elimination proceedure to eliminate x and therefore to find y. Is there a number we can multiply (both sides of) the first equation by and another number to multiply (both sides of) the second equation by to get the coefficients to be opposites? The easiest thing to do is to multiply the first equation by -2 and the second equation by 3 (or 2 and -3, respectively).

We get the resulting system is

$$\begin{cases} -6x + 4y = -12 \\ 6x - 3y = 15 \end{cases}.$$

Adding gives $y = 3$ as we have already seen.

Note. We could have also multiplied the first equation by -10 and the second by 15, but this would give us larger numbers than necessary to work with. If the x term in the first equation is $6x$ and in the second equation is $15x$ what are the smallest numbers you can multiply by to eliminate the x variable upon adding?

Try 3.1. Solve by elimination

$$\begin{cases} x - 2y = -3 \\ 2x - 5y = -8 \end{cases}.$$

Try 3.2. (No longer a challenge) Solve the following by substitution

$$\begin{cases} 3x - 2y = 5 \\ 2x - 3y = -5 \end{cases}.$$

Challenge 3.3.

$$\begin{cases} 3x - 2y = 5 \\ 6x - 4y = 5 \end{cases} .$$

Challenge 3.4.

$$\begin{cases} 3x - 2y = 5 \\ 6x - 4y = 10 \end{cases} .$$

Note. Depending on the form of the equation, one or the other method will be easier. They should both result in the same answer. To find the solution by looking at a graph also has its advantages when an estimate is needed or when the solutions to the equations are not lines.