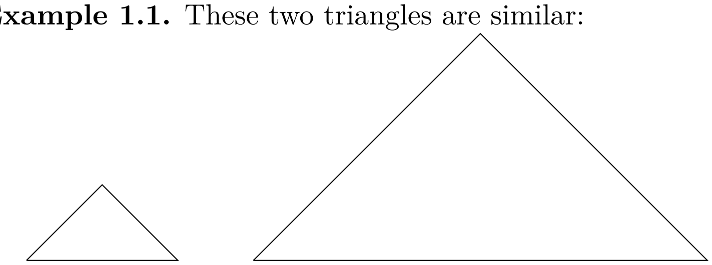
Similar triangles

November 4, 2018

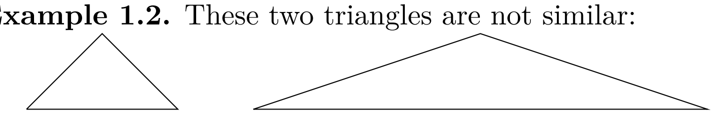
1 Similar Triangles

We say two triangles are **similar** if they are the same shape but perhaps differnt sizes. That is, one could be blown up to match the other.

Example 1.1. These two triangles are similar:

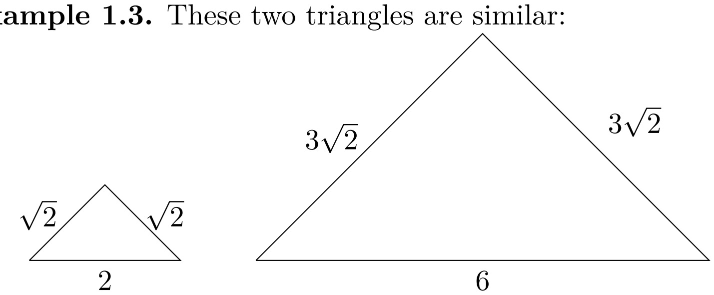


Example 1.2. These two triangles are not similar:



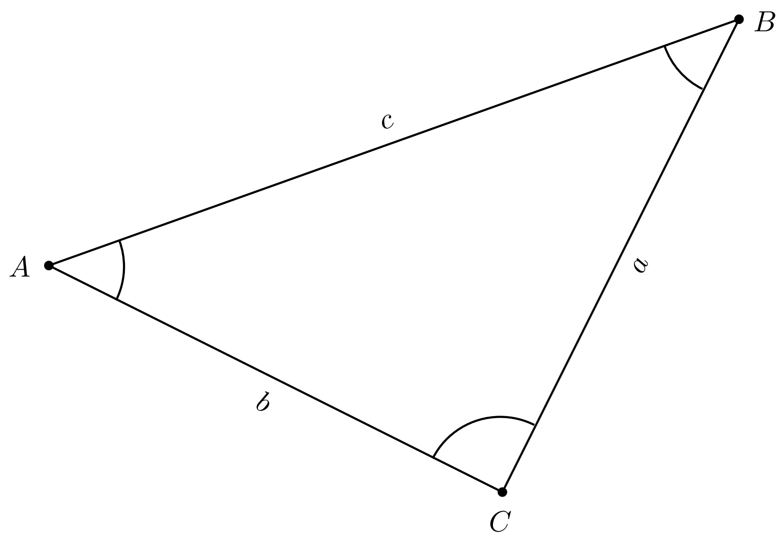
When we have two triangles which are similar, we can multiply the lengths of the sides of one triange each by the same number to get the lengths of the other triangle.

Example 1.3. These two triangles are similar:



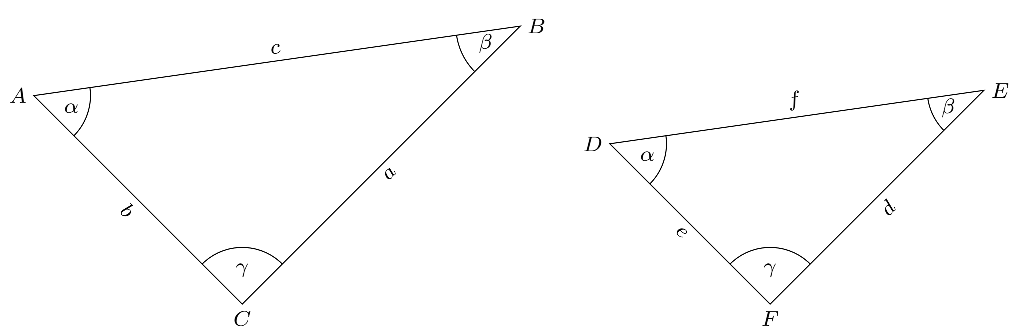
2 Anatomy of triangles

Consider the triangle



We say that the angle A is opposite the side a and the angle B is opposite the side b. Which angle is opposite side c? The sides c and b are adjacent to angle A. The sides a and b are adjacent to angle C. Which sides are adjacent to B?

If two triangles are similar, we have the following situation



The angle A corresponds to the angle D because they have the same measure. The angle B and the angle E correspond.

Try 2.1. Which angle corresponds to angle C?

We say that the sides opposite the corresponding angles correspond. For example, the side c corresponds to the side f because the opposite angles C and F, respectively, have equal measure γ. The side a corresponds to d because angles A and D have equal measure.

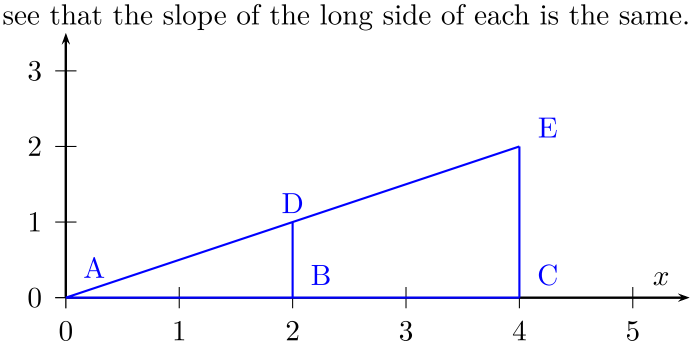
Try 2.2. Which side corresponds to b?

3 ratios of sides

We can multiply each of the lengths of the left triangle by 3 to get the lengths of the other triangle. This also implies that cooresponding ratios of sides are equal since this factor of 3 will cancel. So, in our example,

$$\frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2} \text{ and } \frac{3\sqrt{2}}{3\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}}$$

Consider the two triangles
We see that the slope of the long side of each is the same. That is,

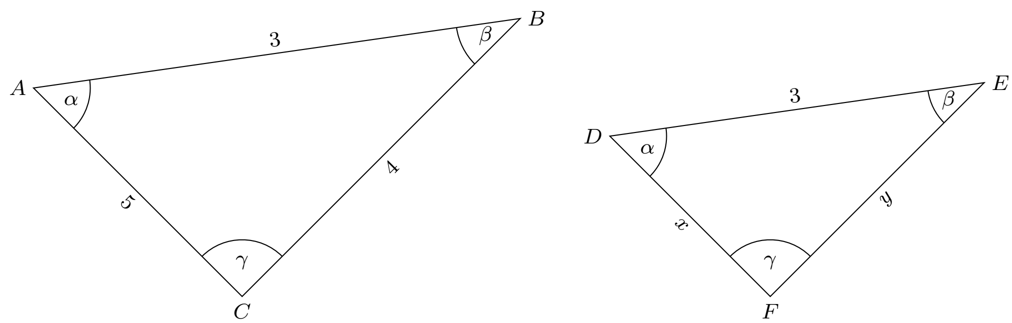


Here the slope of the line segment $\overline{AD} = \frac{|\overline{BD}|}{|\overline{AB}|}$ is equal to the slope of the line segment $\overline{AE} = \frac{|\overline{CE}|}{|\overline{AC}|}$. So these two ratios are equal:

$$\frac{|\overline{BD}|}{|\overline{AB}|} = \frac{|\overline{CE}|}{|\overline{AC}|}.$$

Note. All pictures from this book should be taken as an indication of relationships, not of measure. That is, you can not solve the problems presented here using a ruler or protractor.

Example 3.1. Consider the following situation:



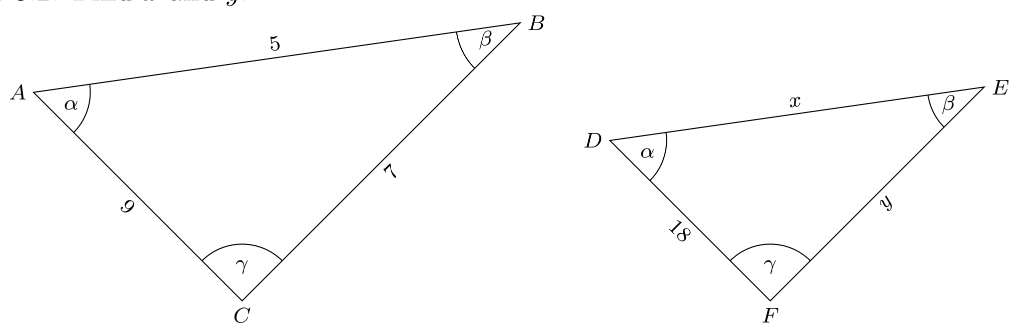
Find x then y. We want to form a ratio of sides of the second triange which involve only x if possible and then note that this ratio is the same with the other triangle. This gives us the equation:

$$\frac{x}{2} = \frac{5}{3}$$

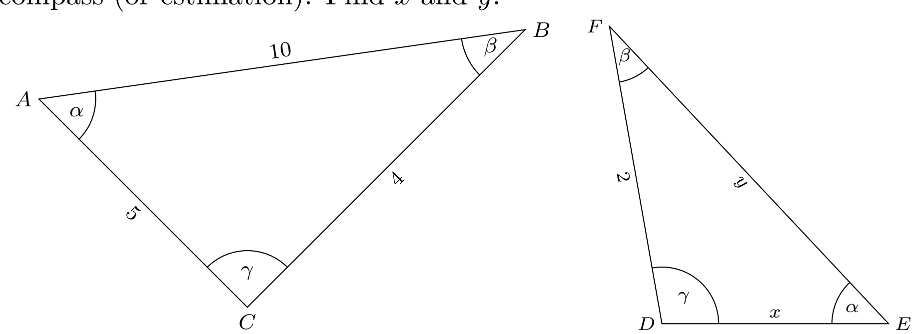
which we can easily solve by multiplying both sides by 2 to see that $x = \frac{10}{3}$.

Try 3.1. Do a similar thing to find that $y = \frac{8}{3}$

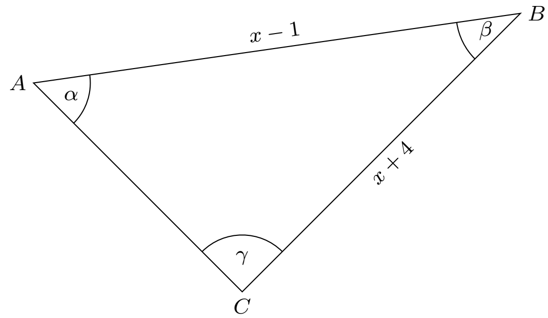
Try 3.2. Find x and y.

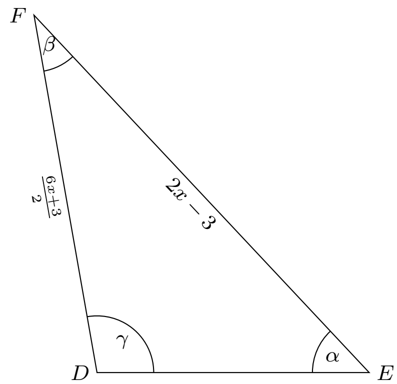


Try 3.3. Be careful in this example to observe the corresponding angles. Also, these pictures should not be understood as representing the actual lengths and angles but only exhibiting the relationships between angles and sides as might be indicated. For instance, the longest side may not look so in the picture. In particular, do not solve this problem using a ruler or compass (or estimation). Find x and y.



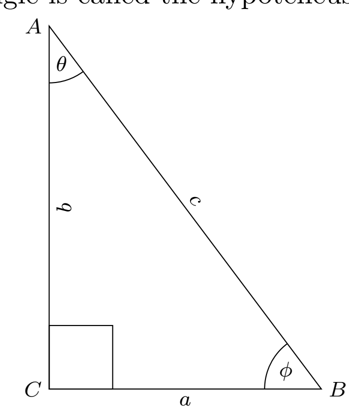
Challenge 3.4. Find x.





3.1 the anatomy of a right triangle

A right triangle is a triangle where one of the angles is 90°. The side opposite the right angle is called the hypoteneuse. Consider the following right triangle.



First, there is the following theorem:

Theorem 3.5 (The Pythagorean Theorem)**.** *Given the diagram above,* $a^2 + b^2 = c^2$.

Try 3.6. If $b = 6$ and $c = 9$, what is a?

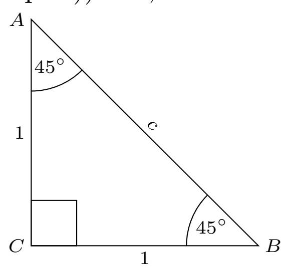
3.2 special triangles

There are two special triangles: the 45-45-90 and the 30-60-90.

Let's discuss each in turn.

3.2.1 the 45-45-90 triangle

This triangle is iscoseles (the sides opposite the equal angles are equal (to see this requires knowledge of congruence however I hope you agree it is plausible by looking at various examples)). So, if we let these equal lengths be length 1, we have the following picture:

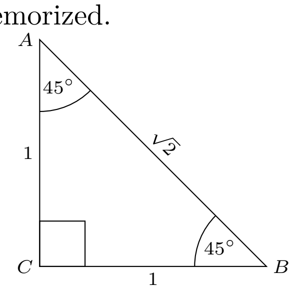


Using the Pythagorean Theorem we can solve for c:

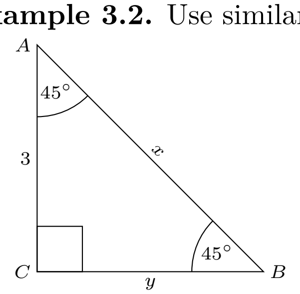
$$1^2 + 1^2 = c^2 \implies 2 = c^2 \implies c = \sqrt{2}$$

since $c \geq 0$.

So we have the following picture of one specific 45-45-90 triangle. All 45-45-90 triangles are similar to this one. You should be able to derive this on the spot, or have this picture memorized.



Example 3.2. Use similarity to find x and y.



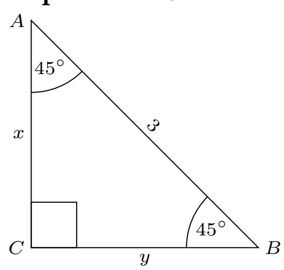
We see that $\sin 45° = \frac{3}{x}$. From the example triangle above, we see that $\sin 45° = \frac{1}{\sqrt{2}}$. So these two ratios are equal (refer to solving rational equations chapter):

$$\frac{3}{x} = \frac{1}{\sqrt{2}} \implies x = 3\sqrt{2}.$$

Note that we could see that this triangle is the same as what you get if you multiply the lengths of each side of the prototype triangle by 3. Doing it the above way, however, will be more useful to general problems.

Try 3.7. Find y.

Example 3.3. Use similarity to find x and y.



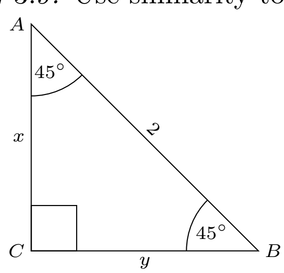
We see that $\sin 45° = \frac{x}{3}$. From the example triangle above, we see that $\sin 45° = \frac{1}{\sqrt{2}}$. So these two ratios are equal:

$$\frac{x}{3} = \frac{1}{\sqrt{2}} \implies x = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Note that this is the same as multiplying each side of the prototype triangle by $\frac{3}{\sqrt{2}}$ (though this is harder to see than the previous example).

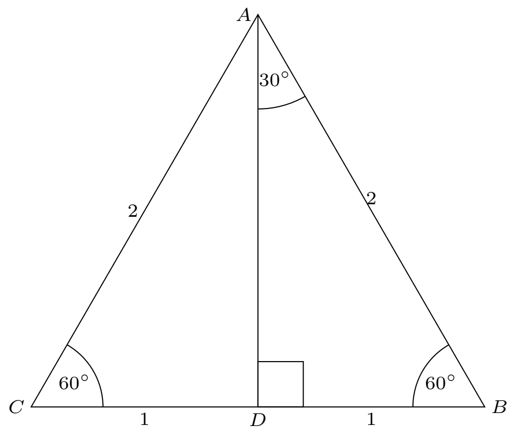
Try 3.8. Find y.

Try 3.9. Use similarity to find x and y.



3.2.2 the 30-60-90 triangle

Consider the equalateral triangel with sides all length 2 units. Then the angle bisector also is a perpendicular bisector the the opposite side. So we have the following picture:



We can use the Pythagorean theorem to find the lenth of AD:

$$1^2 + |AD|^2 = 2^2$$

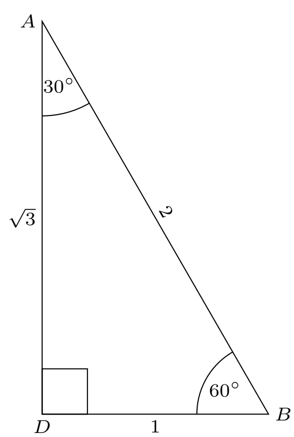
So,

$$|AD|^2 = 3$$

So, we learn that

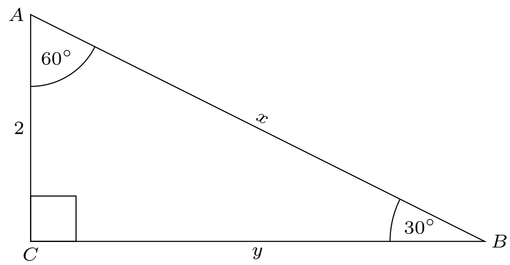
$$|AD| = \sqrt{3}.$$

So, we have the following example (which you should either memorize or be able to derive as we have done):



So for example,

Example 3.4. Find x and y:

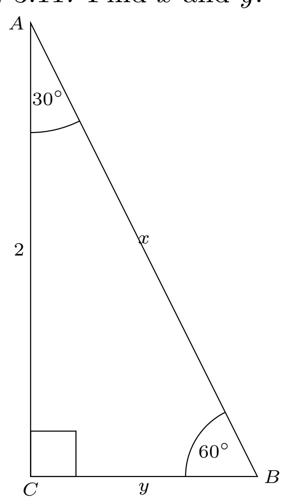


We form a ratio of sides involving just the x variable: $\frac{2}{x}$ and then we equate this with the corresponding ratio of our example triangle above. Note that x is the hypoteneuse and 2 is opposite the 30° angle. So we have

$$\frac{2}{x} = \frac{1}{2} \implies x = 4.$$

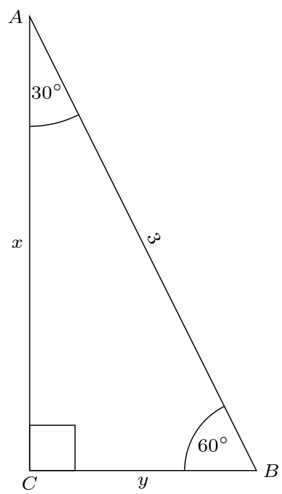
Try 3.10. Find y starting from the ratio $\frac{y}{2}$.

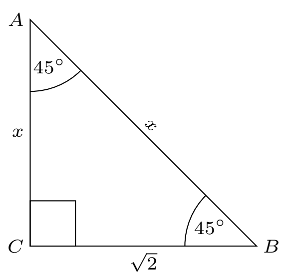
Try 3.11. Find x and y:



3.2.3 mixed examples

Find x and y in the following examples:

Try 3.12. 

Try 3.13. 

3.3 trigonometric ratios

Now, besides the hypoteneuse, there are two sides called legs. With respect to the angle measuring θ the two legs are in different relations: the side with measure a is opposite, while the side with measure b is adjacent (or next to) the angle measuring θ.

Try 3.14. Which side is opposite the angle measuring ϕ? Which is the side adjacent?

There are six ratios of sides that we can form for any triangle. Each ratio is associated is associated with the names sine, cosine, tangent, cosecant, secant, or cotangent. We can name the ratios with respect to the angle θ thusly:

$$\sin\theta = \frac{\text{opposite}}{\text{hypoteneus}} = \frac{a}{c},$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypoteneus}} = \frac{b}{c},$$

and

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b},$$

and their respective reciprocols:

$$\csc\theta = \frac{\text{hypoteneus}}{\text{opposite}} = \frac{c}{a},$$

$$\sec\theta = \frac{\text{hypoteneus}}{\text{adjacent}} = \frac{c}{b},$$

and

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}.$$

Try 3.15. Write down the names of the ratios for the angle ϕ.

Challenge 3.16. Note that the angles θ and ϕ are **co**mplementary ($\theta + \phi = 90°$). Look at the above 12 equalities and describe the relationships between sine and cosine. How about secant and cosecant? And what about tangent and cotangent?

Try 3.17. If $a = 3$, $b = 4$, and $c = 5$, show that the Pythagorean Theorem is satisfied. Note that

$$\sin\theta = \frac{3}{5}.$$

What are the other 5 ratios with respect to θ (include their names as above).

Try 3.18. Use the special triangles you know to find $\cos(60°)$, $\sin(60°)$, $\sin(45°)$, $\tan(30°)$.

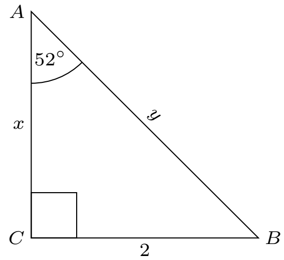
Try 3.19. Your calculator knows the approximate values of these ratios for any reference angle. Check that you are getting the same answer for the 6 trigonometric ratios with reference angles $30°, 60°$, and $45°$. There are 18 computations to verify (though you may simplify your work by taking advantage of the fact that a ratio with a reference angle is the same as the "co-ratio" of the complementary angle).

3.4 general right triangles

We will have only exact values for a few special angles (so far this only includes $60°, 30°$, and $45°$ but we will be able to find others later). When we can use exact computations, we will always opt for that. However, for most angles, approximating is the best we can do. Fortunately, your calculator knows how to approximate ratios associated with all angles. It is an interesinting to know how your calculator knows this but this is well beyond the scope of what we can answer.

Because, for similar triangles, the ratios of corresponding sides are equal, we find that a trigonometric ratio only depends on the angle that is your reference angle.

Example 3.5. Consider the following diagram:



If we want to find x, we form a ratio. It is easier to choose a ratio that corresponds to $\cos, \sin$ or $\tan$ since we will be using our calculator because $52°$ is not a special angle. So, x is the length of the side adjacent to angle A. So, browsing the ratios suggested leads us to either $\tan 52°$ or $\cos 52°$. The ratio $\cos 52°$ involves y as well, so we will use $\tan 52°$:

$$\tan 52° = \frac{2}{x}.$$

We no solve for x (we will use our calculator to evaluate $\tan 52°$ later (you can replace it with a single letter T if that helps you solve the equation). We find that

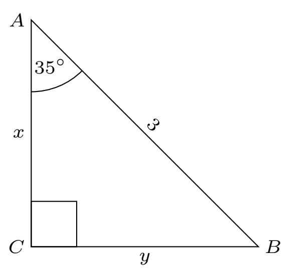
$$x = \frac{2}{\tan 52°}.$$

Now we use our calculator to approximate the answer above: $x \approx 1.5623$.

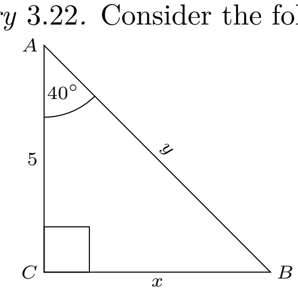
Try 3.20. Find y.

Note. We could also have found the measure of the complementary angle $B : 90 - 52 = 38$, and solved $\frac{x}{2} = \tan 38°$.

Try 3.21. Consider the following diagram and find approximate values for x and y:

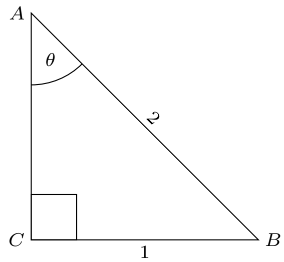


Try 3.22. Consider the following diagram and find approximate values for x and y:



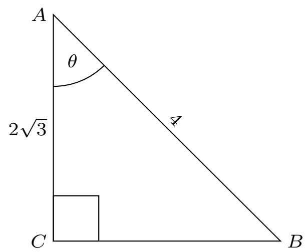
3.5 Finding angles

Example 3.6. Find θ:



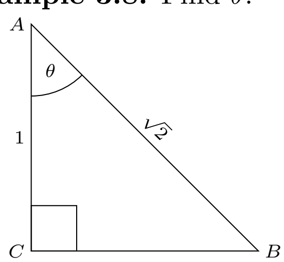
We recognize this triangle as a 30-60-90 triangle whose side opposite to A is 1, so $\theta = 30$.

Example 3.7. Find θ:



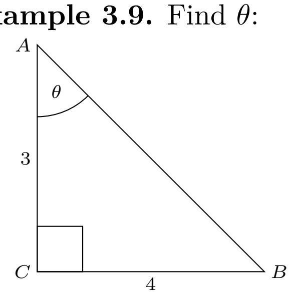
We can divide each side by the same number to get a similar triangle. In this case divide by 2. We then recognize the resulting similar triangle as a 30-60-90 triangle whose side adjacent to A has length $\sqrt{3}$, so $\theta = 30$.

Example 3.8. Find θ:



We recognize this as the 45-45-90 triangle, so $\theta = 45$.

Example 3.9. Find θ:



We don't recognize this triangle as a special triangle. But we can form a ratio. The sides we know are the opposite and the adjacent. So we learn that

$$\tan\theta = \frac{4}{3}.$$

Our calculator knows the answer to this equation $\theta = \tan^{-1}(\frac{4}{3})$, or $\arctan(\frac{4}{3})$, depending on your calculator. This is usually associated to the button tan on your calculator. We find that $\theta \approx 52.1°$.

Note. The sin and $\sin^{-1} = \arcsin$ share the same button on most calculators, just like x^2 and $\sqrt{x}$.

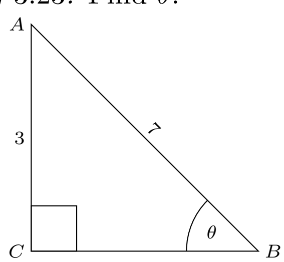
$$x^2 = \frac{1}{2} \implies x = \sqrt{\frac{1}{2}}$$

just like

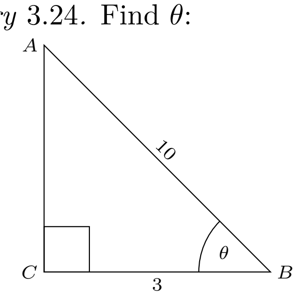
$$\sin x = \frac{1}{2} \implies x = \sin^{-1}(\frac{1}{2}).$$

Note. In the expression $\sin^{-1}\theta$, -1 is NOT an exponent. This, eventhough it may be confusing, is standard notation for what is called the inverse function to the sin function with a certain domain (i.e., it is not the multiplicitive inverse but is instead the compositional inverse). All that has to be remembered here is that $\sin\theta = a \implies \theta = \sin^{-1} a$. The same goes for the other trigonometric functions if we restrict ourselves to "solving triangles". More will be said about this in a later course.

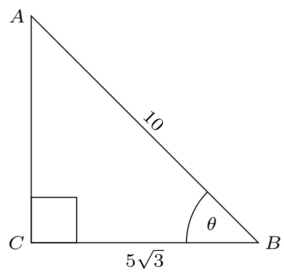
Try 3.23. Find θ:



Try 3.24. Find θ:



Try 3.25. Find θ:

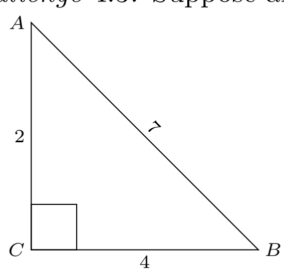


4 Applications

Try 4.1. Suppose you want to measure the height of a tree and the sun is casting a shadow of 15 feet. Suppose also that you are 6 ft tall and are casting a shadow or 4 feet. How tall is the tree?

Try 4.2. Suppose you want to find the height of a house that you have a picture that is taken that shows the house straight on. You know that the house is 40 ft wide. But on the photo, it is 3 inches wide and 8 in tall. How tall is the house in reality?

Challenge 4.3. Suppose an areal photograph of a triangular reservoir is the following

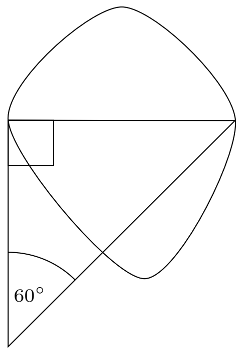


The reservoir is a half a mile on the shortest side. What is the area of the reservoir? What if the reservoir were a rectangle which were half a mile at the base and the photo showed base=$3in$ and length=$5in$, then what would the area be?

Note. The similarity really can be used for any corresponding lengths (they need not be part of a triangle).

Try 4.4. Suppose you are 100 ft from the ground (looking out a window) and you spot a fire. The angle of depression (see figure) is 40°. How far is the fire away from the base of the building you are in?

Try 4.5. Suppose you want to know the distance across a lake. You stand 50 paces off in the direction tangent to a point on the shore so that you have the following picture.



You know that the angle that you make if you point to ends of the shore is 60 degrees. What is the distance across the body of water?