Lines

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1 Introduction to linear equations in two variables and their solutions

Solving a linear equation like $2 - 3x = 7$ is easy. Its answer is a single number and we can easily write it down $x = -\frac{5}{3}$. So we understand what it means to be a solution in this case: A solution is a value for the variable that makes the statement true. And in this case we can write down all solutions explicitly since there is just one solution.

In this chapter we consider the equation with two variables. A solution is a value for each of the variables that makes this statement true.

Consider the equation $2x - 3y = 6$. In this case $(x, y) = (3, 0)$ by which we mean $x = 3$ and $y = 0$ is a solution since $2 \cdot 3 - 3 \cdot 0 = 6$. Whereas $(0, 3)$ is not.

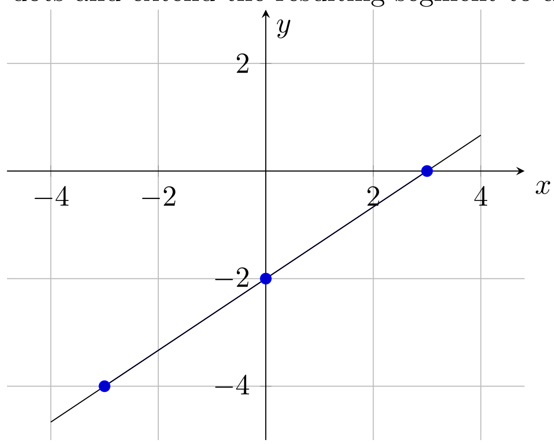
There are infinitely many solutions to an equation of the form $ax + by = c$, where a, b, and c are any numbers but a and b are not both zero. Since there are infinitely many solutions, they can not be written explicitly like the linear equation in one variable. We therefore draw a picture of the solutions.

In the form that the equation $2x-3y = 6$ appears, the solutions are most easily graphed by finding several solutions and connecting them with a curve (it turns out that this will be a straight line and that $ax + by = c$ is the form of the equation of a line). We say that the picture of the solutions of an equation is the **graph of the equation**.

So in this example we find several solutions by guessing, essentially. We look for, for example, a solution of the form $(0, y)$. Plugging this in to the equation will give us a linear equation for y which we can then solve: $2 \cdot 0 - 3y = 6 \implies -3y = 6 \implies y = -2$. So the point $(0, -2)$ is a solution. Now we choose, say, a point of the form $(-3, y)$ – we choose a multiple of 3 since the other two terms are divisible by 3! We then have $-6 - 3y = 6$ so that adding 6 on both sides of the equation gives $-3y = 12$ and so $y = -4$. Therefore, $(-3, -4)$ is also a solution. We see that this can be done for any value we may choose for the x-coordinate.

Try 1.1. Find three other solutions to this equation.

If we plot the points $(3,0),(0,-2)$, and $(-3,-4)$ on the coordinate plane and connect the dots and extend the resulting segment to a line, we get:



Try 1.2. Put your solutions on the above and make sure they also lie on the line.

Try 1.3. In the above fashion, graph $3x+2y=4$.

Challenge 1.4. As an equation with two variables, graph $x=7$.

Challenge 1.5. As an equation with two variables, graph $y=-3$.

It turns out that the equation for any line can be written in **standard form**: $ax+by=c$, where a,b, and c are some numbers.

2 The slope and slope-intercept form

If $ax+by=c$ and $b\neq 0$, then we can solve this equation for y. For example,

$$2x-3y=6 \implies -3y=-2x+6 \implies y=\frac{-2x+6}{-3} \implies y=\frac{2}{3}x-2.$$

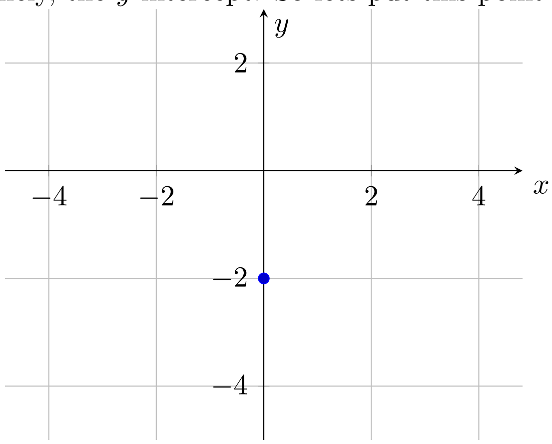
With this form $y=\frac{2}{3}x-2$, it is easy to find solutions since you can just plug in any value for x and evaluate the resulting right hand side to find the corresponding value for y.

Note also that if we increase the x value by 3, the y value increases by 2 since $\frac{2}{3}(x+3)-2=\frac{2}{3}x-2+2$. That is to say the graph is a line with slope $\frac{2}{3}=\frac{\text{rise}}{\text{run}}$.

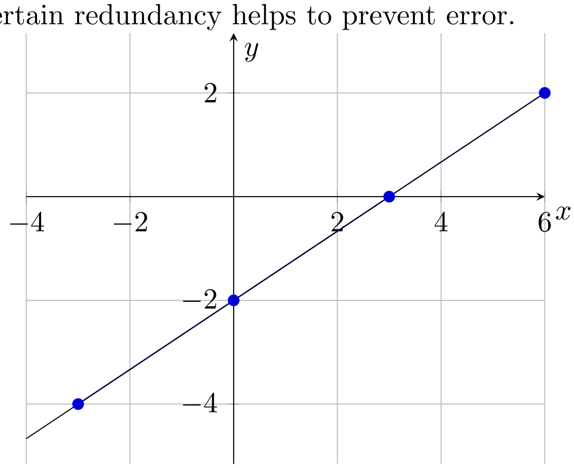
In general if you have a line written in the form $y=mx+b=\frac{\text{rise}}{\text{run}}x+b$, an you increase the x by the run, then the y increases the value by the rise. $m=\frac{\text{rise}}{\text{run}}$ is called the slope of the line. Also note that in this form it is easy to find the y-intercept, that is, the solution of the form $(0,y)$ since by replacing x with zero leaves you with $y=m\cdot 0+b=b$. So, from this form, the slope and the intercept are easily read off the equation (a clear correspondence of the information in the algebraic equation and the geometry of the line!). For this reason form $y=mx+b$ is called the **slope-intercept** form of the line.

Try 2.1. Write $3x + 2y = 4$ in the form $y = mx + b$, the slope-intercept form. Read off the slope and y-intercept of the line.

Going back to our example $y = \frac{2}{3}x - 2$, let us note that the slope is $\frac{2}{3}$ and the y-intercept is $(0, -2)$. With this information of the intercept and the slope we can graph the line. Note that two points determine a line. We already know one point on the line, namely, the y-intercept. So lets put this point on our graph:

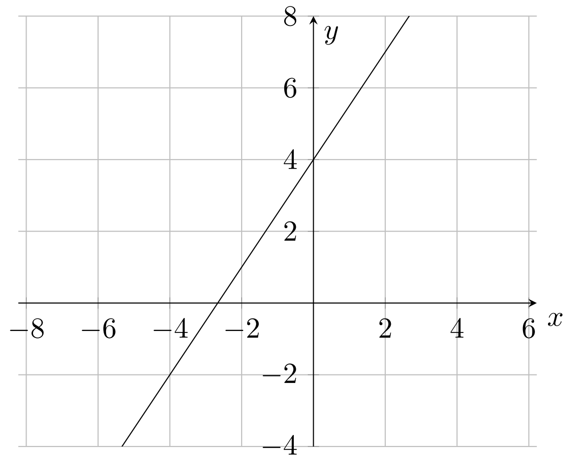


Now we know that if we increase the x value for a solution by the run (the denominator of the slope) then we increase the y coordinate by the rise (increasing by a negative number gives a decrease). In other words, let's take our point $(0, -2)$ and add the run$= 3$ to 0 and the rise$= 2$ to -2 to get the new solution $(3, 0)$. Doing the same to $(3, 0)$ gives us the new solution $(6, 2)$. Also, subtracting the run and the rise from the x and y coordinates of the solution also gives a new solution. So, $(0 - 3, -2 - 2) = (-3, -4)$ is also a solution. While we only need two of these to graph the line, the picture of the solutions to the equation, we use more here to illustrate and it is also useful in sketching the straight line. Besides, a certain redundancy helps to prevent error.



Try 2.2. Graph the line $3x + 2y = 4$ using the slope and y-intercept you found above. It should agree with the graph you just got (but don't look at the earlier graph, this is being used to check your answer!).

Try 2.3. Look at the graph below and identify the slope and the y-intercept. Use this to write down the equation of the line. To find the slope, identify two points on the line whose coordinates you know (on the grid) and ask what you have to add (may be negative) to the x and y-coordinates of one point to arrive at the other point.



3 a word about the slope

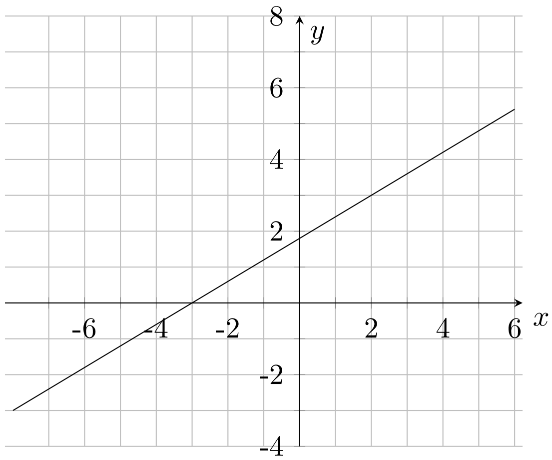
We will get the same slope no matter which two point on the line we deal with! (x_2, y_2) are on the line then $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Try 3.1. Why doesn't it matter which order the points are in?

Try 3.2. Find the slope of the line containing $(91004, -230)$ and $(1, 100)$. Why would this be difficult to do by looking at the picture?

4 point-slope form

In the previous section we studied the slope-intercept form of the line. However, sometimes this isn't so clear. For instance, in



the y-coordinate will have to be estimated to use the slope-interecept of the line. It is, however possible to proceed in a different way. Let us note from the graph that $(-3,0)$ is on the line as is $(2,3)$, and that the slope is $\frac{3}{5}$. So, using the formula for the slope, for any point (except $(-3,0)$) on the line

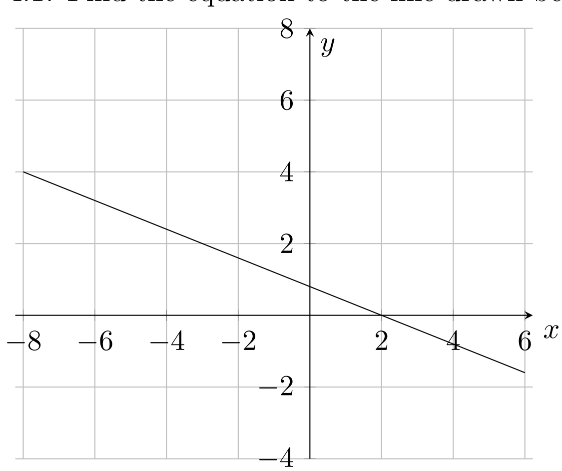
$$\frac{y-0}{x-(-3)} = \frac{3}{5}$$

or

$$y - 0 = \frac{3}{5}(x - (-3)).$$

In general, $y - y_1 = m(x - x_1)$ where (x_1, y_1) is on the line and m is the slope. This form is called the **point-slope** form.

Try 4.1. Find the equation to the line drawn below



Try 4.2. Write the equation you found above in slope-intercept form.

5 Summary of formulas

One needs two pieces of information to determine the equation for a line:

ingredients	the name of the form	formula
plot points	standard form	$ax + by = c$
a point and the y-intercept	the slope-intercept form	$y = mx + b$
a point (x_1, y_1) and the slope	point-slope form	$y - y_1 = m(x - x_1)$.
two points		Use these two points to determine the slope then use the point-slope form.

6 Applications

Try 6.1. Recall that the freezing temperature of water is $0°C$ or $32°F$ where as the boiling point of water is $100°C$ or $412°F$. The Fahrenheit scale is linearly related to the Celsius scale, that is, we can write a linear equation relating Celcius degrees (C) to Fahrenheit degrees (F). So the above information tells us that two points are on this line: $(0, 32)$ and $(100, 412)$. Use this to write an equation that relates the two temperature scales. Use it to convert $20°C$ to Fahrenheit and to convert $100°F$ into Celsius.

Try 6.2. What is the slope of a line perpendicular to $y = mx$? Try some examples (different values for m) and see if you can figure out what is going on.

Try 6.3. The part of the line connecting two points (a, b) and (c, d) is called the **line segment** connecting (a, b) and (c, d). What is the midpoint of this line segment? Keep this important result in mind.

Example 6.1. We will use the results in the previous two problems to find the formula for the perpendicular bisector of a line segment. Please complete these before reading this example. Also, try to do it on your own first before looking at the solution.

Find an equation for the perpendicular bisector for the line segment connecting $(1, 1)$ and $(-3, 2)$ (the line perpendicular to the segment which passes through the midpoint).

We need to find enough information to determine line which could be two points, or a point and a slope. We only have one point which we know it goes through (the midpoint of the given segment) and we have information about the slope (it is perpendicular to the given segment). So we will aim at finding these two pieces of information explicitly and then using the point-slope form for the line. We have

$$\text{the slope of the given segment} = \frac{2-1}{-3-1} = -\frac{1}{4}$$

so,

$$m = \text{the slope of the perpendicular bisector} = 4.$$

The perpendicular bisector also passes through the midpoint of the given line segment, which is

$$\left(\frac{1+-3}{2}, \frac{1+2}{2}\right) = \left(-1, \frac{3}{2}\right).$$

So, using the point slope form of the line we see that the line for the perpendicular bisector has the equation

$$y - \frac{3}{2} = 4(x - (-1)).$$

This can be written as

$$y = 4x + \frac{11}{2}$$

or

$$-8x + 2y = 11.$$

Check that the above forms are equivalent.

Try 6.4. Find an equation for the perpendicular bisector for the line segment joining $(-1, 4)$ and $(-2, -3)$.

Try 6.5. A sprinkler must be placed so that it is equally distant from two plants. If the plants are located at $(0, 0)$ and $(1, 4)$, what is the equation of line it must it be placed on?

Challenge 6.6. If there are three plants located at $(0, 0)$, $(1, 4)$, and $(1, 6)$. Where must a sprinkler be placed so that it can cover the plants with the least water pressure?