Fractions

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1 Fractions

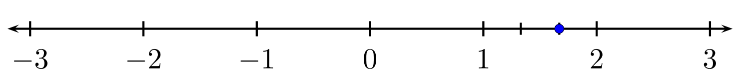
Fractions are used regularly in the English language. For example, a dollar is divided into 10 dimes and time is often given in terms of quarters of hours. Fractions like this are examples of what are called rational numbers.

Rational numbers are numbers of the form p/q, where p and q are integers. For example, $\frac{-3}{4}$, $\frac{14}{7}$, -12, and $\frac{24}{15}$ are rational numbers, and π, $\sqrt{2}$, and $\frac{2}{\sqrt{3}}$ are not rational numbers.

Question 1.1. Is $\frac{-3}{\sqrt{4}}$ rational?

Each rational number has an infinite number of representations, but only one place on the number line. One way to see where to put a rational number on a number line is to first reduce the fraction and then write the fraction as a mixed number.

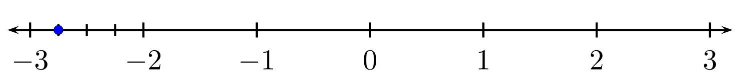
Example 1.1. Locate $25/15$ on the number line. Note that $\frac{25}{15} = \frac{5 \cdot 5}{5 \cdot 3} = \frac{5}{3}$ (remember that to reduce a fraction first write the numerator and denominator in factored form) And writing this in the form of a mixed number, $\frac{5}{3} = 1\frac{2}{3}$. So we see that the number $\frac{25}{15}$ is between 1 and 2. We divide this interval up into 3 pieces (the denominator) and count out 2 of them (the numerator).



Try 1.2. Find $-\frac{72}{12}$ on the number line following the example above.

Try 1.3. Find $\frac{125}{-30}$ on the number line following the example above.

Try 1.4. Write the following number as a rational number



Try 1.5. Which number is greater? $\frac{15}{21}$ or $\frac{16}{24}$? Draw a picture and explain.

Challenge 1.6. Can you think of a number to divide the intervals into that will be convenient to plot each of the two points above?

Challenge 1.7. Above we suggested to first reduce the fraction, then to turn it into a mixed number. This can be done in the other order. Consider $\frac{189}{147}$. First write this as a mixed number, then reduce. Do the same problem by reducing first, then writing as a mixed number. Are there advantages to doing it one way over the other? What might you do to reduce $\frac{147}{189}$?

2 Operations with fractions

2.1 Multiplying and dividing

Fractions can be multiplied and divided. We will give an example as a reminder (Even more examples may be found in Arithmetic|Algebra).

$$\frac{2}{15} \cdot \frac{3}{14} = \frac{2 \cdot 3}{15 \cdot 14} = \frac{2 \cdot 3}{5 \cdot 3 \cdot 2 \cdot 7} = \frac{1}{35}$$

Note that we did not immediately multiply the numbers in the numerator and the denominator but waited until the last step. Why? This will be important later!

$$\frac{2}{21} \div \frac{3}{14} = \frac{\frac{2}{21}}{\frac{3}{14}} = \frac{\frac{2}{21} \cdot \frac{14}{3}}{\frac{3}{14} \cdot \frac{14}{3}} = \frac{2}{21} \cdot \frac{14}{3} = \frac{2 \cdot 2 \cdot 7}{7 \cdot 3 \cdot 3} = \frac{4}{9}.$$

Note that some people may write:

$$\frac{2}{21} \div \frac{3}{14} = \frac{2}{21} \cdot \frac{14}{3} = \frac{2 \cdot 2 \cdot 7}{7 \cdot 3 \cdot 3} = \frac{4}{9}.$$

2.2 Adding and subtracting

Just like with counting numbers, we can add and subtract fractions. The issue is to be comparing like items. How did you compare the two fractions $\frac{15}{21}$ or $\frac{16}{24}$ above? One way to make these like is to find a common denominator. We see $\frac{2}{5}$ and $\frac{3}{4}$ are difficult to compare. But since 5 and 4 both divide 20, we can write these two fractions in an easily comparable form.

$$\frac{2}{5} = \frac{2 \cdot 4}{5 \cdot 4} = \frac{8}{20}$$

Do the same for the other fraction so that the denominator is also 20.

$$\frac{3}{4}$$

Now adding these two equivalent fractions is easy. Do it here!

Example 2.1. Subtract

$$\frac{3}{20} - \frac{4}{35}.$$

We see that 20 and 35 are not directly comparable. However they both divide $20 \cdot 35$. However it is convenient for the purpose of later simplification to find the smallest number that is divisible by both 20 and 35. To see what this is we can factor $20 = 2 \cdot 2 \cdot 5$ and $35 = 5 \cdot 7$. We see that the 5 is a common factor between the numbers. 20 and 35 both divide $2 \cdot 2 \cdot 5 \cdot 7$. This is the smallest number that will do (go ahead and try to find a smaller one). So,

$$\frac{3}{20} - \frac{4}{35} = \frac{3 \cdot 7}{20 \cdot 7} - \frac{4 \cdot 2 \cdot 2}{35 \cdot 2 \cdot 2} = \frac{21 - 16}{2 \cdot 2 \cdot 5 \cdot 7} = \frac{5}{2 \cdot 2 \cdot 5 \cdot 7} = \frac{1}{2 \cdot 2 \cdot 7} = \frac{1}{28}.$$

2.3 Applications

Example 2.2. Write the following scenerio in terms of fractions (where one dollar is 1) and compute the indicated quantity (make sure this coincides with what you think the answer should be).

If you have 2 dollars and 3 quarters and a friend gives you 1 dollar and 3 dimes, how much money do you then have?

Example 2.3. It takes Jamel 2 hours and 45 minutes to do a task. The boss wants it to take $\frac{2}{3}$ of the time. How much time should Jamel aim for? (Try to estimate the answer before computing so that you have a value to compare with your answer).

Example 2.4. If a rectangular room has a width of $5\frac{1}{2}$ m and a length of $7\frac{2}{3}$ m, what is the perimeter of the room? What is the area?

Note. For more practice on rational numbers consult the book Arithmetic|Algebra.

Next up: We will now consider expressions of the form p/q where the p and q are polynomials. Keep in mind the examples done in this section since the same ideas are used!