The Symmetric Group and Fair Division: Does Knowledge Matter?

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Amuse-bouche

Let $T = \{3, 4, 5\} \subset \{1, \ldots, 6\}$ and consider two operations on T:

- $C(T) = \{1, \ldots, 6\} \setminus \{3, 4, 5\} = \{1, 2, 6\}$ (complement),
- $F(T) = \{7-5, 7-4, 7-3\} = \{2, 3, 4\}$ ("flip").

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Although $C(T) \neq F(T)$, their sums are equal:

$$1 + 2 + 6 = 9 = 2 + 3 + 4.$$

Is this true for every three-element subset of $\{1, \ldots, 6\}$?

Mathematics Inspired by Social Sciences

- Networks: six degrees of separation, strength of weak ties, small world graphs. *Huge networks can be closely knit.*
- Game theory: modeling interactions of decision makers whose choice effect each other; Prisoners' Dilemma, Chicken. *Greed need not lead to the corporate good.*
- Voting theory: Arrow, Borda, Condorcet; Shapley-Shubik, Banzhaf power indices. *Not every vote is equal.*
- Fair division: Cake-cutting of very heterogenous cakes, moving knives. *People value things differently.*

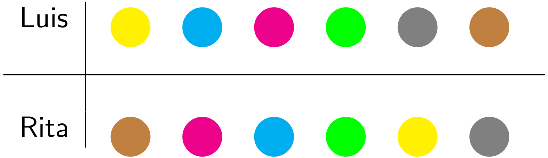
Fair Division of Indivisible Items

Suppose Luis & Rita have to split a collection of six plates. They will take turns selecting plates with Luis going first, so that each will end up with three plates.

What makes this interesting is that they may have different preferences.

Fair Division of Indivisible Items

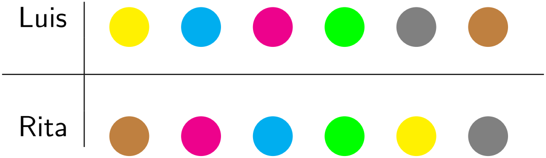
Let their preferences be these lists, starting with their favorites.



- What selection procedure do they use to get the best possible collection of three plates?
- How does depend on whether they know each others' preferences?

Naïve versus Naïve

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.



Luis	Rita

Naïve versus Naïve turn 1

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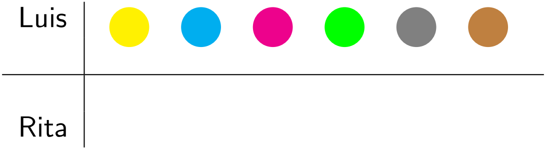
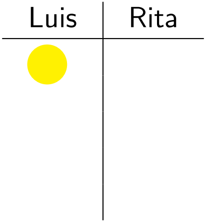
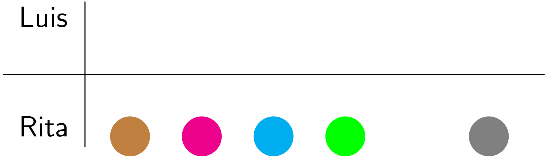
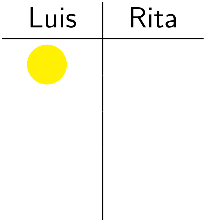




Naïve versus Naïve turn 2

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.





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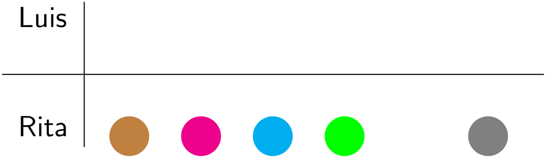
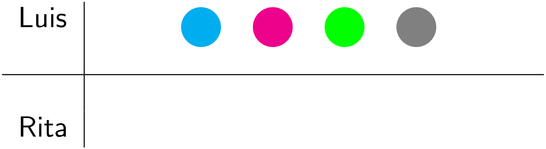


Naïve versus Naïve turn 3

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.



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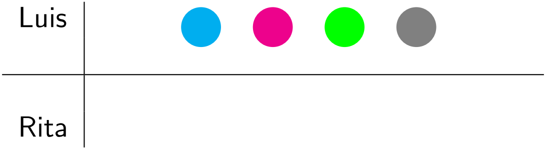
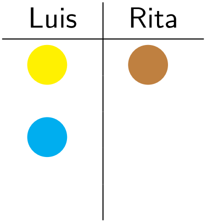
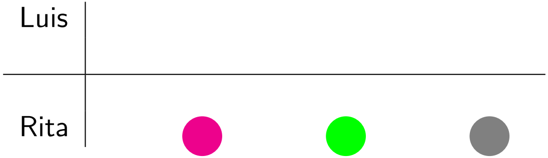
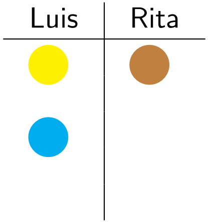




Naïve versus Naïve turn 4

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.





Naïve versus Naïve turn 4

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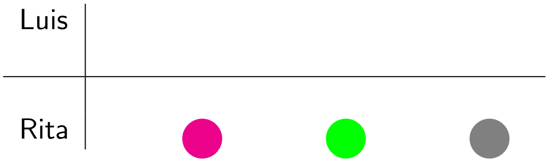
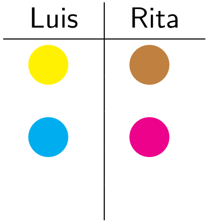
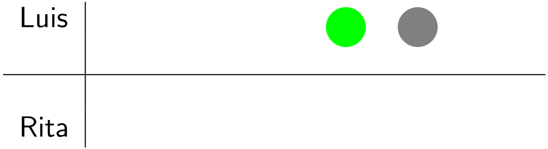
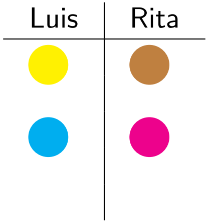




Naïve versus Naïve turn 5

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.





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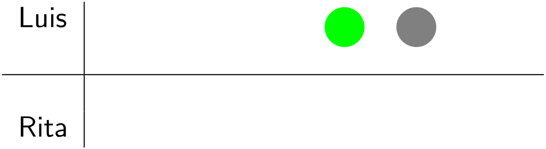


Naïve versus Naïve turn 6

Suppose they do not know each other's preference. Then, at each turn, the player chooses his or her favorite of the available plates.

Strategic versus Strategic

Now suppose they do know each other's preferences. How can they use this knowledge?

Luis 

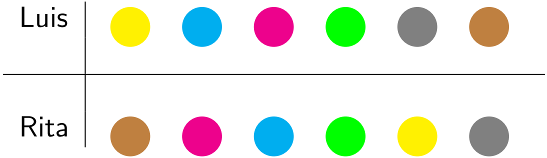
Rita will end up with the brown plate; might as well take it last and try to do better with her earlier turns.

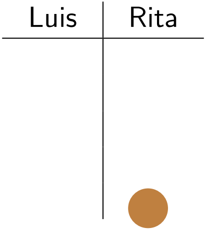
Kohler and Chandrasekaharan, *Operations Research* 1971

Repeating this "bottom-up" approach gives the optimal outcome for both players.

Strategic versus Strategic turn 6

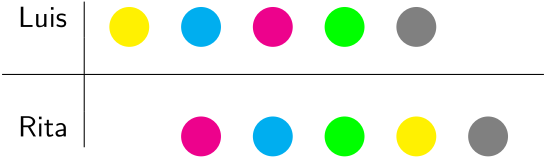
Suppose they do know each other's preference. Work backwards.

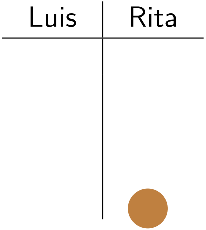




Strategic versus Strategic turn 5

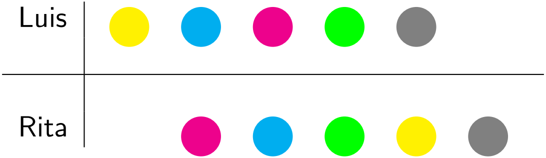
Suppose they do know each other's preference. Work backwards.

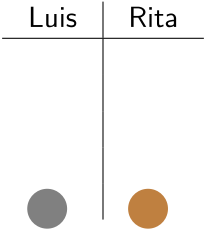




Strategic versus Strategic turn 5

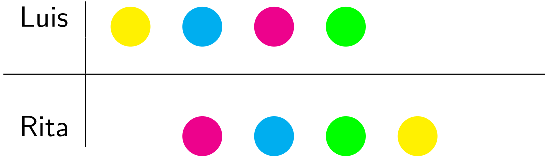
Suppose they do know each other's preference. Work backwards.

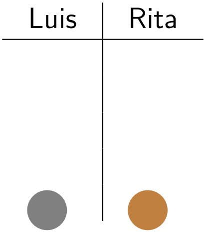




Strategic versus Strategic turn 4

Suppose they do know each other's preference. Work backwards.





Strategic versus Strategic turn 4

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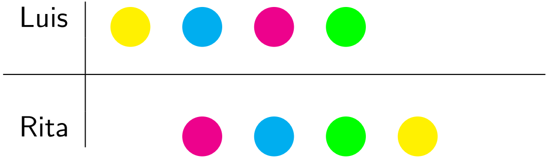
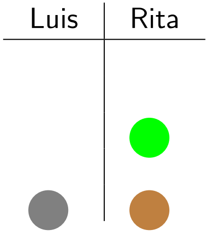
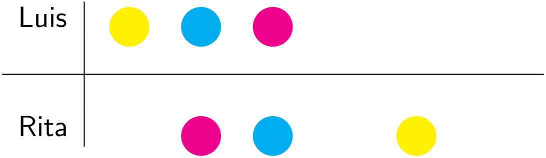
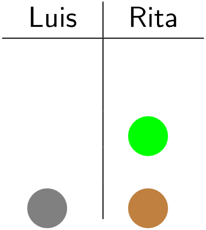




Strategic versus Strategic turn 3

Suppose they do know each other's preference. Work backwards.





Strategic versus Strategic turn 3

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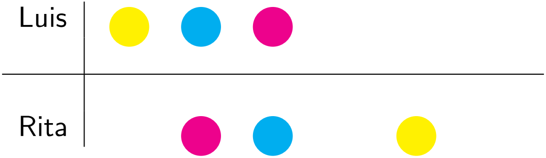
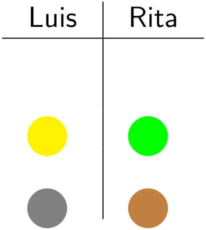
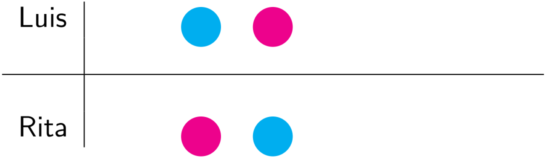
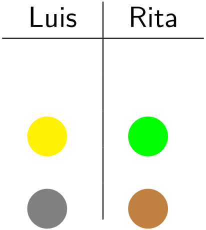




Strategic versus Strategic turn 2

Suppose they do know each other's preference. Work backwards.





Strategic versus Strategic turn 2

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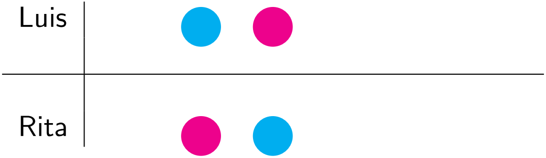
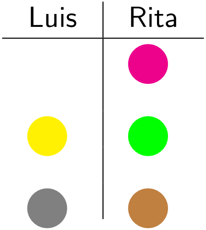
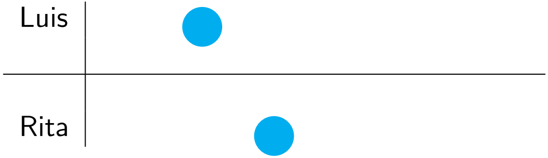
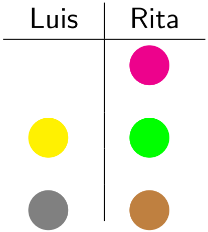




Strategic versus Strategic turn 1

Suppose they do know each other's preference. Work backwards.





Strategic versus Strategic turn 1

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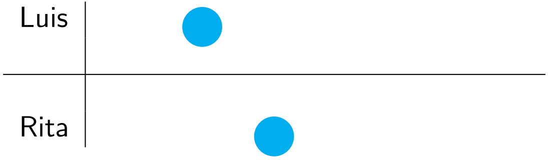
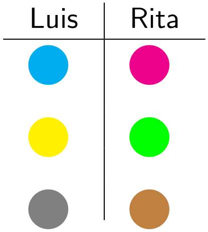
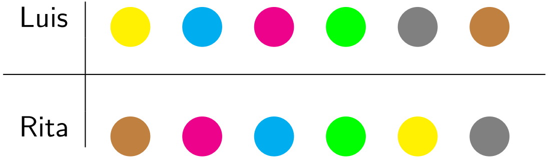
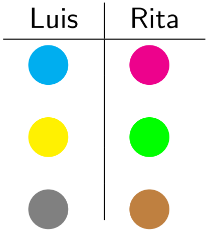




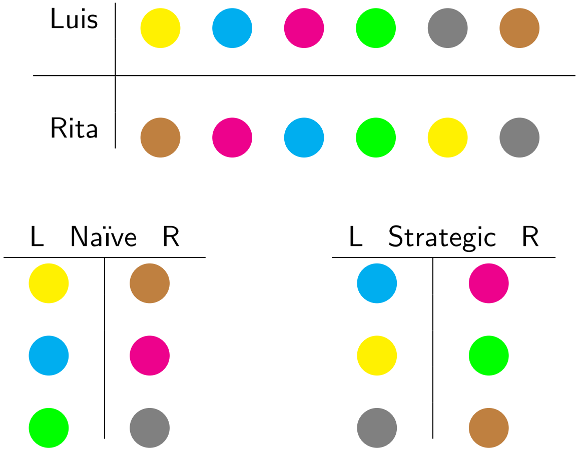
Strategic versus Strategic from the top

Suppose they do know each other's preference. Work backwards.



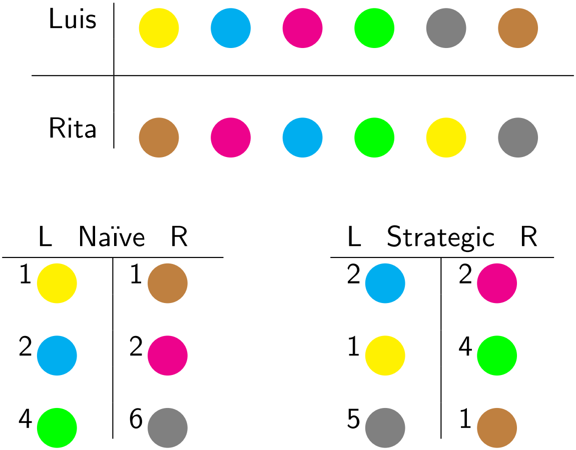


Comparison



Here, Rita does better with open knowledge, Luis worse.

Comparison



Here, Rita does better with open knowledge, Luis worse.
(Lower sums are better, Rita $1 + 2 + 6 = 9 > 1 + 2 + 4 = 7$.)

Questions

- Does Rita always do better with open knowledge?
- If not always, does she do better on average with open knowledge?
- What are the extreme cases?
- How many possible outcomes does each player have?
- Minimizing preference sum makes sense, but are there other good measures? What about other motivations?
- Colors are nice, but what known mathematical structures can we bring to bear?

Permutations

Rather than deal with $2n$ colors, name the objects $1, \ldots, 2n$ according to Luis' preferences.

Then Rita's preferences

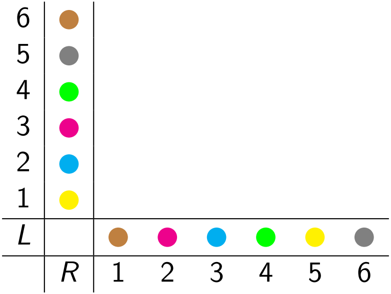
correspond to the permutation $\pi = (6, 3, 2, 4, 1, 5)$.

Cons: Puts results in terms of Luis' preferences.

Pros: Lots of potential tools.

Permutation Diagrams

There is one visual tool for permutations that addresses the labeling disparity and will be the vehicle for our major proof.



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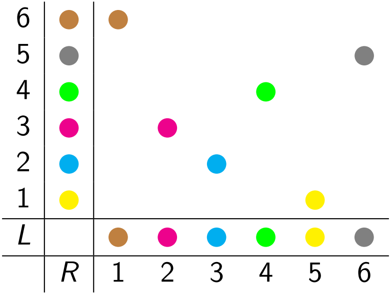

Points at $(1,6),(2,3),(3,2),(4,4),(5,1),(6,5)$.

Permutation Diagrams

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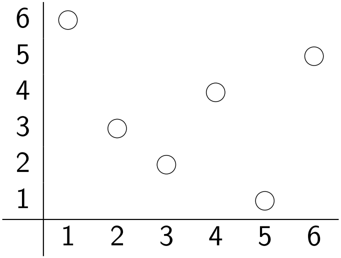


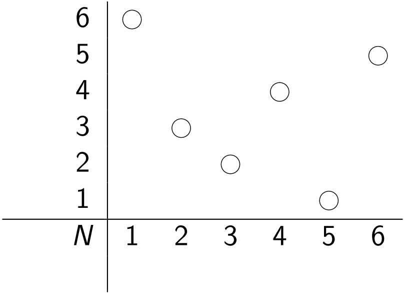
Permutation diagram for $\pi = (6, 3, 2, 4, 1, 5)$.

Algorithms Revisited: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

L	R

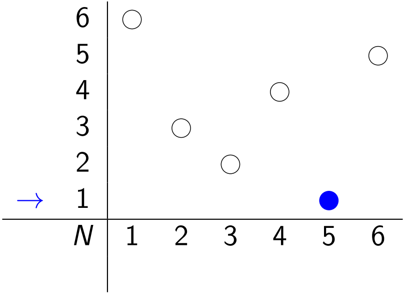


Algorithms Revisited: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

L	R
1	

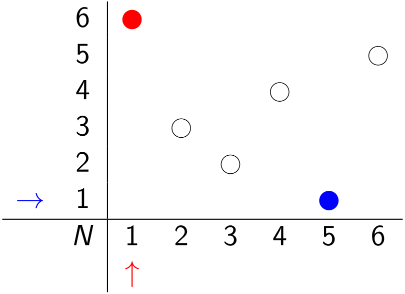


Algorithms Revisited: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $\quad 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, \quad 5$

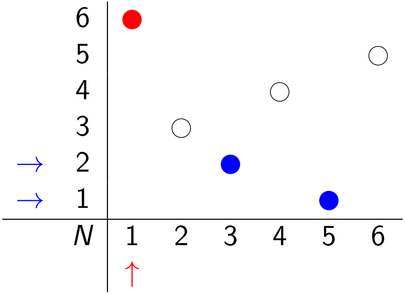
L	R
1	6



Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $2, 3, 4, 5$
Rita: $3, 2, 4, \quad 5$

L	R
1	6
2	

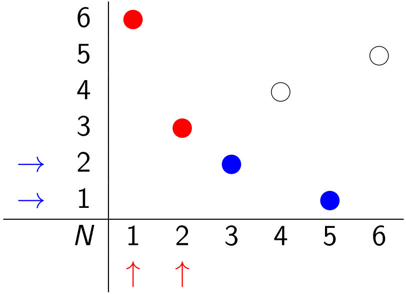


Algorithms Revisited: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $3, 4, 5$
Rita: $3, \quad 4, \quad 5$

L	R
1	6
2	3

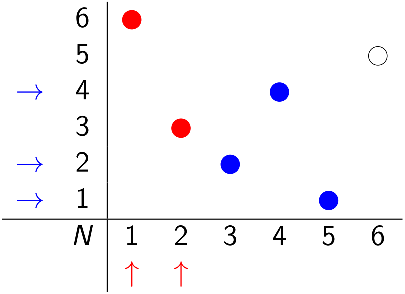


Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $4, 5$

Rita: $4, \quad 5$

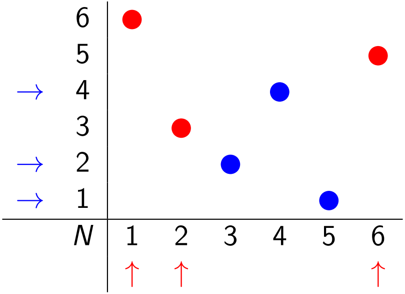
L	R
1	6
2	3
4	



Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: 5
Rita: 5

L	R
1	6
2	3
4	5
7	14

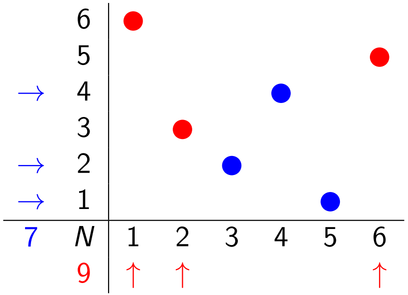


Algorithms Revisited: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

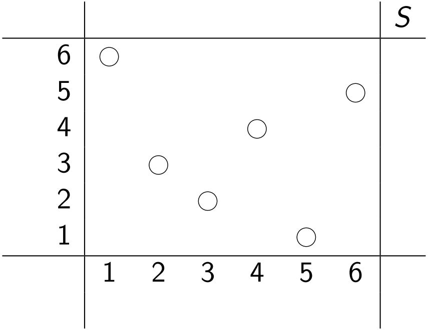
L	R	
1	6	$= \pi(1)$
2	3	$= \pi(2)$
4	5	$= \pi(6)$
7		9



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

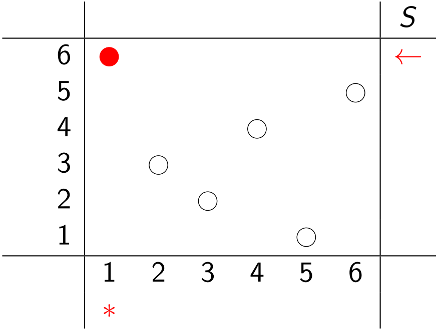
L	R



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

L	R
	$6 = \pi(1)$

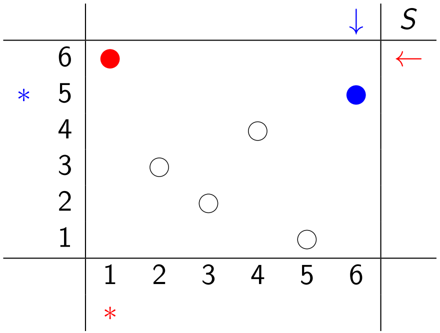


Algorithms Revisited: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: 1, 2, 3, 4, 5
Rita: 3, 2, 4, 1, 5

L	R
5	$6 = \pi(1)$

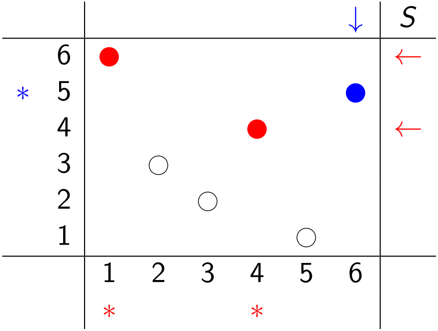


Algorithms Revisited: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4$
Rita: $3, 2, 4, 1$

L	R
	$4 = \pi(4)$
5	$6 = \pi(1)$

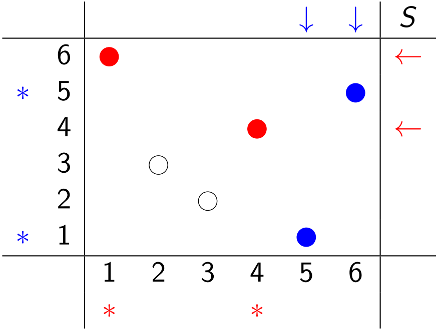


Algorithms Revisited: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3$
Rita: $3, 2, \quad 1$

L	R
1	$4 = \pi(4)$
5	$6 = \pi(1)$

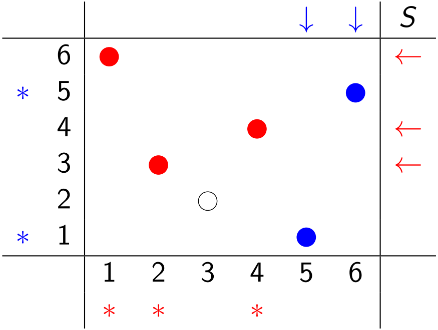


Algorithms Revisited: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $2, 3$
Rita: $3, 2$

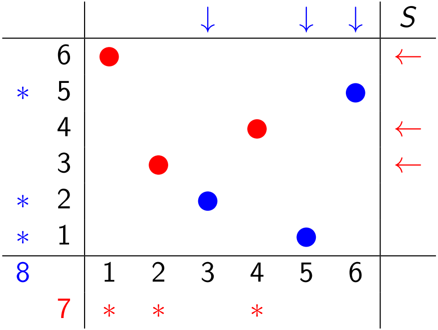
L	R
	$3 = \pi(2)$
1	$4 = \pi(4)$
5	$6 = \pi(1)$



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

Luis: $1, 2, 3, 4, 5, 6$
Rita: $6, 3, 2, 4, 1, 5$

L	R
2	$3 = \pi(2)$
1	$4 = \pi(4)$
5	$6 = \pi(1)$
8	7



An Old Chestnut

"Add up the numbers from 1 to 100 and be quiet until you finish!"

$$S = 1 + 2 + \cdots + 99 + 100$$

An Old Chestnut

"Add up the numbers from 1 to 100 and be quiet until you finish!"

$$
\begin{array}{rrrrrr}
S = & 1\,+ & 2\,+ & \cdots\,+ & 99\,+ & 100 \\
S = & 100\,+ & 99\,+ & \cdots\,+ & 2\,+ & 1 \\
\hline
2S = & 101\,+ & 101\,+ & \cdots\,+ & 101\,+ & 101
\end{array}
$$

$$S = \frac{100 \cdot 101}{2} = 5050$$

An Old Chestnut

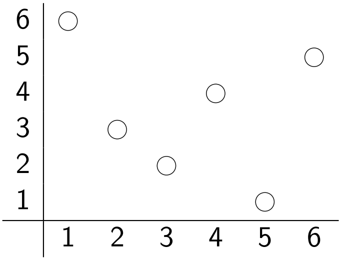
"Add up the numbers from 1 to n and be quiet until you finish!"

$$\begin{array}{rrrrrr} S = & 1\, + & 2\, + & \cdots\, + & n-1\, + & n \\ S = & n\, + & n-1\, + & \cdots\, + & 2\, + & 1 \\ \hline 2S = & n+1\, + & n+1\, + & \cdots\, + & n+1\, + & n+1 \end{array}$$

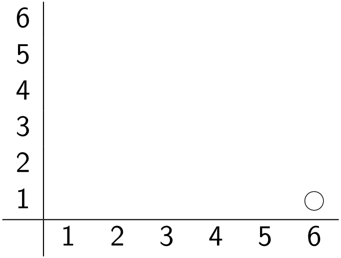
$$S = \frac{n(n+1)}{2}$$

Permutation Diagram Operations

Find the inverse of $\pi = (6, 3, 2, 4, 1, 5)$:



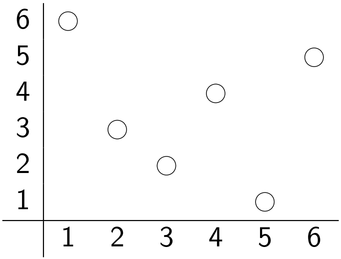
π includes $(1, 6)$,

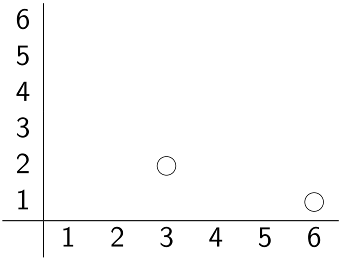


π^{-1} includes $(6, 1)$.

Permutation Diagram Operations

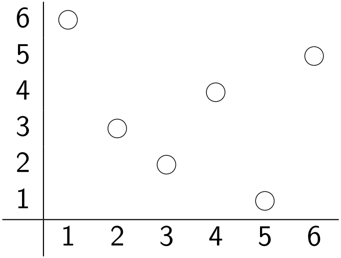
Find the inverse of $\pi = (6, 3, 2, 4, 1, 5)$:

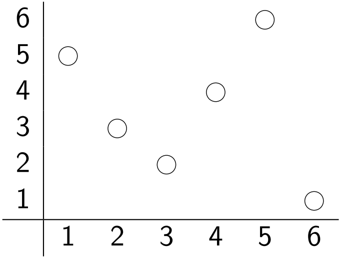




π includes $(2, 3)$, π^{-1} includes $(3, 2)$.

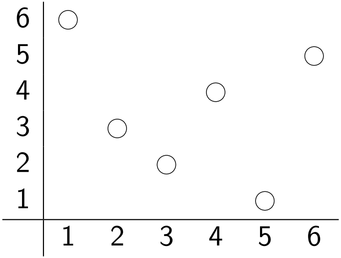
Find the inverse of $\pi = (6, 3, 2, 4, 1, 5)$:

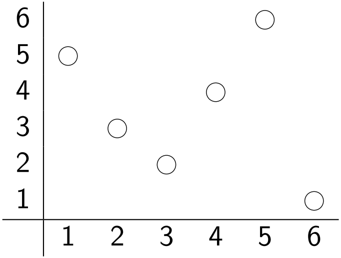




Permutation Diagram Operations

Find the inverse of $\pi = (6, 3, 2, 4, 1, 5)$:

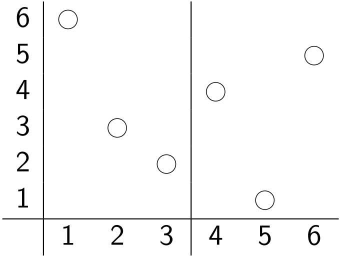


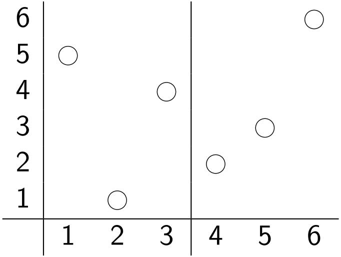


The inverse is $\pi^{-1} = (5, 3, 2, 4, 6, 1)$; reflection across diagonal.

Permutation Diagram Operations

Horizontal reflection:

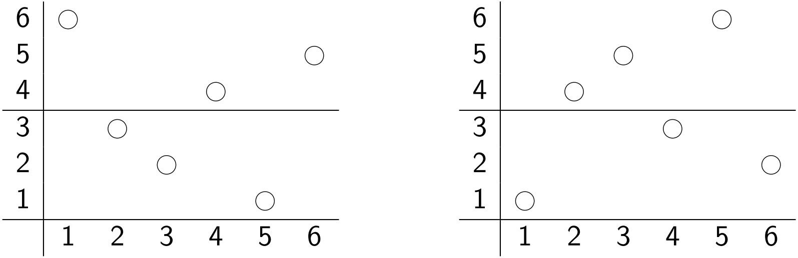




$H(6, 3, 2, 4, 1, 5) = (5, 1, 4, 2, 3, 6)$, "reversal"

Permutation Diagram Operations

Vertical reflection:



$V(6,3,2,4,1,5) = (1,4,5,3,6,2)$, “$7 - \pi(i)$”

Permutation Diagram Operations

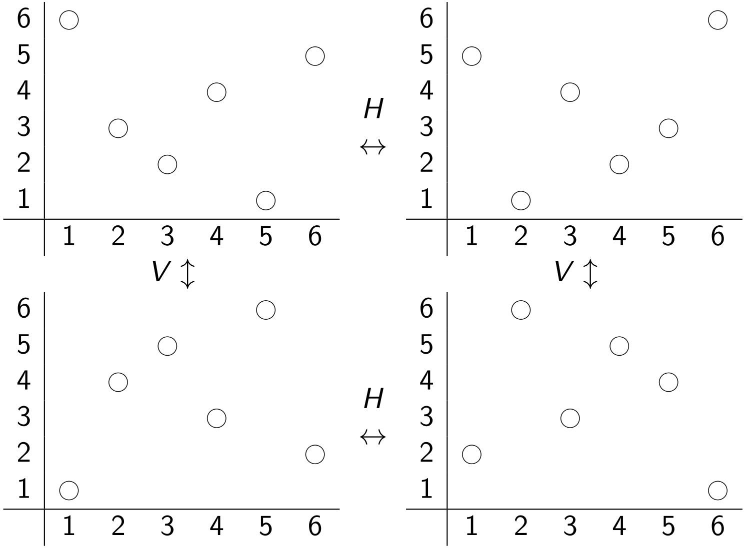
Egge, *Annals of Combinatorics* 2007

Interaction of inverse, H, V, and all symmetries of the square are used to study various "pattern avoiding permutations."

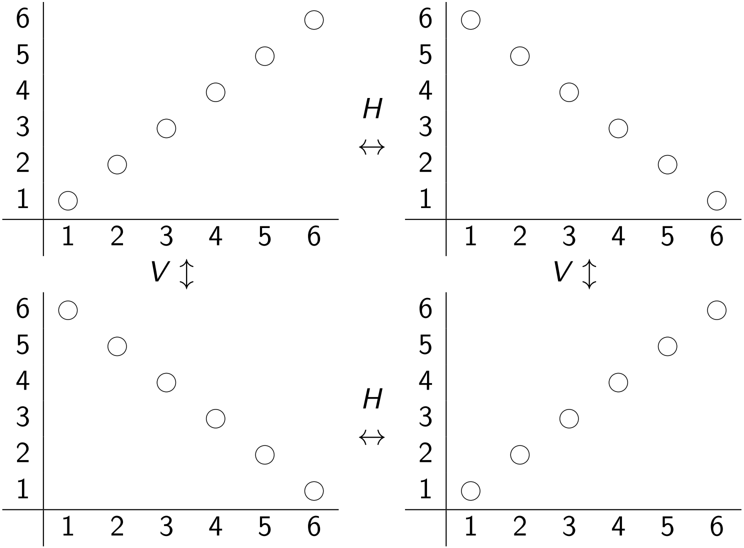
Some basic facts:

- $H(\pi) \neq \pi$ for all $\pi \in S_n$,
- $V(\pi) \neq \pi$ for all $\pi \in S_n$,
- $H \circ V = V \circ H$, a 180° rotation of the permutation diagram.

$H \circ V = V \circ H$ example

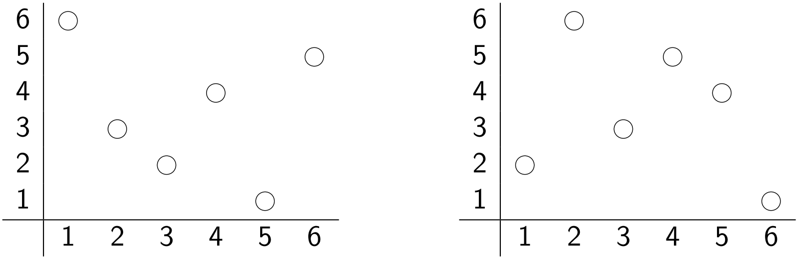


$(H \circ V)(\pi) = \pi$ example



Connection

Idea: Relate the selection results of π and $(H \circ V)(\pi)$.



$(H \circ V)(6,3,2,4,1,5) = (2,6,3,5,4,1)$, 180° rotation.

Table Comparison

$\pi = (6, 3, 2, 4, 1, 5)$ || $HV(\pi) = (2, 6, 3, 5, 4, 1)$

Naïve (π)

1	$6 = \pi(1)$
2	$3 = \pi(2)$
4	$5 = \pi(6)$
7	9

Strategic (π)

2	$3 = \pi(2)$
1	$4 = \pi(4)$
5	$6 = \pi(1)$
8	7

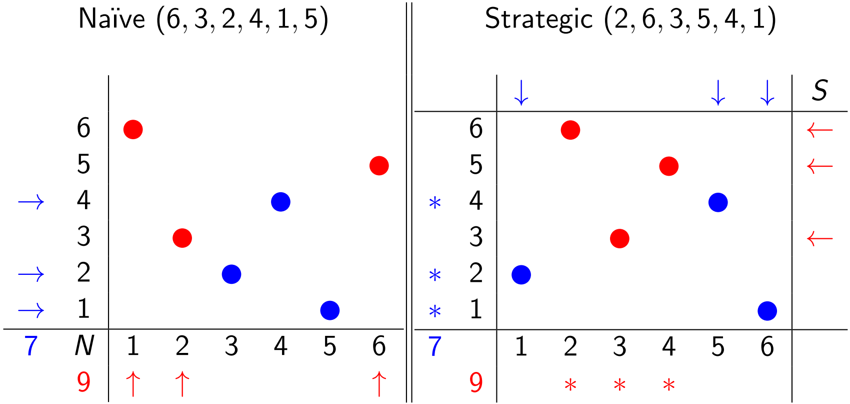
Naïve ($HV(\pi)$)

1	$2 = \pi(1)$
3	$6 = \pi(2)$
4	$5 = \pi(4)$
8	7

Strategic ($HV(\pi)$)

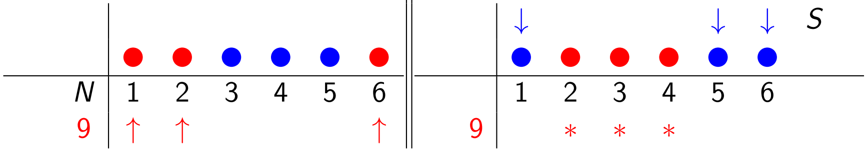
2	$3 = \pi(3)$
4	$5 = \pi(4)$
1	$6 = \pi(2)$
7	9

Diagram Comparison



Project down to x-axis for Rita results.

Naïve $(6, 3, 2, 4, 1, 5)$ ‖ Strategic $(2, 6, 3, 5, 4, 1)$

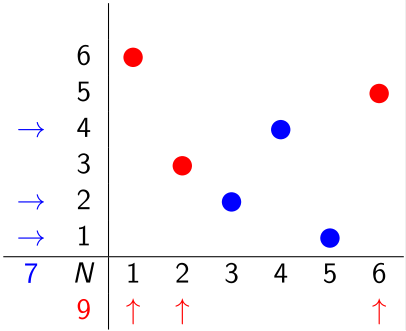


Rita's result for naïve π is $\{1, 2, 6\} = C(\{3, 4, 5\})$.
Rita's result for strategic $HV(\pi)$ is $\{2, 3, 4\} = F(\{3, 4, 5\})$.

In the horizontal reflection, the x-coordinates of the dots are reversed. (Vertical reflection has no effect in this projection.)
In the mutually strategic algorithm, blue works along the x-axis (backwards), so colors are swapped.

Diagram Comparison

Naïve $(6, 3, 2, 4, 1, 5)$



Strategic $(2, 6, 3, 5, 4, 1)$

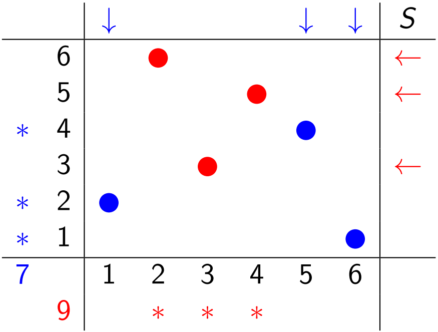
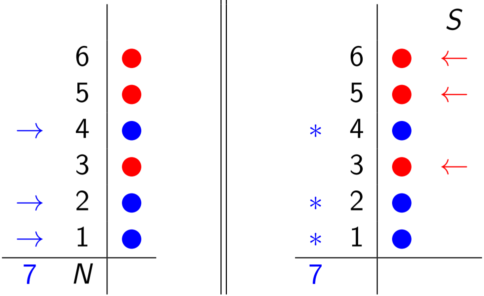


Diagram Comparison, Luis

Naïve $(6, 3, 2, 4, 1, 5)$ || Strategic $(2, 6, 3, 5, 4, 1)$



$$C(\{3, 5, 6\}) = \{1, 2, 4\} = \{7 - 6, 7 - 5, 7 - 3\} = F(\{3, 5, 6\})$$

Comparison Theorem

Write $P_N(\pi)$ for a player's n objects resulting from the mutually naïve strategy when Rita's preferences are given by $\pi \in S_{2n}$.

Comparison Theorem

For each $\pi \in S_{2n}$, there is $T \in \{1, \ldots, 2n\}$ with $|T| = n$ such that

$$P_N(\pi) = C(T) \text{ and } P_S((H \circ V)(\pi)) = F(T).$$

Proof Idea: The diagram for $(H \circ V)(\pi)$ is the 180° rotation of the π diagram. The strategic procedure from the upper right of $(H \circ V)(\pi)$ is equivalent to the naïve procedure from the lower left with player roles reversed. So

$$P_S((H \circ V)(\pi)) = (F \circ C)(P_N(\pi)).$$

Setting $T = C(P_N(\pi))$ satisfies the theorem statement.

Set Theory Lemma

Lemma

For $T \subset \{1, \ldots, 2n\}$ with $|T| = n$, the sums $\Sigma C(T) = \Sigma F(T)$.

Example: Let $T = \{3, 4, 5\}$. Since $T \cup C(T) = \{1, \ldots, 6\}$, we have

$$\begin{aligned}
\Sigma C(T) &= \Sigma\{1, \ldots, 6\} - \Sigma T \\
&= \frac{6 \cdot 7}{2} - \Sigma T \\
&= 3 \cdot 7 - (3 + 4 + 5) \\
&= \ (7-3) + (7-4) + (7-5) \ = \ \Sigma F(T).
\end{aligned}$$

Set Theory Lemma

Lemma

For $T \subset \{1, \ldots, 2n\}$ with $|T| = n$, the sums $\Sigma C(T) = \Sigma F(T)$.

Proof: Let $T = \{t_1, \ldots, t_n\}$, so $F(T) =$ " $2n + 1 - T$ " $= \{2n + 1 - t_1, \ldots, 2n + 1 - t_n\}$. Since $T \cup C(T) = \{1, \ldots, 2n\}$,

$$\begin{aligned}
\Sigma C(T) &= \Sigma\{1, \ldots, 2n\} - \Sigma T \\
&= \frac{2n(2n+1)}{2} - \Sigma T \\
&= n(2n+1) - \sum_{i=1}^{n} t_i \\
&= \sum_{i=1}^{n} 2n + 1 - t_i \;=\; \Sigma F(T).
\end{aligned}$$

Big Answer

Main Theorem

With each $\pi \in S_{2n}$ equally likely as Rita's preference, measuring outcomes by preference sums, on average neither mutual strategy offers an advantage to either player.

So does knowledge matter? In this case, NO!

Proof Idea: Over the set $\{\pi \mid \pi \neq (H \circ V)(\pi)\}$, each pair π and $(H \circ V)(\pi)$ together has no net effect on the strategy choice since $P_N(\pi) = C(T)$ and $P_S(H \circ V)(\pi) = F(T)$ for some T, and $\Sigma C(T) = \Sigma F(T)$. Conclude $\Sigma P_N(\pi) = \Sigma P_S(H \circ V)(\pi)$.

For $\{\pi \mid \pi = (H \circ V)(\pi)\}$, Comparison Theorem still holds and shows $P_N(\pi) = P_S(\pi)$, clearly no advantage.

The rank of a permutation is a non-negative integer measuring "how different" it is from the identity $(1, \ldots, k)$. Ranks range from 0 to $\binom{k}{2}$.

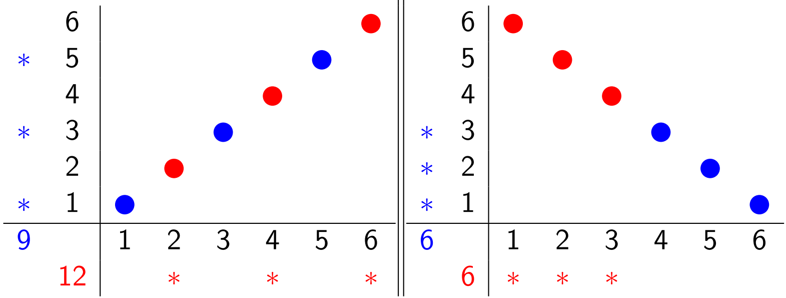
Stronger Theorem

Fix a rank r. With each $\pi \in S_{2n}$ of rank r equally likely as Rita's preference, measuring outcomes by preference sums, on average neither mutual strategy offers an advantage to either player.

Follows from showing that $\text{rank}((H \circ V)(\pi)) = \text{rank}(\pi)$.

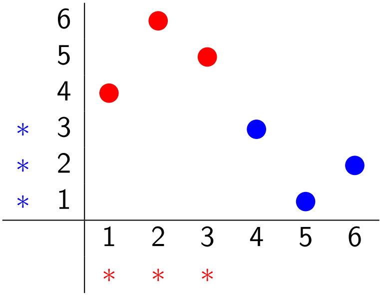
Algorithms on Extreme Cases

(1, 2, 3, 4, 5, 6), rank 1 || (6, 5, 4, 3, 2, 1), rank 15



(Note that Rita's worst is worse than Luis'.)

$(4, 6, 5, 3, 1, 2)$



Only $(1, \ldots, 2n)$ for Rita's preference gives both players their worst possible outcomes in both algorithms.

There are $(n!)^2$ preferences that give both players their best possible outcomes (top n choices) in both algorithms.

With no correlation of preferences, outcomes are better for both than you might initially guess.

BH, Taking Turns, *COLLEGE MATHEMATICS JOURNAL* **41**(4) 289–297 (September 2010).

article ideas, interest in refereeing: `cmj@maa.org`

So Much More

- Catalan numbers via 2-column Young tableaux, intervals in the permutation posets under the left & right weak Bruhat orders, . . .
- Alternating turns gives the first player (Luis) a large advantage. What are the fairest orders?
  - LRRL
  - LRLRRL
  - LLRRRLRL
  - LRRLLRLRRL
  - LLRRRRLLRLLR
  - LLLRRRRLLRRRLL

  Length 12 and 14 orders and further work by William Keith and Chelsea Grindatti (student).
- Motivation other than greed? Spite, altruism, "the common good" or "envy-free" divisions . . .