

# Review of Calculus for MAT 2680 – Differential Equations

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## 1 Derivatives

### 1.1 Definition and Notation

Let  $y = f(x)$ .

(1) The derivative is defined to be  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

(2) All of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$

(3) All of the following are equivalent notations for derivative evaluated at  $x = a$ .

$$f'(a) = y'(a) = y'|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{dy}{dx} \right|_{x=a}$$

(4) All of the following are equivalent notations for the second derivative

$$(f'(x))' = f''(x) = y'' = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2} = \frac{d}{dx^2}(f(x))$$

### 1.2 Interpretation of the Derivative

Let  $y = f(x)$ .

(1)  $m = f'(a)$  is the slope of the tangent line to  $y = f(x)$  at  $x = a$  and the equation of the tangent line at  $x = a$  is given by  $y = f(a) + f'(a)(x - a)$ .

(2)  $f'(x)$  is the instantaneous rate of the change of  $f(x)$ .

(3) If  $f(x)$  is the position of an object at time  $x$ , then  $f'(x)$  is the velocity of the object and  $|f'(x)|$  is the speed.

(4) If  $f'(x) > 0$  for all  $x$  in an interval  $I$ , then  $f(x)$  is increasing on the interval  $I$ .

(5) If  $f'(x) < 0$  for all  $x$  in an interval  $I$ , then  $f(x)$  is decreasing on the interval  $I$ .

(6) If  $f'(x) = 0$  for all  $x$  in an open interval  $I$ , then  $f(x)$  is constant on the interval  $I$ .

### 1.3 Basic Properties and Formulas

Let  $f(x)$  and  $g(x)$  be differentiable functions (the derivative exists) and  $c$  and  $n$  be constants.

- (1)  $(c \cdot f(x))' = c \cdot f'(x)$
- (2)  $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- (3) Product rule:  $(f \cdot g)' = f' \cdot g + f \cdot g'$
- (4) Quotient rule:  $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$
- (5) Chain rule:  $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

### 1.4 Common Derivatives

- |  |   |
|--|---|
| (1) (constant)' = 0  | (8) $(\sec x)' = \sec x \tan x$                 |
| (2) Power rule: $(x^n)' = n \cdot x^{n-1}$                                       | (9) $(\cot x)' = -\csc^2 x$                     |
| (3) $(e^x)' = e^x$ . In general, $(a^x)' = a^x \cdot \ln(a)$ .                   | (10) $(\csc x)' = -\csc x \cot x$               |
| (4) $(\ln(x))' = \frac{1}{x}$ . In general, $(\log_a x)' = \frac{1}{x \ln(a)}$ . | (11) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$  |
| (5) $(\sin x)' = \cos x$   | (12) $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$ |
| (6) $(\cos x)' = -\sin x$  | (13) $(\tan^{-1} x)' = \frac{1}{1+x^2}$         |
| (7) $(\tan x)' = \sec^2 x$   |   |

## 2 Integrals

### 2.1 Definitions

- (1) Indefinite Integral:  $\int f(x) dx = F(x) + C$ , where  $F(x)$  is an anti-derivative of  $f(x)$  and  $C$  is a constant.
- (2) Definite Integral:  $\int_a^b f(x) dx$  is defined to be the signed area of the region bounded by  $y = f(x)$ ,  $x = a$ ,  $x = b$  and  $x$ -axis.
- (3) Fundamental Theorem of Calculus: If  $f(x)$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a),$$

where  $F(x)$  is an anti-derivative of  $f(x)$ .

## 2.2 Basic Properties and Formulas

Let  $f(x)$  and  $g(x)$  be differentiable functions (the derivative exists) and  $c$  and  $n$  be constants.

$$(1) \int c \cdot f(x) dx = c \cdot \int f(x) dx$$

$$(2) \int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$$

$$(3) \int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$$

$$(4) \int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

## 2.3 Common Integrals (compare with Common Derivatives)

$$(1) \int k dx = kx + C$$

$$(2) \text{ Power rule } (n \neq -1): \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$(3) \text{ Power rule } (n = -1): \int \frac{1}{x} dx = \ln|x| + C$$

$$(4) \int \frac{1}{ax+b} dx = \frac{\ln|ax+b|}{a} + C$$

$$(5) \int e^x dx = e^x + C; \int e^{kx} dx = \frac{1}{k}e^{kx} + C.$$

$$(6) \int \sin x dx = -\cos x + C;$$
$$\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + C$$

$$(7) \int \cos x dx = \sin x + C;$$
$$\int \cos(kx) dx = \frac{1}{k} \sin(kx) + C$$

$$(8) \int \sec^2 x dx = \tan x + C$$

$$(9) \int \sec x \tan x dx = \sec x + C$$

$$(10) \int \csc^2 x dx = -\cot x + C$$

$$(11) \int \csc x \cot x dx = -\csc x + C$$

$$(12) \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C$$

$$(13) \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$

## 2.4 Standard Integration Techniques

• The following types of problems can be solved by integration by substitution:  $\int f(x) dx = \int g(u) du$ , where  $u = u(x)$  and  $du = u'(x) dx$ .

$$(1) \int x^2 \cos(x^3) dx$$

Set  $u = x^3$ . Then  $du = (x^3)' dx = 3x^2 dx$  and

$$\int x^2 \cos(x^3) dx = \int \cos u \cdot \frac{1}{3} du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3) + C.$$

$$(2) \int x^2 \sqrt{x^3 + 5} dx$$

Set  $u = x^3 + 5$ . Then  $du = (x^3 + 5)' dx = 3x^2 dx$  and

$$\int x^2 \sqrt{x^3 + 5} dx = \int \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} + C = \frac{2}{9} (x^3 + 5)^{3/2} + C.$$

- The following types of problems can be solved by integration by parts  $\int u dv = uv - \int v du$ .

$$(1) \int (3x + 1)e^{2x} dx$$

Set  $u = 3x + 1$ ,  $dv = e^{2x} dx$ . Then  $du = (3x + 1)' dx = 3 dx$  and  $v = \int e^{2x} dx = \frac{1}{2}e^{2x}$ .

$$\begin{aligned} \int (3x + 1)e^{2x} dx &= \int (3x + 1) \cdot e^{2x} dx = \int u \cdot dv = uv - \int v du \\ &= (3x + 1) \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 3 dx \\ &= \frac{1}{2}(3x + 1)e^{2x} - \frac{3}{2} \int e^{2x} dx = \frac{1}{2}(3x + 1)e^{2x} - \frac{3}{4}e^{2x} + C \\ &= \frac{3}{2}xe^{2x} - \frac{1}{4}e^{2x} + C. \end{aligned}$$

$$(2) \int x^3 \ln x dx$$

Set  $u = \ln x$ ,  $dv = x^3 dx$ . Then  $du = (\ln x)' dx = \frac{1}{x} dx$  and  $v = \int x^3 dx = \frac{1}{4}x^4$ .

$$\begin{aligned} \int x^3 \ln x dx &= \int (\ln x) \cdot x^3 dx = \int u \cdot dv = uv - \int v du \\ &= (\ln x) \cdot \frac{1}{4}x^4 - \int \frac{1}{4}x^4 \cdot \frac{1}{x} dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

$$(3) \int (2x + 1) \sin(x) dx$$

Set  $u = 2x + 1$ ,  $dv = \sin x dx$ . Then  $du = (2x + 1)' dx = 2 dx$  and  $v = \int \sin x dx = -\cos x$ .

$$\begin{aligned} \int (2x + 1) \sin(x) dx &= \int (2x + 1) \cdot \sin x dx = \int u \cdot dv = uv - \int v du \\ &= (2x + 1) \cdot (-\cos x) - \int (-\cos x) \cdot 2 dx \\ &= -(2x + 1) \cdot (\cos x) + 2 \int \cos x dx \\ &= -(2x + 1) \cdot (\cos x) + 2 \sin x + C \end{aligned}$$

- The integration of rational functions can be solved by partial fraction decomposition.

$$(1) \int \frac{3x + 5}{(x - 1)(x + 3)} dx$$

Find constants  $A, B$  so that  $\frac{3x + 5}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}$ . Multiply the common denominator  $(x - 1)(x + 3)$ :

$$3 \cdot x + 5 = A(x + 3) + B(x - 1) = (A + B) \cdot x + (3A - B)$$

Compare coefficients:

$$\begin{cases} A + B = 3 \\ 3A - B = 5 \end{cases}$$

Solve the system of equation:  $A = 2, B = 1$ . Thus

$$\begin{aligned} \int \frac{3x + 5}{(x - 1)(x + 3)} dx &= \int \frac{A}{x - 1} + \frac{B}{x + 3} dx = \int \frac{2}{x - 1} + \frac{1}{x + 3} dx \\ &= 2 \ln |x - 1| + \ln |x + 3| + C \end{aligned}$$

$$(2) \int \frac{7x^2 + 13x}{(x - 1)(x^2 + 4)} dx$$

Find constants  $A, B, C$  so that

$$\frac{7x^2 + 13x}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}.$$

The rest of the solution is similar to (1). For your reference, the answer is posted below.

$$\int \frac{7x^2 + 13x}{(x - 1)(x^2 + 4)} dx = 4 \ln |x - 1| + \frac{3}{2} \ln(x^2 + 4) + 8 \tan^{-1} \left( \frac{x}{2} \right) + C$$