Ordinary Differential Equations Notes **July 2020**

Bernoulli Equations

Dr. S. Singh

We will study Bernoulli equations of the shape

$$y' + p(x)y = f(x)y^n$$

where n is any real number except for 0 or 1.

Observation 1: For $n = 0$, the equation is of the form $y' + p(x)y = f(x)$ and it stands out as a linear first order equation.

Observation 2: For $n = 1$, the equation is of the form $y' + p(x)y = f(x)y$; we can rearrange this equation to get $y' = (f(x) - p(x))y$, or $\frac{dy}{y} = (f(x) - p(x))dx$ which yields to the separation of variables technique. It is also useful to note that this is a linear first order equation.

We will use the three step approach outlined below to solve $y' + p(x)y = f(x)y^n$.

We begin with the **first step** by solving the homogenous part of the equation $y' + p(x)y = 0$. Separating variables will give $\frac{dy}{y} = -p(x)dx$ or $\int \frac{dy}{y} = -\int p(x)dx$ and integrating gives $\ln|y| = -\int p(x)dx$. After solving for y, we get $|y| = e^{-\int p(x)dx}$. Pick one of the solutions, say $y_1 = e^{-\int p(x)dx}$.

In the **second step**, we let $y = uy_1$, where $u = u(x)$ and differentiate y to get $y' = u'y_1 + uy_1'$. Now substitute for y' and y into Bernoulli's equation to obtain an equation in terms of u and x that can be solved by separation of variables. Once you have solved for u do not forget to find y by multiplying u and y_1. Recall that $y = uy_1$.

A **third step** is required if we are given an initial condition such as $y(x_0) = y_0$. It is recommended that you find the value of the constant with the expression that you obtained for $u(x)$ to avoid extraneous solutions. From $y(x_0) = y_0$ or the point (x_0, y_0) and $y(x) = u(x)y_1(x)$, it follows that $y(x_0) = u(x_0)y_1(x_0)$ or $u(x_0) = y_0/y_1(x_0)$.

Recommended review from calculus: Integration by parts. You may access the text book CalculusVolume2 for an extensive review of integration. Before going to the next page, use integration by parts to evaluate the integral $\int xe^{-x/2}dx$ which we will use in our example. The expected result is given below.

$$\int xe^{-x/2}dy = -2xe^{-x/2} - 4e^{-x/2} + c.$$

Example 1 Solve the initial value equation

$$y' - y = xy^{1/2},\ \ y(0) = 6.$$

We start by observing that $y' - y = xy^{1/2}$ is a Bernoulli equation of the form $y' + p(x)y = f(x)y^n$, where $p(x) = -1$, $f(x) = x$ and $n = 1/2$.

We now proceed with the **first step** by solving the homogenous equation $y' - y = 0$. We can use the separation of variables technique to solve $y' - y = 0$ directly.

Instead, we will use the outline above and pick $y_1 = e^{-\int p(x)dx} = e^{\int 1dx} = e^{x+c} = e^c e^x = c_1 e^x$. Choose some $c_1 \neq 0$. Since the Bernoulli equation contains the term $y^{1/2}$ or equivalently the $\sqrt{y}$, we must choose $c_1 > 0$ and for convenience we will let $c_1 = 1$. We now have a solution, namely $y_1 = e^x$.

Proceeding to the **second step**, we form $y = uy_1$, and substitute for y_1 get $y = ue^x$. Taking its first derivative, gives us $y' = u'e^x + ue^x$. We will now substitute for y and y' into the equation $y' - y = xy^{1/2}$, to transform it into a separable equation.

$y' - y = xy^{1/2}$ becomes $u'e^x + ue^x - ue^x = x(ue^x)^{1/2}$, and on simplifying, we get $u'e^x = xu^{1/2}e^{x/2}$. We can now separate the variables and simplify to get $u^{-1/2}du = xe^{-x/2}dx$; integrating both sides of $\int u^{-1/2}du = \int xe^{-x/2}dx$, gives us

$$2\sqrt{u} = -2xe^{-x/2} - 4e^{-x/2} + c.$$

We will now use the initial condition, $y(0) = 6$ or more precisely $u(0) = 6$. **Why**? Notice that $u(x) = y(x)e^x$, and substituting $x = 0$ gives us $u(0) = y(0)e^0 = 6$.

To find the value of c, substitute for $x = 0$ and $u = 6$ into the equation $2\sqrt{u} = -2xe^{-x/2} - 4e^{-x/2} + c$, we see that $2\sqrt{6} = -2(0)e^0 - 4e^0 + c$ or $2\sqrt{6} = -4 + c$ and we get that

$$c = 4 + 2\sqrt{6}.$$

We now have the equation

$$2\sqrt{u} = -2xe^{-x/2} - 4e^{-x/2} + 4 + 2\sqrt{6}$$

We will divide by 2 and square both sides of the equation to solve for u to obtain

$$u = (-xe^{-x/2} - 2e^{-x/2} + 2 + \sqrt{6})^2.$$

Since $y = uy_1$, we get that

$$y = e^x(-xe^{-x/2} - 2e^{-x/2} + 2 + \sqrt{6})^2,$$

notice that $e^x = (e^{x/2})^2$, and it follows that

$$\begin{aligned}
y &= e^x[-xe^{-x/2} - 2e^{-x/2} + 2 + \sqrt{6}]^2 \\
&= [e^{x/2})^2(-xe^{-x/2} - 2e^{-x/2} + 2 + \sqrt{6}]^2 \\
&= [e^{x/2}(-xe^{-x/2} - 2e^{-x/2} + 2 + \sqrt{6})]^2, \text{now distribute and simplify to get} \\
&= [-x - 2 + (2 + \sqrt{6})e^{x/2}]^2 \\
&= [(2 + \sqrt{6})e^{x/2} - x - 2]^2
\end{aligned}$$

From above we see that

$$y = [(2+\sqrt{6})e^{x/2} - x - 2]^2.$$

Further resources and comments:
There are other approaches that can be used to solve these equations. Additional problems can be found in `TrenchText` and detailed solutions are provided to the even problems in `StudentSolutionsManual`.