Math 2675: Calculus III

Sample Questions 2: Dot Product, Cross Product, Planes **Professor:** Caner Koca

1. If $\vec{v}$ and $\vec{w}$ make an angle of $60°$, and $\vec{v} \cdot \vec{w} = 2$ and $\|\vec{v}\| = 4$, find the length of $\vec{w}$.
   Solution: We know $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$. Hence, $2 = 4\|\vec{w}\| \cos(30°)$. Thus, $2 = 4\|\vec{w}\|\frac{1}{2}$. Therefore, $\|\vec{w}\| = 1$.

2. Find the cross product of $\langle 1, 2, 3\rangle$ and $\langle 2, 1, -3\rangle$.
   Solution: We know:
   $$\vec{v} \times \vec{w} = \Big\langle v_2w_3 - w_2v_3 \quad , \quad -(v_1w_3 - w_1v_3) \quad , \quad v_1w_2 - w_1v_2 \Big\rangle$$
   . Plug in the values:
   $$\vec{v} \times \vec{w} = \Big\langle 2 \cdot (-3) - 1 \cdot 2, -(1 \cdot (-3) - 2 \cdot 3), 1 \cdot 1 - 2 \cdot 2 \Big\rangle = \langle -8, 5, -3\rangle$$

3. Find the sine of the angle betwee the two vectors in the previous question.
   Solution: We know $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$. Note that
   $$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \qquad \|\vec{w}\| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14} \qquad \|\vec{v} \times \vec{w}\| = \sqrt{(-8)^2 + 5^2 + (-3)^2} = \sqrt{98} = 7\sqrt{2}$$
   Hence: $7\sqrt{2} = \sqrt{14} \cdot \sqrt{14} \cdot \sin\theta$. From here: $\sin\theta = \frac{7\sqrt{2}}{14} = \frac{\sqrt{2}}{2}$.

4. Find the ANGLE between the two vectors above.
   Solution: We know $\sin\theta = \frac{\sqrt{2}}{2}$. Therefore, $\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right)$. This can be accepted as a final answer, but actually we know more: the angle $45°$ has $\sin(45°) = \frac{\sqrt{2}}{2}$. Therefore $\theta = 45° = \frac{\pi}{4}$.

5. Find the equation of the plane through the origin and having normal vector $\langle 2, 3, -5\rangle$.
   Answer: $2(x - 0) + 3(y - 0) - 5(z - 0) = 0$, or after simplification $2x + 3y - 5z = 0$.