Linear Algebra
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Topic: (Reduced) Row Echelon Form, Pivot Positions, Systems of
Linear Equations and Linear Independence.

Idea: Pivot Positions of a matrix tell us about the information it represents.

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A matrix can represent the following:

- + a system of linear equations
  - + Coefficient matrix ($A\mathbf{x}=\mathbf{b}$)
  - + Augmented matrix
- + a list of column vectors
- + a linear transformation ($T_A(\mathbf{v})=A\mathbf{v}$)
- + etc.

Notice that we have already connected a list of vectors to a system of linear equations by the notion of linear independence.

That is, the columns of A are linearly dependent iff the equation $A\mathbf{x}=\mathbf{0}$, has a solution where $\mathbf{x}$ is not zero (a non-trivial solution). Otherwise the columns of A are linearly independent.

We will now look at row operations on matrices. Our initial goal is to solve a system of linear equations, but we will end up learning about linear independence as well.

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EXAMPLE (AUGMENTED MATRIX)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

THERE IS ONLY ONE SOLUTION SINCE $x_3 = 2$ AND ONE CAN BACK-SUBSTITUTE FOR UNIQUE VALUES OF x_2 AND x_1.

WHAT ABOUT THIS MATRIX?

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

THERE IS NO SOLUTION SINCE $0 \neq 4$

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These are both examples of matrices in row echelon form.
For the definition look at section 1.2 of the book (page 13).

The locations of the leading entries of a matrix in row echelon form are called pivot positions.

EXAMPLE

$$\begin{bmatrix} 5 & 3 & 6 & -7 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 PIVOT POSITIONS.

If a matrix, A, can be produced by a sequence of elementary row operations (see sect 1.1) on a matrix B, that is, by multiplying B on the left by a product of elementary matrices (sect 2.2), then we say A is row equivalent to B. (A~B)

The pivot positions of a matrix, A, are the locations in a matrix, B, row equivalent to A which hold the lead coefficients of B.

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EXAMPLE

$$A = \begin{bmatrix} 0 & 3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

PIVOT POSITIONS (SEE NEXT PAGE)

$E_{SWAP(1,4)}$

$$\sim_{SWAP} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot A$$

ROW 1 + ROW 2

$$\sim_{REPLACE} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} E_{SWAP} \cdot A$$

$$\sim_{REPLACE} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

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$$\sim_{\text{SCALE}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$-5 \cdot \text{Row}2 + \text{Row}$

$$\sim_{\text{REPLACE}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\sim_{\text{SWAP}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\text{REPLACE}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOT POSITION

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NOTICE THAT IF A WERE AN AUGMENTED MATRIX THEN IT WOULD REPRESENT:

$$\begin{cases} -3x_2 - 6x_3 + 4x_4 = 9 \\ -1x_1 - 2x_2 - 1x_3 + 3x_4 = 1 \\ -2x_1 - 3x_2 + 3x_4 = -1 \\ x_1 + 4x_2 + 5x_3 - 9x_4 = -7 \end{cases}$$

NOTICE FURTHER BY THESE OPERATIONS THIS SYSTEM IS EQUIVALENT TO

$$\begin{cases} x_1 + 4x_2 + 5x_3 - 9x_4 = -7 \\ x_2 + 2x_3 - 3x_4 = -3 \\ -5x_4 = 0 \end{cases}$$

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THE VARIABLES CORRESPONDING TO THE PIVOT COLUMN (COLUMNS WITH PIVOT POSITIONS) ARE CALLED BASIC VARIABLES. THE OTHERS ARE FREE. HERE x_3 IS FREE.

FACTS ABOUT PIVOT POSITIONS

A: AUGMETED MATRIX OF A SYSTEM OF LINEAR EQUATIONS

THE SYSTEM HAS NO SOLUTION IFF THE AUGMENTED COLUMN IS A PIVOT COLUMN.

ANY SOLUTION IS UNIQUE IFF EVERY COLUMN BUT THE LAST IS A PIVOT COLUMN

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CONCLUSION

THE COLUMNS OF A ARE L.I. IF THERE IS A PIVOT POSITION IN EVERY COLUMN OF A.

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