Linear Algebra
11/5/2012

Topic: Practice computing determinants

Two big methods for m-by-m matrices:
- Cofactor expasion
- Forward elimination

These methods can be combined along with
- 2-by-2: ad-bc
- 3-by-3: \\\-/// method.

EXAMPLE MATRIX

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 4 & -5 \end{bmatrix}$$

Nov 5-8:37 AM

COFACTOR EXPANSION

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 3 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 4 & -5 \end{vmatrix} = \sum_{i=1}^{4} a_{ij}(-1)^{i+j} D_{ij} \quad j=2 \quad \text{(FIXED COLUMN)}$$

LOTS OF ZEROES

$$= \sum_{j=1}^{4} a_{ij}(-1)^{i+j} D_{ij} \quad \text{(FIXED ROW)}$$

D_{ij}: DETERMINANT OF THE MATRIX WITH ROW i AND COLUMN j DELETED

$$= a_{12}(-1)^3 \cdot \begin{vmatrix} 3 & 1 & 4 \\ 0 & 1 & 3 \\ 2 & 4 & -5 \end{vmatrix} + a_{22}(-1)^4 \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 4 & -5 \end{vmatrix} + \dots$$

$$= 0\cdot(-) + (-2)^{*} \cdot 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 4 & -5 \end{vmatrix} + 0\cdot(-) + 0(-)$$

FIXED ROW

$$\sum_{j=1}^{3} b_{ij}(-1)^{i+j} D_{ij}$$

$$= (-2)\left(0(-1)^3\cdot(-1) + 1(-1)^4 \boxed{\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}} + 3(-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}\right)$$

$$= (-2)\left(\boxed{(-5-4)} + (-3)(4+2)\right) = (-2)\left((-9) + (-3)(6)\right) = (-2)(-27) = 54$$

ad-bc

Nov 5-8:44 AM

FORWARD ELIMINATION (NO SCALING)

$$\begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -9 \end{vmatrix}$$

$$= 1 \cdot (-2) \cdot 1 \cdot (-9) = 18$$

I USED THESE FACTS

det(REPLACE) = 1

det (SCALE BY r) = r

MATRICES OF THE FORM

$$\begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \\ 0 & & & 1 \end{bmatrix}$$ ARE PRODUCTS OF "REPLACE"s

$$\begin{bmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & \ddots \; z \end{bmatrix} \begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \\ 0 & & & \ddots \; 1 \end{bmatrix} = \begin{bmatrix} a & b & c & ** \\ 0 & & & \ddots \; z \end{bmatrix}$$

NOTE ALSO

det(swap) = -1

Nov 5-9:03 AM

2nd FORWARD ELIM EXAMPLE

$$\begin{vmatrix} 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 & -1 \\ 4 & 1 & 0 & 1 & -1 \end{vmatrix} \overset{\text{SWAP}}{=} - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 & -1 \\ 4 & 1 & 0 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 & -1 \\ 3 & 0 & 1 & 1 & -1 \\ 4 & 1 & 0 & 1 & -1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 & -1 \\ 0 & 0 & -2 & 4 & -4 \\ 4 & 1 & 0 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 & -1 \\ 0 & 0 & -2 & 4 & -4 \\ 0 & 1 & -4 & 5 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 & -4 \\ 0 & 1 & -4 & 5 & -5 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 & -4 \\ 0 & 0 & -4 & 6 & -6 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -4 & 6 & -6 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix}$$

$$\overset{\text{SWAP}}{=} + \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = (1 \cdot 1 \cdot (-2) \cdot (-2) \cdot (-2)) = (-8)$$

Nov 5-9:17 AM

3-BY-3 TRICK

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ -2 & 0 & 1 \end{vmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & -1 & -3 & 0 & -1 & -3 \\ -2 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\underbrace{1\cdot(-1)\cdot 1 + 2(-3)(-2) + 3\cdot 0\cdot 0}_{} - \underbrace{1(-3)\cdot 0 - 2\cdot 0\cdot 1 - 3\cdot(-1)\cdot(-2)}_{}$$

$$-1 + 12 + 0 - 0 - 0 - 6 = 5 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ -2 & 0 & 1 \end{vmatrix} = 5$$

WARNING!!!

THIS DOES NOT WORK FOR MATRICES THAT ARE ARE NOT 3x3.

Nov 5-9:26 AM

MIXED APPROACH

$$\begin{vmatrix} 1 & 2 & 0 & -3 \\ 2 & 3 & 1 & -2 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 3 & 2 \end{vmatrix} \underset{\text{REPLACE}}{=} \begin{vmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 3 & 2 \end{vmatrix} \underset{\text{COFACTOR EXPANSION}}{=}$$

$$1 \cdot \begin{vmatrix} -1 & 1 & 4 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} + 0 + 0 + 0 = \ldots$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 4 & -1 & 1 & 4 \\ -1 & 2 & 1 & -1 & 2 & 1 \\ -1 & 3 & 2 & -1 & 3 & 2 \end{array}\right]$$

$$(-1)(2)(2) + 1 \cdot 1 \cdot (-1) + 4(-1)3 - (-1) \cdot 1 \cdot 3 - 1 \cdot (-1) \cdot 2 - 4 \cdot 2 \cdot (-1)$$

$$-4 - 1 - 12 + 3 + 2 + 8 = -4$$

Nov 5-9:35 AM

COMPUTE THE DETERMINANT

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1+b_1+c_1 \\ a_2 & b_2 & c_2 & a_2+b_2+c_2 \\ a_3 & b_3 & c_3 & a_3+b_3+c_3 \\ a_4 & b_4 & c_4 & a_4+b_4+c_4 \end{vmatrix} = 0$$

$\searrow$ MATRIX IS NOT INVERTIBLE

$\downarrow$

COLUMNS ARE LINEARLY DEPENDENT $\rightarrow$ LINEAR COMBINATION OF THE FIRST 3 COLUMNS. $\uparrow$

Nov 5-9:43 AM