Linear Algebra
10/24/2012

Topic: Determinants and permutations

Idea: Determinants can be used to understand permutations.

Definition: A (finite) permutation is bijection from a (finite) set to itself.

Example: Set A = {1, 2, 3, 4, 5, 6, 7, 8}

Not a permutation	A permutation
1|-> 2	1|-> 6
2|-> 1	2|-> 1
3|-> 4	3|-> 4
4|-> 2	4|-> 2
5|-> 7	5|-> 7
6|-> 8	6|-> 8
7|-> 3	7|-> 3
8|-> 5	8|-> 5

Oct 24-8:29 AM

Definition: A transposition is a permutation that fixes all but two elements in the set.

Observation: Every permutation can be constructed by a sequence of transpositions.

Example:
1|-> 6, 2|-> 1, 3|-> 4, 4|-> 2, 5|-> 7, 6|-> 8, 7|-> 3, 8|-> 5

(1,2,3,4,5,6,7,8)|-(1 6)->(6,2,3,4,5,1,7,8)|-(1 8)-> (8,2,3,4,5,1,7,6)
|-(1 5)->(5,2,3,4,8,1,7,6) |-(1 7)-> (7,2,3,4,8,1,5,6)
|-(1 3)-> (3,2,7,4,8,1,5,6) |-(1 4)-> (4,2,7,3,8,1,5,6)
|-(1 2)-> (2,4,7,3,8,1,5,6) Seven transpositions

(1,2,3,4,5,6,7,8)|-(5 8)->(1,2,3,4,8,6,7,5)|-(3 7)->(1,2,7,4,8,6,3,5)
|-(6 8)->(1,2,7,4,8,5,3,6) |-(6 7)->(1,2,7,4,8,3,5,6)
** |-(3 4)->(1,2,4,7,8,5,3,6) |-(1 5)->(8,2,4,7,1,5,3,6)
** |-(3 4)->(8,2,7,4,1,5,3,6) |-(1 5)->(1,2,7,4,8,5,3,6)
|-(1 2)->(2,1,7,4,8,3,5,6) |-(2 4)->(2,4,7,1,8,3,5,6)
|-(4 6)->(2,4,7,3,8,1,5,6) Eleven transpositions

Oct 24-8:50 AM

Observation: A permutation of 8 elements can be represented as a sequence of transpositions, using no more that 7 transpositions.

Question: Can every such permutation be represented with exactly 7 transpositions?

Example:
1|->2, 2|->1, 3|->4,4|->3, 5|->6, 6|->5, 7|->8, 8|->7

After a length of time we think this might be impossible.

Theorem: Every finite permutation can be represented by a sequence of transpositions. The number of transpositions used for a given permutation will always have the same parity(odd-ness/even-ness). Thus every finite permutation can be classified as either even or odd.

Oct 24-9:11 AM

A <u>permutation matrix</u> is an invertible 0-1 matrix with exactly one 1 in each row, or equivalently in each column.

Equivalently, a permutation matrix is an invertible 0-1 matrix whose inverse is also a 0-1 matrix.

A permutation matrix is called such because it permutes the coordinates of vectors of the appropriate size when multiplying them on the left (or right).

Examples: Identity matrix, swap matrices (transposition matrices).

Proof of theorem:
Notice that a swap matrix reverses orientation and does not affect volume. Thus the determinant of a swap matrix is -1.
Suppose we row reduced a permutation matrix, P.
$rref(P)=E_nE_{n-1}...E_2E_1P = I$ So $P^{-1}=E_nE_{n-1}...E_2E_1$
Since P was 0-1 and had exactly one 1 in each column. Each E_i can be assumed to be a swap.
The inverse of each E_i is itself. So $P=E_1E_2...E_{n-1}E_n$.
So the determinant $|P|=|E_1E_2...E_{n-1}E_n|=(-1)^n$.
So |P| is -1 if n is odd, and |P| is 1 if n is even.
Since determinant is an invariant of P, the parity of n is invariant also of P.

Oct 24-9:34 AM

Oct 24-9:44 AM

Oct 24-9:20 AM