Linear Algebra
10/23/2012
Topic: Calculating Determinants

Idea: determinants can be computed directly or by using the fact taht |AB|=|A||B|.

Notice that shearing a parallelogram (or a parallelpiped) doss not change its area (or volume) (use integration to compute volume). Shear transformations also don't change orientation. As a result matrices of the following forms have determinant 1.

$$\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ * & & & & 1 \end{bmatrix}$$

UPPER TRIANGULAR LOWER TRIANGULAR
ONES ON THE DIAGONAL.

Specifically, elementary replace matrices have determinant 1.

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A clearer example of determinant is that of an elementary scalar matrix. If one coordinate is scaled by a real number r, then the volume of the unit (hyper) cube is scaled by |r|, the absolute value of r. Moreover if r is negative then such a scaling will reverse orrientation. As a result the determinant of an elementary scalar mstrix is the value by which it scales.

One can then see that a product of elementary scalars matrices will result in a matrix that is diagonal and has a determinant is the product of the scaling factors. As a result, one can see the equality:

$$\begin{vmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{vmatrix} = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n$$

DIAGONAL PRODUCT

λ: GREEK LOWER CASE 'LAMBDA'.

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Combining these realizations one can compute:

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \text{ SO}$$

$$\begin{vmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{vmatrix} = \prod_{i=1}^{n} \lambda_i$$

UPPER TRIANGULAR

AND SIMILARLY FOR LOWER TRIANGULAR MATRICES.

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There is another way to compute determinants. $A: n \times n$

$$\det(A) = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{j+i} \cdot M_{ij} \quad \text{(NOTE } j \text{ IS FIXED)}$$

WHERE M_{ij} IS THE DETERMINANT OF THE MATRIX RESULTING FROM DELETING THE i^{th} ROW AND j^{th} COLUMN OF A.

$A = [a_{ij}]$

THIS IS RELATED TO:

$$\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{j+i} \cdot M_{ij} \quad \text{(HERE } i \text{ IS FIXED)}$$

BY THE FACT THAT $\underline{\det(A^T) = \det(A)}$

WE WILL SEE WHY THIS IS TRUE LATER.

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EXAMPLE

most 0's

(1,2) POSITION $1+2=3$

$2+2=4$

$$\begin{vmatrix} 1 & -2 & 3 & 0 \\ 4 & -1 & 2 & 1 \\ -3 & 0 & 4 & -1 \\ 1 & 0 & -1 & 1 \end{vmatrix} = (-1)^3(-2)\begin{vmatrix} 4 & 2 & 1 \\ -3 & 4 & -1 \\ 1 & -1 & 1 \end{vmatrix} - (-1)^4(-1)\begin{vmatrix} 1 & 3 & 0 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

MULTIPLICATION BY 0 ZERO

$$= 2\begin{vmatrix} 4 & 2 & 1 \\ -3 & 4 & -1 \\ 1 & -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 & 0 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$= \ldots$ TO BE CONTINUED

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RECALL

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

WE'LL SEE

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ NEXT TIME.

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