Linear Algebra
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Linear Transformations and homogeneous systems of linear equations.

Idea: A linear transformation is invertible if it is bijective.

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Review of Functions:
A function, f, from D to C is a relation on D and C (a set of pairs (x,y) with x in D and y in C) such that any x in D occurs on the left of exactly one pair. We write f:D->C.

D is the set of inputs, and is called the domain of f.
C is the set of possible outputs, codomain (or range) of f.
The set of all actual outputs of f is called the image of f and it is contained as a subset of C.

A function is injective (or one-to-one) if each y in C occurs on the right of at most one pair.

A function is surjective (or onto) if the image equals the codomain.

A function is bijective if it is both surjective and injective.

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Examples:

$x \mapsto 2^x$: $\mathbf{R} \to \mathbf{R}^{>0}$ is surjective.

$x \mapsto 2^x$: $\mathbf{R} \to \mathbf{R}$ is not surjective.

$x \mapsto x^2$: $\mathbf{R}^{>0} \to \mathbf{R}^{>0}$ is bijective.

$x \mapsto x^2$: $\mathbf{R} \to \mathbf{R}$ is not injective.

If A: nxm then

$T_A : \mathbf{R}^m \to \mathbf{R}^n$

Question: when is T_A injective, surjective or bijective?

Answer comes from the pivot positions of A.

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Claim: T_A is injective if and only if A**x**=**0** only has the trivial solution.

Recall Homogeneous systems of linear equations (Section 1.5 of Lay)

A system of linear equations equivalent to a matrix equation of the form A**x**=**0** is called a homogeneous system. Notice:

- if A**v**=**0** then so too rA**v**=A(r**v**)=**0** and
- if A**u**=**0** also then A**u**+A**v**=A(**u**+**v**)=**0**.
- Consequently, linear combinations of solutions to a homogeneous system of linear equations are also solutions.
- Also **0** is always a solution. We call it the trivial solution. Any other solution is non-trivial.

Proof of claim: (contrapositive)
(only if): It suffices to show that if the homogeneous system has a non-trivial solution, then there are distinct **v** and **u** such that A**v**=**b**, and A**u**=**b**. Assume A**v**=**b, t**ake **w** non-zero such that A**w**=**0**. Then, A**w**+A**v**=A(**w**+**v**)=**b**. (more than necessary but this presents a nice picture, see page 45)
(if):It suffices to show that if A**v**=**b**, and A**u**=**b** then the homogeneous system has a non-trivial solution. Assume **v** is not equal to **u** and they are both solutions to A**x**=**b**. Then A**v**-A**u**=A(**v**-**u**)=**0**, but **v**-**u** is not zero.

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Corollary: T_A is injective if and only if A has a pivot position in every column. (see boxed statement on page 43)

Using theorem 4 (a&d) on page 37 we can get this corollary: T_A is surjective if and only if A has a pivot position in every row.

Recall the codomain of T_A.

Definitions:

A function, f:D->C, is <u>left invertible</u> if there is a function, g:C->D, such that g(f(x))=x, for any x in D. (g is a <u>left inverse</u>)

A function, f:D->C, is <u>right invertible</u> if there is a function,

g:C->D, such that f(g(y))=y, for any y in C. (g is a <u>right inverse</u>)

A function, f:D->C, is <u>invertible</u> if there is a function, g:C->D, which is both a left and a right inverse of f.

Facts:

Functions are: left invertible if and only if they are injective,

right invertible if and only if they are surjective.

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The same fact holds for linear transformations. To see this we need to see that the functional inverse of a linear transformation is also a linear transformation.

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