New York City College of Technology
City University of New York

Introduction to Calculus II

Preparation for MAT 1575: Single Variable Calculus II

Christopher Chan
Satyanand Singh
Department of Mathematics

Table on Contents

Section 1: Indefinite Integrals

Integration formulas:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1 \qquad \text{(Power rule of antiderivatives)}$$

$$\int \cos(kx+b)\,dx = \frac{1}{k}\sin(kx+b) + c, \qquad k \neq 0$$

$$\int \sin(kx+b)\,dx = -\frac{1}{k}\cos(kx+b) + c, \qquad k \neq 0$$

$$\int e^{kx+b}\,dx = \frac{1}{k}e^{kx+b} + c, \qquad k \neq 0$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int kf(x) \pm cg(x)\,dx = \int kf(x)\,dx \pm \int cg(x)\,dx \qquad \text{(Linearity of Integrals)}$$

1. Let c be an arbitrary constant. $F(x)$ is an antiderivative of $f(x)$, if $F'(x) = f(x)$. The general antiderivative of $f(x)$, is denoted by the following indefinite integral:

$$\int f(x)\,dx = F(x) + c$$

Example: Evaluate the following integral

$$\int x^3\ dx$$

$$\int x^3\ dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$$

2. Indefinite Integrals of a Sum.

Example: Evaluate the following integral

$$\int (\cos(x) + 2x^5)\,dx = \int \cos(x)\ dx + \int 2x^5\,dx$$

$$\int (\cos(x) + 2x^5)\,dx = \sin(x) + \frac{2x^6}{6} + c$$

$$\int (\cos(x) + 2x^5)\,dx = \sin(x) + \frac{x^6}{3} + c$$

3. Indefinite Integrals of a Difference.

Example: Evaluate the following integral

$$\int (11x^3 - 3\sin(x))\,dx = \int 11x^3\,dx - \int 3\sin(x)\,dx$$

$$\int (11x^3 - 3\sin(x))\,dx = \frac{11x^4}{4} - [-3\cos(x)] + c$$

$$\int (11x^3 - 3\sin(x))\,dx = \frac{11x^4}{4} + 3\cos(x) + c$$

4. Evaluate the following integrals.

a) $\int 45x^4 \, dx$

b) $\int \frac{2x^6}{5} \, dx$

c) $\int (x^{-1} + 5\sqrt{x}\,)dx$

d) $\int (x^5 + \sqrt[5]{x}\ )\ dx$

e) $\int (sec^2(x) + x^3)\, dx$

f) $\int (4x^{-2} - \frac{1}{\sqrt{x}})\, dx$

Section 2: Definite Integrals

5. Let $F(x)$ be an antiderivative of $f(x)$; Note: $F'(x) = f(x)$. If $f(x)$ is continuous and integrable on the interval (a, b), the definite integral of f from a to b is denoted by:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example: Evaluate the following definite integrals

$$\int_1^2 x^2\,dx$$

$$\int_1^2 x^2\,dx = \frac{x^{2+1}}{2+1}\Big|_1^2$$

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2$$

$$\int_1^2 x^2\,dx = \frac{(2)^3}{3} - \frac{(1)^3}{3}$$

$$\int_1^2 x^2\,dx = \frac{8}{3} - \frac{1}{3}$$

$$\int_1^2 x^2\,dx = \frac{7}{3}$$

6. Evaluate the following definite integrals.

a) $\int_0^3 (x^4 + 3x)\, dx$

b) $\int_0^{5\pi} 2\sin(x/5)\, dx$

c) $\int_1^4 \sqrt{x}\ dx$

d) $\int_{-3}^{3}\left(3+\sqrt{x^3}\right)dx$

e) $\int_{0}^{2\pi}\left(\frac{x}{3}-\cos(2x)\right)\,dx$

f) $\int_{-4}^{-1}\left(3x^{-2}-\frac{1}{x}\right)\,dx$

Section 3: Integration by Substitution

Integration by the method of variable substitution consists of the following general steps.

- Choose a variable. Generally, u is a common choice and leads to the general expression, integration by "$u = g(x)$ substitution" for the "innermost" term of a composition of a function.

Compute $du = g^{/}(x)\,dx$ so that $\int f(g(x))\,g^{/}(x)\,dx = \int f(u)\,du$

- Substitute all terms of the original integrand with the expressions involving u and du.
- Evaluate the integral involving the u and du expressions.
- Replace all substitutions of u with the original variable expression.

Integration by Substitution of Indefinite Integrals

Example: Evaluate the following indefinite integral using u substitution

$$\int (x^5 + 100)^{50}(5x^4)\,dx$$

Observe that $\frac{d}{dx}\,(x^5 + 100) = 5x^4$, which is a part of the integrand.

Step 1: Choose a variable, let us use the variable u and Let $u = x^5 + 100$

Step 2: Compute for du. Since $u = x^5 + 100$, then $du = 5x^4dx$ by the Power Rule of differentiation.

Step 3: Substitute under the integral sign using u and du.

$$\int (x^5 + 100)^{50}(5x^4)\,dx = \int u^{50}du$$

Step 4: Evaluate:

$$\int u^{50}du = \frac{u^{51}}{51} + c$$

Step 5: Replace u with original expression.

$$\frac{u^{51}}{51} + c = \frac{(x^5 + 100)^{51}}{51} + c$$

Therefore $\int (x^5 + 100)^{50}(5x^4)\,dx = \frac{(x^5+100)^{51}}{51} + c.$

Example: Evaluate the following indefinite integral using variable substitution

$$\int (2\sin(x) - 7)^{15} \cos(x)\, dx$$

Notice: $\frac{d}{dx}(2\sin(x) - 7) = 2\cos(x)$.

Step 1: Let $u = 2\sin(x) - 7$

Step 2: Compute du. Since $u = 2\sin(x) - 7$, then $du = 2\cos(x)\, dx$. Notice that we want $\cos(x)dx$ as the du, not $2\cos(x)$, so we divide out the 2 in the u and du expression to get $du/2 = \cos(x)\, dx$.

Step 3: Substitute expression using u and du to obtain

$$\int (2\sin(x) - 7)^{15} \cos(x)\, dx = \frac{1}{2}\int u^{15} du$$

Step 4: Evaluate:

$$\frac{1}{2}\int u^{15} du = \frac{1}{2}\left(\frac{u^{16}}{16}\right) + c = \frac{u^{16}}{32} + c$$

Step 5: Replace u with original expression.

$$\frac{u^{16}}{32} + c = \frac{(2\sin(x) - 7)^{16}}{32} + c$$

Therefore $\int (2\sin(x) - 7)^{15} \cos(x)\, dx = \frac{(2\sin(x)-7)^{16}}{32} + c$.

7. Evaluate the following Indefinite Integrals using substitution method (u – substitution):

a) $\int x^4\sqrt{x^5-9}\ dx$

b) $\int x^3(x^4-3)^{-\frac{2}{3}}\,dx$

c) $\int sin^3(x)\cos(x)\,dx$

d) $\int \frac{sin\sqrt{x}}{\sqrt{x}}\, dx$

e) $\int \frac{2\sin(x)}{\sqrt{\cos(x)}}\, dx$

f) $\int \frac{1}{\sqrt{1+x}}\, dx$: Hint: Let $u = 1 + x$ or $u^2 = 1 + x$.

Integration by Substitution of Definite Integrals

Like the method of substitution for indefinite integrals, definite integral is very similar. The only difference is that the limits of integration must also change to correspond to the new variable. Let $u = g(x)$.

$$\int_a^b f(g(x))\, g^{/}(x)\, dx = \int_{u(a)}^{u(b)} f(u) du$$

Example: Evaluate the following indefinite integral using variable substitution

$$\int_1^2 3x^2\sqrt{x^3+1}\ dx$$

Notice: $\frac{d}{dx}\,(x^3+1) = \ 3x^2$, which is a part of the integrand.

Step 1: Choose a variable, let us use the variable u and Let $u = \ x^3 + 1$

Step 2: Compute for du. Since $u = x^3 + 1$, then $du = \ 3x^2 dx$ by the Power Rule of differentiation.

Step 3: Substitute expression using u and du.

$$\int_1^2 3x^2\sqrt{x^3+1}\ dx = \int_{u(1)}^{u(2)} \sqrt{u}\, du$$

Step 4: Substitute for expression $u(x)$ for each limit of the integral. Since $u = x^3 + 1$, then $u(1) = 4$ for when $x = 1$ and $u(2) = 9$ for when $x = 2$.

$$\int_1^2 3x^2\sqrt{x^3+3}\ dx = \int_{u(1)}^{u(2)} \sqrt{u}\, du = \int_4^9 \sqrt{u}\, du$$

Step 5: Evaluate:

$$\int_4^9 \sqrt{u}\, du = \frac{2x^{\frac{3}{2}}}{3}\Big|_4^9 = \frac{2(9)^{\frac{3}{2}}}{3} - \frac{2(4)^{\frac{3}{2}}}{3} = \frac{54}{3} - \frac{16}{3} = \frac{38}{3}$$

$$\int_1^3 3x^2\sqrt{x^3+3}\ dx = \frac{38}{3}.$$

8. Evaluate the following definite Integrals using u – substitution:

a) $\int_0^1 x\sqrt{x^2+2}\ dx$

b) $\int_{-1}^1 \frac{x}{(x^2+1)^2}\ dx$

c) $\int_1^2 \frac{6}{\sqrt{x}}\ dx$

d) $\int_0^{\pi} sin^2(x)\, dx$

e) $\int_{\pi}^{2\pi} \frac{cos\sqrt{x}}{\sqrt{x}}\, dx$

f) $\int_0^{\frac{\pi}{2}} cos^3(x) sin(x)\, dx$

Section 4: Area Computation Between Curves

The area between two curves is determined by the region enclosed between; the point(s) of intersection of the two functions and the general shapes of the curves. We will illustrate the simple case where the limits of integration correspond to the x coordinates of the intersections. Such that $f(x) \leq g(x)$ on $[a, b]$ and $a \leq b$. Note there are other cases to consider but once you master this you can easily extend this to more complicated curves. **It is extremely important to first sketch the curves to establish the region of integration and to choose the upper and lower curves correctly.**

$$\int_a^b \left(g(x) - f(x)\right) dx$$

Finding the area between two curves consists of the general steps.

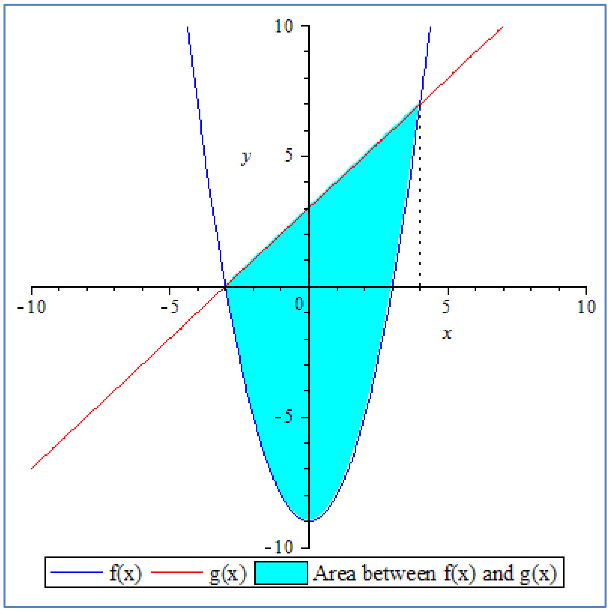
- Sketch the curves and find limits of integration by setting the two functions equal to each other and solve for the variable.
- Set up the function to be integrated. **(Upper curve – Lower curve).**
- Evaluate the integral to obtain the area.

Area Between Two Curves

Example: Find the area between the two curves.

$$g(x) = x + 3$$

$$f(x) = x^2 - 9$$



Step 1: Sketch and find limits of integration. Set the two functions equal to each other and solve for the variable.

$$f(x) = g(x)$$

$$x^2 - 9 = x + 3$$

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

$$x = -3, x = 4$$

The limit of integration is from $a = -3$ to $b = 4$.

Step 2: Set up the integral. **Upper curve – Lower curve**

$$\int_a^b \big(g(x) - f(x)\big)\, dx = \int_a^b \big((x+3) - (x^2 - 9)\big)\ dx$$

The problem now becomes:

$$\int_{-3}^{4} \big((x+3) - (x^2 - 9)\big)\, dx$$

Step 3: Evaluate.

$$\int_{-3}^{4} \big((x+3) - (x^2 - 9)\big)\ dx = \int_{-3}^{4} (-x^2 + x + 12)\ dx = -\frac{x^3}{3} + \frac{x^2}{2} + 12x\ \Big|_{-3}^{4}$$

$$\left(-\frac{x^3}{3} + \frac{x^2}{2} + 12x\right)\Big|_{-3}^{4} = \left[-\frac{(4)^3}{3} + \frac{(4)^2}{2} + 12(4)\right] - \left[-\frac{(-3)^3}{3} + \frac{(-3)^2}{2} + 12(-3)\right] = \left(\frac{104}{3} + \frac{45}{2}\right)$$

$$\left(\frac{104}{3} + \frac{45}{2}\right) = \frac{343}{6}$$

Thus the area enclosed between $f(x)$ and $g(x)$ is $\frac{343}{6}$.

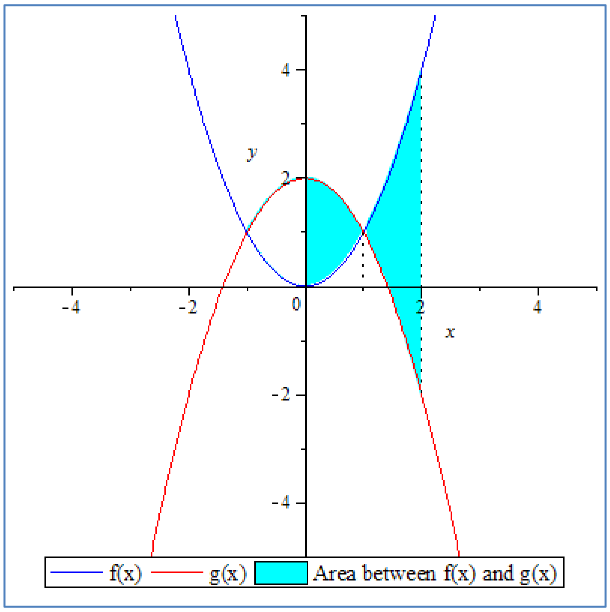
Area Between Curves that Cross with Boundaries

The general idea of finding area between curves is the same as what we have just previously shown. However, in order to obtain the area of curves that cross, the summation of the areas we want is needed over different intervals.

Example: Find the area between the two curves for $0 \leq x \leq 2$.

$$g(x) = 2 - x^2$$

$$f(x) = x^2$$



Notice we only want the area for where x is between 0 and 2, but there is some intersection where the two curves cross.

We already know how to set up the integral and the function to integrate. The problem again is to choose the limits of integration correctly.

We set the two equations equal to each other and solve for x. That is: $x^2 = 2 - x^2$. As a result, we get $x = -1, x = 1$. However, we are only concerned with the original problem with bounds between 0 and 2. Since only one value of x, for when $x = 1$ is between 0 and 2, our limits of integration would be from 0 to 1, and 1 to 2. **Notice the change in the upper and lower curves in these intervals**.

Now that we have the limits of integration, we can fit it into our formula.

$$\int_a^b \big(g(x) - f(x)\big)\, dx + \int_b^c \big(f(x) - g(x)\big)\, dx$$

$$\int_0^1 \big((2 - x^2) - x^2\big)\, dx + \int_1^2 \big(x^2 - (2 - x^2)\big)\, dx = \int_0^1 (2 - 2x^2)\, dx + \int_1^2 (2x^2 - 2)\, dx$$

Evaluate as we did earlier to get

$$\int_0^1 2 - 2x^2\, dx + \int_1^2 2x^2 - 2\, dx = \left(2x - \frac{2x^3}{3}\right)\Big|_0^1 + \left(\frac{2x^3}{3} - 2x\right)\Big|_1^2 = \frac{4}{3} + \frac{8}{3} = 4$$

Therefore the area between $f(x)$ and $g(x)$ where $0 \le x \le 2$ is 4.

9. Find the area between the curves on the given intervals.

   a) $f(x) = x^3, g(x) = x, 0 \le x \le 1$

b) $f(x) = x^2 - 3, g(x) = 5 - x^2, -2 \leq x \leq 2$

c) $f(x) = \sin(x)\,,\ g(x) = x^2,\ 0 \leq x \leq 1$

10. Find the areas enclosed between the curves in each case.

a) $f(x) = 7 - x^2,\ g(x) = x^2 - 1$

b) $f(x) = x^2, g(x) = \sqrt{x}$

c) $f(x) = x, g(x) = x^2 - 16$

Section 5: Integration by Parts

Many integrals cannot be evaluated using the basic formulas or integration by variable substitution. We introduce another method to help evaluate some of those integrals; the method of integration by parts.

Example:

$$\int x\, sin(x)\, dx$$

There are no simplifications or obvious substitution we can that would help in this case. We would use integration by parts which is denoted by the formula

$$\int u\, dv = uv - \int v\, du$$

Now let us fit the question to our formula

$$\int x\, sin(x)\, dx = \int u\, dv$$

Notice we are working with four "objects" (u, v, du, dv). We are given u and dv, we need to carefully choose which part of our function will be u (something to differentiate to obtain du) and which will be dv (something to integrate to obtain v). Poor choices of u and dv will most likely lead to more complications.

Let $u = x$ and $dv = \sin(x)dx$
Then $du = 1dx$ and $v = -\cos(x)$
Recall:

$$\int x\, sin(x)\, dx = \int u\, dv = uv - \int v\, du$$

Thus

$$uv - \int v\, du = -xcos(x) - \int (-\cos(x)\; dx$$

$$= -xcos(x) + \int \cos(x)\, dx = -xcos(x) + \sin(x) + c$$

Therefore

$$\int x\, sin(x)\, dx = \sin(x) - xcos(x) + c$$

11. Evaluate the following integrals using integration by parts.

a) $\int x\cos(x)\,dx$

b) $\int \ln(x)\,dx$

c) $\int \frac{\ln(x)}{x}\,dx$

d) $\int x\, sec^2(x) dx$

e) $\int x sin(5x)\, dx$

f) $\int tan^{-1}(x)\, dx$

Section 6: Integration by Trigonometric Substitution

When finding the area of a circle, ellipse or hyperbolas integrals containing terms of the forms below are encountered.

$$\sqrt{a^2 - x^2},\ \sqrt{a^2 + x^2},\ \text{or}\ \sqrt{x^2 - a^2}\ ,\ \text{for}\ a > 0$$

A trigonometric substitution can often be made to evaluate such integrals.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\ sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - sin^2\theta = cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\ tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + tan^2\theta = sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\ sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}\ or\ \pi \leq \theta < \frac{3\pi}{2}$	$sec^2\theta - 1 = tan^2\theta$

Trigonometric Identities

Reciprocal Functions	Pythagorean Theorem
$\sin u = \frac{1}{\csc u}$	$sin^2u + cos^2u = 1$
$cos\ u = \frac{1}{\sec u}$	$tan^2u + 1 = sec^2u$
$\tan u = \frac{1}{\cot u}$	$cot^2u + 1 = csc^2u$

Example: Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{x^2\sqrt{4\left(\frac{x^2}{4}+1\right)}}\,dx$$

We factored out the 4 to have $\frac{x^2}{4}+1$ under the radical. This form closely resembles the trigonometric function $tan^2\theta+1$. So let us set the two equal to each other.

$$\frac{x^2}{4}+1 = tan^2\theta+1$$

Then

$$\frac{x^2}{4} = tan^2\theta$$

$$x^2 = 4\,tan^2\theta$$

$x = \pm 2\,tan\theta$ (Note: we will use $x = 2\,tan\theta$; both works)

$$dx = 2\,sec^2\theta\,d\theta$$

Now that we have converted all of our x terms into θ form, we can now directly substitute it into our original formula.

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{x^2\sqrt{4\left(\frac{x^2}{4}+1\right)}}\,dx = \int \frac{2\,sec^2\theta}{4\,tan^2\theta\sqrt{4\,(tan^2\theta+1)}}\,d\theta$$

Notice

$$tan^2\theta+1 = sec^2\theta$$

$$\int \frac{2\,sec^2\theta}{4\,tan^2\theta\sqrt{4\,(tan^2\theta+1)}}\,d\theta = \int \frac{2\,sec^2\theta}{4\,tan^2\theta\sqrt{4\,(sec^2\theta)}}\,d\theta$$

$$\int \frac{2\,sec^2\theta}{4\,tan^2\theta\sqrt{4\,(sec^2\theta)}}\,d\theta = \int \frac{2\,sec^2\theta}{4\,tan^2\theta(2sec\theta)}\,d\theta = \int \frac{sec\theta}{4tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{cos\theta}{sin^2\theta}\,d\theta$$

Then

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \frac{1}{4}\int \frac{cos\theta}{sin^2\theta}\,d\theta$$

Now we can integrate. The method of variable substitution would work.

Let $u = sin\theta$, then

$$\frac{1}{4}\int \frac{cos\theta}{sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int u^{-2}du$$

$$\frac{1}{4}\int u^{-2}du = \frac{1}{4}\left(-u^{-1}\right) + c = -\frac{1}{4u} + c$$

Now replace $u = sin\theta$

$$-\frac{1}{4u} + c = -\frac{1}{4sin\theta} + c$$

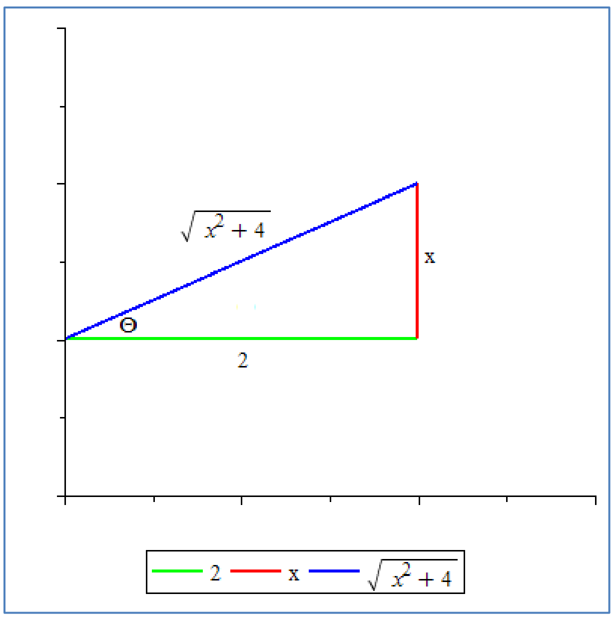
But remember, we started with x terms, not θ. We must convert back to x expression.

Note that earlier we solved for

$$x = 2tan\theta$$

$$\frac{x}{2} = tan\theta$$

Recall the Pythagorean Theorem (SOH CAH TOA)



Thus

$$-\frac{1}{4sin\theta} + c = -\frac{\sqrt{x^2+4}}{4} + c$$

Therefore

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4} + c$$

12. Evaluate the following integrals

a) $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

b) $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

c) $\int \frac{x^2}{\sqrt{x^2+9}}\, dx$

d) $\int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$

e) $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

f) $\int_{\sqrt{2}}^2 \frac{1}{x^3\sqrt{x^2-1}}\,dx$

Section 7: Integration by Partial Fraction

The method of partial fractions is used to integrate some rational functions, where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. We do this by expressing it as a sum of simpler fractions.

Example: Evaluate the following integral. Note the degree of the polynomial in the numerator is 0 and that in the denominator is 2. **Always factor the denominator whenever possible before proceeding.**

$$\int \frac{1}{x^2 + x - 2}\, dx$$

Note:

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

Thus

$$1 = A(x+2) + B(x-1)$$

Solve for A and B by choosing appropriate values for x.

If $x = 1$, notice that the B term disappears and we get $1 = 3A$ or $A = \frac{1}{3}$
Similarly if we choose $x = -2$, then $1 = 3B$ or $B = -\frac{1}{3}$.

Substitute the values for A and B and evaluate the following integral

$$\int \left(\left(\frac{1}{3}\right)\left(\frac{1}{x-1}\right) + \left(-\frac{1}{3}\right)\left(\frac{1}{x+2}\right)\right) dx$$

$$\int \left(\left(\frac{1}{3}\right)\left(\frac{1}{x-1}\right) + \left(-\frac{1}{3}\right)\left(\frac{1}{x+2}\right)\right) dx = \frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx$$

$$\frac{1}{3}\int \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx = \frac{\ln(x-1)}{3} - \frac{\ln(x+2)}{3} + c$$

$$\int \frac{1}{x^2 + x - 2}\, dx = \frac{\ln(x-1)}{3} - \frac{\ln(x+2)}{3} + c = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c$$

13. Evaluate the following integrals

a) $\int \frac{x}{x-6}\,dx,$ Hint: $\frac{x}{x-6} = \frac{x-6+6}{x-6} = 1 + \frac{6}{x-6}$

b) $\int \frac{x-9}{(x+5)(x-2)}\,dx$

c) $\int \frac{2x-1}{x^2-x-6}\, dx$

d) $\int \frac{x-1}{x^2+3x+2}\, dx$

e) $\int \frac{x^2+x-5}{x^2-1}\,dx$: Hint: Use long division before applying partial fractions

f) $\int \frac{x^2-x+6}{x^3+3x}\,dx$

Section 8: Volumes by Disk

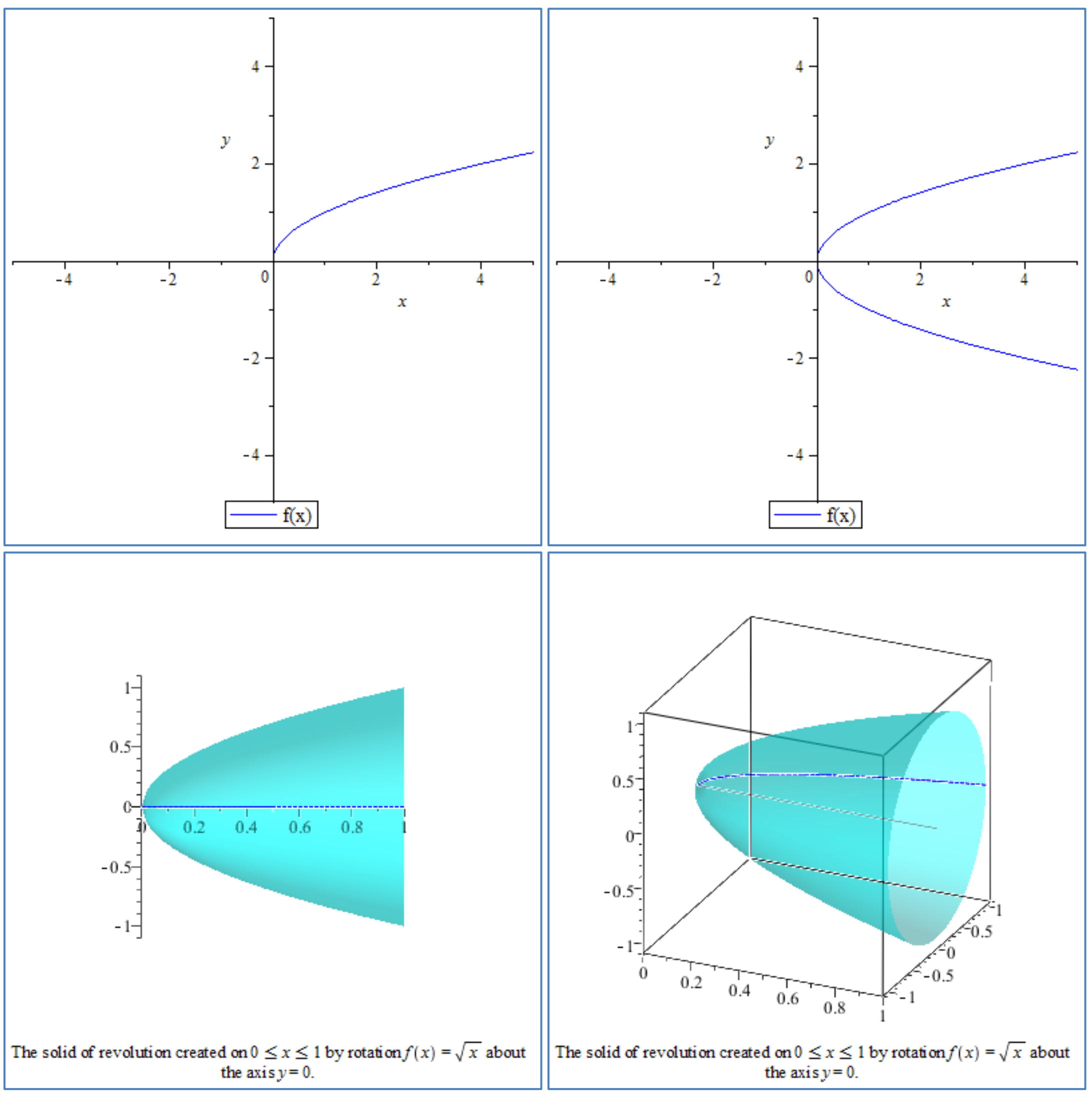
For a continuous function $f(x)$ on the interval $[a,b]$, for $\leq x \leq b$, we can revolve it around the x-axis to generate a solid. We can rotate about any line to generate solids but we will concentrate on the rotation about the x-axis which can easily be extended to other lines. We can compute the volume of the disk obtained by slicing vertically to obtain a disk with volume that is given by: $dV \approx \pi(f(x))^2\ \Delta x$, where the radius $r = f(x)$. The volume can be computed by the following integral:

$$V = \int_a^b \pi(radius)^2(thickness)$$

$$V = \int_a^b \pi(f(x))^2\ dx$$

Example:

Find the volume of the region bounded by $f(x) = \sqrt{x}$ on the interval $[0,1]$ about the x-axis.



The solid of revolution created on $0 \leq x \leq 1$ by rotation $f(x) = \sqrt{x}$ about the axis $y = 0$.

The solid of revolution created on $0 \leq x \leq 1$ by rotation $f(x) = \sqrt{x}$ about the axis $y = 0$.

$$V = \int_a^b \pi(f(x))^2 \, dx = \int_0^1 \pi\left(\sqrt{x}\right)^2 dx = \int_0^1 \pi x \, dx$$

$$\int_0^1 \pi x \, dx = \frac{\pi x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

$$V = \int_0^1 \pi\left(\sqrt{x}\right)^2 dx = \frac{\pi}{2}$$

14. Find the volume of the solid obtained by revolving about the region about the given line using the method of disk

a) $y = x^2, y = 0, x = 3,$ about the x-axis

b) $y = 9 - x^2, y = 0, x = 0$, about the x-axis

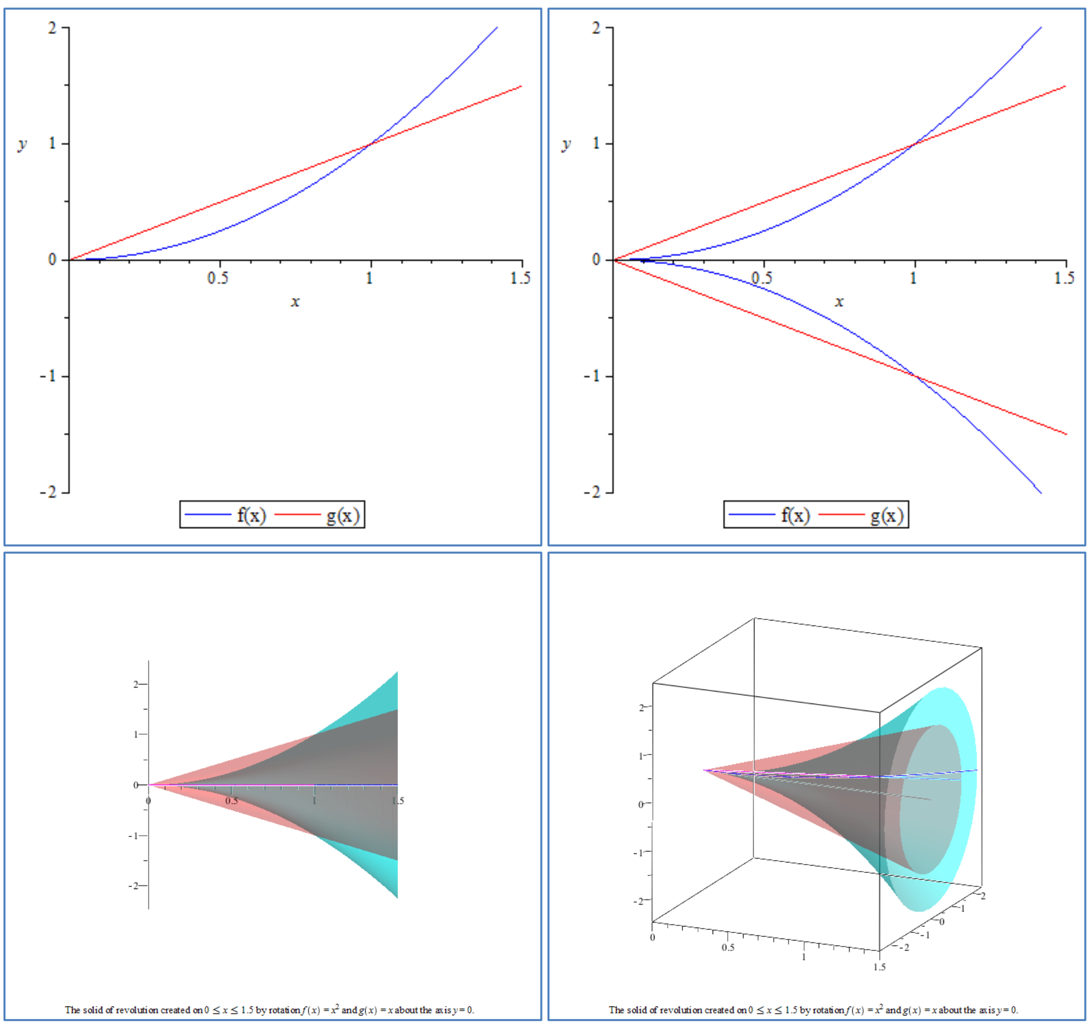
c) $y = x^3, y = 8, x = 0$, about the y-axis

Section 9: Volumes by Washers

A washer is typically defined as a solid with some cavity in the middle. To compute the area of a washer is similar to computing the volume of a disk, except there are now two radii instead of one. We would go about solving it by subtracting the inner radius from the outer radius and integrating as before.

Example:
Find the volume of the region bounded by $f(x) = x^2, g(x) = x$ on the interval [0,1] about the x-axis. Notice if we revolve about the x-axis, the inner radius would be $g(x)$, and outer is $f(x)$.

$$V = \int_a^b \pi(outer\ radius)^2(thickness) - \int_a^b \pi(inner\ radius)^2(thickness)$$
$$V = \int_a^b \pi\big(g(x)\big)^2 dx - \int_a^b \pi\big(f(x)\big)^2 dx$$



$$V = \int_a^b \pi\big(g(x)\big)^2 dx - \int_a^b \pi\big(f(x)\big)^2 dx = \int_0^1 \pi(x)^2 dx - \int_0^1 \pi(x^2)^2 dx = \int_0^1 \pi(x^2 - x^4) dx$$

$$\int_0^1 \pi(x^2 - x^4) dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]\frac{2\pi}{15}$$

$$V = \int_0^1 \pi(x)^2 dx - \int_0^1 \pi(x^2)^2 dx = \frac{2\pi}{15}$$

15. Find the volume of the solid obtained by revolving about the region about the given line using the washer method.

    a) $y = \sqrt{x}, y = x^2$ about the x-axis

b) $y = \frac{x^2}{4}, y = 1$, about the x-axis

c) $y = \sqrt{x}, y = x$ about the $y = 8$

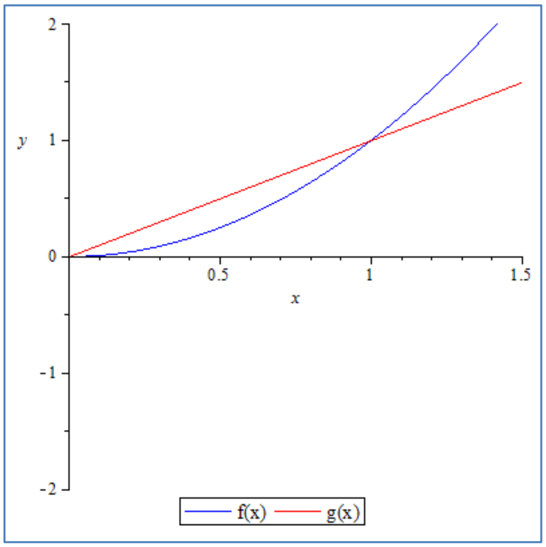
Section 10: Volumes by Cylindrical Shells

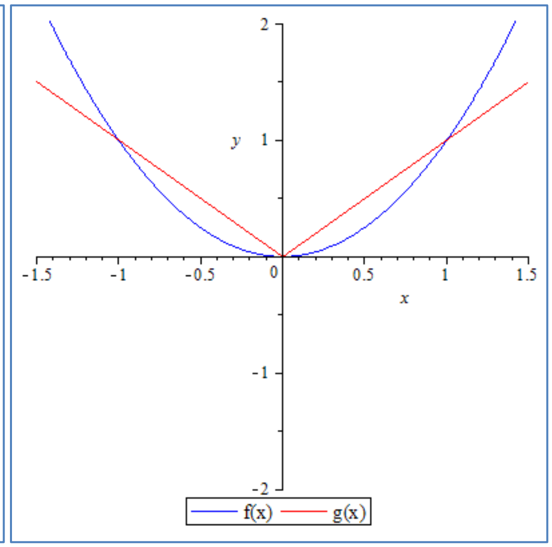
Let us use the same functions from the washer example and revolve them around the y-axis.

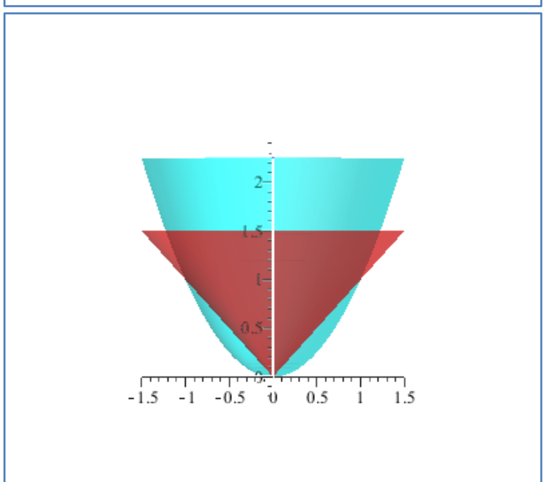
Example: Let $f(x) = x^2, g(x) = x$ bounded by the interval [0,1] about the y-axis.

$$V = \int_a^b 2\pi(radius)(height)(thickness)$$

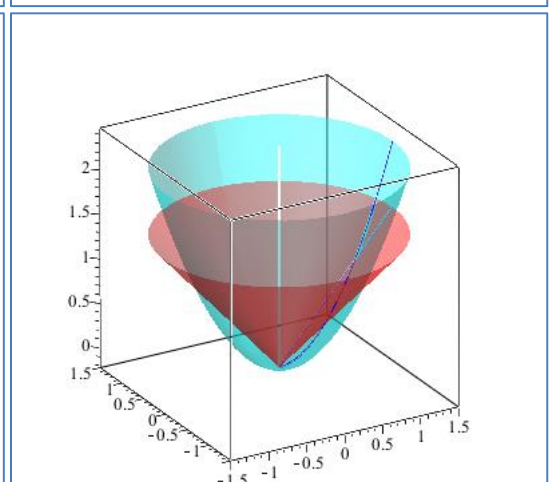
$$V = \int_a^b 2\pi r(g(x) - f(x))dx$$







The solid of revolution created on $0 \le x \le 1.5$ by rotation $f(x) = x^2$ and $g(x) = x$ about the axis $x = 0$.



The solid of revolution created on $0 \le x \le 1.5$ by rotation $f(x) = x^2$ and $g(x) = x$ about the axis $x = 0$.

$$\int_a^b 2\pi r(g(x) - f(x))dx = \int_0^1 2\pi x(x - x^2)dx = \int_0^1 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right) dx$$

$$\int_0^1 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1 = \frac{\pi}{6}$$

$$V = \int_0^1 2\pi x(x - x^2)dx = \frac{\pi}{6}$$

16. Find the volume of the solid obtained by revolving about the region about the given line using the method of cylindrical shell

a) $y = \sqrt[3]{x}, x = 0, x = 8$ about the x-axis

b) $y = x, y = -x, x = 2$, about the y-axis

c) $y = 4 - x^2, y = 0$, about the $x = 3$

Section 11: Sequences of Real Numbers

A sequence can be viewed as a list of numbers written in a specific order.

Example: A sequence of numbers.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

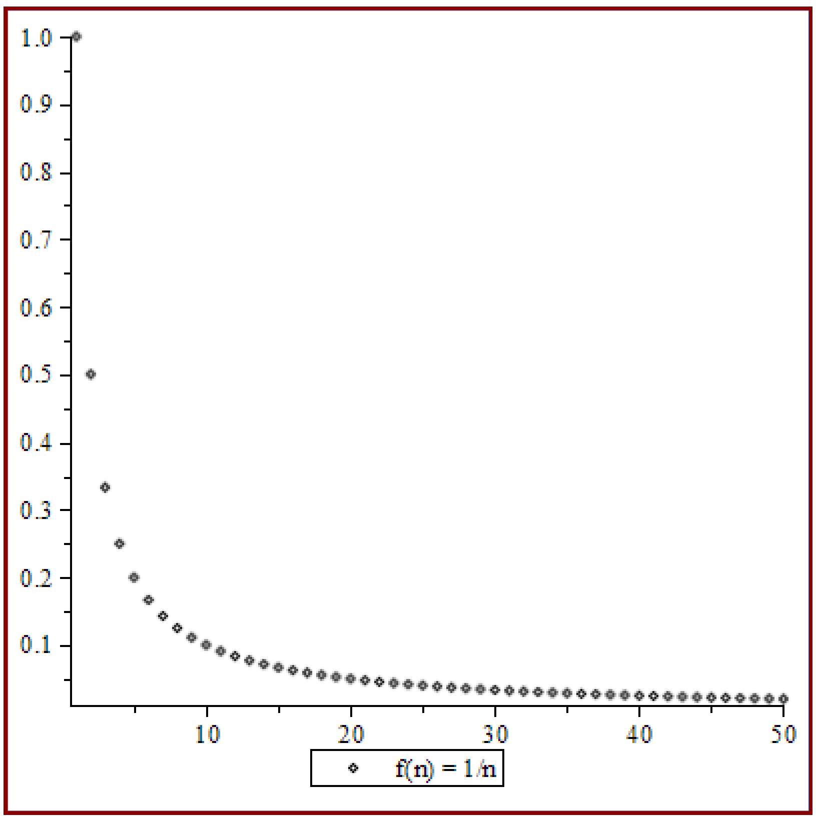
This is a sequence of values of the function $f(n) = \frac{1}{n}$ for $n = 1, 2, 3, 4, 5, 6 \dots$

The values of $a_n = f(n)$ are called the terms of the sequence and n is called their index.

Example:

$$a_1 = 1, \qquad a_2 = \frac{1}{2}, \qquad a_3 = \frac{1}{3}, \qquad a_4 = \frac{1}{4}$$

If we graph this sequence $\{a_n\}_{n=1}^{\infty}$ as n becomes larger and larger, notice a_n gets closer and closer to 0, visually. We can say this sequence **converges** to 0.



Comment: Formally a sequence $\{a_n\}_{n=n_0}^{\infty}$ **converges** to L and is denoted by $\lim_{n\to\infty} a_n = L$, if for all $\epsilon > 0$, there exists some integer N, such that $|a_n - L| < \epsilon$, for all $n > N$. Else we say the sequence **diverges**.

Properties of Limits

Identity	Condition
$\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$	$\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both converge
$\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$	$\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both converge
$\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$	$\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both converge
$\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$	$\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both converge and $\lim_{n\to\infty} b_n \neq 0$

Example:

Determine whether the sequence $\left\{\frac{7n-4}{3n+1}\right\}_{n=1}^{\infty}$ converges.

We will first evaluate the limit below:

$$\lim_{n\to\infty}\frac{7n-4}{3n+1}$$

If we were to simply substitute ∞ for n, our solution would be $\frac{7(\infty)-4}{7(\infty)+1} = \frac{\infty}{\infty}$

Instead, divide out the n to get $\lim_{n\to\infty}\frac{7n-4}{7n+1}\cdot\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{7-\frac{4}{n}}{7+\frac{1}{n}} = \frac{7-\left(\frac{4}{\infty}\right)}{7+\left(\frac{1}{\infty}\right)} = \frac{7}{3}$

Therefore $\left\{\frac{7n-4}{3n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{7}{3}$.

Example:

Determine whether the sequence $\left\{\frac{7n^2-4}{3n+1}\right\}_{n=1}^{\infty}$ converges.

We will first evaluate the limit below:

$$\lim_{n\to\infty}\frac{7n^2-4}{3n+1}$$

We can attempt the same method as before. And divide out an n (the highest power of the denominator) from the top and bottom. Note: We can also use L'Hopitals rule to evaluate these limits.

$$\lim_{n\to\infty}\frac{7n^2-4}{3n+1}=\lim_{n\to\infty}\frac{7n-\frac{4}{n}}{3+\frac{1}{n}}=\frac{7(\infty)-\frac{4}{\infty}}{3+\frac{1}{\infty}}=\frac{\infty-0}{3+0}=\infty$$

Therefore $\left\{\frac{7n^2-4}{3n+1}\right\}_{n=1}^{\infty}$ diverges.

Example:

Determine whether the sequence $\{(-1)^n\}_{n=1}^{\infty}$ converges.

We will first evaluate the limit below:

$$\lim_{n\to\infty}(-1)^n$$

If we write out the terms of the solution, we will have

$$\{-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,\dots\}$$

Notice the terms oscillates between -1 and 1, so this sequence does not converge, it diverges.

17. Write out the first six terms of the sequences whose nth terms are given below. Note: The sequences are $\{a_n\}_{n=1}^{\infty}$.

a) $a_n = (-1)^n n^2$

b) $a_n = \frac{1}{n^3}$

c) $a_n = \frac{2n-1}{5}$

d) $a_n = \left(\frac{-1}{2}\right)^n$

18. Determine whether the sequences with their nth terms stated below converges. Note: The sequence is $\{a_n\}_{n=1}^{\infty}$.

a) $a_n = \frac{5n^2+2}{3n^2-1}$

b) $a_n = \frac{5n+2}{3n^2-1}$

c) $a_n = (-1)^n \left(\frac{n+1}{n^2+10}\right)$

d) $a_n = \frac{e^n+2}{e^{2n}-3}$

Section 12: Infinite Series

A partial sum for any sequence $\{a_k\}_{k=1}^{\infty}$, is the addition of the terms within the sequence. Let us denote the partial sums by $S_1, S_2, S_3, \ldots S_n$

$S_1 = a_1$
$S_2 = S_1 + a_2 = a_1 + a_2$
$S_3 = S_2 + a_3 = a_1 + a_2 + a_3$
$S_4 = S_3 + a_4 = a_1 + a_2 + a_3 + a_4$

$S_n = S_{n-1} + a_n = a_1 + a_2 + a_3 + \cdots a_{n-1} + a_n$

Infinite Series are infinite sums. Let us denote the sum of the series by S.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S$$

Example: Let $a_k = \frac{1}{k}$ for $k = 1, 2, 3, 4, 5, 6, \ldots$

$$S = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Suppose we want the 3rd partial sum of the series

$$S_3 = \sum_{n=1}^{3} a_n$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_3 = \frac{11}{6}$$

If the sequence of partial sums $S_N = \sum_{k=1}^{n} a_k$ converges to a number S, then the series $\sum_{k=1}^{\infty} a_k$ also **converges** to S. Else $\{S_n\}_{n=1}^{\infty}$ diverges $\left(\lim_{n\to\infty} S_n \text{ does not have a limit}\right)$, then the series S **diverges**.

19. Find the first six partial sums of

a) $\sum_{k=1}^{\infty} \frac{1}{2^k}$

b) $\sum_{k=1}^{\infty} \frac{1}{n}$

Section 13: Convergence and divergence of a series

Important tests for convergence and divergence

The n'th Term Test:

$\lim_{n\to\infty} a_n \neq 0$ implies that $\sum_{n=1}^{\infty} a_n$ **diverges**.

Example:

$\sum_{n=1}^{\infty} \frac{3n-5}{2n+1}$ diverges since $\lim_{n\to\infty} a_n = \frac{3}{2}$

The p - Series Test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is **convergent** if and only if $p > 1$.

Example:

$\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ **converges** since $p = 1.2$

The Integral Test:

For $f(x)$ continuous and decreasing with $f(n) = a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ **is convergent** if and only if $\int_1^{\infty} f(x)dx$ **converges**.

Example:

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ **converges** since $\frac{1}{1+x^2}$ this function is continuous, decreasing and $\int_1^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{4}$

The Geometric Series Test:

$a_n = ar^{n-1}$; $\sum_{n=1}^{\infty} ar^{n-1}$ **converges** if and only if $|r| < 1$.

Example:

$\sum_{k=1}^{\infty} \frac{5}{7^k}$ **converges** since $r = \frac{1}{7}$.

Important tests for convergence and divergence cont'd.

The Alternating Series Test:

Let $b_n = (-1)^n a_n$ or $b_n = (-1)^{n-1} a_n$ and $a_n > 0$. If $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing then $\sum_{n=1}^{\infty} b_n$ **converges.**

Example:

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ **converges** since $\lim_{n\to\infty} a_n = 0$, $a_n > 0$ and a_n is decreasing

The Root Test:

Consider $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$; If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ **converges absolutely**;

If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ **diverges**.

Example:

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n+1}{5n}\right)^n$ **converges absolutely** since $|a_n| = \left(\frac{3n+1}{5n}\right)^n$ and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = \frac{3}{5} < 1$

The Ratio Test:

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$; If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ **converges absolutely**;

If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ **diverges**.

Example:

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$ **converges absolutely** since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} = L < 1.$

The Comparison Test:

For $0 \le a_n \le b_n$; If $\sum_1^{\infty} a_n$ **diverges** then $\sum_1^{\infty} b_n$ **diverges**; If $\sum_1^{\infty} b_n$ **converges** then $\sum_1^{\infty} a_n$ **converges**.

Example:

$\sum_{n=1}^{\infty} \frac{1}{n^5+7}$ **converges.** Choose $b_n = \frac{1}{n^5}$

20. Classify the following series as convergent or divergent using one of the eight tests outlined above.

a) $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

b) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

c) $\sum_{n=1}^{\infty} \frac{3n}{e^n}$

d) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$

e) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^5}\right)$

f) $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$

g) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$