$\int \tan(x)\sec^2(x)\,dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$\int u\,du$

$\frac{u^2}{2} + C$

$\frac{\tan^2(x)}{2} + C$

$\int \tan(x)\sec^2(x)\,dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$\int \sec(x)\cdot\sec(x)\tan(x)\,dx$

$\int u\,du$

$\frac{u^2}{2} + C$

$\frac{\sec^2(x)}{2} + C$

$\sec^2(x) - 1 = \tan^2(x)$

* Recall that $\tan^2(x) + 1 = \sec^2(x)$

1 is a possible value for C.

$\rightarrow$ $\frac{\tan^2(x)}{2} + C$ and $\frac{\sec^2(x)}{2} + C$ are vertical translations of each other.

are in the same family of antiderivatives.

$$\int \csc(x)\,dx$$

$$\int \csc(x)\,\frac{\csc(x)+\cot(x)}{\csc(x)+\cot(x)}\,dx$$

$$\int \frac{\csc^2(x)+\csc(x)\cot(x)}{\csc(x)+\cot(x)}\,dx$$

$$\int \underbrace{\frac{1}{\csc(x)+\cot(x)}}_{u}\,\underbrace{\left(\csc^2(x)+\csc(x)\cot(x)\right)}_{-du}\,dx$$

$u=\csc(x)+\cot(x)$

$du=\left(-\csc(x)\cot(x)-\csc^2(x)\right)dx$

$\quad=-\left(\csc(x)\cot(x)+\csc^2(x)\right)dx$

$-du=\left(\csc(x)\cot(x)+\csc^2(x)\right)dx$

$$-\int \frac{1}{u}\,du$$

$$-\ln|u|+c$$

$$-\ln|\csc(x)+\cot(x)|+c$$

$\int \sec(x)\,dx$ by u-sub.

$$\int \frac{\sec(x)}{1}\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$u=\sec(x)+\tan(x)$

$du=\left(\sec(x)\tan(x)+\sec^2(x)\right)dx$

$$\int \frac{1}{\sec(x)+\tan(x)}\cdot\left(\sec^2(x)+\sec(x)\tan(x)\right)dx$$

$$\int \frac{1}{u}\,du$$

$$\ln|u|+c$$

$$\int \sec(x)\,dx=\ln|\sec(x)+\tan(x)|+c \quad *$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

Observe $x^2-9 \sim \sec^2\theta - 1$

For Pythagorean trig identity, need constant $= 1$

Let $x^2 = 9\sec^2\theta$

$* \; x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\longrightarrow \sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x} = \frac{a}{h}$

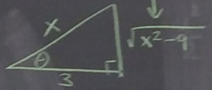
$a^2+b^2=c^2$

$a^2+3^2=x^2$

$a^2 = x^2-9$

$a = \sqrt{x^2-9}$

$\downarrow$



$\rightarrow \sin\theta = \frac{a}{h} = \frac{\sqrt{x^2-9}}{x}$

Recall

$\cos^2\theta + \sin^2\theta = 1$

$\rightarrow 1-\sin^2\theta = \cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{1}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}\,(3\sec\theta\tan\theta\,d\theta)$$

$$\int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$\int \frac{\cancel{3}\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\cancel{\sqrt{9}}\sqrt{\sec^2\theta-1}}$$

$$\frac{1}{9}\int \frac{\cancel{\tan\theta}\,d\theta}{\sec\theta\cancel{\sqrt{\tan^2\theta}}}$$

$$\frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$

$$\frac{1}{9}\int \cos\theta\,d\theta$$

$$\frac{1}{9}\sin\theta + C$$

Need in terms of x

$$\frac{1}{9}\cdot\frac{\sqrt{x^2-9}}{x} + C$$

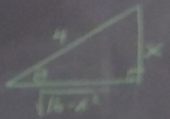
$$\boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$

$$\int \frac{dx}{x^2\sqrt{16-x^2}}$$

$x^2 = 16\sin^2\theta$

$x = 4\sin\theta \longrightarrow \sin\theta = \frac{x}{4}$

$dx = 4\cos\theta\, d\theta$



$\cot\theta = \frac{a}{o} = \frac{\sqrt{16-x^2}}{x}$

Recall

$\cos^2\theta + \sin^2\theta = 1$

$\longrightarrow 1 - \sin^2\theta = \cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$$= \int \frac{4\cos\theta\, d\theta}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}$$

$$= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{16}\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{16}\int \frac{\cancel{\cos\theta}\, d\theta}{\sin^2\theta\,\cancel{\sqrt{\cos^2\theta}}}$$

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{16}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{16}\cot\theta + C$$

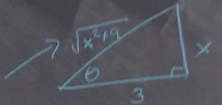
$$= -\frac{1}{16}\cdot\frac{\sqrt{16-x^2}}{x} + C = \boxed{-\frac{\sqrt{16-x^2}}{16x} + C}$$

$$\int \frac{x^2\,dx}{\sqrt{x^2+9}}$$

$x^2 = 9\tan^2\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$



$$\int \frac{9\tan^2\theta \cdot 3\sec^2\theta\,d\theta}{\sqrt{9\tan^2\theta + 9}}$$

$$\frac{27}{\sqrt{9}}\int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sqrt{\tan^2\theta+1}}$$

$$9\int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}}$$

$$9\int \tan^2\theta\sec\theta\,d\theta$$

$$9\int \tan^2\theta\sec\theta\,d\theta = 9\int(\sec^2\theta-1)\sec\theta\,d\theta$$

$$= 9\int \sec^3\theta - \sec\theta\,d\theta$$

$$= 9\int \sec^3\theta\,d\theta - 9\int\sec\theta\,d\theta$$

$$= 9\left(\sec\theta\tan\theta - \int\sec\theta\tan^2\theta\,d\theta\right) - 9\ln|\sec\theta+\tan\theta|$$

$$= 9\left(\frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2}\right) - 9\ln|\sec\theta+\tan\theta|$$

other board

$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{9}{2}\left(\frac{\sqrt{x^2+9}}{3}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right| + C = \boxed{\frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + C}$$

Recall

$\cos^2\theta + \sin^2\theta = 1$

$\rightarrow 1 - \sin^2\theta = \cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$\int \sec^3\theta\,d\theta =$

$\int \sec\theta\sec^2\theta\,d\theta$

$u = \sec\theta$ $\quad dv = \sec^2\theta\,d\theta$

$du = \sec\theta\tan\theta\,d\theta$ $\quad v = \tan\theta$

$$\int \sec^3\theta \, d\theta = \int \sec\theta \, \sec^2\theta \, d\theta \qquad u = \sec\theta \qquad dv = \sec^2\theta \, d\theta$$

$$uv - \int v \, du \qquad du = \sec\theta \tan\theta \, d\theta \qquad v = \tan\theta$$

$$= \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta$$

$$= \sec\theta \tan\theta - \int (\sec^3\theta - \sec\theta) \, d\theta$$

$$= \sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| - \int \sec^3\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta = \frac{\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|}{2}$$

$$\int_0^1 \frac{x\,dx}{\sqrt{4-x^2}}$$

via trig sub

$x^2 = 4\sin^2\theta$

$x = 2\sin\theta \longrightarrow 1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\frac{1}{2} = \sin\theta \quad \theta = \frac{\pi}{6}$

$0 = 2\sin\theta$

$0 = \sin\theta \quad \theta = 0$

$$\int_0^{\frac{\pi}{6}} \frac{2\sin\theta\ 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$\frac{4}{\sqrt{4}}\int_0^{\frac{\pi}{6}} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$2\int_0^{\frac{\pi}{6}} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$2\int_0^{\frac{\pi}{6}} \sin\theta\, d\theta$$

$$-2\cos\theta\Big|_0^{\frac{\pi}{6}}$$

$$\left(-2\cos\left(\frac{\pi}{6}\right)\right) - \left(-2\cos(0)\right)$$

$$-\cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right) + 2(1)$$

$$\boxed{2-\sqrt{3}}$$

$$\int\left(\frac{3}{x-1}+\frac{1}{x+2}\right)dx$$

$$\int\frac{3}{x-1}\,dx+\int\frac{1}{x+2}\,dx$$

$$3\ln|x-1|+\ln|x+2|+C$$

$$\frac{x+2}{x+2}\cdot\frac{3}{x-1}+\frac{1}{x+2}\cdot\frac{x-1}{x-1}$$

LCD: $(x-1)(x+2)$

$$\frac{3(x+2)+(x-1)}{(x-1)(x+2)}$$

$$\frac{3x+6+x-1}{(x-1)(x+2)}$$

$$\frac{4x+5}{(x-1)(x+2)}$$

$$\int\frac{4x+5}{x^2+x-2}\,dx = 3\ln|x-1|+\ln|x+2|+C$$

$$\int\frac{4x+5}{(x-1)(x+2)}\,dx = \int\left(\frac{3}{x-1}+\frac{1}{x+2}\right)dx$$

How?

Make Partial Fractions

$$\frac{4x+5}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2} = \frac{3}{x-1}+\frac{1}{x+2}$$

$$\frac{4x+5}{\cancel{(x-1)(x+2)}} = \frac{A(x+2)+B(x-1)}{\cancel{(x-1)(x+2)}}$$

Let $x=-2$

$$4(-2)+5 = \cancel{A(-2+2)}+B(-2-1)$$

$$-3 = -3B$$

$$1 = B$$

Let $x=1$

$$4(1)+5 = A(1+2)+\cancel{B(1-1)}$$

$$9 = 3A$$

$$3 = A$$

$$\int \frac{x^2+x-5}{x^2-1}\,dx$$

$\deg(n) \geq \deg(d)$

$\rightarrow$ polynomial division is required

$$\begin{array}{r} 1 \\ x^2-1 \overline{\big)\, x^2 + x - 5} \\ -(x^2 \quad\;\; - 1) \\ \hline x - 4 \end{array}$$

$$= \int \left(1 + \frac{x-4}{x^2-1}\right) dx$$

$$= x + \int \left(\frac{-\frac{3}{2}}{x-1} + \frac{\frac{5}{2}}{x+1}\right) dx$$

$$= \boxed{x - \frac{3}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + C}$$

$$\frac{x-4}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x-4 = A(x+1) + B(x-1)$$

$$x-4 = Ax + A + Bx - B$$

$$1x - 4 = (A+B)x + (A-B)$$

grouping coefficients of x and Cs

$$\begin{array}{l} A+B=1 \\ A-B=-4 \\ \hline 2A \quad = -3 \end{array}$$

$$A = -\frac{3}{2}$$

$\rightarrow A+B=1$

$$-\frac{3}{2}+B=1$$

$$B=\frac{5}{2}$$

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x}+\frac{-1x-1}{x^2+3}\right)dx$$

$$=\int\left(\frac{2}{x}-\frac{1x}{x^2+3}-\frac{1}{x^2+3}\right)dx$$

$$\boxed{=2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+C}$$

$$\int \frac{x}{x^2+3}\,dx \qquad u=x^2+3$$

$$du = 2x\,dx$$

$$\int \frac{x}{u}\,\frac{du}{2x} \qquad \frac{du}{2x}=dx$$

$$\frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}\ln|u|$$

$$\frac{1}{2}\ln|x^2+3|$$

$$\int \frac{1}{x^2+3}\,dx \sim \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)$$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x}+\frac{Bx+C}{x^2+3}$$

↑ Cannot be factored

$$x^2-x+6 = A(x^2+3)+(Bx+C)x$$

Let $x=0$

$$(0)^2-(0)+6 = A((0)^2+3)+\cancel{(B(0)+C)(0)}$$

$$6=3A$$

$$A=2$$

$$x^2-x+6 = 2(x^2+3)+(Bx+C)x$$

$$x^2-x+6 = 2x^2+6+Bx^2+Cx$$

$$1x^2-1x+6 = (2+B)x^2+Cx+6$$

$$2+B=1 \rightarrow B=-1$$

$$C=-1$$

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$$

$$= \int \frac{\frac{1}{15}}{x-4} + \frac{-\frac{1}{15}x - \frac{6}{15}}{x^2+2x+6}\,dx$$

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{x^2+2x+6}\,dx$$

Finish this

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+1}{x^2+2x+6} + \frac{5}{x^2+2x+6}\,dx$$

$u = x^2+2x+6$

$du = (2x+2)\,dx$

$du = 2(x+1)\,dx$

$\frac{du}{2} = (x+1)\,dx$

Finish the rest.

CT's

x^2+2x+6

$x^2+2x+1-1+6$

$(x+2)^2+5$

$(x+2)^2+(\sqrt{5})^2$

$$\frac{2}{(x-4)(x^2+2x+6)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+2x+6}$$

$$2 = A(x^2+2x+6) + (Bx+C)(x-4)$$

Let $x = 4$

$2 = A(4^2+2\cdot 4+6) + \cancel{(B(4)+C)(4-4)}$

$2 = 30A$

$\frac{1}{15} = A$

$0x^2+0x+2 = \frac{1}{15}x^2 + \frac{2}{15}x + \frac{6}{15} + Bx^2 + Cx - 4Bx - 4C$

$0x^2+0x+2 = \left(\frac{1}{15}+B\right)x^2 + \left(\frac{2}{15} - 4B + C\right)x + \left(\frac{6}{15} - 4C\right)$

$\frac{1}{15} + B = 0$

$B = -\frac{1}{15}$

$\frac{2}{5} - 4C = 2$

$-4C = \frac{8}{5}$

$C = -\frac{2}{5}$

$$5\int \frac{1}{x^2+2x+6}\,dx$$

$$5\int \frac{1}{(x+2)^2+5}\,dx$$

$$5\int \frac{1}{5u^2+5}\,\sqrt{5}\,du$$

$$\sqrt{5}\int \frac{1}{u^2+1}\,du$$

$$\sqrt{5}\,\arctan(u)$$

$$\sqrt{5}\,\arctan\left(\frac{x+2}{\sqrt{5}}\right)$$

Let $(x+2)^2 = 5u^2$

$$x+2 = \sqrt{5}\,u \rightarrow u = \frac{x+2}{\sqrt{5}}$$

$$dx = \sqrt{5}\,du$$

$$dx$$