$$\int_{-3}^{3} x^2\,dx = \left.\frac{x^3}{3}\right|_{-3}^{3}$$

$$= \frac{(3)^3}{3} - \frac{(-3)^3}{3}$$

$$= 18$$

$$\int_{3}^{-3} x^2\,dx = \left.\frac{x^3}{3}\right|_{3}^{-3}$$

$$= \frac{(-3)^3}{3} - \frac{(3)^3}{3}$$

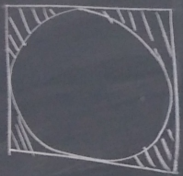
$$= -18$$

$$* \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_{-3}^{0} x^2\,dx + \int_{0}^{3} x^2\,dx = 18$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Recall $\int_a^b f(x)\,dx$ is signed area between graph of $f(x)$ and x-axis over interval $[a,b]$



How to find area of shaded region.

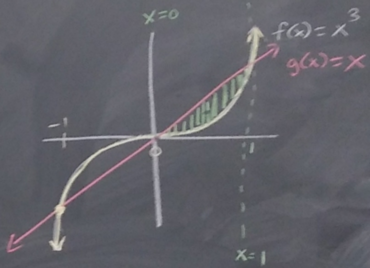
1. Find area of square
2. Find area of circle

→ 3. Area larger − Area Smaller

Find Area $f(x)=x^3$

$g(x)=x$

$x\in[0,1]$



Observe. over $[0,1]$

$x > x^3$

$$A=\int_a^b (\text{larger} - \text{smaller})\,dx$$

$$=\int_0^1 (x - x^3)\,dx$$

$$=\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Bigg|_0^1 = \left(\frac{1}{2}-\frac{1}{4}\right) - \cancel{\left(\frac{0}{2}-\frac{0}{4}\right)} = \frac{1}{4}$$

Note: check for intersection points

$f(x)=g(x)$

$x^3 = x$

$x^3 - x = 0$

$x(x^2-1)=0$

$x(x+1)(x-1)=0$

$x=-1, 0, 1$

omit

not in $[0,1]$

$f(x) = x^2 - 3$ $\quad [-2, 2]$

$g(x) = 5 - x^2$

1. Check for intersection points

$f(x) = g(x)$

$x^2 - 3 = 5 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = -2, 2$

intersection points are interval bounds

2. Which functions are larger over intervals

$5 - x^2 > x^2 - 3$ over $[-2, 2]$

Test over $[-2, 2]$ $\quad g(0) = 5 - (0)^2 = 5$

Let $x = 0$ $\quad f(0) = (0)^2 - 3 = -3$

$g(0) > f(0)$

$\rightarrow g(x) > f(x)$

3.) $A = \int_{-2}^{2} \left(\overset{g(x)}{(5 - x^2)} - \overset{f(x)}{(x^2 - 3)} \right) dx$

$A = \int_{-2}^{2} (8 - 2x^2)\, dx$

$A = \left(8x - \frac{2x^3}{3} \right) \Big|_{-2}^{2}$

$A = \left(8(2) - \frac{2(2)^3}{3} \right) - \left(8(-2) - \frac{2(-2)^3}{3} \right)$

$A = \boxed{\frac{64}{3}}$

Area between $f(x) = x^2$

$g(x) = \sqrt{x}$

no bounds

1. Intersection Points

$f(x) = g(x)$

$x^2 = \sqrt{x}$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x(x-1)\underline{\underline{(x^2+x+1)}} = 0$

x is imaginary

$\boxed{\begin{matrix} x = 0 \\ x = 1 \end{matrix}}$

2. larger over interval

interval $[0,1]$

$\rightarrow \sqrt{x} > x^2$ over $[0,1]$

3. Integral

$$\int_0^1 (\sqrt{x} - x^2)\,dx$$

$$\int_0^1 (x^{\frac{1}{2}} - x^2)\,dx$$

$$\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right)\Bigg|_0^1$$

$$\left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1$$

$$= \left(\frac{2(1)^{\frac{3}{2}}}{3} - \frac{(1)^3}{3}\right) - \cancel{\left(\frac{2(0)^{\frac{3}{2}}}{3} - \frac{(0)^3}{3}\right)} = \boxed{\frac{1}{3}}$$

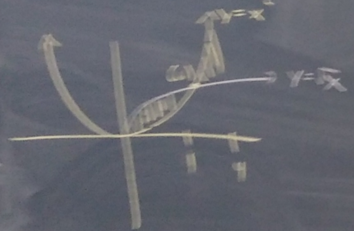
$f(x) = x^2$ $[0, 4]$

$g(x) = \sqrt{x}$

1. Intersection Points

From earlier: intersection points

$x = 0$, $x = 1$



2. Which is larger over interval

2 separate intervals

$[0,1]$ $\sqrt{x} > x^2$

$[1,4]$ $x^2 > \sqrt{x}$

3. Area

$$\int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^4 (x^2 - \sqrt{x})\,dx$$

$$\frac{1}{3} + \left(\frac{x^3}{3} - \frac{2x^{\frac{3}{2}}}{3}\right)\Bigg|_1^4$$

$$\frac{1}{3} + \left(\left(\frac{(4)^3}{3} - \frac{2(4)^{\frac{3}{2}}}{3}\right) - \left(\frac{(1)^3}{3} - \frac{2(1)^{\frac{3}{2}}}{3}\right)\right)$$

$$\frac{1}{3} + \left(\frac{48}{3} - \left(-\frac{1}{3}\right)\right)$$

$$\boxed{\frac{50}{3}}$$

Recall Chain Rule

Let f be a function of u

u be a function of (x)

$(f(u)) = f'(u) \cdot u'$

$= f'(u)\, du$

$$\int f(u)\, du$$

$u = g(x)$ $\quad$ $du = g'(x)$

$$\int x^4 \sqrt{x^5-9}\, dx$$

$u = x^5 - 9$

$du = 5x^4\, dx$

$\frac{du}{5} = x^4\, dx$

$$\int \sqrt{u}\, \frac{du}{5}$$

$$\frac{1}{5}\int u^{\frac{1}{2}}\, du$$

$$\frac{1}{5} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + C$$

$$\boxed{\frac{2}{15}\left(x^5-9\right)^{\frac{3}{2}} + C}$$

$$\int x^4 \sqrt{x^5-9}\, dx$$

Observation: $\frac{d}{dx}(x^5-9) = 5x^4$

$x^4 \sim \frac{5x^4}{5}$

$u = x^5 - 9$ $\quad$ $du = 5x^4\, dx$

$\frac{du}{5x^4} = dx$

$$\int x^4 \sqrt{x^5-9}\, dx = $$

$$= \int \cancel{x^4} \sqrt{u}\, \frac{du}{5\cancel{x^4}}$$

$$= \frac{1}{5}\int u^{\frac{1}{2}}\, du$$

$$= \frac{1}{5} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \boxed{\frac{2}{15}\left(x^5-9\right)^{\frac{3}{2}} + C}$$

$$\int \tan(x)\,dx$$

$$\int \frac{\sin(x)}{\cos(x)}\,dx$$

$$\int \frac{1}{\cos(x)} \cdot \sin(x)\,dx$$

$$\int \frac{1}{u} \cdot \cancel{\sin(x)} \cdot \frac{du}{\cancel{-\sin(x)}}$$

$$-\int \frac{1}{u}\,du$$

$$-\ln|u| + c$$

$$\boxed{\begin{gathered} -\ln|\cos(x)| + c \\ \vdots \\ \ln|\sec(x)| + c \end{gathered}}$$

$$u = \cos(x)$$

$$du = -\sin(x)\,dx$$

$$\frac{du}{-\sin(x)} = dx$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| + c$$

$$= \ln|\sec(x)| + c$$

$$\int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int \frac{1}{\sqrt{1+x^2}}\,x\,dx$$

$$\int \frac{1}{\sqrt{u}}\,\cancel{x}\,\frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$\frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$$

$$\cancel{\frac{1}{2}}\,\frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}+C$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$\boxed{\begin{aligned} &= (1+x^2)^{\frac{1}{2}}+C \\ &= \sqrt{1+x^2}+C \end{aligned}}$$

$$\int \frac{x}{1+x^2}\,dx$$

$$\int \frac{1}{1+x^2}\cdot x\,dx$$

$$\int \frac{1}{u}\cdot \cancel{x}\,\frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}\ln|u|+C$$

$$\boxed{\frac{1}{2}\ln|1+x^2|+C}$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$$

$$u = x^{\frac{1}{2}}$$

$$\int \sin(x^{\frac{1}{2}}) \cdot x^{-\frac{1}{2}}\,dx$$

$$du = \frac{1}{2} x^{-\frac{1}{2}}\,dx$$

$$\int \sin(u)\ x^{-\frac{1}{2}} \cdot \frac{2\,du}{x^{-\frac{1}{2}}}$$

$$\frac{2\,du}{x^{-\frac{1}{2}}} = dx$$

$$2\int \sin(u)\,du$$

$$2(-\cos(u)) + c$$

$$-2\cos(x^{\frac{1}{2}}) + c$$

$$\boxed{-2\cos(\sqrt{x}) + c}$$

$$\int e^{ax+b}\,dx$$

$u = ax+b$

$du = a\,dx$

$\frac{du}{a} = \frac{a\,dx}{a}$

$\frac{du}{a} = dx$

$$\int e^{u}\,\frac{du}{a}$$

$$\frac{1}{a}\int e^{u}\,du$$

$$\frac{1}{a}e^{u} + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Let a, b be constants

$$\int \frac{1}{ax+b}\,dx$$

$u = ax+b$

$du = a\,dx$

$\frac{du}{a} = \frac{a\,dx}{a}$

$\frac{du}{a} = dx$

$$\int \frac{1}{u}\cdot\frac{du}{a}$$

$$\frac{1}{a}\int \frac{1}{u}\,du$$

$$\frac{1}{a}\ln|u| + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int f(ax+b)\,dx$$

$$\frac{1}{a} F(ax+b) + C$$

$$\int \cos(ax+b)\,dx$$

$u = ax+b$

$du = a\,dx$

$\frac{du}{a} = dx$

$$\int \cos(u)\,\frac{du}{a}$$

$$\frac{1}{a}\int \cos(u)\,du$$

$$\frac{1}{a}\sin(u) + C$$

$$\frac{1}{a}\sin(ax+b) + C$$

$$\int_0^1 x\sqrt{x^2+2}\,dx$$

$u = x^2+2$ → $u(1) = (1)^2+2 = 3$

$du = 2x\,dx$

$\frac{du}{2x} = dx$ → $u(0) = (0)^2+2 = 2$

$$= \int_2^3 u^{\frac{1}{2}} \cdot x \cdot \frac{du}{2x}$$

$$= \frac{1}{2}\int_2^3 u^{\frac{1}{2}}\,du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{\frac{3}{2}} \Big|_2^3$$

$$= \frac{1}{3} u^{\frac{3}{2}} \Big|_2^3$$

$$= \frac{1}{3}(3)^{\frac{3}{2}} - \frac{1}{3}(2)^{\frac{3}{2}} = \frac{\sqrt{27} - \sqrt{8}}{3}$$

$$\int_0^1 x\sqrt{x^2+2}\,dx$$

$u = x^2+2$

$du = 2x\,dx$

$$\frac{1}{3}u^{\frac{3}{2}}$$

Note: didn't change limits of integration

$$\frac{1}{3}(x^2+2)^{\frac{3}{2}}\Big|_0^1$$

$$\frac{1}{3}((1)^2+2)^{\frac{3}{2}} - \frac{1}{3}((0)^2+2)^{\frac{3}{2}}$$

$$\frac{1}{3}(3)^{\frac{3}{2}} - \frac{1}{3}(2)^{\frac{3}{2}}$$

$$\frac{\sqrt{27} - \sqrt{8}}{3}$$