9. Find the derivative using combination rules

a) $y = -4x^2(2x^3 - 14)^4$

$f = -4x^2$	$g = (2x^3-14)^4$
$f' = -8x$	$g' = 4(2x^3-14)^3 \cdot 6x^2 = 24x^2(2x^3-14)^3$

$u = 2x^3 - 14 \rightarrow f'(u) \cdot u' = 4u^3 \cdot u'$

$f(u) = u^4$

$= 4(2x^3-14)^3 \cdot (2x^3-14)'$

$= 4(2x^3-14)^3 \cdot (6x^2)$

$= 24x^2(2x^3-14)^3$

$y' = f'g + fg'$

$y' = -8x(2x^3-14)^4 + (-4x^2)\,24x^2(2x^3-14)^3$

b) $y = x^3 \cdot \sqrt{x^2+3}$ $= x^3(x^2+3)^{\frac{1}{2}}$

$f = x^3$	$g = (x^2+3)^{\frac{1}{2}}$
$f' = 3x^2$	$g' = x(x^2+3)^{-\frac{1}{2}}$

$g' = h'(u)\,u'$

$= \frac{1}{2}u^{\frac{1}{2}-1} \cdot u'$

$= \frac{1}{2}u^{-\frac{1}{2}} \cdot u'$

$= \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(x^2+3)'$

$= \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x = x(x^2+3)^{-\frac{1}{2}}$

$\rightarrow u = x^2+3$

$\rightarrow h = u^{\frac{1}{2}}$

$y' = f'g + fg'$

$y' = 3x^2(x^2+3)^{\frac{1}{2}} + x^4(x^2+3)^{-\frac{1}{2}}$

c) $y = \frac{\sqrt{x^2+3}}{x^3}$

$f = (x^2+3)^{\frac{1}{2}}$	$g = x^3$
$f' = x(x^2+3)^{-\frac{1}{2}}$	$g' = 3x^2$

$u = x^2+3$

$h(u) = u^{\frac{1}{2}} \rightarrow f' = h'(u) \cdot u'$

$= \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot (x^2+3)'$

$= \frac{1}{\cancel{2}}(x^2+3)^{-\frac{1}{2}}(\cancel{2}x)$

$= x(x^2+3)^{-\frac{1}{2}}$

$y' = \frac{f'g - fg'}{g^2}$

$y' = \frac{x^4(x^2+3)^{-\frac{1}{2}} - 3x^2(x^2+3)^{\frac{1}{2}}}{x^6}$

SECTION 4 SUPPLEMENTARY EXERCISES

1) Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

c) $y = 6\sqrt{x} - \sqrt[3]{x}$

d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

e) $s(t) = t^2 + \frac{5}{t^2}$

f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$ (Do not use quotient rule!)

g) $f(x) = \dfrac{5x^2 - 2x + 1}{x}$ (Do not use quotient rule!)

2) Find the derivative using the Product or Quotient Rule.

a) $h(t) = (4t + 3)(t - 7)$

b) $y = 3x\sqrt{x + 5}$

c) $p(x) = \frac{x+5}{x^2-9}$

d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative using the Chain Rule and combination rules.

a) $v(x) = (2 - 4x)^{100}$

b) $v(x) = -x^3(2 - 4x)^{100}$

c) $y = \sqrt{x^2 + 3x + 4}$

4) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2, -10)$ using the differentiation formulas, and find the equation of the tangent line.

5) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2, -8)$ using the differentiation formulas, and find the equation of the tangent line.

6) Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x - 6}$ at the point $(5, -1)$ using the differentiation formulas, and find the equation of the tangent line.

Section 5: The Derivative of Trigonometric Functions

Derivative of all six Trigonometric Functions	
Sine	$\frac{d}{dx}\sin(x) = \cos(x)$
Cosine	$\frac{d}{dx}\cos(x) = -\sin(x)$
Tangent	$\frac{d}{dx}\tan(x) = \sec^2(x)$
Cotangent	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
Secant	$\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(x)$
Cosecant	$\frac{d}{dx}\csc(x) = -\csc(\mathrm{x})\cot(x)$

Example: Find the derivative of the trigonometric function $f(x) = 2x^3\cos(5x)$ using the rules of differentiation.

A derivative that requires a combination of the Product Rule and the Chain Rule

Product Rule: $(fg)' = f'g + g'f$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}(2x^3\cos(x)) = \left[\frac{d}{dx}(2x^3)\right]\cos(5x) + 2x^3\left[\frac{d}{dx}\cos(5x)\right]$$

$$= 6x^2\cos(5x) + 2x^3(-\sin(5x)\cdot 5)$$

$$= 6x^2\cos(5x) - 10x^3\sin(5x)$$

Exercise 1: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $y = 2\sin x$

$y' = 2\frac{d}{dx}(\sin(x))$

$y = 2\cos(x)$

b) $y = \sin(2x)$

$x \to u = 2x \to f = \sin(u)$

$\frac{dy}{dx} = f'(u)\cdot u'$

$= \cos(u)\cdot u'$

$= \cos(2x)\cdot(2x)'$

$= 2\cos(2x)$

$y = \sin(2x)$

$= 2\sin(x)\cos(x)$

$f: 2\sin(x)$	$g: \cos(x)$
$f': 2\cos(x)$	$g': -\sin(x)$

$y' = f'g + fg'$

$y' = 2\cos(x)\cos(x) + 2\sin(x)(-\sin(x))$

$y' = 2\cos^2(x) - 2\sin^2(x)$

$y' = 2(\cos^2(x) - \sin^2(x))$

$y' = 2\cos(2x)$

c) $y = x\sin(x)$

$f: x$	$g: \sin(x)$
$f': 1$	$g' = \cos(x)$

$\frac{dy}{dx} = 1\cdot\sin(x) + x\cos(x)$

d) $y = \sin(x^2)$

x

$\to u = x^2$

$\to f(u) = \sin(u)$

$y' = f'(u)\cdot u'$

$= \cos(x^2)\cdot 2x$

$y' = 2x\cos(x^2)$

e) $y = \sin^2(x) = (\sin(x))^2$

x

$\to u = \sin(x)$

$\to f(u) = u^2$

$y' = f'(u)\cdot u'$

$= 2u\cdot u'$

$= 2\sin(x)\cdot\cos(x)$

$= \sin(2x)$

f) $y = \sin^2(x^2) = (\sin(x^2))^2$

x

$\to v = x^2$

$\to u(v) = \sin(v)$

$\to f(u) = u^2$

$\frac{dy}{dx} = f'(u)\cdot u'$

$= 2u\cdot u'$

$= 2\sin(v)\cdot\cos(v)\cdot v'$

$= 2\sin(x^2)\cos(x^2)\cdot 2x$

$= 4x\sin(x^2)\cos(x^2)$

$= 2x\sin(2x^2)$

Exercise 2: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $r(x) = x\cos(2x^2)$

$f: x$	$g: \cos(2x^2)$
$f': 1$	$g': -\sin(2x^2)\cdot 4x$

$r'(x) = 1\cdot\cos(2x^2) + x(-4x\sin(2x^2))$

$= \cos(2x^2) - 4x^2\sin(2x^2)$

b) $g(x) = \tan\left(\frac{3x}{4}\right)$

x

$\rightarrow u = \frac{3}{4}x$

$\rightarrow f(u) = \tan(u)$

$\frac{dg}{dx} = f'(u)\cdot u'$

$= \sec^2(u)\cdot u'$

$= \sec^2(\frac{3}{4}x)\cdot\frac{3}{4}$

$= \frac{3}{4}\sec^2(\frac{3}{4}x)$

c) $k(x) = \csc x \cdot \cot x$

$f: \csc(x)$	$g: \cot(x)$
$f': -\csc(x)\cot(x)$	$g' = -\csc^2(x)$

$k'(x) = -\csc(x)\cot^2(x) - \csc^3(x)$

d) $h(x) = \frac{\cos(2x)}{\sin(x)+1}$

e) $f(x) = \sqrt{\sin x + 5}$

Exercise 3: Show $\frac{d}{dx}\tan x = \sec^2 x$. (Hint, rewrite $\tan x = \frac{\sin x}{\cos x}$)

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \quad \leftarrow f, \leftarrow g$$

$$= \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1^2}{\cos^2(x)}$$

$$= \left(\frac{1}{\cos(x)}\right)^2$$

$$= (\sec(x))^2$$

$$= \sec^2(x) \qquad \text{QED}$$

$f: \sin(x)$	$g: \cos(x)$
$f': \cos(x)$	$g': -\sin(x)$

$$\frac{f'g - fg'}{g^2}$$

$* \cos^2(x) + \sin^2(x) = 1$

SECTION 5 SUPPLEMENTARY EXERCISES

1. $v(x) = \tan\left(\sqrt{x^3 + 2}\right)$
2. $n(x) = 5\cos^3(x) - \sin(2x)$
3. $g(x) = 2x^2\sec^2(8x)$
4. $f(x) = \frac{\sin(x)\sec(5x)}{3x^2}$
5. $y = \frac{\cos(x)}{2\sin(-3x)}$
6. $y = \sqrt{\sin^2(x) + 5}$

Section 6: Higher Order Derivatives

One can determine higher order derivatives by finding the derivative of derivatives. For example, the second derivative is the derivative of the derivative of the function (or the first derivative). The third derivative is the derivative of the second derivative, etc. The higher order derivatives are denoted by

$$y' = \frac{dy}{dx},\quad y'' = \frac{d^2y}{dx^2},\quad y''' = \frac{d^3y}{dx^3},\quad y^{(4)} = \frac{d^4y}{dx^4}, \ldots,\quad y^{(n)} = \frac{d^ny}{dx^n}$$

or in function notation $f', f'', f''', f^{(4)}, \ldots, f^{(n)}$.

Exercise 1. Find $f', f'', f''', f^{(4)}$ of the polynomial function $f(x) = x^3 - 2x^2 - 5x + 6$. What is the pattern for 4th and higher order derivatives?

$f(x)=x^3-2x^2-5x+6$

$f'(x)=3x^2-4x-5$ $\qquad f^{(n)}(x)=0\ ,\ n\geq 4$

$f''(x)=6x-4$

$f'''(x)=6$

$f^{(4)}=0$

Exercise 2. Find $y', y'', y''', y^{(4)}$ of the rational function $y = \frac{1}{x}$. What is the pattern for 4th and higher order derivatives?

$y=x^{-1}$

$y=x^{-1}$

$y'=-1x^{-2}$

$y''=2x^{-3}$

$y'''=-6x^{-4}$

$y^{(4)}=24x^{-5}$

$y=1x^{-1}$

$y'=1\cdot(-1)x^{-2} = -1!\ x$

$y''=(-1)(-2)x^{-3} = 2!$

$y'''=(-1)(-2)(-3)x^{-4}=-3!$

$y^{(4)}=(-1)(-2)(-3)(-4)x^{-5}=4!$

$y^{(n)}=(-1)^n\, n!\ x^{-1-n}$

Exercise 3. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$, of the sine function $y = \sin x$. What is the pattern for 4th and higher order derivatives?

$y = \sin(x)$

$\frac{dy}{dx} = \cos(x) \longrightarrow \frac{d^{4n+1}y}{dx^{4n+1}} = \cos(x)$

$\frac{d^2y}{dx^2} = -\sin(x) \longrightarrow \frac{d^{4n+2}y}{dx^{4n+2}} = -\sin(x)$

$\frac{d^3y}{dy^3} = -\cos(x) \longrightarrow \frac{d^{4n+3}y}{dx^{4n+3}} = -\cos(x)$

$\frac{d^4y}{dy^4} = \sin(x) \longrightarrow \frac{d^{4n}y}{dy^{4n}} = \sin(x)$, $n \in \mathbb{N}$

$n \in \{1, 2, 3, \dots\}$

SECTION 6 SUPPLEMENTARY EXERCISES

1. Find $f', f'', f''', f^{(4)}$ of the polynomial function $f(x) = x^4 + 8x^3 - 9x^2 - 15x + 89$. What is the pattern for 4th and higher order derivatives?

2. Find $y', y'', y''', y^{(4)}$ of the radical function $y = \sqrt{x}$. What is the pattern for 4th and higher order derivatives?

3. Find $g', g'', g''', g^{(4)}$ of the polynomial function $g(x) = \tan x$. What is the pattern for 4th and higher order derivatives?

Section 7: Derivatives and the Shape of a Graph

Critical Points

Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.

In other words, a point c is a critical point if it satisfies the following:

- c is an interior point in the domain of f.
- $f(c)$ is defined.
- Either $f'(c) = 0$ or $f'(c)$ is undefined.

Intervals of Increasing and Decreasing

i. If $f'(x) > 0$ on an open interval, then f is increasing on the interval.

ii. If $f'(x) < 0$ on an open interval, then f is decreasing on the interval.

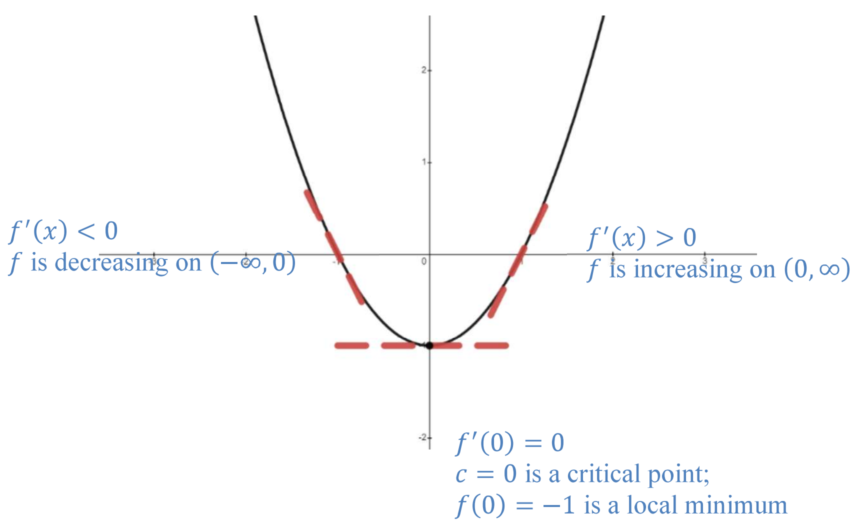
First Derivative Test

Suppose that f is a continuous function over an interval containing a critical point c. If f is differentiable on the interval, except possibly at point c then $f(c)$ satisfies one of the following descriptions:

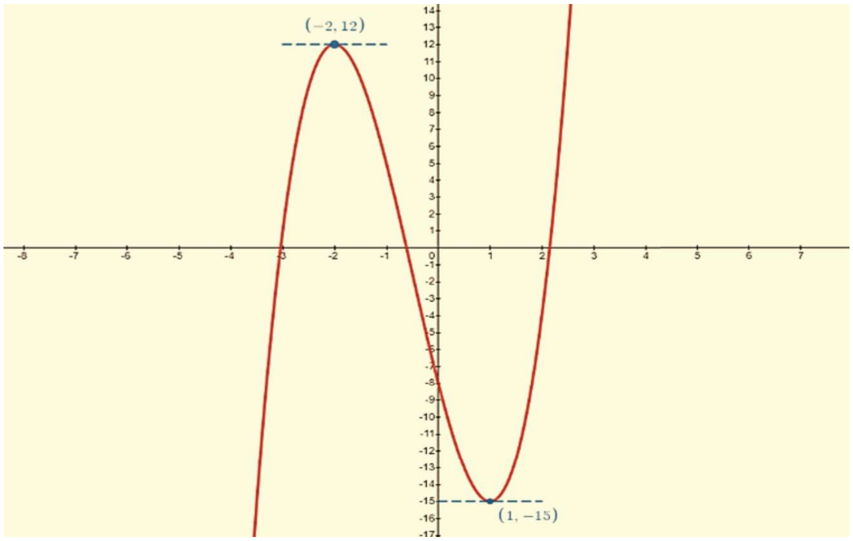
i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.

ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.

iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.



Exercise 1. Curve analysis of a polynomial function: $f(x) = 2x^3 + 3x^2 - 12x - 8$



- Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$
- Show Critical points are $c = -2$ and $c = 1$.
- The point $(-2, 12)$ is a local maximum; the point $(1, -15)$ is a local minimum
- Show the function is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$ and decreasing on the interval $(-2, 1)$

$f(x) = 2x^3 + 3x^2 - 12x - 8$

$f'(x) = 6x^2 + 6x - 12$

Critical Points $f'(x) = 0$, $f'(x)$ undefined

$f'(x) = 0$

$6x^2 + 6x - 12 = 0$

$6(x^2 + x - 2) = 0$

$6(x+2)(x-1) = 0$

$6 \neq 0$ $\quad x+2=0$ or $x-1=0$

$x = -2$ $\quad x = 1$

$\rightarrow$ CP @ $x \in \{-2, 1\}$

$f'(x)$ is undefined on $x \in \varnothing$

First Derivative Test

Sign Diagram of $f'(x)$

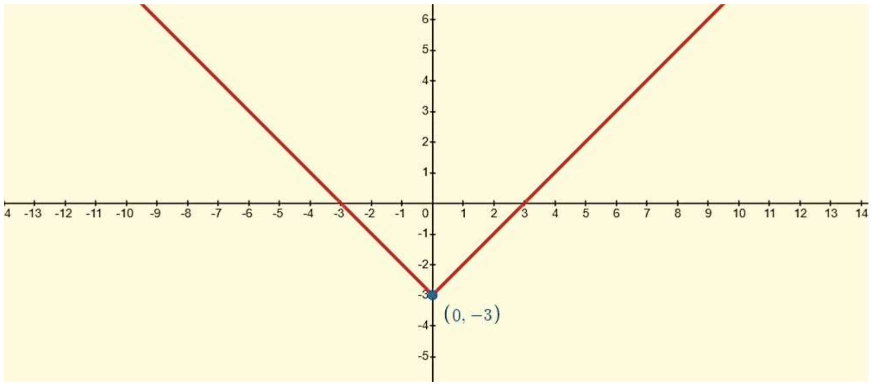
Factors of $f'(x)$	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,\infty)$
6	$+$	$+$	$+$	$+$	$+$
$x+2$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
	$\nearrow$		$\searrow$		$\nearrow$

$\therefore$ max @ $x = -2$ $\quad$ min @ $x = 1$

$f(-2) = 12$ $\quad$ $f(1) = -15$

$(-2, 12)$ $\quad$ $(1, -15)$

Exercise 2. Curve analysis of an absolute value function: $f(x) = |x| - 3$



- Domain = $(-\infty, \infty)$; Range = $(-3, \infty)$
- Show the critical point is $c = 0$. The derivative $f'(0)$ is undefined.

  Why is $f'(0)$ undefined? Hint: What is $f'(x)$ on the interval $(-\infty, 0)$? What is $f'(x)$ on the interval $(0, \infty)$? Is it true that $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$?
- The point $(0, -3)$ is a local minimum; there is no local maximum.
- Show the function is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$

$f(x) = |x| - 3$

$$f(x) := \begin{cases} -x-3, & x \in (-\infty, 0) \\ x-3, & x \in [0, \infty) \end{cases}$$

$$f'(x) := \begin{cases} -1, & x \in (-\infty, 0) \\ 1, & x \in (0, \infty) \end{cases}$$

CP $f'(x) = 0$, $f'(x)$ undefined

$$f'(x) := \begin{cases} -1, & x \in (-\infty, 0) \\ 1, & x \in (0, \infty) \end{cases}$$

$f'(x) \neq 0$

$f'(x)$ is undefined @ $x = 0$

$\rightarrow$ CP @ $x = 0$

$f(0) = -3$ $\quad (0, -3)$

First Derivative Test

	$(-\infty, 0)$	0	$(0, \infty)$
-1	$-$		
1			$+$
$f'(x)$	$-$	U	$+$
	$\searrow$		$\nearrow$

As $f(x)$ is defined @ $x = 0$

local minimum @ $x = 0$