Section 1: Finding Limits Graphically and Numerically

Finding Limits Graphically and Numerically

Definition of limits:

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of x (except for $x = a$) approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.

$$\lim_{x \to a} f(x) = L$$

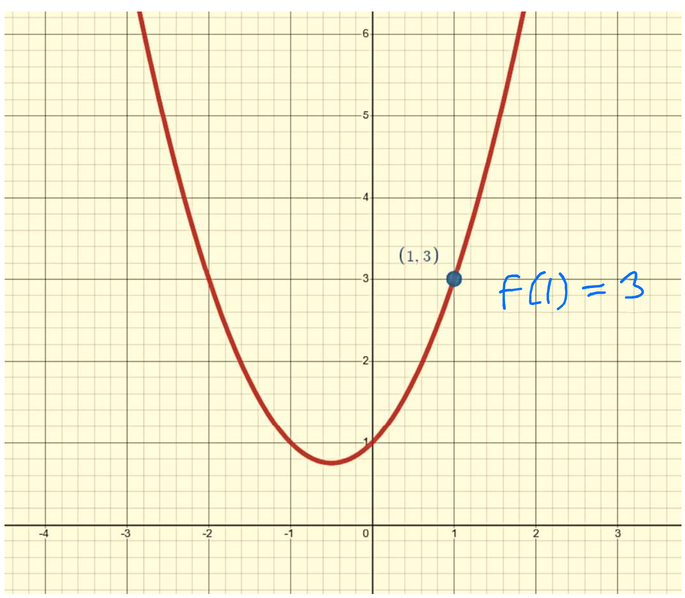
Remark about limit:

1. The function $f(x)$ does not need to be defined at a. The emphasis is on the word "approach."
2. Another key point is that "all" values of the function $f(x)$ must approach the same number L. This means $f(x)$ must approach L whether x is approaching a from the left or from the right.
3. If a function $f(x)$ is continuous at a, then the limit of $f(x)$ at a is $f(a)$.

Example 1: Find

$$\lim_{x \to 1} x^2 + x + 1$$

A graphical method shows the limit of $f(x) = x^2 + x + 1$ as x approaches 1 is 3.



A numerical method shows the same result.

x Approaches 1 from the left → ← x Approaches 1 from the right

x	.75	.9	.99	.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.170	2.970	2.997	3	3.003	3.030	3.310	3.813

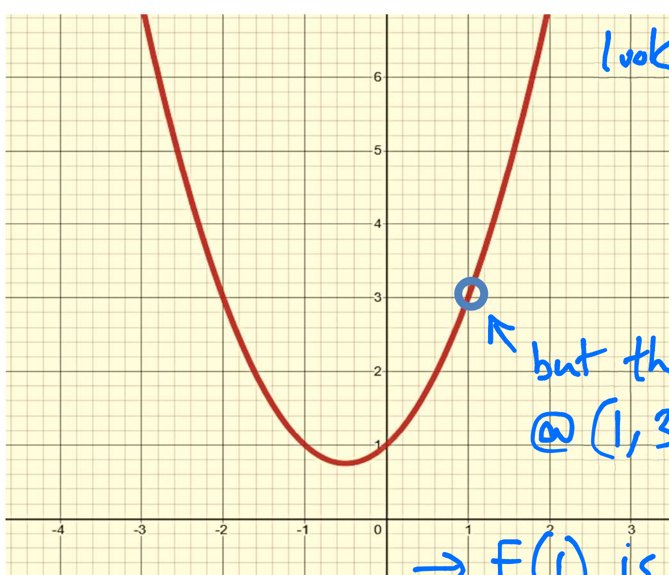
$f(x)$ Approaches 3 → ← $f(x)$ Approaches 3

Furthermore, because $f(x)$ is continuous at 1, we can simply substitute 1 into the function to determine that the limit is $f(1) = 3$.

Example 2: Find

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

The graphical method shows the same graph. However, $f(x)$ is not defined at 1, so we must assume the graph at the point 1 does not exist, or there is a "hole". We use an open circle to represent that the function is not defined at a particular point.



looks like

$y = x^2 + x + 1$

but there is a hole @ $(1, 3)$

→ $f(1)$ is undefined

The numerical method shows the values of $f(x)$ approaches 3 as x approaches 1, with the value at exactly 1 being undefined. In other words, $f(1)$ is undefined, but

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

x Approaches 1 from the left → ← x Approaches 1 from the right

x	.75	.9	.99	.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.170	2.970	2.997	Error	3.003	3.030	3.310	3.813

$f(x)$ Approaches 3 → ← $f(x)$ Approaches 3

$x - 1 = 0$

$x = 1$

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 1 & 1 & 0 & 0 & -1 \\ & \downarrow & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \\ & x^2 & x & c & \end{array} \leftarrow \text{Remainder}$$

$\rightarrow 1x^2 + 1x + 1$

Furthermore, if we factor and reduce the expression algebraically

$$\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

This explains why the limit in example 1 is equal to the limit in example 2. Or

"$\frac{0}{0}$" indeterminate form

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 3$$

→ there is a hole @ $x=1$

We say $f(x)$ has a "removable discontinuity" at 1.

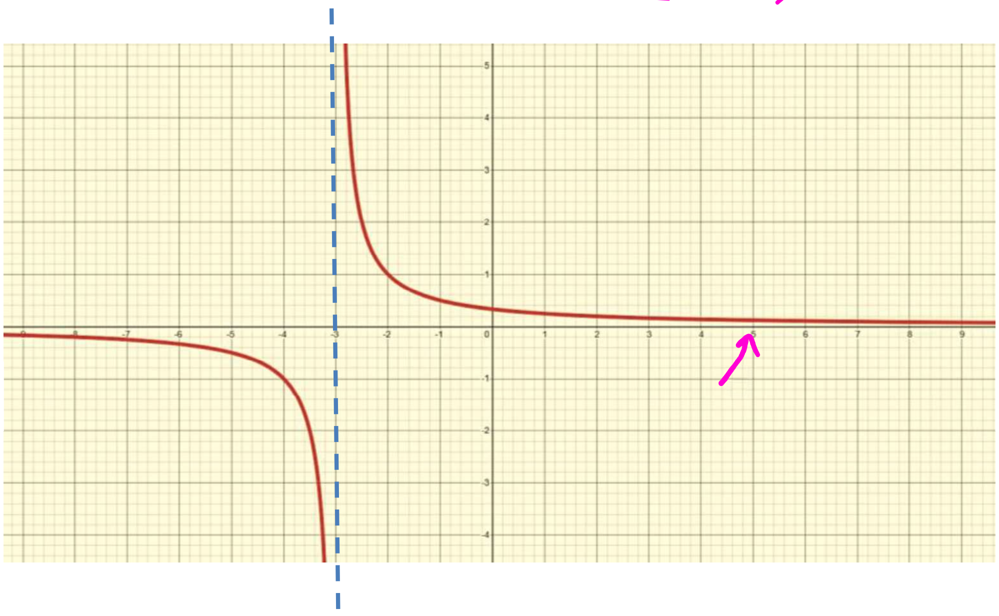
Keep in mind, although the two limits are equal, the two functions $\frac{x^3-1}{x-1}$ and $x^2 + x + 1$ are not the same and don't have the same domain. The two functions behave and graph "almost" the same, except at the point $x = 1$.

Exercise 1: Find

$$\lim_{x\to 5}\frac{x-5}{x^2-2x-15}$$

a) Use the graph to estimate the limit.

look @ f(x) when x=5



b) Create a table with values on both sides as x approaches 5.

x Approaches 5 from the left

x Approaches 5 from the right

x	4.9	4.99	4.999	5	5.001	5.01	5.1
$f(x)$	.127	.125	.125	Error	.125	.125	.123

$f(x)$ Approaches ?

$f(x)$ Approaches ?

.125 ?

c) Use algebraic method to factor and reduce the expression, then find the limit.

$$\lim_{x\to 5}\frac{x-5}{x^2-2x-15}=\lim_{x\to 5}\frac{x-5}{(x+3)(x-5)}=\lim_{x\to 5}\frac{1}{x+3}=\frac{1}{(5)+3}=\frac{1}{8}$$

$$=.125$$

Exercise 2: Find

$$\lim_{x\to -3} \frac{x-5}{x^2-2x-15}$$

a) Use the graph to estimate the limit.

$\lim_{x\to -3^-} \frac{x-5}{x^2-2x-15} = -\infty$ from left side

$\lim_{x\to -3^+} \frac{x-5}{x^2-2x-15} = +\infty$ from the right

Left limit ≠ right limit

→ limit DNE

b) Create a table with values on both sides as x approaches -3.

x Approaches -3 from the left | x Approaches -3 from the right

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$	-10	-100	-1000	Error	100	100	10

$f(x)$ Approaches ? $-\infty$ | $f(x)$ Approaches ? $+\infty$

no limit

c) What about using the algebraic method to find the limit?

$\lim_{x\to -3} \frac{x-5}{x^2-2x-15} = \frac{-8}{0}$

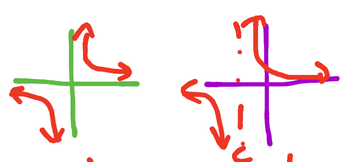
$\rightarrow \frac{\text{nonzero constant}}{0} \rightarrow$ VA

left: $\lim_{x\to -3^-} \frac{x-5}{x^2-2x-15} = \lim_{x\to -3^-} \frac{1}{x+3} = -\infty$

right: $\lim_{x\to -3^+} \frac{x-5}{x^2-2x-15} = \lim_{x\to -3^+} \frac{1}{x+3} = \infty$

→ limit DNE

d) What is your interpretation of $\lim_{x\to -3} \frac{x-5}{x^2-2x-15}$?



$y = \frac{1}{x}$ $y = \frac{1}{x-c}$ is horizontal translation

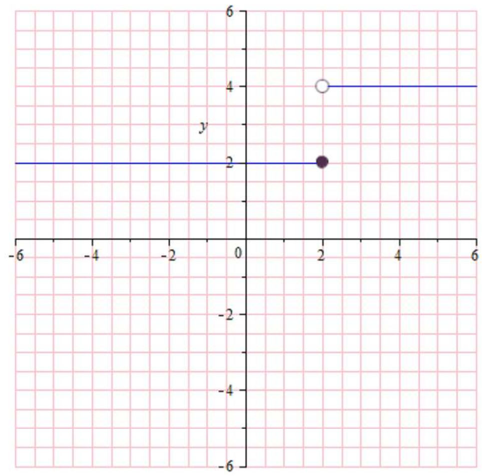
In summary, the function $f(x) = \frac{x-5}{x^2-2x-15}$ is not continuous at $x = 5$ and $x = -3$. From Exercise 1 and 2, we see that the limit exists at $x = 5$ but does not exist at $x = -3$. We say $f(x)$ has a "removable discontinuity" at $x = 5$ and an "infinite discontinuity" at $x = -3$.

Limits That Fail to Exist

Behavior that differs from the right and left

The graph of the function $f(x) = \begin{cases} 4 & x > 2 \\ 2 & x \le 2 \end{cases}$ has a value of 4 when x approaches 2 from the right and a value of 2 when x approaches 2 from the left.

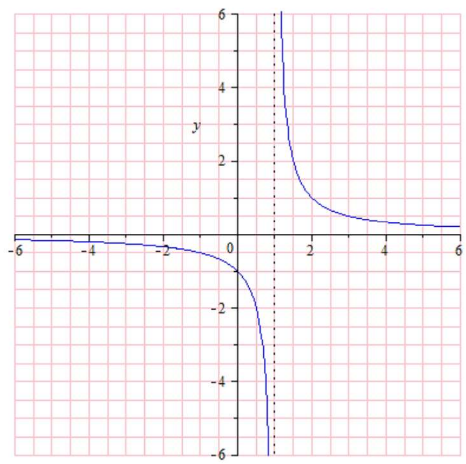
Hence, $\lim_{x \to 2} f(x)$ does not exist



Unbounded behavior

The graph of the function $f(x) = \frac{1}{x-1}$ exhibits unbounded behavior at $x = 1$.

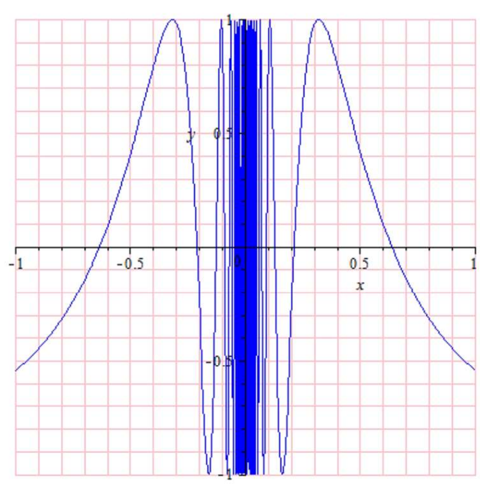
Hence, $\lim_{x \to 1} \frac{1}{x-1}$ does not exist



Oscillating behavior:

The graph of the function $f(x) = \sin\frac{1}{x}$ exhibits oscillating behavior at $x = 0$.

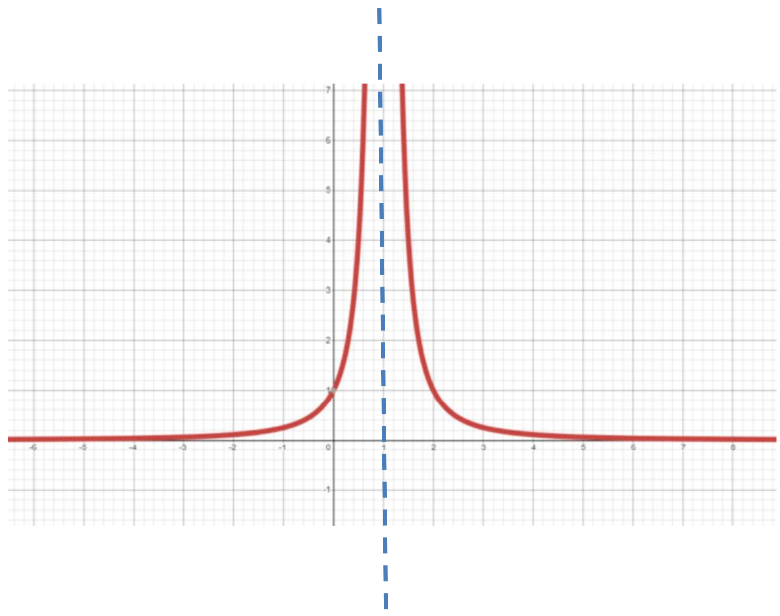
Hence, $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist



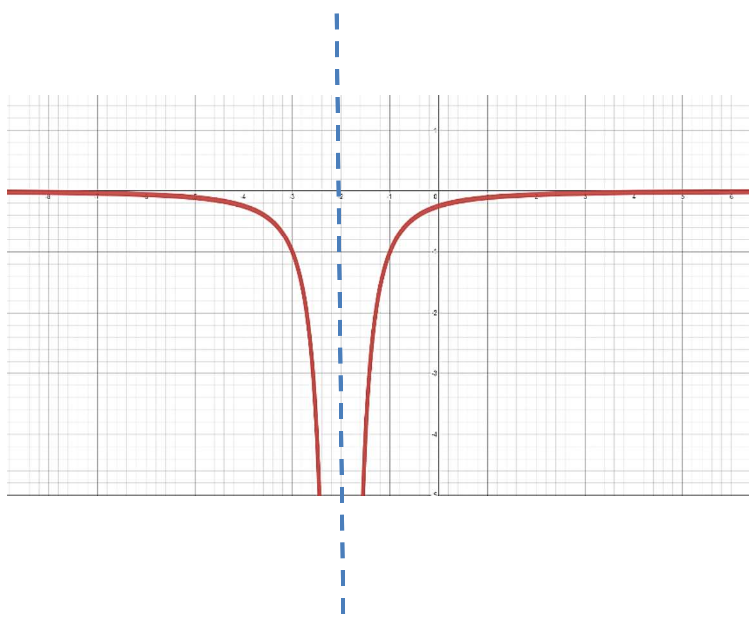
Infinite Limits

Infinite limits are technically limits that do not exist. However, sometimes we say the limit is ∞ if the limit of $f(x)$ as x approaches a is ∞ from the right and the left; or the limit is $-\infty$ if the limit of $f(x)$ as x approaches a is $-\infty$ from the right and the left.

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$



$$\lim_{x \to -2} \frac{-1}{(x+2)^2} = -\infty$$



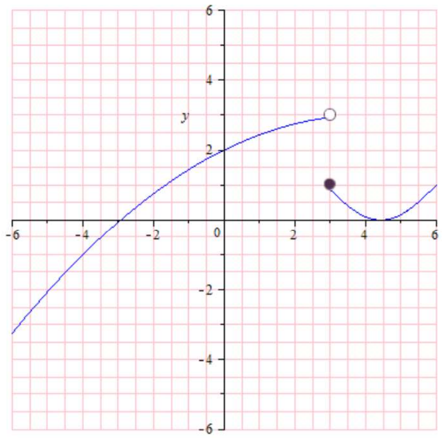
One-Sided Limits

Sometimes we may be interested in knowing about the limit behavior from either the left or the right, even if the limit does not exist.

One-Sided Limits

Left-hand limit: $\lim_{x\to a^-} f(x)$ is the limit of $f(x)$ as x approaches a from the left

Right-hand limit: $\lim_{x\to a^+} f(x)$ is the limit of $f(x)$ as x approaches a from the right



Example: The left hand limit at $x = 3$:

$$\lim_{x\to 3^-} f(x) = 3$$

The right hand limit at $x = 3$:

$$\lim_{x\to 3^+} f(x) = 1$$

$\lim_{x\to 3} f(x)$ DNE because $3 \neq 1$

The $\lim_{x\to a} f(x) = \mathrm{L}$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \mathrm{L}$.

Explain:

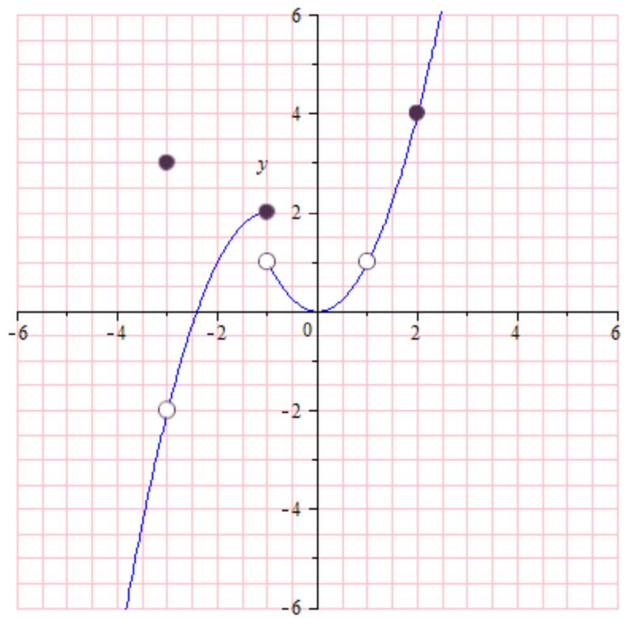
The limit exists if and only if left-limit equals right-limit

Continuity at a Point

A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

Exercise 3: Find the limits and values.



$f(2)$ $= 4$

$\lim_{x \to 2^+} f(x)$ $= 4$

$\lim_{x \to 2^-} f(x)$ $= 4$

$\lim_{x \to 2} f(x)$ $= 4$

$f(-1)$ $= 2$

$\lim_{x \to -1^+} f(x)$ $= 1$

$\lim_{x \to -1^-} f(x)$ $= 2$

$\lim_{x \to -1} f(x)$ DNE

$f(-3)$ $= 3$

$\lim_{x \to -3^+} f(x)$ $= -2$

$\lim_{x \to -3^-} f(x)$ $= -2$

$\lim_{x \to -3} f(x)$ $= -2$

$f(1)$ undefined

$\lim_{x \to 1^+} f(x)$ $= 1$

$\lim_{x \to 1^-} f(x)$ $= 1$

$\lim_{x \to 1} f(x)$ $= 1$

Exercise 4: Find the limits and values.

$f(-7) = -1$

$\lim_{x\to -7^+} f(x) = -1$

$\lim_{x\to -7^-} f(x) = -1$

$\lim_{x\to -7} f(x) = -1$

$f(-6)$ undefined

$\lim_{x\to -6^+} f(x) = 3$

$\lim_{x\to -6^-} f(x) = -\infty$

$\lim_{x\to -6} f(x)$ DNE

$f(-3) = 3$

$\lim_{x\to -3^+} f(x) = 3$

$\lim_{x\to -3^-} f(x) = 3$

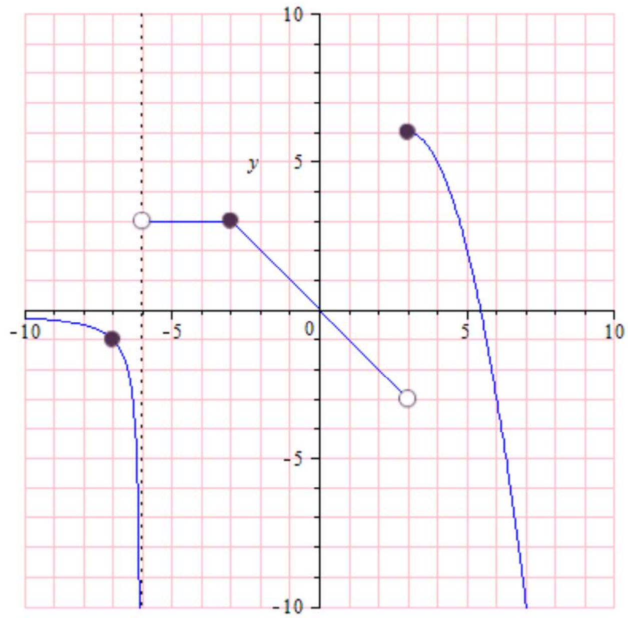
$\lim_{x\to -3} f(x) = 3$

$f(3) = 6$

$\lim_{x\to 3^+} f(x) = 6$

$\lim_{x\to 3^-} f(x) = -3$

$\lim_{x\to 3} f(x)$ DNE

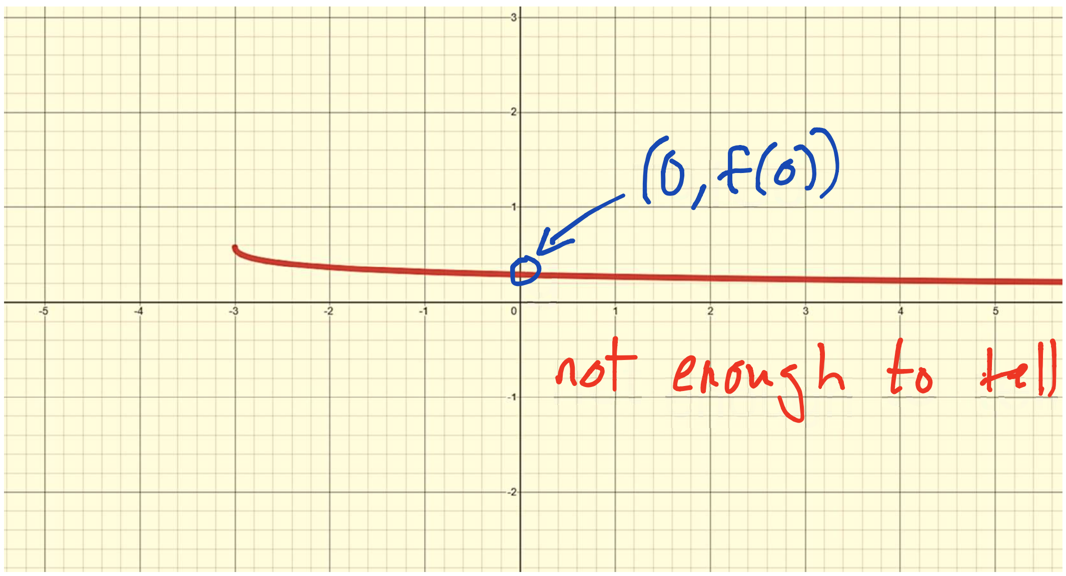


Exercise 5: Find

$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x} = \frac{\text{"}0\text{"}}{0}$$

$\rightarrow$ hole @ $x=0$

a) Use the graph method to estimate the limit.



b) Use the numerical method to find the limit.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	.2911	.2889	.28869...	—	.28865...	.2884	.2863

We know limit exists, but not enough.

c) In the next section, we discuss how to evaluate the limits analytically and algebraically. We will demonstrate how to simplify the function $f(x) = \frac{\sqrt{x+3}-\sqrt{3}}{x}$ to a form in which we can compute the limit algebraically.

SECTION 1 SUPPLEMENTARY EXERCISES

1) Determine the limit by either the graph method or the numerical method.

a) $\lim\limits_{x \to 0} \frac{\sin x}{x}$ $= 1$ $\lim\limits_{x \to 0} \frac{\cos(x) - 1}{x} =$

b) $\lim\limits_{x \to -4} \frac{1}{x+4}$

c) $\lim\limits_{x \to 2} \frac{1}{(x-2)^2}$

2) Find the limits and values.

$f(2)$

$\lim\limits_{x \to 2^+} f(x)$

$\lim\limits_{x \to 2^-} f(x)$

$\lim\limits_{x \to 2} f(x)$

$f(-1)$

$\lim\limits_{x \to -1^+} f(x)$

$\lim\limits_{x \to -1^-} f(x)$

$\lim\limits_{x \to -1} f(x)$

$f(-3)$

$\lim\limits_{x \to -3^+} f(x)$

$\lim\limits_{x \to -3^-} f(x)$

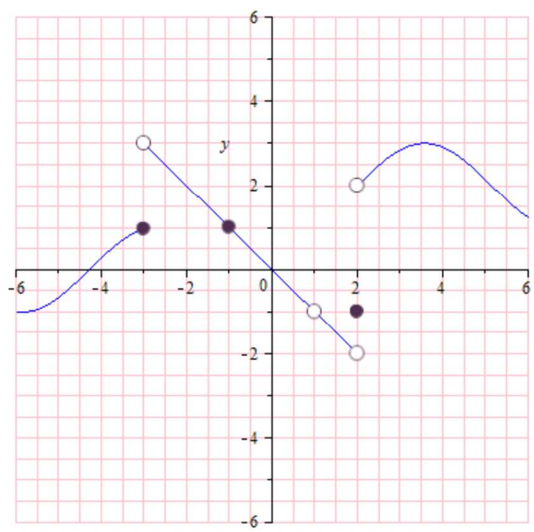
$\lim\limits_{x \to -3} f(x)$

$f(1)$

$\lim\limits_{x \to 1^+} f(x)$

$\lim\limits_{x \to 1^-} f(x)$

$\lim\limits_{x \to 1} f(x)$



3) Find the limits and values.

$f(7)$

$\lim\limits_{x \to 7^+} f(x)$

$\lim\limits_{x \to 7^-} f(x)$

$\lim\limits_{x \to 7} f(x)$

$f(5)$

$\lim\limits_{x \to 5^+} f(x)$

$\lim\limits_{x \to 5^-} f(x)$

$\lim\limits_{x \to 5} f(x)$

$f(1)$

$\lim\limits_{x \to 1^+} f(x)$

$\lim\limits_{x \to 1^-} f(x)$

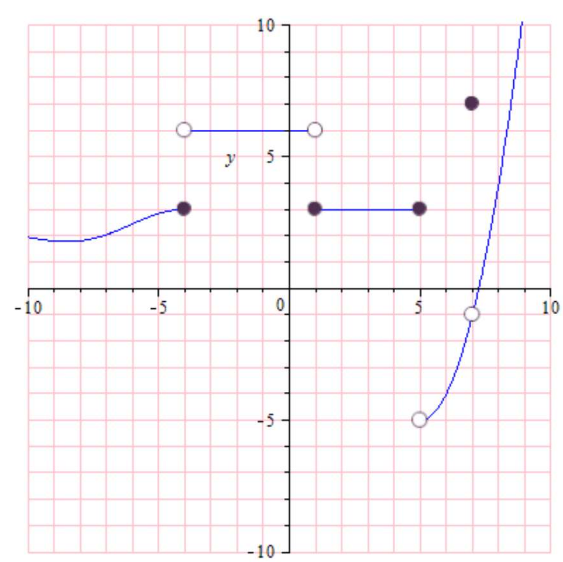
$\lim\limits_{x \to 1} f(x)$

$f(-4)$

$\lim\limits_{x \to -4^+} f(x)$

$\lim\limits_{x \to -4^-} f(x)$

$\lim\limits_{x \to -4} f(x)$



Section 2: Evaluating Limits Analytically

Properties of Limits

Suppose *c* is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists.

Limit Properties			**Example:** Let $f(x)=2x^2$ and $g(x)=x$
1.	Sum	$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$	
2.	Difference	$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$	
3.	Scalar multiple	$\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x)$	
4.	Product	$\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\bullet\lim_{x\to a} g(x)$	
5.	Quotient	$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$	
6.	Power	$\lim_{x\to a}[f(x)]^n=[\lim_{x\to a} f(x)]^n$ where *n* is a positive integer	
7.		$\lim_{x\to a} c=c$	
8.		$\lim_{x\to a} x=a$	
9.		$\lim_{x\to a} x^n=a^n$ where *n* is a positive integer	
10.		$\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ where *n* is a positive integer If *n* is even, we assume that a > 0	
11.	Root	$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$ where *n* is a positive integer If *n* is even, we assume that $\lim_{x\to a} f(x)>0$	

Strategies for Finding Limits

If the function f is continuous at a, we can substitute directly, and the limit is $f(a)$.

If the function f is not continuous at a, we can perform algebraic manipulations to derive an equivalent function g where $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Method 1: Direct Substitution – when f is continuous at a

Exercise 1: Find the limit.

a) $\lim_{x \to -1} \frac{x-5}{x^2-3x} = \frac{(-1)-5}{(-1)^2-3(-1)} = \frac{-6}{4} = -\frac{3}{2}$

b) $\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x} = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

Method 2: Dividing Out Technique – Factor and divide out any common factors

Helpful Factoring Formulas:	**Difference of Two Squares**	$a^2 - b^2 = (a+b)(a-b)$
	Square of a Binomial	$a^2 + 2ab + b^2 = (a+b)^2$ $a^2 - 2ab + b^2 = (a-b)^2$
	Sum of Two Cubes **Difference of Two Cubes**	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Exercise 2: Find the limit.

a) $\lim\limits_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

$= \lim\limits_{x \to -5} \frac{(x-2)(x+5)}{x+5}$

$= \lim\limits_{x \to -5} (x-2)$

$= (-5) - 2 = -7$

b) $\lim\limits_{x \to -5} \frac{x^2+10x+25}{x+5} = \text{"}\frac{0}{0}\text{"}$

$= \lim\limits_{x \to -5} \frac{(x+5)^2}{x+5}$

$= \lim\limits_{x \to -5} (x+5)$

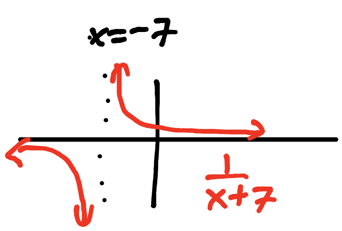
$= (-5) + 5 = 0$

Edit $x \to 7$

c) $\lim\limits_{x \to -7} \frac{x-7}{x^2-49}$

$= \lim\limits_{x \to -7} \frac{(-7)-7}{(-7)^2-49} = \frac{-14}{0}$

$\lim\limits_{x \to -7} \frac{\cancel{(x-7)}\ 1}{\cancel{(x-7)}(x+7)}$



$\lim\limits_{x \to -7} \frac{1}{x+7}$

→ Check both sides

Left

$\lim\limits_{x \to -7^-} \frac{1}{x+7} = -\infty$

Right

$\lim\limits_{x \to -7^+} \frac{1}{x+7} = \infty$

$\lim\limits_{x \to -7} \frac{1}{x+7}$ DNE

d) $\lim\limits_{x \to 2} \frac{x^3 - 8}{x - 2}$

Method 3: Rationalizing Technique – If the function has a radical expression in the numerator, rationalize the numerator by multiplying in the numerator and denominator by the "conjugate of the numerator."

Exercise 3: Find the limit.

a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

$$\frac{\sqrt{x+3}-\sqrt{3}}{x} = \frac{\sqrt{x+3}-\sqrt{3}}{x} \cdot \frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}$$

(multiply by conjugate of numerator)

$$a^2-b^2 = (a+b)(a-b)$$

conjugates of each other

$$= \frac{(\sqrt{x+3})^2-(\sqrt{3})^2}{x(\sqrt{x+3}+\sqrt{3})}$$

$$= \frac{x+\cancel{3}-\cancel{3}}{x(\sqrt{x+3}+\sqrt{3})}$$

$$= \frac{\cancel{x}}{\cancel{x}(\sqrt{x+3}+\sqrt{3})}$$

$$= \frac{1}{\sqrt{x+3}+\sqrt{3}}$$

$$\rightarrow \lim_{x\to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}}$$

$$= \frac{1}{\sqrt{0+3}+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

b) $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$

Method 4: The LCD Technique - Combining fractions in the numerator using the Least Common Denominator (LCD)

Exercise 4: Find the limit.

a) $\lim_{x \to 0} \dfrac{\frac{1}{x+5} - \frac{1}{5}}{\frac{x}{1}} = \lim_{x \to 0} \dfrac{\frac{1}{x+5} - \frac{1}{5}}{x} \left(\dfrac{5(x+5)}{5(x+5)} \right)$

LCD: $5(x+5)$

$$= \lim_{x \to 0} \frac{5 - (x+5)}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{\cancel{5} - x - \cancel{5}}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{-\cancel{x}}{5\cancel{x}(x+5)}$$

$$= \lim_{x \to 0} -\frac{1}{5(x+5)}$$

$$= -\frac{1}{5((0)+5)}$$

$$= -\frac{1}{25}$$

b) $\lim_{x \to 2} \dfrac{\frac{1}{x-4} + \frac{1}{2}}{x-2}$