Applications of Calculus

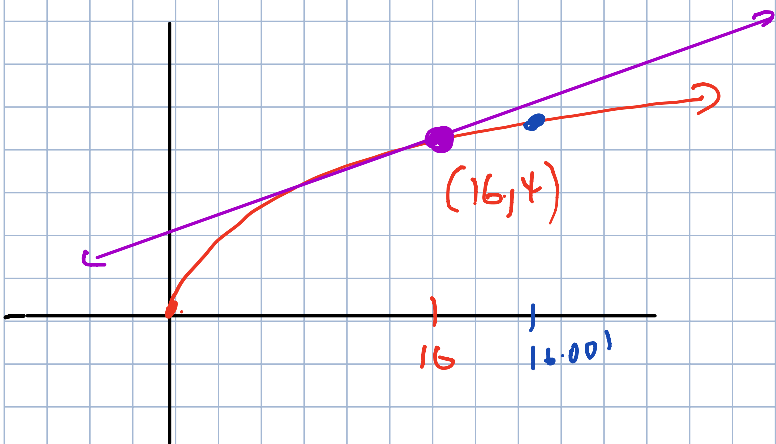
1. Linear Approximation

Approximate $\sqrt{16.001} \approx 4.00012499805$

"square root function"

tangent to $y=\sqrt{x}$ @ $x=16$

closest number that we know the square root of.



Find the tangent line

1. Find derivative

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \leftarrow \text{derivative function}$$

$$f'(16) = \frac{1}{2}(16)^{-\frac{1}{2}} = \frac{1}{2\sqrt{16}} = \frac{1}{8} \leftarrow \text{derivative @ a point}$$

2. Tangent line

$x_1 =$ "nice value"

$f(16) = \sqrt{16} = 4$

$$y = f'(x_1)(x - x_1) + y_1$$

$$y = f'(16)(x - 16) + 4$$

$$y = \frac{1}{8}(x - 16) + 4$$

$$y = \frac{1}{8}x + 2 \leftarrow$$

Linear approximation of $\sqrt{x}$ @ $x = 16$

3. Plug value for x in linear approximation equation

$$y = \frac{1}{8}x + 2 \qquad x = 16.001$$

$$y = \frac{1}{8}(16.001) + 2$$

$$y = 4.000125$$

approximate value of $\sqrt{16.001}$ using linear approximation.

$$\sqrt{15.998} \approx 3.99974999219$$

Approximate $\sqrt{15.998}$

using our equation...

$$y = \frac{1}{8}(15.998) + 2 = 3.99975$$

Approximate: $\sqrt{100} = 10$

use our equation

$$y = \frac{1}{8}(100) + 2 = 14.5$$

* not such a good approximation because 100 is far from 16; tangent line equation is centered at $x = 16$

Approximate $\sqrt[3]{26.98} = 2.99925907\overline{6}$

Nearest nice number with a cube root: is 27

1. Find derivative
2. Tangent line
3. Plug in 26.98 for x.

1. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$f'(27) = \frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{27})^2} = \frac{1}{3(9)} = \frac{1}{27}$

2. $y = f'(x_1)(x - x_1) + f(x_1)$ $\qquad \sqrt[3]{27} = 3$

$= f'(27)(x - 27) + 3$

$= \frac{1}{27}(x - 27) + 3$

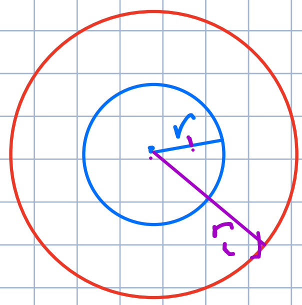
$= \frac{1}{27}x - 1 + 3$

$y = \frac{1}{27}x + 2$

3. $y = \frac{1}{27}(26.98) + 2$

$y = 2.999259\overline{259}$

$\boxed{\sqrt[3]{26.98} \approx 2.999\overline{259}}$



Formulas for circle

Area of circle:

$$A = \pi r^2$$
$$A(r) = \pi r^2$$

Derivative of Area

$$A'(r) = \pi \cdot 2r$$
$$A'(r) = 2\pi r$$
$$A'(r) = C(r)$$

Circumference

$$C = 2\pi r$$
$$C(r) = 2\pi r$$

derivative of area is the circumference

Related Rates of Change (over time)

$$A = \pi r^2$$

A is a function of time
r is a " " "

* We want to take the rate of change over time
We will take derivatives with respect to t

* π is a constant

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

The radius of a circular oil slick expands at a rate of 3 m/min.

$$\frac{dr}{dt} = 3\,\frac{m}{min}$$

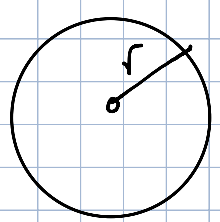
(a) How fast is the area of the oil slick increasing when the radius is 23 m?

$\frac{dA}{dt} = \square\ m^2/min$

$r = 23m$

(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 5 mins?

$\frac{dA}{dt} = \square\ m^2/min$



$$\frac{dr}{dt} = 3\,\frac{m}{min}$$

$$\frac{dA}{dt} = ?$$

a.) $A = \pi r^2$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(23m)\left(3\frac{m}{min}\right)$$

$$\frac{dA}{dt} = 138\pi\ \frac{m^2}{min}$$

b.) $r = \left(3\frac{m}{min}\right)(5\ min) = 15m$

$$A = \pi r$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

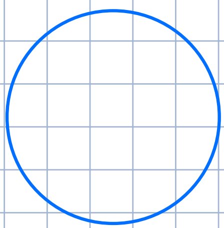
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(15m)\left(3\frac{m}{min}\right)$$

$$\frac{dA}{dt} = 90\pi\ \frac{m^2}{min}$$

Helium is pumped into a spherical balloon at a rate of 3 cubic feet per second. How fast is the radius increasing after 2 minutes?
Note: The volume of a sphere is given by $V = (4/3)\pi r^3$.

Rate of change of radius (in feet per second) = $\square$



$$\frac{dV}{dt} = 3\,\frac{ft^3}{s}$$

@ $t=0$, $V=0$

$t=120$, $V=360\ ft^3$

$$\frac{dr}{dt} = ?$$

in two minutes

120 seconds

How do we find r?

$$V = \frac{4}{3}\pi r^3$$

$$360 = \frac{4}{3}\pi r^3$$

$$\frac{360\cdot 3}{4\pi} = r^3$$

$$\frac{270}{\pi} = r^3$$

$$\sqrt[3]{\frac{270}{\pi}} = r$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2\,\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}$$

$$3 = 4\pi\left(\sqrt[3]{\frac{270}{\pi}}\right)^2\frac{dr}{dt}$$

$$\frac{3}{4\pi}\left(\sqrt[3]{\frac{\pi}{270}}\right)^2\ \frac{ft}{s} = \frac{dr}{dt}$$

A 15 ft ladder leans against a wall. The bottom of the ladder is 5 ft from the wall at time $t = 0$ and slides away from the wall at a rate of $1 \mathrm{ft}/\mathrm{sec}$.

Find the velocity of the top of the ladder at time $t = 2$.

The velocity of ladder at time $t = 2$ is ☐ ft/sec.

$a^2 + b^2 = c^2$

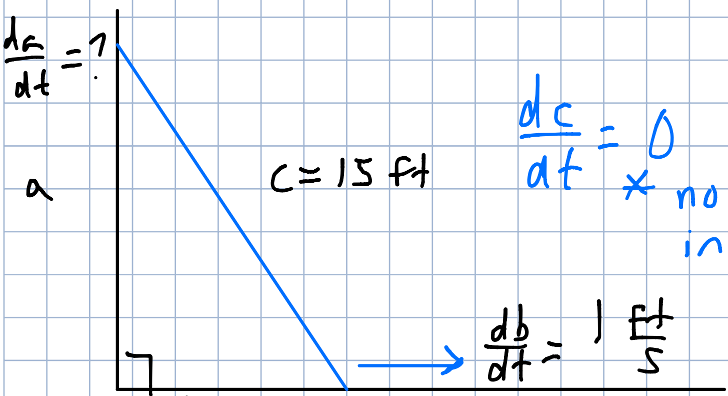
$a^2 + (7)^2 = (15)^2$

$a^2 + 49 = 225$

$a^2 = 176$

$a = \sqrt{176}$

$a = 4\sqrt{11}$



$\frac{dc}{dt} = 0$ * no change in $\dot{c}$.

$b = 5$ ft, $t = 0$

* $b = 7$ ft, $t = 2$

$a^2 + b^2 = c^2$

$\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt} c^2$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

* every term is divisible by 2

$a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}$

$4\sqrt{11}\frac{da}{dt} + 7(1) = 15(0)$

$4\sqrt{11}\frac{da}{dt} + 7 = 0$

$\frac{da}{dt} = -\frac{7}{4\sqrt{11}} \frac{ft}{s}$

$\frac{da}{dt} = -\frac{7\sqrt{11}}{44} \frac{ft}{s}$

height is decreasing

Graph $y = x^3 - 3x^2 - 1$ Domain: $(-\infty, \infty)$

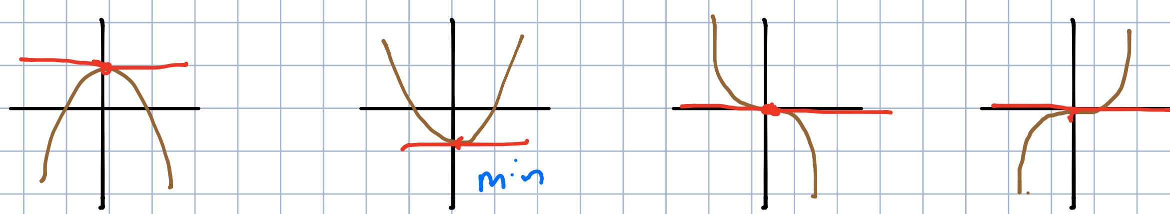
$y' = 3x^2 - 6x$ ← First Derivative

Set $y=0$

$$0 = 3x^2 - 6x$$
$$0 = 3x(x-2)$$
$$x \in \{0, 2\}$$

horizontal points of inflection.

$y' = 0$ max



$y' = 3x^2 - 6x$ Test values for x

$-\infty$		0		2		∞
	$x=-1$		$x=1$		$x=3$	
	$y' = +9$		$y' = -3$		$y' = +9$	
	$y' > 0$		$y' < 0$		$y' > 0$	

loc max @ $x = 0$

loc min $x = 2$

If we want to find points, substitute into original equation

$$y = x^3 - 3x^2 - 1$$

$y(0) = -1$

max @ $(0, -1)$

$y(2) = -5$

min $(2, -5)$

$y' = 3x^2 - 6x$

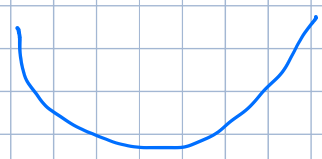
$y'' = 6x - 6$ ⇐ 2nd derivative

set $y'' = 0$ $\qquad$ $0 = 6x - 6$

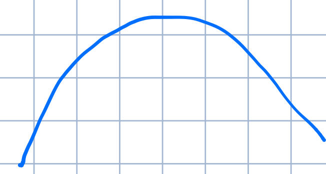
$\vdots$

$x = 1$

$y'' > 0$



$y'' < 0$

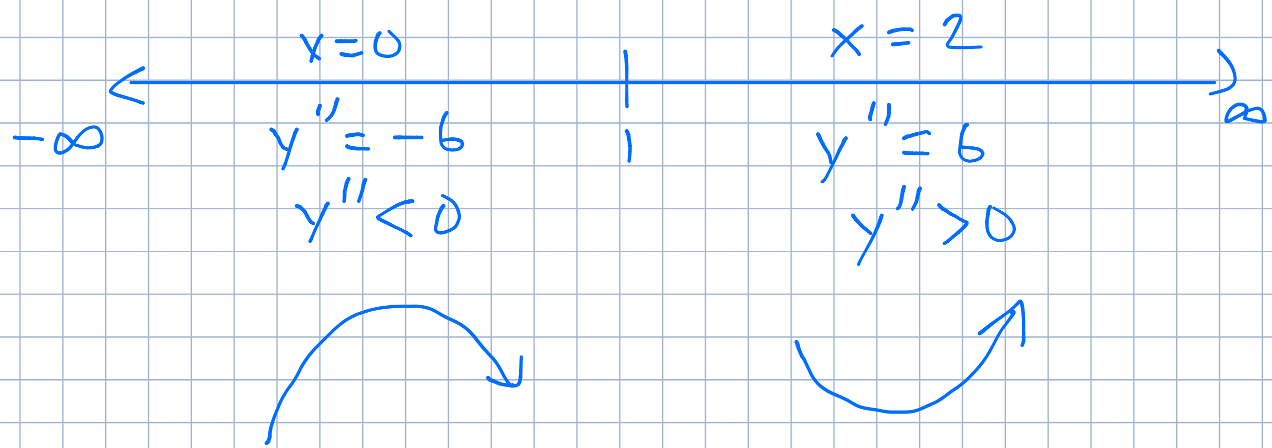


curvature - concavity

concave up

concave down

$y'' = 6x - 6$



Point of inflection

@ $x = 1$

$y = x^3 - 3x^2 - 1$

$y(1) = -3$ $\qquad$ $(1, -3)$

end behavior

$$\lim_{x\to-\infty} x^3 - 3x^2 - 1 = -\infty$$

$$\lim_{x\to\infty} x^3 - 3x^2 - 1 = \infty$$

Find..

x-int

y-int (0,-1)