New York City College of Technology

Head Start to Calculus I

A Preparatory Workshop for MAT 1475 Calculus I

Dr. Sandie Han
Dr. Janet Liou-Mark
Dr. Urmi Ghosh-Dastidar
Department of Mathematics

Contributors:
Christopher Chan, Yiming Yu, and Karmen Yu

Table of Contents

In Memory of our beloved friend and colleague

Professor Janet Liou-Mark

Whose devotion and contribution to the education of City Tech students will be remembered

This workbook is created to provide students with a review and an introduction to Calculus I before the actual Calculus course. Studies have shown that students who attend a preparatory workshop for the course tend to perform better in the course. The Calculus workshop bears no college credits nor contributes towards graduation requirement. It may not be used to substitute for nor exempt from the Calculus requirement.

The Calculus workshop meets four days, three hours a day, for a total of 12 hours during the week before the start of the semester. The workshop is facilitated by instructors and/or peer leaders.

Section 1: Finding Limits Graphically and Numerically

Finding Limits Graphically and Numerically

Definition of limits:

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number *L* as the values of x (except for $x = a$) approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.

$$\lim_{x \to a} f(x) = L$$

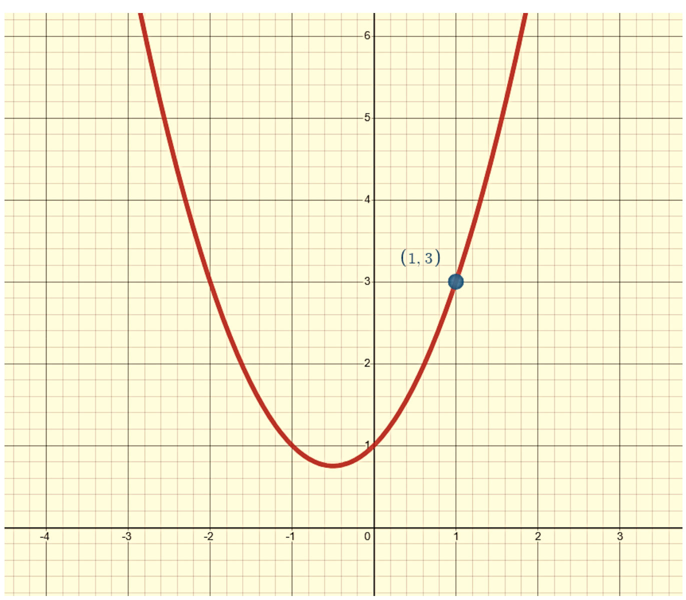
Remark about limit:

1. The function $f(x)$ does not need to be defined at a. The emphasis is on the word "approach."
2. Another key point is that "all" values of the function $f(x)$ must approach the same number L. This means $f(x)$ must approach L whether x is approaching a from the left or from the right.
3. If a function $f(x)$ is continuous at a, then the limit of $f(x)$ at a is $f(a)$.

Example 1: Find

$$\lim_{x \to 1} x^2 + x + 1$$

A graphical method shows the limit of $f(x) = x^2 + x + 1$ as x approaches 1 is 3.



A numerical method shows the same result.

x Approaches 1 from the left → ← x Approaches 1 from the right

x	.75	.9	.99	.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.170	2.970	2.997	3	3.003	3.030	3.310	3.813

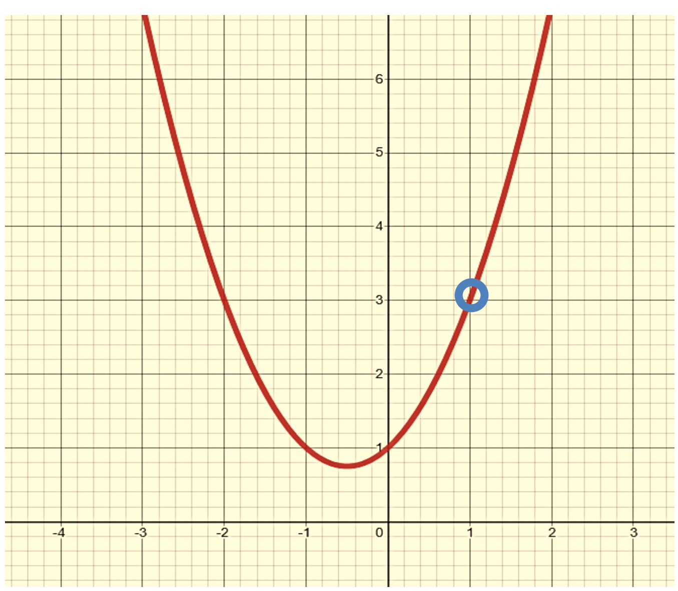
$f(x)$ Approaches 3 → ← $f(x)$ Approaches 3

Furthermore, because $f(x)$ is continuous at 1, we can simply substitute 1 into the function to determine that the limit is $f(1) = 3$.

Example 2: Find

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$$

The graphical method shows the same graph. However, $f(x)$ is not defined at 1, so we must assume the graph at the point 1 does not exist, or there is a "hole". We use an open circle to represent that the function is not defined at a particular point.



The numerical method shows the values of $f(x)$ approaches 3 as x approaches 1, with the value at exactly 1 being undefined. In other words, $f(1)$ is undefined, but

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

x Approaches 1 from the left → ← x Approaches 1 from the right

x	.75	.9	.99	.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.170	2.970	2.997	Error	3.003	3.030	3.310	3.813

$f(x)$ Approaches 3 → ← $f(x)$ Approaches 3

Furthermore, if we factor and reduce the expression algebraically

$$\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1$$

This explains why the limit in example 1 is equal to the limit in example 2. Or

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 3$$

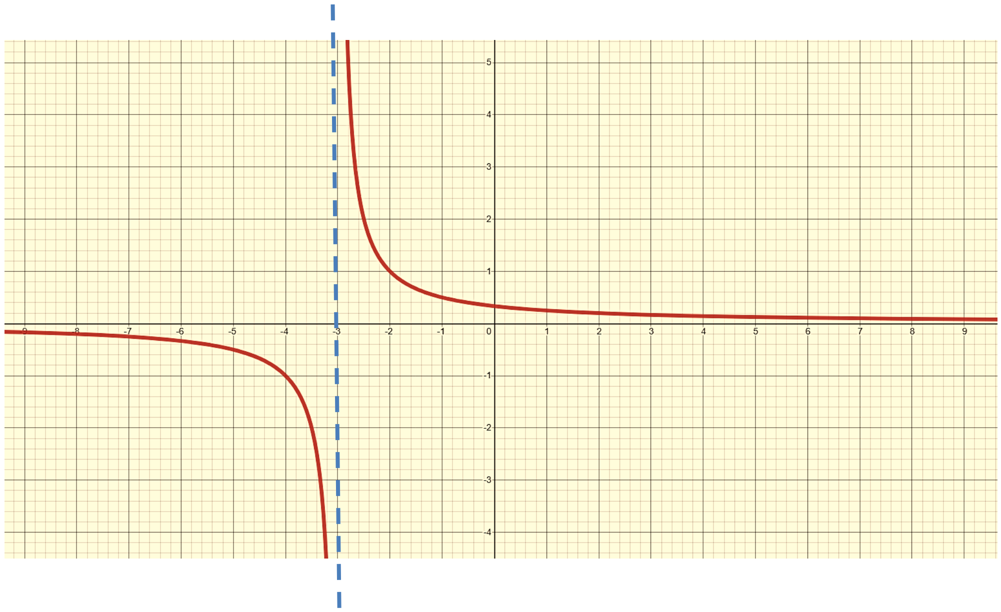
We say $f(x)$ has a "removable discontinuity" at 1.

Keep in mind, although the two limits are equal, the two functions $\frac{x^3-1}{x-1}$ and $x^2 + x + 1$ are not the same and don't have the same domain. The two functions behave and graph "almost" the same, except at the point $x = 1$.

Exercise 1: Find

$$\lim_{x \to 5} \frac{x - 5}{x^2 - 2x - 15}$$

a) Use the graph to estimate the limit.



b) Create a table with values on both sides as x approaches 5.

x Approaches 5 from the left

x Approaches 5 from the right

x	4.9	4.99	4.999	5	5.001	5.01	5.1
$f(x)$							

$f(x)$ Approaches ?

$f(x)$ Approaches ?

c) Use algebraic method to factor and reduce the expression, then find the limit.

Exercise 2: Find

$$\lim_{x\to -3} \frac{x-5}{x^2-2x-15}$$

a) Use the graph to estimate the limit.

b) Create a table with values on both sides as x approaches -3.

x Approaches -3 from the left

x Approaches -3 from the right

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$							

$f(x)$ Approaches ?

$f(x)$ Approaches ?

c) What about using the algebraic method to find the limit?

d) What is your interpretation of $\lim_{x\to -3} \frac{x-5}{x^2-2x-15}$?

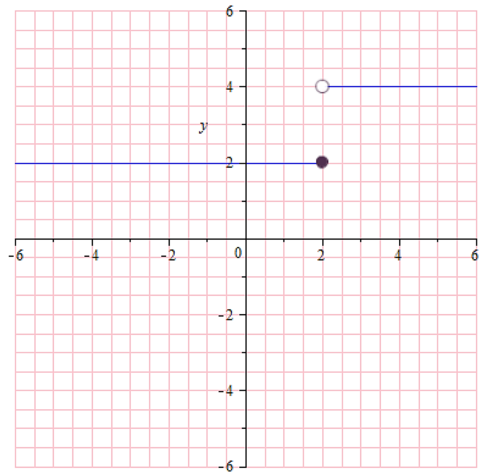
In summary, the function $f(x) = \frac{x-5}{x^2-2x-15}$ is not continuous at $x = 5$ and $x = -3$. From Exercise 1 and 2, we see that the limit exists at $x = 5$ but does not exist at $x = -3$. We say $f(x)$ has a "removable discontinuity" at $x = 5$ and an "infinite discontinuity" at $x = -3$.

Limits That Fail to Exist

Behavior that differs from the right and left

The graph of the function $f(x) = \begin{cases} 4 & x > 2 \\ 2 & x \leq 2 \end{cases}$ has a value of 4 when x approaches 2 from the right and a value of 2 when x approaches 2 from the left.

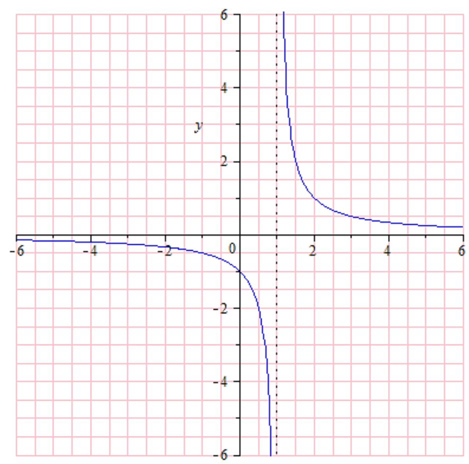
Hence, $\lim_{x\to 2} f(x)$ does not exist



Unbounded behavior

The graph of the function $f(x) = \frac{1}{x-1}$ exhibits unbounded behavior at $x = 1$.

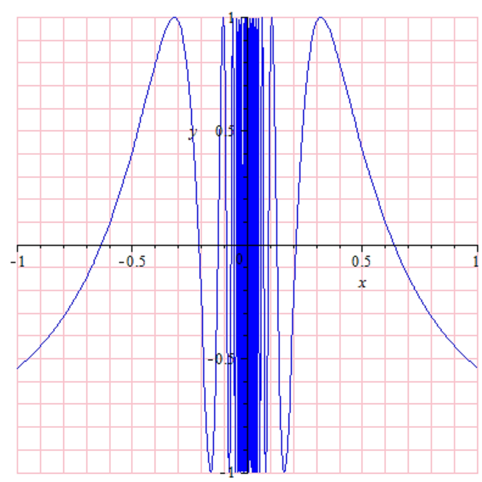
Hence, $\lim_{x\to 1} \frac{1}{x-1}$ does not exist



Oscillating behavior:

The graph of the function $f(x) = \sin\frac{1}{x}$ exhibits oscillating behavior at $x = 0$.

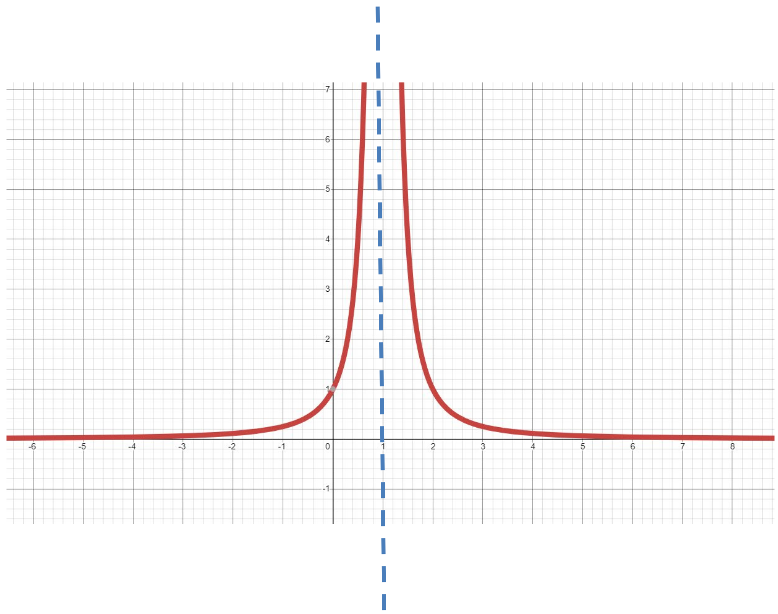
Hence, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist



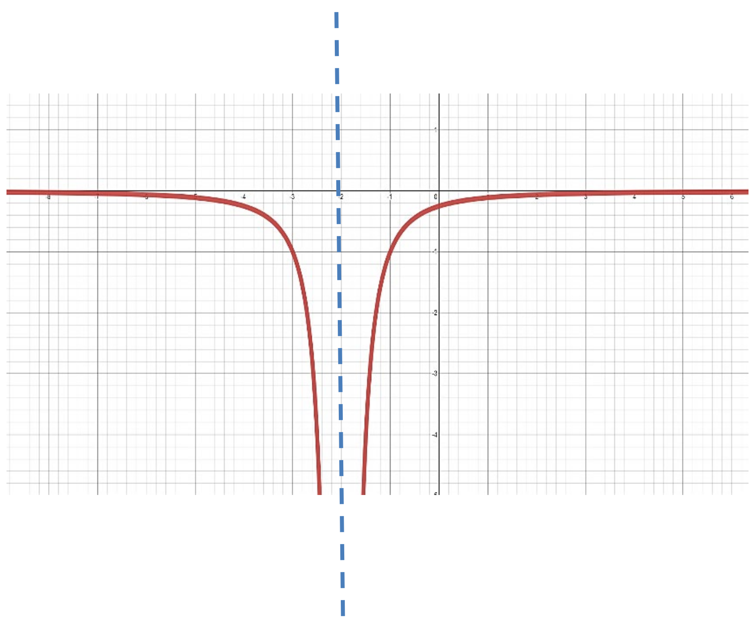
Infinite Limits

Infinite limits are technically limits that do not exist. However, sometimes we say the limit is ∞ if the limit of $f(x)$ as x approaches a is ∞ from the right and the left; or the limit is $-\infty$ if the limit of $f(x)$ as x approaches a is $-\infty$ from the right and the left.

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$



$$\lim_{x \to -2} \frac{-1}{(x+2)^2} = -\infty$$



One-Sided Limits

Sometimes we may be interested in knowing about the limit behavior from either the left or the right, even if the limit does not exist.

One-Sided Limits

Left-hand limit: $\lim_{x \to a^-} f(x)$ is the limit of $f(x)$ as x approaches a from the left

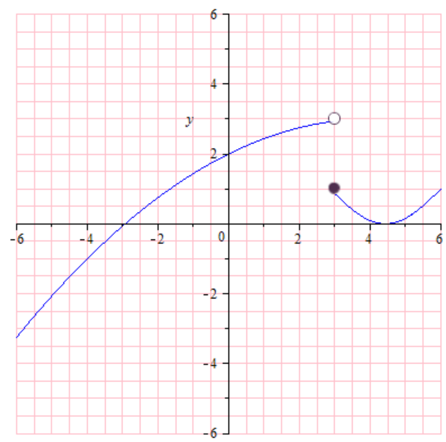
Right-hand limit: $\lim_{x \to a^+} f(x)$ is the limit of $f(x)$ as x approaches a from the right

Example: The left hand limit at $x = 3$:

$$\lim_{x \to 3^-} f(x) = 3$$

The right hand limit at $x = 3$:

$$\lim_{x \to 3^+} f(x) = 1$$



The $\lim_{x \to a} f(x) = \mathrm{L}$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \mathrm{L}$.

Explain:

____

____

Continuity at a Point

A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Exercise 3: Find the limits and values.

$f(2)$

$\lim_{x\to 2^+} f(x)$

$\lim_{x\to 2^-} f(x)$

$\lim_{x\to 2} f(x)$

$f(-1)$

$\lim_{x\to -1^+} f(x)$

$\lim_{x\to -1^-} f(x)$

$\lim_{x\to -1} f(x)$

$f(-3)$

$\lim_{x\to -3^+} f(x)$

$\lim_{x\to -3^-} f(x)$

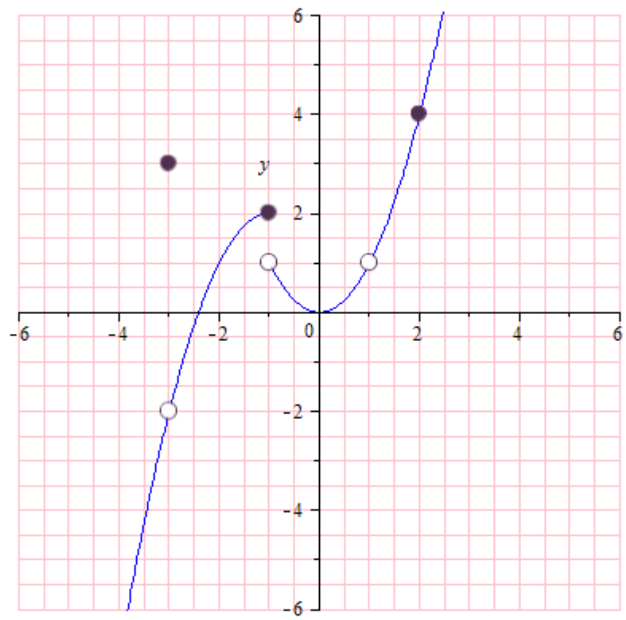
$\lim_{x\to -3} f(x)$

$f(1)$

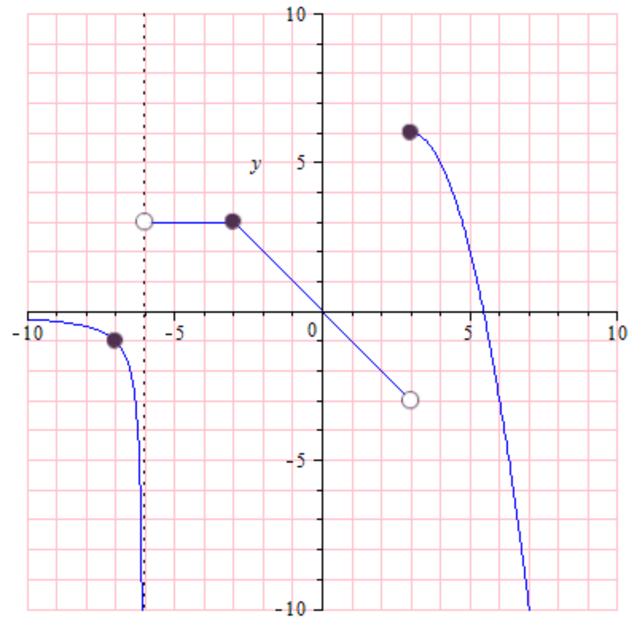
$\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1} f(x)$



Exercise 4: Find the limits and values.



$f(-7)$

$\lim_{x\to -7^+} f(x)$

$\lim_{x\to -7^-} f(x)$

$\lim_{x\to -7} f(x)$

$f(-6)$

$\lim_{x\to -6^+} f(x)$

$\lim_{x\to -6^-} f(x)$

$\lim_{x\to -6} f(x)$

$f(-3)$

$\lim_{x\to -3^+} f(x)$

$\lim_{x\to -3^-} f(x)$

$\lim_{x\to -3} f(x)$

$f(3)$

$\lim_{x\to 3^+} f(x)$

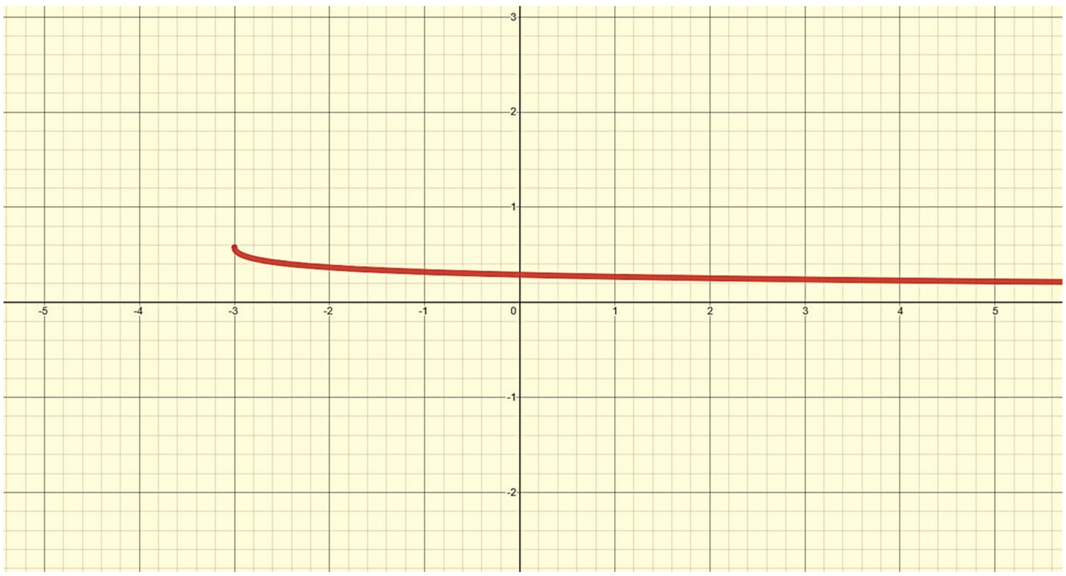
$\lim_{x\to 3^-} f(x)$

$\lim_{x\to 3} f(x)$

Exercise 5: Find

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

a) Use the graph method to estimate the limit.



b) Use the numerical method to find the limit.

x	– 0.1	– 0.01	– 0.001	0	0.001	0.01	0.1
$f(x)$							

c) In the next section, we discuss how to evaluate the limits analytically and algebraically. We will demonstrate how to simplify the function $f(x) = \frac{\sqrt{x+3}-\sqrt{3}}{x}$ to a form in which we can compute the limit algebraically.

SECTION 1 SUPPLEMENTARY EXERCISES

1) Determine the limit by either the graph method or the numerical method.

a) $\lim_{x \to 0} \frac{\sin x}{x}$

b) $\lim_{x \to -4} \frac{1}{x+4}$

c) $\lim_{x \to 2} \frac{1}{(x-2)^2}$

2) Find the limits and values.

$f(2)$

$\lim_{x \to 2^+} f(x)$

$\lim_{x \to 2^-} f(x)$

$\lim_{x \to 2} f(x)$

$f(-1)$

$\lim_{x \to -1^+} f(x)$

$\lim_{x \to -1^-} f(x)$

$\lim_{x \to -1} f(x)$

$f(-3)$

$\lim_{x \to -3^+} f(x)$

$\lim_{x \to -3^-} f(x)$

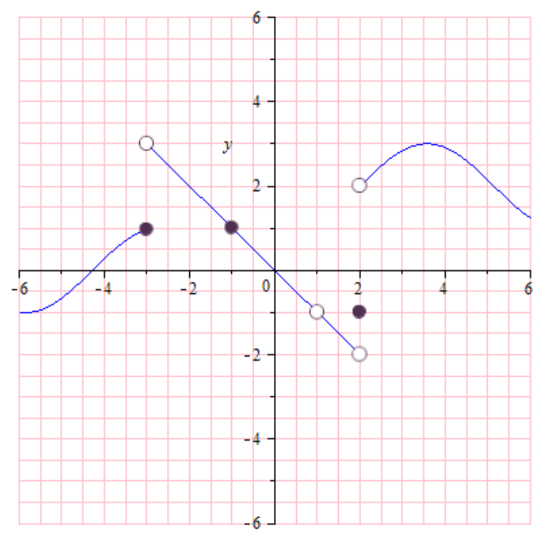
$\lim_{x \to -3} f(x)$

$f(1)$

$\lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1} f(x)$



3) Find the limits and values.

$f(7)$

$\lim_{x \to 7^+} f(x)$

$\lim_{x \to 7^-} f(x)$

$\lim_{x \to 7} f(x)$

$f(5)$

$\lim_{x \to 5^+} f(x)$

$\lim_{x \to 5^-} f(x)$

$\lim_{x \to 5} f(x)$

$f(1)$

$\lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^-} f(x)$

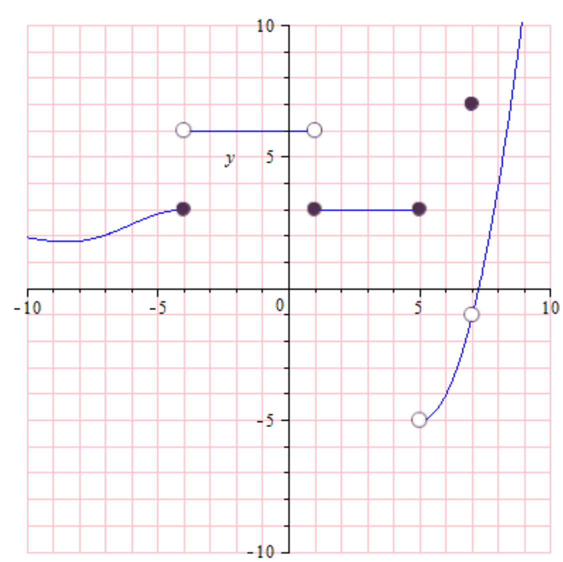
$\lim_{x \to 1} f(x)$

$f(-4)$

$\lim_{x \to -4^+} f(x)$

$\lim_{x \to -4^-} f(x)$

$\lim_{x \to -4} f(x)$



Section 2: Evaluating Limits Analytically

Properties of Limits

Suppose *c* is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists.

Limit Properties			Example: Let $f(x)=2x^2$ and $g(x)=x$
1.	Sum	$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$	
2.	Difference	$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$	
3.	Scalar multiple	$\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x)$	
4.	Product	$\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\bullet\lim_{x\to a} g(x)$	
5.	Quotient	$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$	
6.	Power	$\lim_{x\to a}[f(x)]^n=[\lim_{x\to a} f(x)]^n$ where *n* is a positive integer	
7.		$\lim_{x\to a} c=c$	
8.		$\lim_{x\to a} x=a$	
9.		$\lim_{x\to a} x^n=a^n$ where *n* is a positive integer	
10.		$\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ where *n* is a positive integer If *n* is even, we assume that a > 0	
11.	Root	$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$ where *n* is a positive integer If *n* is even, we assume that $\lim_{x\to a} f(x)>0$	

Strategies for Finding Limits

If the function f is continuous at a, we can substitute directly, and the limit is $f(a)$.

If the function f is not continuous at a, we can perform algebraic manipulations to derive an equivalent function g where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Method 1: Direct Substitution – when f is continuous at a

Exercise 1: Find the limit.

a) $\lim_{x\to -1} \frac{x-5}{x^2-3x}$

b) $\lim_{x\to \frac{\pi}{2}} \frac{\sin x}{x}$

Method 2: Dividing Out Technique – Factor and divide out any common factors

Helpful Factoring Formulas:

Difference of Two Squares $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

Square of a Binomial $\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$

Sum of Two Cubes $\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$

Difference of Two Cubes $\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

Exercise 2: Find the limit.

a) $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

b) $\lim_{x \to -5} \frac{x^2+10x+25}{x+5}$

.

c) $\lim_{x \to -7} \frac{x - 7}{x^2 - 49}$

d) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

Method 3: Rationalizing Technique – If the function has a radical expression in the numerator, rationalize the numerator by multiplying in the numerator and denominator by the "conjugate of the numerator."

Exercise 3: Find the limit.

a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$ $= \text{"}\frac{0}{0}\text{"}$

$$\frac{\sqrt{x+3}-\sqrt{3}}{x} = \frac{\sqrt{x+3}-\sqrt{3}}{x}\cdot\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}$$

$$= \lim_{x\to 0}\left(\frac{\sqrt{x+3}-\sqrt{3}}{x}\right)\left(\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right)$$

$$= \lim_{x\to 0}\frac{(\sqrt{x+3})^2-(\sqrt{3})^2}{x(\sqrt{x+3}+\sqrt{3})}$$

$$= \lim_{x\to 0}\frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})}$$

$$= \lim_{x\to 0}\frac{x}{x(\sqrt{x+3}+\sqrt{3})}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}}$$

$$= \frac{1}{\sqrt{(0)+3}+\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \boxed{\frac{\sqrt{3}}{6}}$$

Recall $a-b$ and $a+b$ are conjugates.

$(a-b)(a+b) = a^2 - ab + ab - b^2$

$(a-b)(a+b) = a^2 - b^2$

* Difference of squares

b) $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$ $= \text{"}\frac{0}{0}\text{"}$ ← if we substitute $x=4$

→ simplify.

$$= \lim_{x\to 4}\left(\frac{\sqrt{x+3}-3}{x-4}\right)\cdot\left(\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}\right)$$

$$= \lim_{x\to 4}\frac{(\sqrt{x+5})^2-(3)^2}{(x-4)(\sqrt{x+3}+3)}$$

$$= \lim_{x\to 4}\frac{x+5-9}{(x-4)(\sqrt{x+3}+3)}$$

$$= \lim_{x\to 4}\frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x+3}+3)}$$

$$= \lim_{x\to 4}\frac{1}{\sqrt{x+3}+3}$$

$$= \frac{1}{\sqrt{(4)+5}+3}$$

$$= \frac{1}{6}$$

Method 4: The LCD Technique - Combining fractions in the numerator using the Least Common Denominator (LCD)

Exercise 4: Find the limit.

a) $\lim_{x \to 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x} = \text{"}\frac{0}{0}\text{"}$ LCD

complex fraction
"fractions within fractions"

Simplify complex fraction

1. Find LCD of all denominators

LCD of 1, x+5, 5

$5(x+5)$

2. Multiply the numerator & denominator by LCD.

3. Simplify

$$= \lim_{x \to 0} \left(\frac{\frac{1}{x+5} - \frac{1}{5}}{x} \right) \left(\frac{5(x+5)}{5(x+5)} \right)$$

$$= \lim_{x \to 0} \frac{\frac{5(x+5)}{x+5} - \frac{5(x+5)}{5}}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{5 - (x+5)}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{5 - x - 5}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{-x}{5x(x+5)}$$

$$= \lim_{x \to 0} \frac{-1}{5(x+5)}$$

$$= -\frac{1}{5(0)+5}$$

$$= \boxed{-\frac{1}{25}}$$

b) $\lim_{x \to 2} \frac{\frac{1}{x-4} + \frac{1}{2}}{x-2}$

LCD: $2(x-4)$

$$= \lim_{x \to 2} \frac{\frac{1}{x-4} + \frac{1}{2}}{x-2} \left(\frac{2(x-4)}{2(x-4)} \right)$$

$$= \lim_{x \to 2} \frac{2 + (x-4)}{2(x-2)(x-4)}$$

$$= \lim_{x \to 2} \frac{(x-2)}{2(x-2)(x-4)}$$

$$= \lim_{x \to 2} \frac{1}{2(x-4)}$$

$$= \frac{1}{2(2)-4} = \boxed{-\frac{1}{4}}$$

SECTION 2 SUPPLEMENTARY EXERCISES

Find the limit.

a) $\lim_{x\to -4}(-x^3+5x^2)$

b) $\lim_{x\to 6}\sqrt{2x+4}$

c) $\lim_{x\to -1}\frac{x^2+6x-7}{x+2}$

d) $\lim_{x\to 0}\frac{x^2+7x}{x}$

e) $\lim_{x\to 5}\frac{x^2-3x-10}{x^2-8x+15}$

f) $\lim_{x\to 0}\frac{\frac{1}{x-6}+\frac{1}{6}}{3x}$

g) $\lim_{x\to 0}\frac{\sqrt{x+6}-\sqrt{6}}{x}$

h) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1}}{x}$

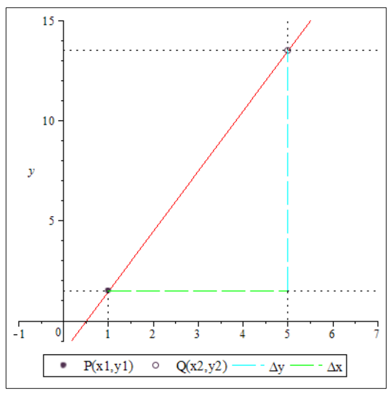
Section 3: The Derivative of a Function

Calculus is primarily the study of change. The basic focus on calculus is divided in to two categories, Differential Calculus and Integral Calculus. In this section we will introduce differential calculus, the study of rate at which something changes.

Consider the example at which x is to be the independent variable and y the dependent variable. If there is any change Δx in the value of x, this will result in a change Δy in the value of y. The resulting change in y for each unit of change in x remains constant and is called the slope of the line.

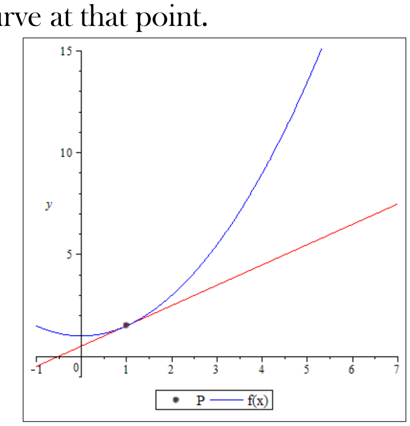
The slope of a straight line is represented as:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in the } y \text{ coordinate}}{\text{Change in the } x \text{ coordinate}}$$



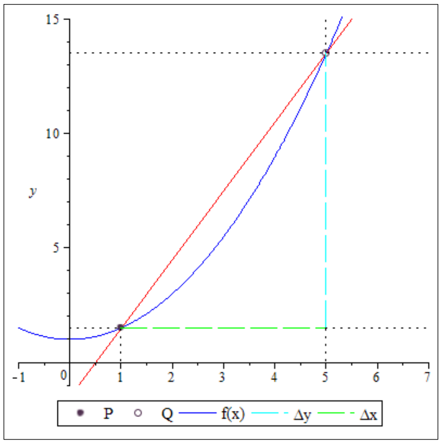
TANGENT LINES

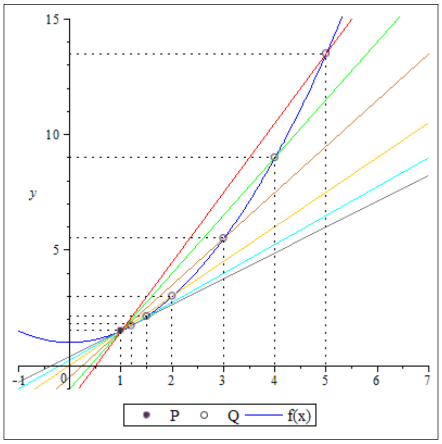
Calculus is concerned with the rate of change that is not constant. Therefore, it is not possible to determine a slope that satisfies every point of the curve. The question that calculus presents is: "What is the rate of change at the point P?" And we can find the slope of the tangent line to the curve at point P by the method of differentiation. A tangent line at a given point to a plane curve is a straight line that touches the curve at that point.



SECANT LINES

Like a tangent line, a secant line is also a straight line; however a secant line passes through two points of a given curve.





Therefore we must consider an infinite sequence of shorter intervals of Δx, resulting in an infinite sequence of slopes. We define the tangent to be the limit of the infinite sequence of slopes. The value of this limit is called the derivative of the given function.

$$\text{The slope of the tangent at P} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Secant line graph
https://www.desmos.com/calculator/pzlb0a2z3v

Go to Desmos link above to see how secant line works. The red dot represents point P, the blue dot represents point Q. Slide the blue dot Q towards the red dot P to see how the secant line PQ becomes the tangent line at P when the distance between PQ approaches zero.

Tangent Line graph
https://www.desmos.com/calculator/dxg5fjxwb7

Go to Desmos link above to see how tangent line changes as it traverses along a function f.

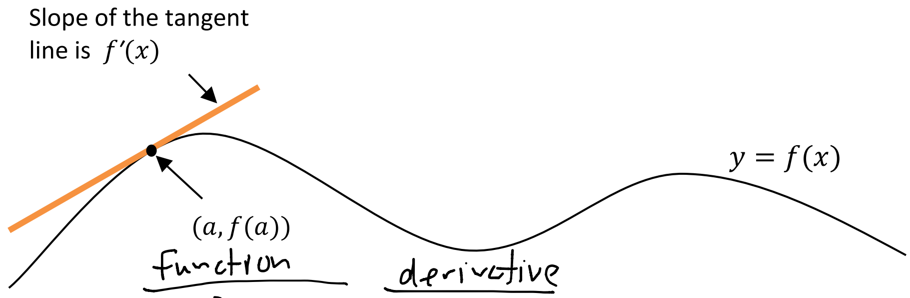
In your own words, what is the tangent line to a function?

____

____

What does derivative of $f(x)$ mean graphically?

It is the slope (m) of the tangent line of the graph $\boldsymbol{y = f(x)}$ at the point $(a, f(a))$



derivative notations:

Function	derivative
$f(x)$	$f'(x)$
y	y' or $\frac{dy}{dx}$

THE DEFINITION OF THE DERIVATIVE

THE DIFFERENCE QUOTIENT

To find the slope of the tangent line to the function $y = f(x)$ at, we must choose a point of tangency, $(x, f(x))$ and a second point $(x + h, f(x + h))$, where $h = \Delta x$.

The slope of the tangent line, or the derivative of a function f is defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{(x + h) - h} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example 1: Find the derivative of the function $f(x) = 7x + 11$ using the definition of derivative.

Let $f(x) = 7x + 11$
And $f(x + h) = 7(x + h) + 11$

By the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{7(x + h) + 11 - (7x + 11)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{7x + 7h + 11 - 7x - 11}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{7h}{h} = \lim_{h \to 0}(7) = 7$$

Therefore, the derivative of $f(x)$ is 7.

Example 2: Find the derivative of the function $g(x) = 3x^2 + 6x - 9$ using the definition of derivative.

Let $g(x) = 3x^2 + 6x - 9$
And $g(x+h) = 3(x+h)^2 + 6(x+h) - 9$

By the definition of the derivative:

$$g'(x) = \lim_{h \to 0} \frac{(3(x+h)^2 + 6(x+h) - 9) - (3x^2 + 6x - 9)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 6x + 6h - 9 - 3x^2 - 6x + 9}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 6x + 6h - 9 - 3x^2 - 6x + 9}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 + 6h}{h}$$

$$g'(x) = \lim_{h \to 0} (6x + 3h + 6)$$

$$g'(x) = 6x + 3(0) + 6$$

$$g'(x) = 6x + 6$$

Therefore the derivative of $g(x)$ is $6x + 6$.

Exercise 1: Find the derivative of the following functions using the definition of derivative.

a) $f(x) = 5x + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(5x+5h+2) - (5x+2)}{h}$$

$$= \lim_{h \to 0} \frac{5x + 5h + 2 - 5x - 2}{h}$$

$$= \lim_{h \to 0} \frac{5h}{h}$$

$f'(x) = 5$

$f(x) = 5x+2$
$f(x+h) = 5(x+h)+2$
$= 5x+5h+2$

* If $f(x) = mx+b$
$f'(x) = m$

slope of the line is its derivative

b) $f(x) = 3x^2 - 4x$

c) $f(x) = \sqrt{x}$

$f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+(0)} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x}$$

d) $f(x) = \frac{1}{x}$

$f(x) = \frac{1}{x}$ $\quad f(x+h) = \frac{1}{x+h}$

Complex fraction

LCD: $x(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h}\right)\left(\frac{x(x+h)}{x(x+h)}\right)$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{x - x - h}{hx(x+h)}$$

$$= \lim_{h \to 0} -\frac{h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$f'(x) = -\frac{1}{x(x+0)}$$

$$f'(x) = -\frac{1}{2x}$$

Example 3. Find the slope of the tangent line to the curve $f(x) = x^2 + 2$ at the point $(-1,3)$ using the definition of the derivative, and find the equation of the tangent line.

By definition, the slope of the tangent line at any point is given by $f'(x)$.
Therefore $f'(x)$ equals to the following:

$$m = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$m = \lim_{h\to 0}\frac{((x+h)^2 + 2) - (x^2 + 2)}{h}$$
$$m = \lim_{h\to 0}\frac{(x^2 + 2xh + h^2 + 2) - x^2 - 2}{h}$$
$$m = \lim_{h\to 0}\frac{2xh + h^2}{h}$$
$$m = \lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$m = \lim_{h\to 0}(2x+h)$$
$$m = 2x + 0$$
$$m = 2x$$

Now this is the slope of the tangent at a point $(x, f(x))$ of the graph. Since the line is tangent at $(-1,3)$, we have to evaluate m at $(-1,3)$. Therefore, $m = 2(-1) = -2$.
The slope of the tangent at $(-1,3)$ is -2.

Equation of a Line Review

Slope-Intercept Form

The equation of any line with slope m and y-intercept b is given by $y = mx + b$

$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}}$ $\qquad$ $b = \text{y} - \text{intercept}$

Point-Slope Form

The equation of the line through (x_1, y_1) with slope m is given by $y - y_1 = m(x - x_1)$

To find the equation of the tangent line to the curve $f(x) = x^2 + 2$ use the point-slope formula to find the equation:

$$y - (3) = -2(x - (-1))$$
$$y - 3 = -2x - 2$$
$$y = -2x + 1$$

The equation of the tangent line to the curve $f(x) = x^2 + 2$ at the point $(-1,3)$ is $y = -2x + 1$.

Exercise 2: Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2,-10)$ using the definition of the derivative, and find the equation of the tangent line.

$f(x+h) = -3(x+h)^2 + (x+h)$

$= -3(x^2+2xh+h^2) + (x+h)$

$= -3x^2 - 6xh - 3h^2 + x + h$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(-3x^2-6xh-3h^2+x+h) - (-3x^2+x)}{h}$

$= \lim_{h\to 0} \frac{-3x^2-6xh-3h^2+x+h+3x^2-x}{h}$

$= \lim_{h\to 0} \frac{-6xh-3h^2+h}{h}$

$f'(x) = \lim_{h\to 0} \frac{h(-6x-3h+1)}{h}$

$= \lim_{h\to 0} (-6x-3h+1)$

$= -6x-3(0)+1$

$f'(x) = -6x+1$ → slope function of $f(x) = -3x^2+x$

Tangent Line (Point-Slope) @ $(2,-10)$ (x_1, y_1)

$y - y_1 = m(x - x_1)$

$y = m(x-x_1) + y_1$

→ $m = f'(2) = -6(2)+1 = -11$

$y = f'(x_1)(x - x_1) + y_1$

$y = f'(2)(x-(2)) + (-10)$

$y = -11(x-2) - 10$

$y = -11x + 22 - 10$

$\boxed{y = -11x + 12}$

Exercise 3: Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2,-8)$ using the definition of the derivative, and find the equation of the tangent line.

$f(x+h) = (x+h)^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$

1
1 1
1 2 1
1 3 3 1

$(2, f(2))$

↓

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3) - x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$

$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$

$= 3x^2 + 3x(0) + (0)^2$

$f'(x) = 3x^2$

Tangent Line @ $(-2,-8)$

$y = f'(x_1)(x - x_1) + y_1$

→ $f'(-2) = 3(-2)^2 = 12$

$y = f'(-2)(x-(-2)) + (-8)$

$y = 12(x+2) - 8$

$y = 12x + 24 - 8$

$\boxed{y = 12x + 16}$

SECTION 3 SUPPLEMENTARY EXERCISES

1. Use the definition of derivative $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to determine the derivative of each function.

   a) $g(x) = \sqrt{1-5x}$

   b) $h(y) = \dfrac{6y}{y+1}$

   c) $p(x) = \sqrt{x} + x$

   d) $k(t) = \dfrac{1}{t^2}$

2. Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x-6}$ at the point $(5,-1)$ using the definition of the derivative, and find the equation of the tangent line.

Section 4: Differentiation Rules

The Differentiation Formulas

Derivative of a Constant Function	$\frac{d}{dx}(c) = 0$	
The Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ where *n* is any real number	
The Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$	
The Sum Rule	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$	
The Difference Rule	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$	
The Product Rule	$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$	Or in prime notation $(fg)' = f'g + g'f$
The Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - \frac{d}{dx}[g(x)] \cdot f(x)}{[g(x)]^2}$	Or in prime notation $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
The Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$	Or, if y is a function of u, and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Find the derivative of each of the following functions using the differentiation rules.

Derivative of a Constant Function: $\frac{d}{dx}(c) = 0$

1a) $f(x) = 60$

$f'(x) = 0$

1b) $y = \frac{8800^{718}}{369} + \frac{40\pi}{47} - 62768.32$

$y' = 0$ $\frac{dy}{dx} = 0$

The Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ where *n* is any real number

2a) $y = x^5$

$\frac{dy}{dx} = 5x^{5-1} = 5x^4$

2b) $g(x) = x^{-6}$

$g'(x) = -6x^{-6-1} = -6x^{-7}$

* $\frac{dy}{dx}$ = derivative of function y

$\frac{d}{dx}(\quad)$ = take derivative of what is in parentheses

The Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

3a) $y = -4x$

$\frac{dy}{dx} = -4\frac{d}{dx}(x^1)$

$= -4(1x^0) = -4(1) = -4$

3b) $p(x) = 3x^7$

$p'(x) = 3 \cdot \frac{d}{dx}(x^7)$

$= 3(7x^{7-1})$

$= 21x^6$

The Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

The Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

4a) $y = 2x^{\frac{1}{2}} + 4x^{\frac{2}{3}}$

$y' = \frac{d}{dx}(2x^{\frac{1}{2}}) + \frac{d}{dx}(4x^{\frac{2}{3}})$

$= 2\frac{d}{dx}(x^{\frac{1}{2}}) + 4\frac{d}{dx}(x^{\frac{2}{3}})$

$= 2(\frac{1}{2}x^{\frac{1}{2}-1}) + 4(\frac{2}{3}x^{\frac{2}{3}-1})$

$= x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{1}{3}}$

4b) $t(x) = 2x^{-5} + 4x + 1$

4c) $y = -2x^{-4} - 5x^2 - 7x$

$\frac{dy}{dx} = -2\frac{d}{dx}(x^{-4}) - 5\frac{d}{dx}(x^2) - 7\frac{d}{dx}(x)$

$= -2(-4x^{-5}) - 5(2x) - 7(1)$

$= 8x^{-5} - 10x - 7$

4d) $h(x) = 4x^{-7} - 3x^{-1}$

Review Properties of Exponents		Examples
Product Rule	$x^m \cdot x^n = x^{m+n}$	$y^8 \cdot y^3 =$ y^{11}
Quotient Rule	$\frac{x^m}{x^n} = x^{m-n}$ where $(x \neq 0)$	$\frac{a^7}{a} =$ a^6
Zero Exponent	$x^0 = 1$ where $(x \neq 0)$	$w^0 =$ 1
Power Rule	$(x^m)^n = x^{m \cdot n}$	$(b^6)^2 =$ b^{12}
Power of a Product	$(x \cdot y)^n = x^n y^n$	$(r^4 t)^3 =$ $r^{12} t^3$
Power of a Quotient	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ where $(y \neq 0)$	$\left(\frac{p^9}{q^2}\right)^5 =$ $\frac{p^{45}}{q^{10}}$
Negative Exponent	$x^{-n} = \frac{1}{x^n}$ where $(x \neq 0)$	$h^{-3} =$ $\frac{1}{h^3}$
Rational Exponents	$a^{\frac{1}{n}} = \sqrt[n]{a}$ where $\sqrt[n]{a}$ is defined on $\mathbb{R}$	$(64)^{\frac{1}{3}} =$ $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$
Rational Exponents	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ where m and n are positive integers and $\sqrt[n]{a}$ is defined on $\mathbb{R}$	$(32)^{\frac{4}{5}} =$ $\sqrt[5]{(32)^4}$ or $\left(\sqrt[5]{32}\right)^4$ $= 2$

5. For each exercise below,
   i. Simplify and rewrite each term as x^n with exponent in the numerator.
   ii. Find the derivative.

Function	Rewriting the exponents	Derivative
$y = \frac{1}{x^2} + \frac{1}{x}$	$y = x^{-2} + x^{-1}$	$y' = -2x^{-3} - x^{-2}$

Function	Rewriting the exponents	Derivative
$y = \frac{1}{\sqrt{x}} - \sqrt{x}$		
$y = \frac{1}{\sqrt[3]{x}} + \left(\sqrt{x}\right)^3$		
$y = \left(x^{\frac{2}{3}}\right)^{\frac{5}{8}}$		
$y = \frac{x^3 + 3x^2 + 6 + 7x^{-3}}{x^2}$		
$y = \frac{2x^3 - x^2 + 3x - 5}{\sqrt{x}}$		

The Product Rule: $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$

or

$$(fg)' = f'g + g'f$$

6a) $y = x^4(2x+3)$

6b) $g(x) = (3x-7)(x^2+6x)$

The Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - \frac{d}{dx}[g(x)] \cdot f(x)}{[g(x)]^2}$

or

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

7a) $y = \frac{x}{3x+1}$

7b) $q(x) = \frac{9x^2}{3x^2 - 2x}$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivative of the composite function $f(g(x))$

Derivative of the outside function f

Derivative of the inside function g

If y is a function of u, and u is a function of x, then the chain rule is $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

8. Find the derivative using the Chain Rule

a) $y = (x^2 + 3)^8$

b) $s(x) = 2(5x - 500)^{1000}$

c) $y = \sqrt{3x + 1}$

d) $y = \dfrac{8}{\sqrt{5x^2+1}}$

9. Find the derivative using combination rules

a) $y = -4x^2(2x^3 - 14)^4$

b) $y = x^3 \cdot \sqrt{x^2 + 3}$

c) $y = \frac{\sqrt{x^2+3}}{x^3}$

SECTION 4 SUPPLEMENTARY EXERCISES

1) Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

c) $y = 6\sqrt{x} - \sqrt[3]{x}$

d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

e) $s(t) = t^2 + \frac{5}{t^2}$

f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$ (Do not use quotient rule!)

g) $f(x) = \dfrac{5x^2 - 2x + 1}{x}$ (Do not use quotient rule!)

2) Find the derivative using the Product or Quotient Rule.

a) $h(t) = (4t + 3)(t - 7)$

b) $y = 3x\sqrt{x + 5}$

c) $p(x) = \frac{x+5}{x^2-9}$

d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative using the Chain Rule and combination rules.

a) $v(x) = (2 - 4x)^{100}$

b) $v(x) = -x^3(2 - 4x)^{100}$

c) $y = \sqrt{x^2 + 3x + 4}$

4) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2,-10)$ using the differentiation formulas, and find the equation of the tangent line.

5) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2,-8)$ using the differentiation formulas, and find the equation of the tangent line.

6) Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x-6}$ at the point $(5,-1)$ using the differentiation formulas, and find the equation of the tangent line.

Section 5: The Derivative of Trigonometric Functions

Derivative of all six Trigonometric Functions	
Sine	$\frac{d}{dx}\sin(x) = \cos(x)$
Cosine	$\frac{d}{dx}\cos(x) = -\sin(x)$
Tangent	$\frac{d}{dx}\tan(x) = \sec^2(x)$
Cotangent	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
Secant	$\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(x)$
Cosecant	$\frac{d}{dx}\csc(x) = -\csc(\mathrm{x})\cot(x)$

Example: Find the derivative of the trigonometric function $f(x) = 2x^3\cos(5x)$ using the rules of differentiation.

A derivative that requires a combination of the Product Rule and the Chain Rule

Product Rule: $(fg)' = f'g + g'f$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}(2x^3\cos(x)) = \left[\frac{d}{dx}(2x^3)\right]\cos(5x) + 2x^3\left[\frac{d}{dx}\cos(5x)\right]$$

$$= 6x^2\cos(5x) + 2x^3(-\sin(5x)\cdot 5)$$

$$= 6x^2\cos(5x) - 10x^3\sin(5x)$$

Exercise 1: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $y = 2\sin x$

b) $y = \sin(2x)$

c) $y = x\sin x$

d) $y = \sin x^2$

e) $y = \sin^2 x$

f) $y = \sin^2(x^2)$

Exercise 2: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $r(x) = x\cos(2x^2)$

b) $g(x) = \tan\left(\frac{3\text{x}}{4}\right)$

c) $k(x) = \csc x \cdot \cot x$

d) $h(x) = \frac{\cos(2\text{x})}{\sin(x)+1}$

e) $f(x) = \sqrt{\sin x + 5}$

Exercise 3: Show $\frac{d}{dx}\tan x = \sec^2 x$. (Hint, rewrite $\tan x = \frac{\sin x}{\cos x}$)

SECTION 5 SUPPLEMENTARY EXERCISES

1. $v(x) = \tan\left(\sqrt{x^3 + 2}\right)$
2. $n(x) = 5\cos^3(x) - \sin(2x)$
3. $g(x) = 2x^2\sec^2(8x)$
4. $f(x) = \frac{\sin(x)\sec(5x)}{3x^2}$
5. $y = \frac{\cos(x)}{2\sin(-3x)}$
6. $y = \sqrt{\sin^2(x) + 5}$

Section 6: Higher Order Derivatives

One can determine higher order derivatives by finding the derivative of derivatives. For example, the second derivative is the derivative of the derivative of the function (or the first derivative). The third derivative is the derivative of the second derivative, etc. The higher order derivatives are denoted by

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad y''' = \frac{d^3y}{dx^3}, \quad y^{(4)} = \frac{d^4y}{dx^4}, \ldots, \quad y^{(n)} = \frac{d^ny}{dx^n}$$

or in function notation $f', f'', f''', f^{(4)}, \ldots, f^{(n)}$.

Exercise 1. Find $f', f'', f''', f^{(4)}$ of the polynomial function $f(x) = x^3 - 2x^2 - 5x + 6$. What is the pattern for 4th and higher order derivatives?

Exercise 2. Find $y', y'', y''', y^{(4)}$ of the rational function $y = \frac{1}{x}$. What is the pattern for 4th and higher order derivatives?

Exercise 3. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4},$ of the sine function $y = \sin x$. What is the pattern for 4th and higher order derivatives?

SECTION 6 SUPPLEMENTARY EXERCISES

1. Find $f', f'', f''', f^{(4)}$ of the polynomial function $f(x) = x^4 + 8x^3 - 9x^2 - 15x + 89$. What is the pattern for 4th and higher order derivatives?

2. Find $y', y'', y''', y^{(4)}$ of the radical function $y = \sqrt{x}$. What is the pattern for 4th and higher order derivatives?

3. Find $g', g'', g''', g^{(4)}$ of the polynomial function $g(x) = \tan x$. What is the pattern for 4th and higher order derivatives?

Section 7: Derivatives and the Shape of a Graph

Critical Points

Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.

In other words, a point c is a critical point if it satisfies the following:

- c is an interior point in the domain of f.
- $f(c)$ is defined.
- Either $f'(c) = 0$ or $f'(c)$ is undefined.

Intervals of Increasing and Decreasing

i. If $f'(x) > 0$ on an open interval, then f is increasing on the interval.

ii. If $f'(x) < 0$ on an open interval, then f is decreasing on the interval.

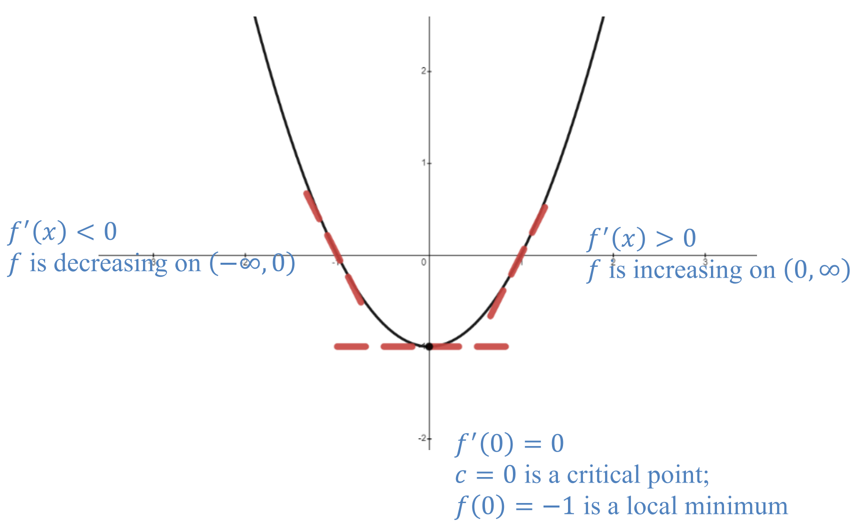
First Derivative Test

Suppose that f is a continuous function over an interval containing a critical point c. If f is differentiable on the interval, except possibly at point c then $f(c)$ satisfies one of the following descriptions:

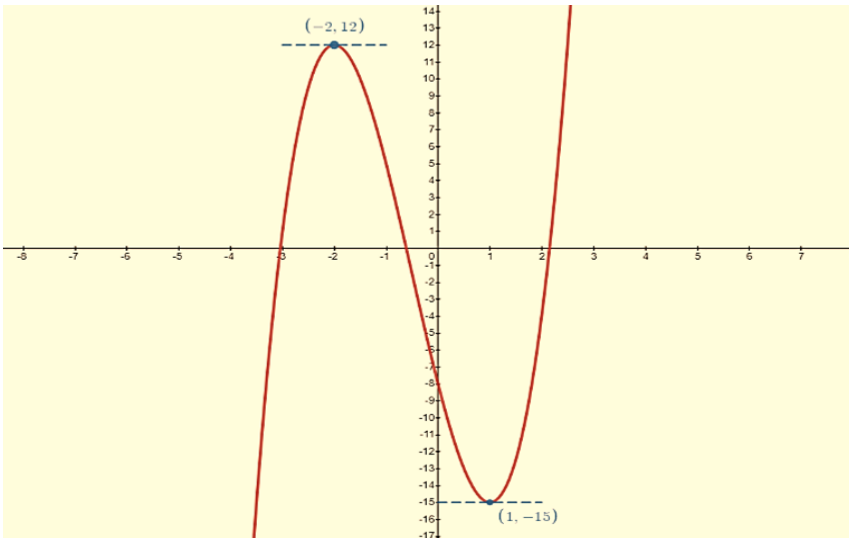
i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.

ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.

iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

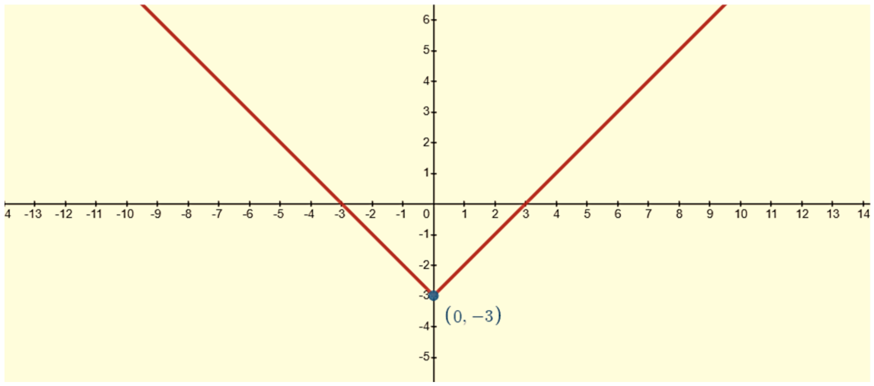


Exercise 1. Curve analysis of a polynomial function: $f(x) = 2x^3 + 3x^2 - 12x - 8$



- Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$
- Show Critical points are $c = -2$ and $c = 1$.
- The point $(-2, 12)$ is a local maximum; the point $(1, -15)$ is a local minimum
- Show the function is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$ and decreasing on the interval $(-2, 1)$

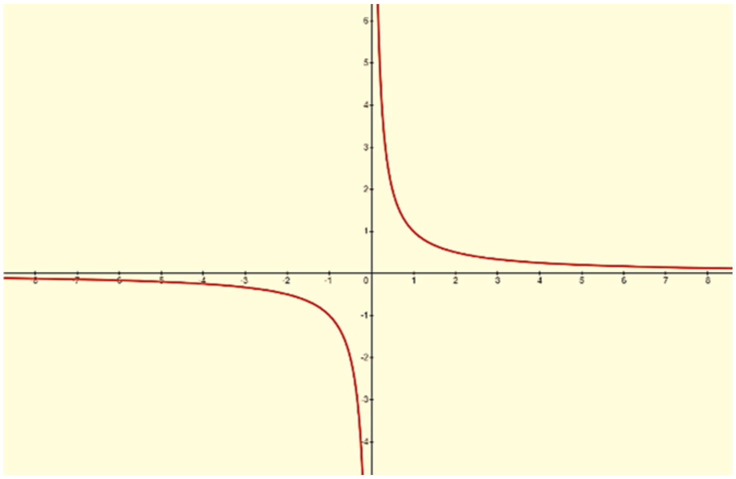
Exercise 2. Curve analysis of an absolute value function: $f(x) = |x| - 3$



- Domain = $(-\infty, \infty)$; Range = $(-3, \infty)$
- Show the critical point is $c = 0$. The derivative $f'(0)$ is undefined.

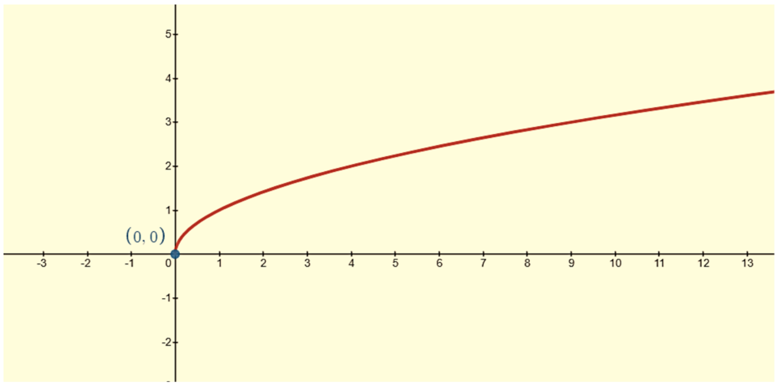
  Why is $f'(0)$ undefined? Hint: What is $f'(x)$ on the interval $(-\infty, 0)$? What is $f'(x)$ on the interval $(0, \infty)$? Is it true that $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$?
- The point $(0, -3)$ is a local minimum; there is no local maximum.
- Show the function is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$

Exercise 3. Curve analysis of a rational function: $f(x) = \frac{1}{x}$



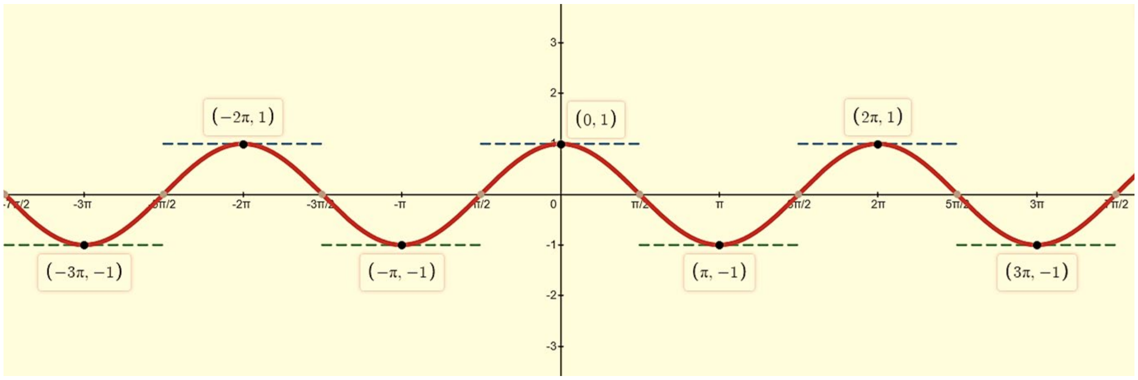
- Domain = $(-\infty, 0) \cup (0, \infty)$; Range = $(-\infty, 0) \cup (0, \infty)$.
- There is no critical point. Why is there no critical point? (Answer: $x = 0$ is not in the domain.)
- Why is there no local maximum or local minimum?
- Show the function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$.
- $y = 0$ is a horizontal asymptote, $x = 0$ is a vertical asymptote.

Exercise 4. Curve analysis of a radical function: $f(x) = \sqrt{x}$



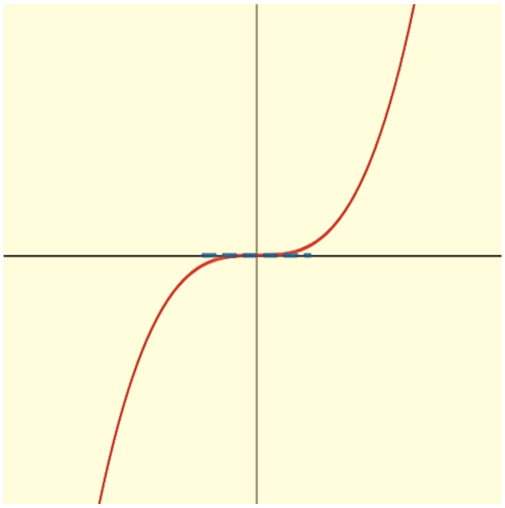
- Domain = $[0, \infty)$; Range = $[0, \infty)$.
- There is no critical point. Why is $x = 0$ not a critical point? (Answer: $x = 0$ is not an interior point in the domain; it is an endpoint.)
- The point $(0, 0)$ an absolute minimum. There is no absolute maximum. There is no local maximum/minimum.
- Show the function is increasing on the interval $(0, \infty)$.

Exercise 5. Curve analysis of a trigonometric function: $f(x) = \cos x$



- Domain = $(-\infty, \infty)$; Range = $[-1, 1]$.
- Show the critical points are $c = k\pi$ where $k = \cdots, -2, -1, 0, 1, 2, \cdots$ are integers.
- Show the local maximums are points $(2k\pi, 1)$; the local minimums are points $\big((2k+1)\pi, -1\big)$, where $k = \cdots, -2, -1, 0, 1, 2, \cdots$ are integers.
- The function is decreasing on the intervals $(2k\pi, (2k+1)\pi)$ and increasing on the intervals $\big((2k+1)\pi, (2k+2)\pi\big)$, where $k = \cdots, -2, -1, 0, 1, 2, \cdots$ are integers

Exercise 6. Curve analysis of $f(x) = x^3$



- Domain = $(-\infty, \infty)$; Range = $(-\infty, \infty)$
- Show Critical point is $c = 0$.
- There is no local maximum or local minimum. Why is there no local maximum or local minimum although there exists a critical point? (Answer: $f'(x) = 3x^2 \geq 0$ for all x in the domain. Since there is no sign change in $f'(x)$ from $x < c$ to $x > c$ (both positive), thus $f(c)$ is neither a local max nor min.)

Are both statements below true? Are they Interchangeable?

Statement 1: If $f'(x) > 0$ on an open interval, then f is increasing on the interval.
Statement 2: If f is increasing on an interval, then $f'(x) > 0$.

Counter Example: The function $f(x) = x^3$ is increasing on the interval $(-\infty, \infty)$, but it's not true that $f'(x) > 0$ on $(-\infty, \infty)$. (Because $f'(0) = 0$, not > 0.)

Statement 1 is true by definition. Statement 2 is false because of the counter example. The two statements are not interchangeable.

SECTION 7 SUPPLEMENTARY EXERCISES

For each function, determine the following. State “none” if there is none.

i. Domain and range
ii. Critical points
iii. Local maximum and local minimum
iv. The intervals where the function is increasing and the intervals where it is decreasing
v. Sketch the graph

1. $f(x) = x^3 + 3x^2 - 9x - 10$

2. $f(x) = |x + 3|$

3. $f(x) = \frac{1}{x^2}$

4. $f(x) = \sin x$

5. $f(x) = e^x$