The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivative of the composite function $f(g(x))$

Derivative of the outside function $f$

Derivative of the inside function $g$

If $y$ is a function of $u$, and $u$ is a function of $x$, then the chain rule is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

8. Find the derivative using the Chain Rule

a) $y = (x^2+3)^8$

How?

$x \to (x^2+3)^8$

$x \to u = x^2+3$

$\to f(u) = u^8$

$$\frac{dy}{dx} = f'(u) \cdot u'$$

$$= 8u^7 \cdot u'$$

$$= 8(x^2+3)^7 \cdot (x^2+3)'$$

$$= 8(x^2+3)^7 (2x)$$

$$\boxed{\frac{dy}{dx} = 16x(x^2+3)^7}$$

b) $s(x) = 2(5x-500)^{1000}$

$u = 5x-500$

$f(u) = u^{1000}$

$$s'(x) = 2f'(u) \cdot u'$$

$$= 2 \cdot 1000u^{999} \cdot u'$$

$$s'(x) = 2 \cdot 1000(5x-500)^{999}(5x-500)'$$

$$= 2000(5x-500)^{999} \cdot 5$$

$$= 10000(5x-500)^{999}$$

c) $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$x \to (3x+1)^{\frac{1}{2}}$

$x \to u = 3x+1$

$\to f(u) = u^{\frac{1}{2}}$

$$y' = f'(u) \cdot u'$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \cdot 3$$

$$= \frac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3$$

$$= \frac{3}{2}(3x+1)^{-\frac{1}{2}}$$

$$\boxed{y' = \frac{3}{2\sqrt{3x+1}}}$$

d) $y = \frac{8}{\sqrt{5x^2+1}} = 8(5x^2+1)^{-\frac{1}{2}}$

$x \to (5x^2+1)^{-\frac{1}{2}}$

$u = 5x^2+1$

$f(u) = u^{-\frac{1}{2}}$

$$y' = 8\frac{d}{dx}(5x^2+1)^{-\frac{1}{2}}$$

$$= 8(f'(u) \cdot u')$$

$$= 8\left(-\frac{1}{2}u^{-\frac{3}{2}} \cdot 10x\right)$$

$$= 8\left(-\frac{1}{2}(5x^2+1)^{-\frac{3}{2}} \cdot 10x\right)$$

$$= -40x(5x^2+1)^{-\frac{3}{2}}$$

$$= -\frac{40x}{(5x^2+1)^{\frac{3}{2}}}$$

9. Find the derivative using combination rules

a) $y = -4x^2(2x^3 - 14)^4$

*use product rule

$f = -4x^2$ $\quad g = (2x^3-14)^4$

$f' = -8x$ $\quad g' = 24x^2(2x^3-14)^3$

$u = 2x^3 - 14$

$h(u) = u^4$

$g' = h'(u)\,u' = 4u^3 \cdot 6x^2 = 4(2x^3-14)^3 \cdot 6x^2$

$y' = f'g + fg'$

$= -8x(2x^3-14)^4 + -4x^2\left(24x^2(2x^3-14)^3\right)$

$\boxed{y' = -8x(2x^3-14)^4 - 96x^4(2x^3-14)^3}$

b) $y = x^3 \cdot \sqrt{x^2+3}$ $= x^3(x^2+3)^{\frac{1}{2}}$

$f = x^3$ $\quad g = (x^2+3)^{\frac{1}{2}}$

$f' = 3x^2$ $\quad g' = x(x^2+3)^{-\frac{1}{2}}$

$x \to (x^2+3)^{\frac{1}{2}}$

$u = x^2+3$

$h(u) = u^{\frac{1}{2}}$

$g' = h'(u) \cdot u' = \frac{1}{2}u^{-\frac{1}{2}} \cdot 2x = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x$

$\frac{dy}{dx} = f'g + fg'$

$= 3x^2(x^2+3)^{\frac{1}{2}} + x^3\left(x(x^2+3)^{-\frac{1}{2}}\right)$

$= 3x^2(x^2+3)^{\frac{1}{2}} + x^4(x^2+3)^{-\frac{1}{2}}$

c) $y = \frac{\sqrt{x^2+3}}{x^3}$ $= \frac{(x^2+3)^{\frac{1}{2}}}{x^3}$

$f = (x^2+3)^{\frac{1}{2}}$ $\quad g = x^3$

$f' = x(x^2+3)^{-\frac{1}{2}}$ $\quad g' = 3x^2$

$u = x^2+3$

$h(u) = u^{\frac{1}{2}}$

$f' = h'(u) \cdot u'$

$= \frac{1}{2}u^{-\frac{1}{2}} \cdot 2x$

$= \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \cdot 2x = x(x^2+3)^{-\frac{1}{2}}$

$y' = \frac{f'g - fg'}{g^2}$

$= x(x^2+3)^{-\frac{1}{2}}x^3 - (x^2+3)^{\frac{1}{2}} \cdot 3x^2$

$\boxed{y' = x^4(x^2+3)^{-\frac{1}{2}} - 3x^2(x^2+3)^{\frac{1}{2}}}$

## SECTION 4 SUPPLEMENTARY EXERCISES

1) Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

   a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

   b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

   c) $y = 6\sqrt{x} - \sqrt[3]{x}$

   d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

   e) $s(t) = t^2 + \frac{5}{t^2}$

   f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$ (Do not use quotient rule!)

   g) $f(x) = \dfrac{5x^2 - 2x + 1}{x}$ (Do not use quotient rule!)

2) Find the derivative using the Product or Quotient Rule.

   a) $h(t) = (4t + 3)(t - 7)$

   b) $y = 3x\sqrt{x + 5}$

   c) $p(x) = \frac{x+5}{x^2-9}$

   d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative using the Chain Rule and combination rules.

   a) $v(x) = (2 - 4x)^{100}$

   b) $v(x) = -x^3(2 - 4x)^{100}$

   c) $y = \sqrt{x^2 + 3x + 4}$

4) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2, -10)$ using the differentiation formulas, and find the equation of the tangent line.

5) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2, -8)$ using the differentiation formulas, and find the equation of the tangent line.

6) Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x-6}$ at the point $(5, -1)$ using the differentiation formulas, and find the equation of the tangent line.

# Section 5: The Derivative of Trigonometric Functions

| Derivative of all six Trigonometric Functions | |
|---|---|
| Sine | $\frac{d}{dx}\sin(x) = \cos(x)$ |
| Cosine | $\frac{d}{dx}\cos(x) = -\sin(x)$ |
| Tangent | $\frac{d}{dx}\tan(x) = \sec^2(x)$ |
| Cotangent | $\frac{d}{dx}\cot(x) = -\csc^2(x)$ |
| Secant | $\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(x)$ |
| Cosecant | $\frac{d}{dx}\csc(x) = -\csc(\mathrm{x})\cot(x)$ |

Example: Find the derivative of the trigonometric function $f(x) = 2x^3\cos(5x)$ using the rules of differentiation.

A derivative that requires a combination of the Product Rule and the Chain Rule

Product Rule: $(fg)' = f'g + g'f$

The Chain Rule: $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)\cdot g'(x)$

$$\frac{d}{dx}(2x^3\cos(x)) = \left[\frac{d}{dx}(2x^3)\right]\cos(5x) + 2x^3\left[\frac{d}{dx}\cos(5x)\right]$$

$$= 6x^2\cos(5x) + 2x^3(-\sin(5x)\cdot 5)$$

$$= 6x^2\cos(5x) - 10x^3\sin(5x)$$

**Exercise 1:** Find the derivative of the following trigonometric functions using the differentiation rules.

a) $y = 2\sin x$

$\frac{dy}{dx} = 2\frac{d}{dx}(\sin(x))$

$\frac{dy}{dx} = 2\cos(x)$

b) $y = \sin(2x)$

using chain rule

$u = 2x$

$f(u) = \sin(u)$

$y' = f'(u) \cdot u'$

$= \cos(u) \cdot 2$

$= 2\cos(2x)$ ✓

$y = \sin(2x) = 2\sin(x)\cos(x)$

using product rule

$f = 2\sin(x)$ $g = \cos(x)$

$f' = 2\cos(x)$ $g' = -\sin(x)$

$\frac{dy}{dx} = f'g + fg'$

$= 2\cos(x)\cos(x) + 2\sin(x)(-\sin(x))$

$= 2(\cos^2(x) - \sin^2(x))$

$= 2\cos(2x)$ ✓

c) $y = x\sin(x)$

| $f = x$ | $g = \sin(x)$ |
|---|---|
| $f' = 1$ | $g' = \cos(x)$ |

$y' = f'g + fg'$

$= 1 \cdot \sin(x) + x\cos(x)$

$= \sin(x) + x\cos(x)$

d) $y = \sin(x^2)$

use chain rule

$u = x^2$

$f(u) = \sin(u)$

$y' = f'(u) \cdot u'$

$= \cos(u) \cdot 2x$

$= \cos(x^2) \cdot 2x$

$= 2x\cos(x^2)$

e) $y = \sin^2(x) = (\sin(x))^2$

chain rule

$u = \sin(x)$

$f(u) = u^2$

$\frac{dy}{dx} = f'(u) \cdot u'$

$= 2u \cdot \cos(x)$

$= \boxed{2\sin(x)\cos(x)}$

$= \sin(2x)$

f) $y = \sin^2(x^2)$

Chain Rule

$x \to \sin^2(x^2)$

$u = x^2$

$g(u) = \sin(u)$

$f(g) = g^2$

$y' = f'(g) \cdot g'(u) \cdot u'$

$= 2g \cdot \cos(u) \cdot 2x$

$= 2g \cdot \cos(x^2)2x$

$= 2\sin(u) \cdot \cos(x^2) \cdot 2x$

$= 2\sin(x^2)\cos(x^2) \cdot 2x$

$= \boxed{4x\sin(x^2)\cos(x^2)}$

$= 2x\sin(2x^2)$

chain rule (identifying out → in)

outermost: $(\sin(x^2))^2$

inner: $\sin(x^2)$

innermost: $x^2$

$y' = 2\sin(x^2)\cos(x^2) \cdot 2x$

$= 4x\sin(x^2)\cos(x^2)$

**Exercise 2:** Find the derivative of the following trigonometric functions using the differentiation rules.

a) $r(x) = x\cos(2x^2)$

a) $r(x) = x\cos(2x^2)$ — Product ~~Chain~~ Rule

$F = x \qquad G = \cos(2x^2)$

$F' = 1 \qquad G' = -4x\sin(2x^2)$

$y' = f'(u) \cdot u'$

$u = 2x^2$

$f(u) = \cos(u)$

$-\sin(2x^2) \cdot 4x$

$-4x\sin(2x^2)$

$r'(x) = f'g + fg'$

$r'(x) = (1)(\cos(2x^2)) + (x)(-4x\sin(2x^2))$

$= \cos(2x^2) - 4x^2\sin(2x^2)$

b) $g(x) = \tan\left(\frac{3x}{4}\right) = \tan\left(\frac{3}{4}x\right)$

$u = \frac{3}{4}x$

$f(u) = \tan(u)$

$g'(x) = f'(u) \cdot u'$

$= \sec^2(u) \cdot \frac{3}{4}$

$= \frac{3}{4}\sec^2\left(\frac{3}{4}x\right)$

c) $k(x) = \csc x \cdot \cot x$

c) $k(x) = \csc(x) \cdot \cot(x)$

$f = \csc(x) \qquad g = \cot(x)$

$f' = -\csc(x)\cot(x) \qquad g' = -\csc^2(x)$

$k'(x) = f'g + fg'$

$= (-\csc(x)\cot(x))(\cot(x)) + (\csc(x))(-\csc^2(x))$

$k'(x) = -\csc(x)\cot^2(x) - \csc^3(x)$

d) $h(x) = \frac{\cos(2x)}{\sin(x)+1}$

e) $f(x) = \sqrt{\sin x + 5}$

**Exercise 3:** Show $\frac{d}{dx}\tan x = \sec^2 x$. (Hint, rewrite $\tan x = \frac{\sin x}{\cos x}$)

$$\frac{d}{dx}\left(\tan(x)\right)$$

$$= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Apply Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'g - fg'}{g^2}$$

| $f = \sin(x)$ | $g = \cos(x)$ |
|---|---|
| $f' = \cos(x)$ | $g' = -\sin(x)$ |

$$\cos^2(x) + \sin^2(x) = 1$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

QED

## SECTION 5 SUPPLEMENTARY EXERCISES

1. $v(x) = \tan\left(\sqrt{x^3 + 2}\right)$
2. $n(x) = 5\cos^3(x) - \sin(2x)$
3. $g(x) = 2x^2\sec^2(8x)$
4. $f(x) = \frac{\sin(x)\sec(5x)}{3x^2}$
5. $y = \frac{\cos(x)}{2\sin(-3x)}$
6. $y = \sqrt{\sin^2(x) + 5}$