# Section 4: Differentiation Rules

### The Differentiation Formulas

| | | |
|---|---|---|
| Derivative of a Constant Function | $\frac{d}{dx}(c) = 0$ | |
| The Power Rule | $\frac{d}{dx}(x^n) = nx^{n-1}$ where *n* is any real number | |
| The Constant Multiple Rule | $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$ | |
| The Sum Rule | $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ | |
| The Difference Rule | $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ | |
| The Product Rule | $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$ | Or in prime notation $(fg)' = f'g + g'f$ |
| The Quotient Rule | $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - \frac{d}{dx}[g(x)]\cdot f(x)}{[g(x)]^2}$ | Or in prime notation $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ |
| The Chain Rule | $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big) \cdot g'(x)$ | Or, if $y$ is a function of $u$, and $u$ is a function of $x$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ |

$e \approx 2.718\ldots$

Find the derivative of each of the following functions using the differentiation rules.

$y = e$

$\frac{dy}{dx} = 0$

Derivative of a Constant Function: $\frac{d}{dx}(c) = 0$

1a) $f(x) = 60$ $\quad f'(x) = 0$

1b) $y = \frac{8800^{718}}{369} + \frac{40\pi}{47} - 62768.32$

$y' = 0$

$y = \pi$

$y' = 0$

The Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ where *n* is any real number

2a) $y = x^5$

$\frac{dy}{dx} = 5x^{5-1}$

$= 5x^4$

2b) $g(x) = x^{-6}$

$g'(x) = -6x^{-6-1}$

$= -6x^{-7}$

The Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

3a) $y = -4x$

$y' = -4\frac{d}{dx}(x^1)$

$= -4(1x^{1-1})$

$= -4\cancel{x^0} = -4$

3b) $p(x) = 3x^7$

$\frac{dp}{dx} = 3\frac{d}{dx}(x^7)$

$= 3 \cdot 7x^6$

$= 21x^6$

The Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

The Difference Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

4a) $y = 2x^{\frac{1}{2}} + 4x^{\frac{2}{3}}$

$y' = \frac{d}{dx}(2x^{\frac{1}{2}}) + \frac{d}{dx}(4x^{\frac{2}{3}})$

$y' = 2\frac{d}{dx}(x^{\frac{1}{2}}) + 4\frac{d}{dx}(x^{\frac{2}{3}})$

$y' = 2(\frac{1}{2}x^{-\frac{1}{2}}) + 4(\frac{2}{3}x^{-\frac{1}{3}})$

$y' = x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{1}{3}}$

$\boxed{y' = \frac{1}{\sqrt{x}} + \frac{8}{3\sqrt[3]{x}}}$

4b) $t(x) = 2x^{-5} + 4x + 1$

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$t'(x) = \frac{d}{dx}(2x^{-5}) + \frac{d}{dx}(4x^1) + \frac{d}{dx}(1)$

$y' = 2\frac{d}{dx}(x^{-5}) + 4\frac{d}{dx}(x^1) + 0$

$y' = 2(-5x^{-5-1}) + 4(1x^{1-1}) + 0$

$y' = 2(-5x^{-6}) + 4(1) + 0$

$y' = -10x^{-6} + 4 + 0$

$t'(x) = -10x^{-6} + 4$

4c) $y = -2x^{-4} - 5x^2 - 7x$

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$\frac{dy}{dx} = 8x^{-5} - 10x^1 - 7$

4d) $h(x) = 4x^{-7} - 3x^{-1}$

$h'(x) = 4\frac{d}{dx}(x^{-7}) - 3\frac{d}{dx}(x^{-1})$

$= 4(-7x^{-8}) - 3(-1x^{-2})$

$\boxed{h'(x) = -28x^{-8} + 3x^{-2}}$

| Review Properties of Exponents | | Examples |
|---|---|---|
| Product Rule | $x^m \cdot x^n = x^{m+n}$ | $y^8 \cdot y^3 = y^{8+3} = y^{11}$ |
| Quotient Rule | $\frac{x^m}{x^n} = x^{m-n}$ where $(x \neq 0)$ | $\frac{a^7}{a^1} = a^{7-1} = a^6$ |
| Zero Exponent | $x^0 = 1$ where $(x \neq 0)$ | $w^0 = 1$ |
| Power Rule | $(x^m)^n = x^{m \cdot n}$ | $(b^6)^2 = b^{6 \cdot 2} = b^{12}$ |
| Power of a Product | $(x \cdot y)^n = x^n y^n$ | $(r^4 t)^3 = (r^4)^3 t^3 = r^{12} t^3$ |
| Power of a Quotient | $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ where $(y \neq 0)$ | $\left(\frac{p^9}{q^2}\right)^5 = \frac{(p^9)^5}{(q^2)^5} = \frac{p^{45}}{q^{10}}$ |
| Negative Exponent | $x^{-n} = \frac{1}{x^n}$ where $(x \neq 0)$ | $h^{-3} = \frac{1}{h^3}$ |
| Rational Exponents | $a^{\frac{1}{n}} = \sqrt[n]{a}$ where $\sqrt[n]{a}$ is defined on $\mathbb{R}$ | $(64)^{\frac{1}{3}} = \sqrt[3]{64} = \sqrt[3]{4^3} = 4$ |
| Rational Exponents | $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ where $m$ and $n$ are positive integers and $\sqrt[n]{a}$ is defined on $\mathbb{R}$ | $(32)^{\frac{4}{5}} = \sqrt[5]{32^4} = \left(\sqrt[5]{32}\right)^4$<br>$\sqrt[5]{(2^5)^4}$ $= 2^4$<br>$\cancel{\sqrt[5]{(2^4)^5}}$ $= 16$<br>$2^4$<br>$16$ |

5. For each exercise below,
   i. Simplify and rewrite each term as $x^n$ with exponent in the numerator.
   ii. Find the derivative.

| Function | Rewriting the exponents | Derivative |
|---|---|---|
| $y = \frac{1}{x^2} + \frac{1}{x}$ | $y = x^{-2} + x^{-1}$ | $y' = -2x^{-3} - x^{-2}$ |

| Function | Rewriting the exponents | Derivative |
|---|---|---|
| $y = \frac{1}{\sqrt{x}} - \sqrt{x}$ | $y = x^{-\frac{1}{2}} - x^{\frac{1}{2}}$ | $y' = -\frac{1}{2}x^{-\frac{1}{2}-1} - \frac{1}{2}x^{\frac{1}{2}-1}$<br>$\boxed{y' = -\frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}$ |
| $y = \frac{1}{\sqrt[3]{x}} + \left(\sqrt[2]{x}\right)^3$ | $y = x^{-\frac{1}{3}} + x^{\frac{3}{2}}$ | $\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{1}{3}-1} + \frac{3}{2}x^{\frac{3}{2}-1}$<br>$\boxed{\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{4}{3}} + \frac{3}{2}x^{\frac{1}{2}}}$ |
| $y = \left(x^{\frac{2}{3}}\right)^{\frac{5}{8}}$ | $y = x^{\frac{10}{24}} = x^{\frac{5}{12}}$ | $y'(x) = \frac{5}{12}x^{\frac{5}{12}-1}$<br>$\boxed{y'(x) = \frac{5}{12}x^{-\frac{7}{12}}}$ |
| $y = \frac{x^3 + 3x^2 + 6 + 7x^{-3}}{x^2}$ | $y = \frac{x^3}{x^2} + \frac{3x^2}{x^2} + \frac{6}{x^2} + \frac{7x^{-3}}{x^2}$<br>$y = x + 3 + 6x^{-2} + 7x^{-5}$ | $y' = 1 + \cancel{0} + 6(-2x^{-2-1}) + 7(-5x^{-5-1})$<br>$\boxed{y' = 1 - 12x^{-3} - 35x^{-6}}$ |
| $y = \frac{2x^3 - x^2 + 3x - 5}{\sqrt{x}}$ | $y = \frac{2x^3}{x^{\frac{1}{2}}} - \frac{x^2}{x^{\frac{1}{2}}} + \frac{3x^1}{x^{\frac{1}{2}}} - \frac{5}{x^{\frac{1}{2}}}$<br>$y = 2x^{3-\frac{1}{2}} - x^{2-\frac{1}{2}} + 3x^{1-\frac{1}{2}} - 5x^{-\frac{1}{2}}$<br>$y = 2x^{\frac{5}{2}} - x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - 5x^{-\frac{1}{2}}$ | $\frac{dy}{dx} =$<br>$\frac{dy}{dx} = 5x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}} + \frac{5}{2}x^{-\frac{3}{2}}$ |

The Product Rule: $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$

or

$(fg)' = f'g + g'f$ $\quad u'v + uv'$

6a) $y = x^4(2x+3)$

| | |
|---|---|
| $f = x^4$ | $g = 2x+3$ |
| $f' = 4x^3$ | $g' = 2$ |

$$\frac{dy}{dx} = f'g + fg'$$
$$= (4x^3)(2x+3) + (x^4)(2)$$
$$= 8x^4 + 12x^3 + 2x^4$$
$$= 10x^4 + 12x^3$$

6b) $g(x) = (3x-7)(x^2+6x)$

$g(x) = (3x-7)(x^2+6x)$

| | |
|---|---|
| $u = 3x-7$ | $v = x^2+6x$ |
| $u' = 3$ | $v' = 2x+6$ |

$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = (3)(x^2+6x) + (3x-7)(2x+6)$$
$$\frac{dy}{dx} = \cancel{3x^2} + \cancel{18x} + \cancel{6x^2} + \cancel{18x} - 14x - 42$$
$$\frac{dy}{dx} = 9x^2 + 36x - 14x - 42$$
$$\frac{dg}{dx} = 9x^2 + 22x - 42$$

The Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - \frac{d}{dx}[g(x)] \cdot f(x)}{[g(x)]^2}$

or

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

*f is always numerator
g is always denominator

7a) $y = \frac{x}{3x+1}$

| | |
|---|---|
| $f = x$ | $g = 3x+1$ |
| $f' = 1$ | $g' = 3$ |

$$y' = \frac{f'g - fg'}{g^2}$$
$$y' = \frac{(1)(3x+1) - (x)(3)}{(3x+1)^2}$$
$$y' = \frac{3x+1-3x}{(3x+1)^2}$$
$$\boxed{y' = \frac{1}{(3x+1)^2}}$$

7b) $q(x) = \frac{9x^2}{3x^2 - 2x}$

| | |
|---|---|
| $f = 9x^2$ | $g = 3x^2 - 2x$ |
| $f' = 18x$ | $g' = 6x-2$ |

$$q'(x) = \frac{f'g - fg'}{g^2}$$
$$= \frac{(18x)(3x^2-2x) - (9x^2)(6x-2)}{(3x^2-2x)^2}$$
$$= \frac{\cancel{54x^3} - 36x^2 \cancel{- 54x^3} + 18x^2}{(3x^2-2x)^2}$$
$$= \frac{-18x^2}{(3x^2-2x)^2} = \frac{-18\cancel{x^2}}{\cancel{x^2}(3x-2)^2}$$
$$= \boxed{-\frac{18}{(3x-2)^2}}$$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivative of the composite function $f(g(x))$

Derivative of the outside function $f$

Derivative of the inside function $g$

If $y$ is a function of $u$, and $u$ is a function of $x$, then the chain rule is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

8. Find the derivative using the Chain Rule

How?

$x \to (x^2+3)^8$

$x \to u = x^2+3$

$\to f(u) = u^8$

a) $y = (x^2+3)^8$

$$\frac{dy}{dx} = f'(u) \cdot u'$$

$$= 8u^7 \cdot u'$$

$$= 8(x^2+3)^7 \cdot (x^2+3)'$$

$$= 8(x^2+3)^7 (2x)$$

$$\boxed{\frac{dy}{dx} = 16x(x^2+3)^7}$$

b) $s(x) = 2(5x-500)^{1000}$

c) $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$x \to (3x+1)^{\frac{1}{2}}$

$x \to u = 3x+1$

$\to f(u) = u^{\frac{1}{2}}$

$$y' = f'(u) \cdot u'$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \cdot 3$$

$$= \frac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3$$

$$= \frac{3}{2}(3x+1)^{-\frac{1}{2}}$$

$$\boxed{y' = \frac{3}{2\sqrt{3x+1}}}$$

d) $y = \frac{8}{\sqrt{5x^2+1}} = 8(5x^2+1)^{-1}$

9. Find the derivative using combination rules

   a) $y = -4x^2(2x^3 - 14)^4$

   b) $y = x^3 \cdot \sqrt{x^2 + 3}$

   c) $y = \frac{\sqrt{x^2+3}}{x^3}$

## SECTION 4 SUPPLEMENTARY EXERCISES

1) Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

c) $y = 6\sqrt{x} - \sqrt[3]{x}$

d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

e) $s(t) = t^2 + \frac{5}{t^2}$

f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$ (Do not use quotient rule!)

g) $f(x) = \dfrac{5x^2 - 2x + 1}{x}$ (Do not use quotient rule!)

2) Find the derivative using the Product or Quotient Rule.

a) $h(t) = (4t + 3)(t - 7)$

b) $y = 3x\sqrt{x + 5}$

c) $p(x) = \frac{x+5}{x^2-9}$

d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative using the Chain Rule and combination rules.

a) $v(x) = (2 - 4x)^{100}$

b) $v(x) = -x^3(2 - 4x)^{100}$

c) $y = \sqrt{x^2 + 3x + 4}$

4) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2, -10)$ using the differentiation formulas, and find the equation of the tangent line.

5) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2, -8)$ using the differentiation formulas, and find the equation of the tangent line.

6) Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x-6}$ at the point $(5, -1)$ using the differentiation formulas, and find the equation of the tangent line.