The Product Rule: $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]\cdot g(x) + \frac{d}{dx}[g(x)]\cdot f(x)$

or

$$(fg)' = f'g + g'f$$

6a) $y = x^4(2x+3)$

Product Rule

$f = 2x+3$	$g = x^4$
$f' = 2$	$g' = 4x^3$

$$y' = f'g + fg'$$
$$y' = 2x^4 + (2x+3)\cdot 4x^3$$
$$y' = 2x^4 + 4x^3(2x+3)$$
$$y = 2x^4 + 8x^4 + 12x^3$$
$$\boxed{y' = 10x^4 + 12x^3}$$

6b) $g(x) = (3x-7)(x^2+6x)$

Product Rule

$f = 3x-7$	$g = x^2+6x$
$f' = 3$	$g' = 2x+6$

$$g'(x) = (3x-7)(2x+6) + 3(x^2+6x)$$

The Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - \frac{d}{dx}[g(x)]\cdot f(x)}{[g(x)]^2}$

or

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

7a) $y = \frac{x}{3x+1}$ f g

$f = x$	$g = 3x+1$
$f' = 1$	$g' = 3$

Quotient Rule

$$\frac{dy}{dx} = \frac{f'g - fg'}{g^2}$$
$$= \frac{(1)(3x+1) - (3)(x)}{(3x+1)^2}$$
$$= \frac{3x+1-3x}{(3x+1)^2}$$
$$\boxed{\frac{dy}{dx} = \frac{1}{(3x+1)^2}}$$

7b) $q(x) = \frac{9x^2}{3x^2-2x}$

$f = 9x^2$	$g = 3x^2-2x$
$f' = 18x$	$g' = 6x-2$

$$q'(x) = \frac{f'g - fg'}{g^2}$$
$$= \frac{18x(3x^2-2x) - 9x^2(6x-2)}{(3x^2-2x)^2}$$
$$= \frac{\cancel{54x^3} - 36x^2 - \cancel{54x^3} + 18x^2}{(3x^2-2x)^2}$$
$$q'(x) = -\frac{18x^2}{(3x^2-2x)^2}$$

$(3x^2-2x)^2$
$(x(3x-2))^2$
$x^2(3x-2)^2$

$$= -\frac{18x^2}{x^2(3x-2)^2}$$
$$q'(x) = -\frac{18}{(3x-2)^2}$$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Derivative of the composite function $f(g(x))$

Derivative of the outside function f

Derivative of the inside function g

If y is a function of u, and u is a function of x, then the chain rule is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

8. Find the derivative using the Chain Rule

a) $y=(x^2+3)^8$

$u = x^2+3$

$f(u) = u^8$

$y' = f'(u) \cdot u'$

$= 8u^7 \cdot u'$

$= 8(x^2+3)^7 \cdot (x^2+3)'$

$= 8(x^2+3)^7 \cdot (2x)$

$y' = 16x(x^2+3)^7$

b) $s(x)=2(5x-500)^{1000}$

$s'(x) = 2\left(1000(5x-500)^{999}(5x-500)'\right)$

$= 2000(5x-500)^{999} \cdot 5$

$= 10000(5x-500)^{999}$

c) $y = \sqrt{3x+1}$

$y = (3x+1)^{\frac{1}{2}}$

$y' = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3x+1)'$

$= \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$

$\boxed{y' = \frac{3}{2}(3x+1)^{-\frac{1}{2}}}$

$= \frac{3}{2\sqrt{3x+1}}$

$= \frac{3\sqrt{3x+1}}{6x+2}$

d) $y = \frac{8}{\sqrt{5x^2+1}} = 8(5x^2+1)^{-\frac{1}{2}}$

$u = 5x^2+1$

$f(u) = u^{-\frac{1}{2}}$

$y' = 8f'(u) \cdot u'$

$= 8\left(-\frac{1}{2}u^{-\frac{3}{2}} \cdot u'\right)$

$= 8\left(-\frac{1}{2}(5x^2+1)^{-\frac{3}{2}}(5x^2+1)'\right)$

$= 8\left(-\frac{1}{2}(5x^2+1)^{-\frac{3}{2}}(10x)\right)$

$\boxed{y' = -40x(5x^2+1)^{-\frac{3}{2}}}$

$y' = -\frac{40x}{\sqrt{(5x^2+1)^3}} = -\frac{40x\sqrt{5x^2+1}}{(5x^2+1)^2}$

9. Find the derivative using combination rules

a) $y = -4x^2(2x^3-14)^4$

Product

$f = -4x^2 \quad g = (2x^3-14)^4$

$f' = -8x \quad g' = 24x^2(2x^3-14)^3$

$g' = 4(2x^3-14)^3 \cdot 6x^2$

$= 24x^2(2x^3-14)^3$

$y' = f'g + fg'$

$= -8x(2x^3-14)^4 + -4x^2 \cdot 24x^2(2x^3-14)^3$

$= -8x(2x^3-14)^4 - 96x^4(2x^3-14)^3$

$= -8x(2x^3-14)^3(2x^3-14+12x^3)$

$= -8x(2x^3-14)^3(14x-14)$

b) $y = x^3 \cdot \sqrt{x^2+3}$

$x^3 \quad (x^2+3)^{\frac{1}{2}}$

$3x^2 \quad x(x^2+3)^{-\frac{1}{2}}$

$y' = 3x^2(x^2+3)^{\frac{1}{2}} + x^4(x^2+3)^{-\frac{1}{2}}$

$g' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$

$g' = x(x^2+3)^{-\frac{1}{2}}$

c) $y = \frac{\sqrt{x^2+3}}{x^3} = x^{-3} \cdot (x^2+3)^{\frac{1}{2}}$

Product Rule:

$f = x^{-3} \quad g = (x^2+3)^{\frac{1}{2}}$

$f' = -3x^{-4} \quad g' = x(x^2+3)^{-\frac{1}{2}}$

$y' = -3x^{-4}(x^2+3)^{\frac{1}{2}} + x^{-2}(x^2+3)^{-\frac{1}{2}}$

QR:

$f = (x^2+3)^{\frac{1}{2}} \quad g = x^3$

$f' = x(x^2+3)^{-\frac{1}{2}} \quad g' = 3x^2$

$y' = \frac{x^4(x^2+3)^{-\frac{1}{2}} - 3x^2(x^2+3)^{\frac{1}{2}}}{(x^3)^2}$

$$\boxed{y' = \frac{x^4(x^2+3)^{-\frac{1}{2}} - 3x^2(x^2+3)^{\frac{1}{2}}}{x^6}}$$

SECTION 4 SUPPLEMENTARY EXERCISES

1) Find the derivative using the Power Rule. Rewrite each term as an exponent if necessary.

a) $f(x) = 5x^{-3} + 3x^{-6} - 2$

b) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

c) $y = 6\sqrt{x} - \sqrt[3]{x}$

d) $y = \frac{2}{\sqrt[3]{x}} + 9x$

e) $s(t) = t^2 + \frac{5}{t^2}$

f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$ (Do not use quotient rule!)

g) $f(x) = \dfrac{5x^2 - 2x + 1}{x}$ (Do not use quotient rule!)

2) Find the derivative using the Product or Quotient Rule.

a) $h(t) = (4t + 3)(t - 7)$

b) $y = 3x\sqrt{x + 5}$

c) $p(x) = \frac{x+5}{x^2-9}$

d) $y = \frac{x^2}{\sqrt{x+8}}$

3) Find the derivative using the Chain Rule and combination rules.

a) $v(x) = (2 - 4x)^{100}$

b) $v(x) = -x^3(2 - 4x)^{100}$

c) $y = \sqrt{x^2 + 3x + 4}$

4) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2, -10)$ using the differentiation formulas, and find the equation of the tangent line.

5) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2, -8)$ using the differentiation formulas, and find the equation of the tangent line.

6) Find the slope of the tangent line to the curve $f(x) = \dfrac{1}{x - 6}$ at the point $(5, -1)$ using the differentiation formulas, and find the equation of the tangent line.

Section 5: The Derivative of Trigonometric Functions

Derivative of all six Trigonometric Functions	
Sine	$\frac{d}{dx}\sin(x) = \cos(x)$
Cosine	$\frac{d}{dx}\cos(x) = -\sin(x)$
Tangent	$\frac{d}{dx}\tan(x) = \sec^2(x)$
Cotangent	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
Secant	$\frac{d}{dx}\sec(x) = \sec(\mathrm{x})\tan(x)$
Cosecant	$\frac{d}{dx}\csc(x) = -\csc(\mathrm{x})\cot(x)$

Example: Find the derivative of the trigonometric function $f(x) = 2x^3\cos(5x)$ using the rules of differentiation.

A derivative that requires a combination of the Product Rule and the Chain Rule

Product Rule: $(fg)' = f'g + g'f$

The Chain Rule: $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)\cdot g'(x)$

$$\frac{d}{dx}(2x^3\cos(x)) = \left[\frac{d}{dx}(2x^3)\right]\cos(5x) + 2x^3\left[\frac{d}{dx}\cos(5x)\right]$$

$$= 6x^2\cos(5x) + 2x^3(-\sin(5x)\cdot 5)$$

$$= 6x^2\cos(5x) - 10x^3\sin(5x)$$

Exercise 1: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $y = 2\sin x$

$y' = 2\cos(x)$

b) $y = \sin(2x)$

$u = 2x$

$f(u) = \sin(u)$

$y' = f'(u) \cdot u'$
$= \cos(u) \cdot u'$
$= \cos(2x) \cdot (2x)'$
$= \cos(2x) \cdot (2)$
$= 2\cos(2x)$

$y = \sin(2x) = 2\sin(x)\cos(x)$

$f = 2\sin(x)$	$g = \cos(x)$
$f' = 2\cos(x)$	$g' = -\sin(x)$

$y' = 2\cos^2(x) - 2\sin^2(x)$

$y' = 2(\cos^2(x) - \sin^2(x))$

$y' = 2\cos(2x)$

c) $y = x\sin x$

$y' = x\cos(x) + \sin(x)$

$f = x$	$g = \sin(x)$
$f' = 1$	$g' = \cos(x)$

d) $y = \sin(x^2)$

$u = x^2$

$f(u) = \sin(u)$

$y' = f'(u) \cdot u'$
$= \cos(u) \cdot u'$
$= \cos(x^2) \cdot (x^2)'$

$y' = 2x \cdot \cos(x^2)$

e) $y = \sin^2(x) = (\sin(x))^2$

$u = \sin(x)$

$f(u) = u^2$

$y' = f'(u) \cdot u'$
$= 2u \cdot u'$
$= 2\sin(x) \cdot (\sin(x))'$
$= 2\sin(x)\cos(x)$
$= \sin(2x)$

f) $y = \sin^2(x^2) = (\sin(x^2))^2$

$u = x^2$

$f(u) = \sin(u)$

$g(f) = f^2$

$y' = g'(f) \cdot f'$
$= 2f \cdot f'$
$= 2\sin(u) \cdot (\sin(u))'$
$= 2\sin(x^2) \cdot (\sin(x^2))'$
$= 2\sin(x^2)\cos(x^2) \cdot (x^2)'$
$= 2\sin(x^2)\cos(x^2) \cdot 2x$
$= 4x\sin(x^2)\cos(x^2) \longrightarrow 2x \cdot 2\sin(x^2)\cos(x^2)$
$= 2x\sin(2x^2)$

$\sin(2a) = 2\sin(a)\cos(a)$

Exercise 2: Find the derivative of the following trigonometric functions using the differentiation rules.

a) $r(x) = x\cos(2x^2)$ $\quad r'(x) = \cos(2x^2) - 4x^2 \sin(2x^2)$

$f = x$	$g = \cos(2x^2)$
$f' = 1$	$g' = -4x\sin(2x^2)$

$g' = -\sin(2x^2) \cdot (2x^2)'$

$g' = -4x \sin(2x^2)$

b) $g(x) = \tan\left(\frac{3x}{4}\right)$ $\quad g'(x) = \sec^2\left(\frac{3}{4}x\right) \cdot \left(\frac{3}{4}x\right)'$

$g'(x) = \frac{3}{4}\sec^2\left(\frac{3}{4}x\right)$

c) $k(x) = \csc x \cdot \cot x$

$f = \csc(x)$	$g = \cot(x)$
$f' = -\csc(x)\cot(x)$	$g' = -\csc^2(x)$

$k'(x) = -\csc(x)\cot^2(x) - \csc^3(x)$

$= -\csc(x)\left(\csc^2(x) - 1\right) - \csc^3(x)$

$= -\csc^3(x) + \csc(x) - \csc^3(x)$

$= \csc(x) - 2\csc^3(x)$

d) $h(x) = \frac{\cos(2x)}{\sin(x)+1}$

$\cos(2x)$	$\sin(x)+1$
$-2\sin(2x)$	$\cos(x)$

$h'(x) = \frac{f'g - fg'}{g^2}$

$= \frac{-2\sin(2x)(\sin(x)+1) - (\cos(2x)\cos(x))}{(\sin(x)+1)^2}$

e) $f(x) = \sqrt{\sin x + 5} = (\sin(x)+5)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(\sin(x)+5)^{-\frac{1}{2}} \cdot (\sin(x)+5)'$

$f'(x) = \frac{1}{2}(\sin(x)+5)^{-\frac{1}{2}} \cdot \cos(x)$