An application of limits and derivatives in modeling a rocket's path.

In May, 2020 SpaceX launched a rocket. For a rocket launched vertically, we can study the rocket's velocity as a function of its distance traveled by using Newton's law of universal gravitation and his second law of motion. It can be shown that the differential equation $v\frac{dv}{dx} = -\frac{gR^2}{(x+R)^2}$ is true, where v is the velocity of the rocket, R is the radius of the earth and x is the rocket's distance from the surface of the earth. We are assuming friction is negligible and the earth is spherical object. If the rocket has an initial speed v_0 then we can show that a solution to the above differential equation has the form $\frac{v^2-v_0^2}{2} + gR = \frac{gR^2}{x+R}$. The mean radius R of the earth is 6371 km. according to NASA. The acceleration due to gravity, $g = 9.875\ m/s^2$.

1. Implicitly differentiate v with respect to x to show that $\frac{v^2}{2} - \frac{v_0^2}{2} + gR = \frac{gR^2}{x+R}$ is a solution to the above differential equation.

2. Use the equation in problem 1, to show that $v = \sqrt{2gR^2\left(\frac{1}{R+x} - \frac{1}{R}\right) + v_0^2}$.

3. Plot the graph of $v(x)$ verses x, when the initial velocity, $v_0 = 700\ m/s$ and find the value of x if it exists when $v(x) = 0$. Repeat this for $v_0 = 12500\ m/s$.

4. By trial and error predict the smallest positive initial velocity v_0 such that $v(x)$ will achieve 0 for every value of x. Give a practical interpretation of this value of v_0.

5. (a) When the rocket reaches the top of its trajectory $x = L$, find its initial velocity, v_0? Hint: Use the formula you derived in problem 2 and consider what the value of the final velocity v at this instant will be.

   (b) Use the expression you obtained in part (a) for the velocity which we will now call v_L and compute the limit, $v_e = \lim_{L\to+\infty} v_L$.

   (c) Compute the numerical value of v_e.

   (d) Give a physical interpretation of v_e. and compare its value with the value of v_0 obtained in problem 4.

   (e) How does your calculated value for v_e, the escape velocity compare with the actual value stated by NASA?

6. **Further Explorations 1**. Go to rocket and look over how the listed functions are used in Desmos to create a picture of a rocket. Create your own picture of a rocket that is launched at an angle in an upwards direction and provide a copy of the code and a printout of your results.

7. **Further Explorations 2**. Find the escape velocity of the moon and other planets. Compared the values to Earth's value and briefly explain how this helps with space explorations.