Module I

Review of Factoring, Radicals, and Quadratic Equations

Section 1.1 Factoring Review

The Greatest Common Factor

The greatest common factor (GCF) for a polynomial is the largest monomial that divides each term of the polynomial.

Factoring the greatest common factor of a polynomial:

1. **Determine the greatest common factor**
2. **Write the answer in factored form.**

The GCF factoring process is the reverse of the Distributive Property

Multiply

The Distributive Property: $\boldsymbol{a(b+c) = ab + ac}$

Factor/Divide

The GCF factoring: $\boldsymbol{ab + ac = a(b+c)}$

Exercise 1: Factor the greatest common factor and express the answer in factored form. Check by multiplying using the Distributive Property.

a) Factor $5p^3 + 15p^2 - 30p$ $\quad \underset{\text{GCF}}{5p}\left(\frac{5p^3}{5p} + \frac{15p^2}{5p} - \frac{30p}{5p}\right) = 5p(p^2+3p-6)$

b) Factor $9a^3b^4 - 6a^2b^3 + 3ab^2$ $\quad 3ab^2\left(\frac{9a^3b^4}{3ab^2} - \frac{6a^2b^3}{3ab^2} + \frac{3ab^2}{3ab^2}\right) = 3ab^2(3a^2b^2 - 2ab + 1)$

c) Factor $5(x+y) - 6x(x+y)$ $\quad (x+y)(5-6x)$

Factoring by Grouping

Before factoring by grouping, first we factor out GCF from all four terms.

Steps in factoring by grouping (Assume there is no GCF)

1. Group pairs of terms and factor each pair.
2. If there is a common binomial factor, then factor it out.
3. If there is no common binomial factor, then interchange the middle two terms and repeat the process over. If there is still no common binomial, then the polynomial cannot be factored.

Factor $4x + 6y + 2xy + 3y^2$ by grouping

Step 1: Group the pairs of terms	$(\mathbf{4x + 6y}) + (\mathbf{2xy + 3y^2})$
Step 2: Factor the GCF from each pair	$2(\mathbf{2x + 3y}) + y(\mathbf{2x + 3y})$
Step 3: Factor the common binomial factor	$(\mathbf{2x + 3y})(\mathbf{2 + y})$

Exercise 2: Factor by grouping. Follow the steps in the table

a) Factor by grouping: $56 + 21k + 8h + 3hk$

b) Factor by grouping: $5x^2 + 40x - xy - 8y$ $= (5x^2 + 40x) + (-xy - 8y)$

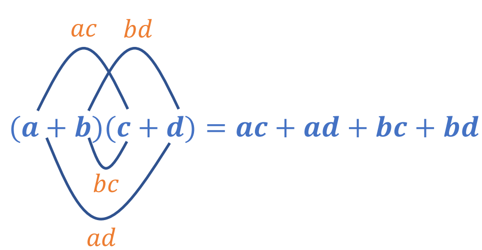
$5x(x+8) + (-y)(x+8)$

		GCF
$5x^2$	$+40x$	$5x$
$-xy$	$-8y$	$-y$
GCF: x	$+8$	

$(5x - y)(x + 8)$

Factoring Trinomials

The FOIL (Front-Outer-Inner-Last) Method: $(a+b)(c+d) = ac + ad + bc + bd$



Given x^2+bx+c
Find 2 numbers whose sum $=b$
product $=c$

Factoring Trinomials with Lead Coefficients of 1

Steps to factoring trinomials with lead coefficient of 1

1. Write the trinomial in descending powers.
2. List the factorizations of the constant (third) term of the trinomial.
3. Pick the factorization where the sum of the factors is the coefficient of the middle term.
4. Check by multiplying the binomials.

Exercise 6: Factor the trinomial with lead coefficient 1 by checking for the correct pair of product and sum

Multiply $(x+m)(x+n)$
$x^2+mx+nx+mn$
$x^2+(m+n)x+mn$
Let $b=m+n$ *b is sum of roots
$c=mn$ *c is product of roots

a) $x^2-8x+12$

$b=-8=\underline{-6}+\underline{-2}$
$c=12=\underline{-6}\cdot\underline{-2}$

$(x-6)(x-2)$

b) $y^2-9y-36$

sum product

$b=-9=\underline{-12}+\underline{3}$
$c=-36=\underline{-12}\cdot\underline{3}$

$(x-12)(x+3)$

c) $x^2+14xy+45y^2$

sum product

$b=14y=\underline{9y}+\underline{5y}$
$c=45y^2=\underline{9y}\cdot\underline{5y}$

$(x+9y)(x+5y)$

d) $a^2+7ab-8b^2$

$b=+7b=\underline{-b}+\underline{8b}$
$c=-8b^2=\underline{-b}\cdot\underline{8b}$

$(a-b)(a+8b)$

Factoring Trinomials with Lead Coefficients other than 1

__Method 1__: Trial and error

1. List the factorizations of the third term of the trinomial.
2. Write them as two binomials and determine the correct combination where the sum of the outer product, ad, and the inner product, bc, is equal to the middle term of the trinomial.

$$(a+b)(c+d) = ac + ad + bc + bd$$

bc

ad

__Method 2__: Factoring by grouping

1. Form the product ac.
2. Find a pair of numbers whose product is ac and whose sum is b.
3. Rewrite the polynomial to be factored so that the middle term bx is written as the sum of the two terms whose coefficients are the two numbers found in step
4. Factor by grouping.

Exercise 7: Factor each trinomial below

$ah^2 + bh + c$

Box Method

a) $2h^2 - 5h - 3$

$2h^2 - 6h + 1h - 3$

$(2h^2 - 6h) + (h - 3)$

$2h(h-3) + 1(h-3)$

$(2h+1)(h-3)$

$a = 2$

$b = -5$

$c = -3$

$b = -5 = \underline{-6} + \underline{1}$

$ac = -6 = \underline{-6} \cdot \underline{1}$

* required steps

$2h^2$	$-6h$	$\rightarrow 2h$
h	-3	$\rightarrow +1$
$\downarrow h$	$\downarrow -3$	

$(2h+1)(h-3)$

b) $2h^2 - h - 3$

$(2h-3)(h+1)$

c) $2h^2 - 5h - 12$

$(h-4)(2h+3)$

d) $3k^2 - 14k - 5$

$(3k+1)(k-5)$

Factor $x^2 - 7 = (x+\sqrt{7})(x-\sqrt{7})$

$x^2 - 10 = (x+\sqrt{10})(x-\sqrt{10})$

$x - 5 = (\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})$

Factoring Special Products

Difference of Two Squares: $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

Exercise 10: Factor each binomial below. Check if it is a difference of two squares.

a) $x^2 - 49 = (x+7)(x-7)$

$x^2 - 7^2$

b) $25 - b^2 = (5+b)(5-b) = -(b-5)(b+5)$

$5^2 - b^2$

c) $36x^2 - 16y^2$ $(6x-4y)(6x+4y)$

$2(3x-2y) \cdot 2(3x+2y)$

$4(3x-2y)(3x+2y)$

d) $100t^2 - 49r^2$ $(10t)^2 - (7r)^2$

$= (10t+7r)(10t-7r)$

e) Factor completely $16x^4 - 81$

$(4x^2)^2 - (9)^2 = (4x^2+9)(4x^2-9)$

$= (4x^2+9)(2x-3)(2x+3)$

f) $x^2 + 36$ $(x+6i)(x-6i)$

What have you noticed about the sum of two squares? not factorable unless we understand imaginary numbers.

Square of a Binomial

$$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$$
$$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$$

Exercise 11: Factor each trinomial below. Check that the first and the third terms are perfect square, and the middle term is $2ab$, then apply the perfect square formula

a) $p^2 + 10t + 25$

b) $9b^2 + 42b + 49$

c) $16a^2 - 40a + 25$

d) $16y^2 - 72y + 81$

Sum of Two Cubes
Difference of Two Cubes

$$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$
$$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

Exercise 12: Factor each trinomial below. Check if they are sum or difference of two cubes

a) Factor $x^3 + 8$

$x^3 + 2^3 = (x+2)(x^2 - x \cdot 2 + 2^2)$

$= (x+2)(x^2 - 2x + 4)$

b) Factor $27y^3 - 1$

$(3y)^3 - (1)^3 = (3y-1)((3y)^2 + 3y \cdot 1 + 1^2)$

$= (3y-1)(9y^2 + 3y + 1)$

c) Factor $64x^3 - 27y^3$

$= (4x)^3 - (3y)^3 = (4x - 3y)((4x)^2 + (4x)(3y) + (3y)^2)$

$= (4x - 3y)(16x^2 + 12xy + 9y^2)$

$64x^3 + (-27)y^3 = (4x)^3 + (-3y)^3$

$= (4x + (-3y))((4x)^2 - (4x)(-3y) + (-3y)^2)$

$= (4x - 3y)(16x^2 + 12xy + 9y^2)$

Cube of a binomial

$$\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3}$$
$$\boldsymbol{a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3}$$

Exercise 13: Factor each polynomial below. Check if they are cube of a binomial.

a) $x^3 - 3x^2 + 3x - 1$

b) $x^3 + 6x^2 + 12x + 8$

c) $x^3 - 9x^2 + 27x - 27$

Mixed Factoring/Factoring Completely

To factor a polynomial, **first factor the greatest common factor**, then consider the number of terms in the polynomial.

I. **Two terms**: Determine if the binomial is one of the following:

Difference of two squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum of two cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of two cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

II. **Three Terms**: Determine if the trinomial is a perfect square trinomial.

a) If the trinomial is a perfect square, then
$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

b) If the trinomial $ax^2 + bx + c$ is not a perfect square, then
   i) if the leading coefficient $a = 1$, then check the product = c and sum = b to determine the correct combination.
   ii) if the leading coefficient $a > 1$, then use trial and error or the ac-method.

III. **Four terms**: Try to factor by grouping.

Exercise 12: Factor each polynomial **completely**. This may mean factoring GCF and/or factoring in two or more steps.

a) Factor completely $3x^4 - 3x^3 - 36x^2$

b) Factor completely $20a - 5a^3$

c) Factor completely $16a^5b - ab$

d) Factor completely $8x^2 - 24x + 18$

e) Factor completely $16x^3y - 40x^2y^2 + 25xy^3$

f) Factor completely $12x^3 + 11x^2 + 2x$

g) Factor completely $7z^2w^2 - 10zw^2 - 8w^2$

Section 1.2: Radicals and the Complex Numbers

Simplifying Radicals

Exercise 1: Simplify each radical below

a. $\sqrt{9}$ $= 3$

b. $\sqrt{18}$ $= \sqrt{9}\cdot\sqrt{2} = 3\sqrt{2}$

c. $\sqrt{25}$ $= 5$

d. $\sqrt{75}$ $= \sqrt{25}\cdot\sqrt{3} = 5\sqrt{3}$

e. $\sqrt{49}$ $= 7$

f. $\sqrt{98}$ $= \sqrt{49}\cdot\sqrt{2} = 7\sqrt{2}$

Exercise 2: Simplify each radical below

$\sqrt{96} = \sqrt{16}\,\sqrt{6} = 4\sqrt{6}$

if x is positive

a. $\sqrt{x^2}$ $= |x|$ — x

b. $\sqrt{x^3}$ $= \sqrt{x^2}\,\sqrt{x} = |x|\sqrt{x}$ — $x\sqrt{x}$

c. $\sqrt{x^4}$ $= |x^2| = x^2$ — x^2

d. $\sqrt{x^5}$ $= \sqrt{x^4}\,\sqrt{x} = x^2\sqrt{x}$ — $x^2\sqrt{x}$

e. $\sqrt{x^6}$ $= |x^3|$ — x^3

f. $\sqrt{x^7}$ $= \sqrt{x^6}\,\sqrt{x} = |x^3|\sqrt{x}$ — $x^3\sqrt{x}$

g. $\sqrt{x^{98}}$ $= |x^{49}|$ — x^{49}

h. $\sqrt{x^{99}}$ $= \sqrt{x^{98}}\,\sqrt{x} = |x^{49}|\sqrt{x}$ — $x^{49}\sqrt{x}$

if x is positive

Write a rule for simplifying a radical of the form $\sqrt{x^n}$

When n is even, $\sqrt{x^n} = x^{\frac{n}{2}}$

When n is odd, $\sqrt{x^n} = x^{\frac{n-1}{2}}\sqrt{x}$

Exercise 3: Simplify each radical (x is positive)

a) $\sqrt{12x^6y^3}$

$\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

$\sqrt{x^6} = x^3$

$\sqrt{y^3} = \sqrt{y^2}\,\sqrt{y} = y\sqrt{y}$

$2x^3y\sqrt{3y}$

b) $3x\sqrt{20x^3y^6z^9}$

$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

$\sqrt{x^3} = x\sqrt{x}$

$\sqrt{y^6} = y^3$

$\sqrt{z^9} = z^4\sqrt{z}$

$= 3x\cdot 2xy^3z^4\sqrt{5xz}$

$= 6x^2y^3z^4\sqrt{5xz}$

Helpful Radical Multiplication Rule:

Square of a Radical: $(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$

Product of radical conjugates: $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

Strategy for rationalizing denominators with radicals:

If the denominator has one term, multiply the numerator and the denominator by the radical:

Example $\frac{2}{\sqrt{a}} = \frac{2}{\sqrt{a}}\cdot\frac{\sqrt{a}}{\sqrt{a}} = \frac{2\sqrt{a}}{a}$

If the denominator has two terms, multiply the numerator and the denominator by the conjugate of the denominator:

Examples a) $\frac{3}{\sqrt{a}+\sqrt{b}} = \frac{3}{\sqrt{a}+\sqrt{b}}\cdot\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3(\sqrt{a}-\sqrt{b})}{a-b}$

Exercise 4: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{9\sqrt{7}}{7}$ $\frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

b) $\frac{3}{2\sqrt{x}}$ $\frac{3\sqrt{x}}{2x}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}} \cdot \frac{\sqrt{10}+\sqrt{6}}{\sqrt{10}+\sqrt{6}}$ $(a-b)(a+b) = a^2-b^2$

$\frac{-12(\sqrt{10}+\sqrt{6})}{(\sqrt{10})^2-(\sqrt{6})^2} = \frac{-12}{4}(\sqrt{10}+\sqrt{6})$

$= -3(\sqrt{10}+\sqrt{6})$

d) $\frac{\sqrt{6}}{5-\sqrt{2}} \cdot \frac{5+\sqrt{2}}{5+\sqrt{2}}$

Complex numbers

We define the imaginary unit $i = \sqrt{-1}$ as the solution to $x^2 = -1$.
The square root of a negative number can be simplified $\sqrt{-b} = \sqrt{-1}\sqrt{b} = i\sqrt{b}$ where $b > 0$

Exercise 5: Simplify the expressions.

a. $\sqrt{-81}$ b. $\sqrt{-75}$ c. $-\sqrt{-49}$ d. $\sqrt{-15}$

Powers of i :

$i^1 = i$	$i^5 =$ ________	$i^9 =$ ________
$i^2 = -1$	$i^6 =$ ________	$i^{10} =$ ________
$i^3 = -i$	$i^7 =$ ________	$i^{11} =$ ________
$i^4 = 1$	$i^8 =$ ________	$i^{12} =$ ________

What is the pattern for powers of i ? Write a rule for it ____

____

Exercise 6: Simplify the product or quotient in terms of i

a) $(\sqrt{-9})(\sqrt{16})$ b) $\frac{\sqrt{-36}}{\sqrt{-9}}$

c) $(\sqrt{-12})(\sqrt{-6})$ d) $\frac{\sqrt{36}}{\sqrt{-2}}$