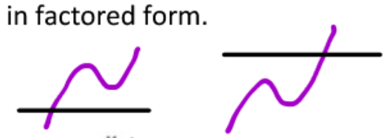
Looks like $x^3 + \dots$

Exercise 3. If $f(x) = x^3 + 2x^2 + 4x + 8$

a) Find a rational root r of the polynomial function using Desmos or graphing calculator.
b) Use synthetic division to divide $f(x)$ by $x - r$.
f) Find the remaining two roots by solving the quadratic function. Express f in factored form. What is the characteristic of the roots? (Multiplicity?)
c) Sketch a graph. How do the roots of the function relate to its graph?



$x^3 + 2x^2 + 4x + 8$

$(x^3 + 2x^2) + (4x + 8)$

$x^2(x+2) + 4(x+2)$

$(x+2)(x^2+4)$

if $(x+2)(x+2i)(x-2i) = 0$

d.

a.) $x = -2$ is a root

$\rightarrow f(-2) = 0$

b.)

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & \downarrow & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

$x^2 + 0x + 4$ remainder

c) $x^2 + 4 = 0$

$x^2 = -4$

$x = \pm 2i$

Exercise 4. If $f(x) = x^3 - x^2 - 10x + 12$

a) Find a rational root r of the polynomial function using Desmos or graphing calculator.
b) Use synthetic division to divide $f(x)$ by $x - r$.
c) Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)
d) Sketch a graph. How do the roots of the function relate to its graph?

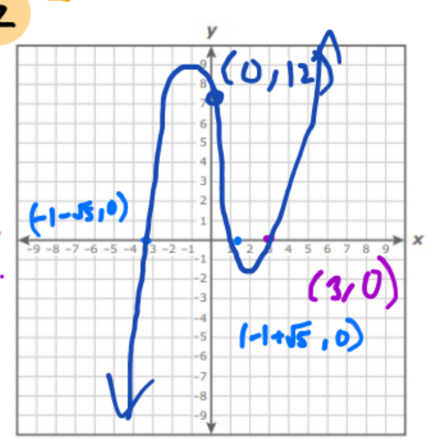
a.) Possible rational roots: $\pm \dfrac{1, 2, 3, 4, 6, 12}{1}$ all factors of constant 12

all factors of leading coefficient 1

$(3, 0) \rightarrow f(3) = 0$

$\therefore$ 3 is a root

$x - 3$ is a factor



b.)

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -10 & 12 \\ & \downarrow & 3 & 6 & -12 \\ \hline & 1 & 2 & -4 & \boxed{0} \end{array}$$

$x^2 + 2x - 4$

c.) $x^2 + 2x - 4 = 0$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} =$$

$x = -1 \pm \sqrt{5}$

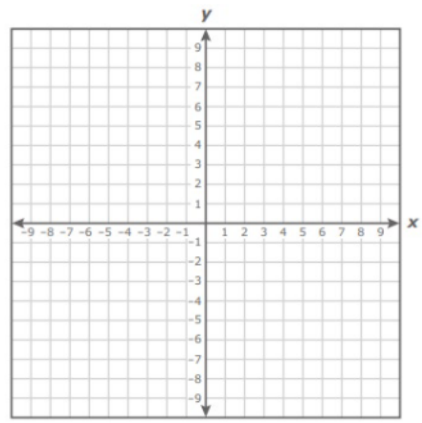
Exercise 5. If $f(x) = x^4 + x^3 - x^2 + x - 2$

a) Find two rational roots r_1 and r_2 of the polynomial function using Desmos or graphing calculator.

b) Apply synthetic division twice: First divide $f(x)$ by $x - r_1$, then divide the resulting quotient by $x - r_2$.

c) Find the remaining two roots by solving the quadratic function. What is the characteristic of the roots? (Rational, irrational, complex?)

d) Sketch a graph. How do the roots of the function relate to its graph?



Exercise 6. Find the polynomial of degree 3 with roots 1, 2, -3, and $f(-1) = 8$.

a) Express the polynomial in factored form $f(x) = a(x - r_1)(x - r_2)(x - r_3)$.

b) Use $f(-1) = 8$ to solve for a.

a.) $f(x) = a(x-1)(x-2)(x+3)$

b.) $f(-1) = a(-1-1)(-1-2)(-1+3)$

$8 = a(-2)(-3)(+2)$

$8 = 12a$

$a = \frac{8}{12} = \frac{2}{3}$

$f(x) = \frac{2}{3}(x-1)(x-2)(x+3)$

Exercise 7. Find a polynomial of degree 4 with real coefficients and roots $2i$ and 3. Root 3 has multiplicity 2.

$\rightarrow$ 4 roots

$0+2i$ $\qquad$ $0-2i$

$r_1 = 2i$ $\qquad$ $r_3 = -2i$

$r_2 = 3$ $\qquad$ $r_4 = 3$

$P(x) = a(x-2i)(x+2i)(x-3)^2$

$= a(x^2+4)(x-3)^2$

SECTION 2.2 SUPPLEMENTARY EXERCISES

For each quadratic function,

i. Determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = (x-3)^2 - 1$

b) $f(x) = -(x-1)^2 + 7$

c) $f(x) = \frac{1}{3}(x-8)^2 - 7$

d) $f(x) = -(x-2)^2 + 9$

e) $f(x) = 6(x+5)^2 + 4$

f) $f(x) = \frac{1}{2}(x+5)^2 - 3$

g) $f(x) = -(x-3)^2 - 1$

h) $f(x) = 4(x+7)^2 + 8$

i) $f(x) = 2x^2 + 4x - 5$

j) $f(x) = -x^2 + 4x - 6$

SECTION 2.3 SUPPLEMENTARY EXERCISES

1. a) Divide $x^3 + 2x^2 - 6x - 9$ by $x - 2$

   b) Divide $x^3 + 2x^2 - 6x - 9$ by $x + 3$

   c) Divide $2x^3 - 3x^2 - 5x + 18$ by $x + 2$

2. Find the zeros of a function

   a) One zero of $f(x) = x^3 - 2x^2 - 9x + 18$ is $x = 2$. Find the other zeros.

   b) One zero of $f(x) = 4x^3 + 9x^2 - 52x + 15$ is $x = -5$. Find the other zeros.

   c) Find all the real zeros for $f(x) = x^4 + 4x^3 - 9x^2 - 36x$.

   d) Find all the real zeros for $f(x) = x^4 - 13x^2 + 36$.

3. Find a polynomial of degree 4 with real coefficients and roots i and $2 + i$.

Module III

Exponential and Logarithmic Functions

Section 3.1 Exponential Functions and Graphs

exponential

$2^x \neq x^2$

power

For any real number x, an exponential function is a function of the form

$$f(x) = c \cdot b^x$$

Where the base b is a positive real number such that $b \neq 1$ and c is a non-zero real number.

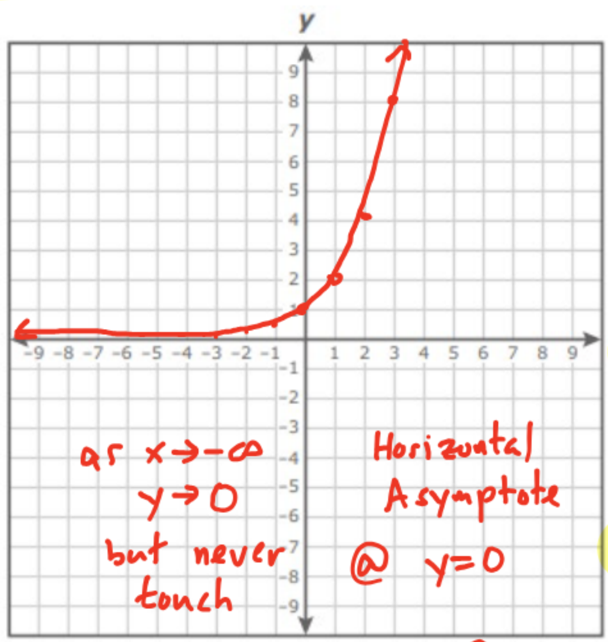
Example 1: Graph the exponential function: $f(x) = 2^x$. What is its domain and range?

x	$y = f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$2^{-1} = \frac{1}{2}$

$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$



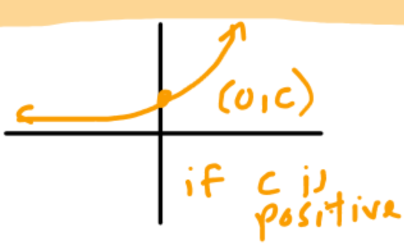
as $x \to \infty$
$y \to \infty$

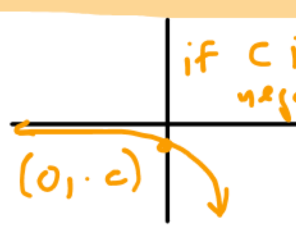
Domain
$(-\infty, \infty)$

Range
$(0, \infty)$

Characteristics of the exponential graph: $y = c \cdot b^x$

- The exponential graph is always increasing if $b > 1$; it is always decreasing if $0 < b < 1$.
- The graph does not cross the x-axis; $y = 0$ is a horizontal asymptote.
- The graph has the y-intercept ~~(0,1).~~ $(0, c)$
- The domain of the exponential function is all real numbers $(-\infty, \infty)$.
- If $c > 0$, the range of the exponential function is $(0, \infty)$. If $c < 0$, the range is $(-\infty, 0)$.





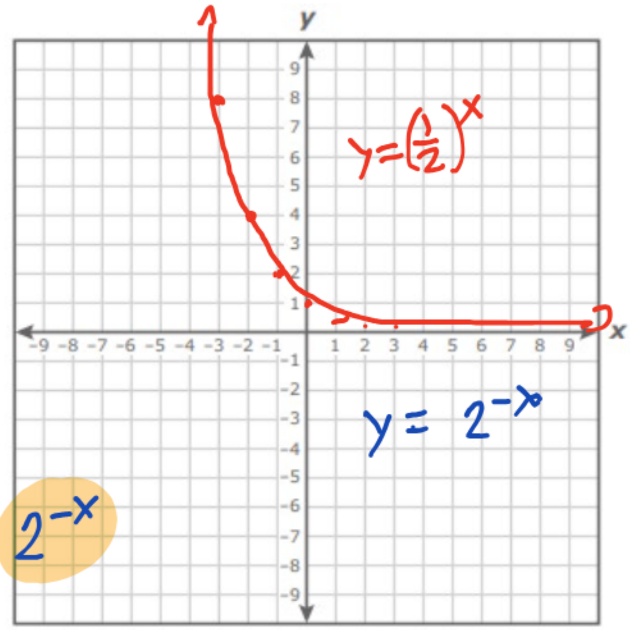
Example 2:

a) Graph the exponential function: $g(x) = \left(\frac{1}{2}\right)^x$. What is its domain and range?

b) Graph the exponential function: $h(x) = 2^{-x}$. What is its domain and range?

c) How do the two graphs compare? Why is it so?

x	$g(x) = \left(\frac{1}{2}\right)^x$	$h(x) = 2^{-x}$
-3	8	8
-2	4	4
-1	2	2
0	1	1
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{8}$	$\frac{1}{8}$



Domain $(-\infty, \infty)$

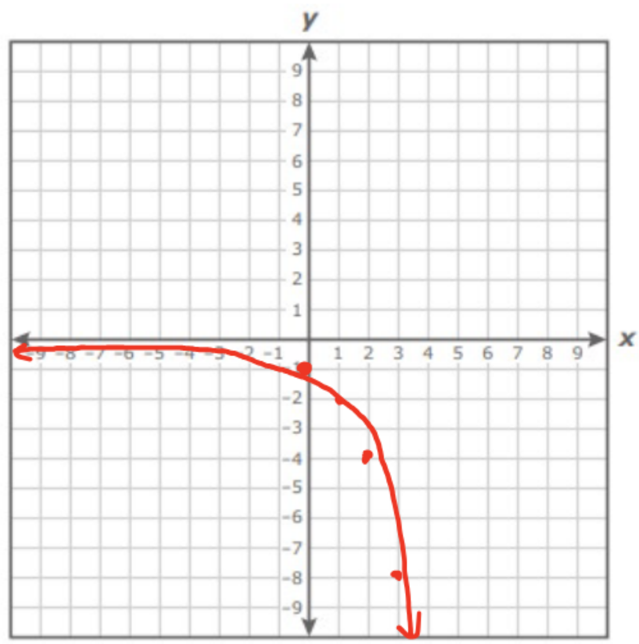
Range $(0, \infty)$

Prove: $\left(\frac{1}{2}\right)^x = 2^{-x}$

$2^{-x} = 2^{-1 \cdot x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

Example 3: Graph the exponential function: $k(x) = -2^x$. What is its domain and range?

x	$y = k(x)$
-3	$-\frac{1}{8}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{2}$
0	-1
1	-2
2	-4
3	-8



Domain: $(-\infty, \infty)$

Range: $(-\infty, 0)$

* $-f(x)$ is $f(x)$ reflected over x-axis

"switch positive and negative."

Section 3.2 Logarithmic Functions and Graphs

If $b > 0$ and $b \neq 1$, then $y = \log_b x \quad \Leftrightarrow \quad x = b^y$

Exercise 1: Evaluate the logarithms and rewrite each logarithmic relation as an exponential relation.

Evaluate the logarithm	Rewrite in exponential form
$\log_3 9 = ?$	$\log_3 9 = 2 \quad \Leftrightarrow \quad 3^2 = 9$
$\log_3 (1) = 0$	$3^0 = 1$
$\log_3 \left(\frac{1}{9}\right) =$ 2	$3^{-2} = \frac{1}{9}$
$\log_2 (8) =$ 3	$2^3 = 8$
$\log_8 2 = \frac{1}{3}$	$8^{\frac{1}{3}} = 2$
$\log_{1/2} 2 = -1$	$\left(\frac{1}{2}\right)^{-1} = 2$

$\sqrt[3]{8} = 8^{\frac{1}{3}}$

$\left(\frac{1}{2}\right)^x = 2$

$2^{-1x} = 2^1$

$-x = 1$

$x = -1$

Exercise 2: Evaluate the logarithms.

a) $\log_{10} 1 = 0$

b) $\mathrm{Log}_{10} 10 = 1$

c) $\log_{10} 100 = 2$

d) $\log_{10} 1000 = 3$

e) $\log_{10} \left(\frac{1}{10}\right) = -1$

f) $\log_{10} \left(\frac{1}{100}\right) = -2$

g) $\log_{10} \left(\frac{1}{1000}\right) = -3$

$\log(x) = \log_{10}(x)$

"common logarithm"

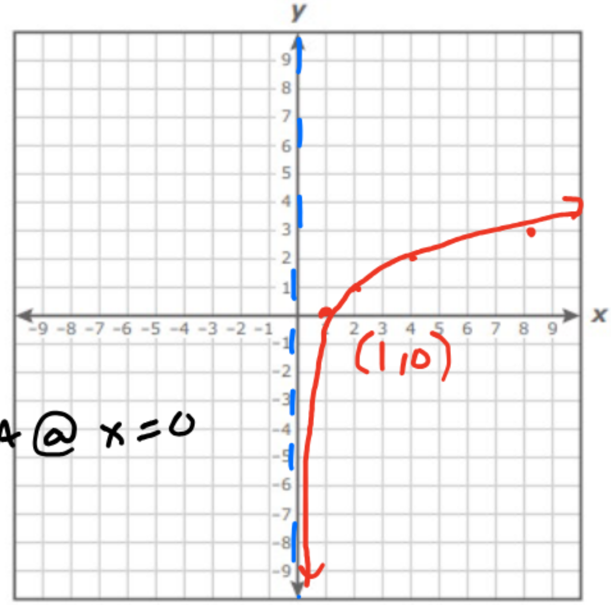
Exercise 3: Graph the logarithmic function: $y = \log_2 x$.
Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$$y = \log_2 x \iff x = 2^y$$

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

VA @ $x = 0$



* Functions and Inverse Functions swap domain and range

* Graphs of functions & Inverse Functions are reflections over line $y=x$

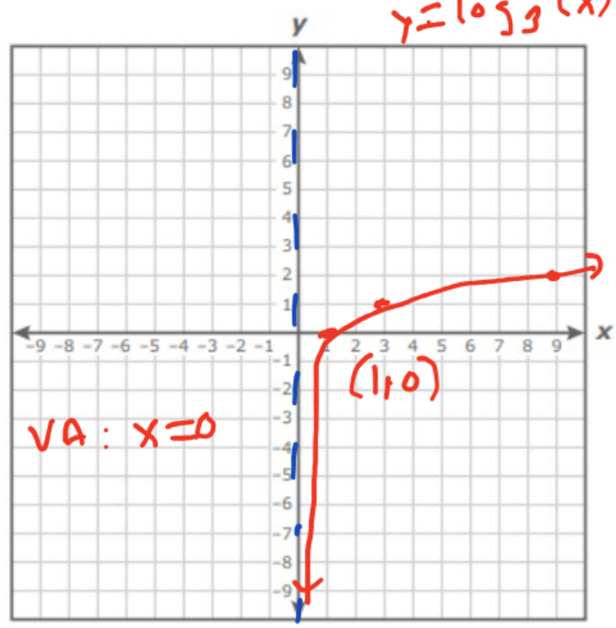
Exercise 4: Graph the logarithmic function: $y = \log_3 x$.
Convert logarithm to exponential form; complete the table of values and graph. What is its domain and range?

$y = \log_3(x)$

$$y = \log_3 x \iff x = 3^y$$

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



Characteristics of the logarithmic function: $f(x) = \log_b x$

- The function does not cross the y-axis; $x = 0$ is a vertical asymptote.
- The function has the x-intercept $(1, 0)$.
- The domain of the exponential function is $(0, \infty)$.
- The range of the exponential function is $(-\infty, \infty)$.

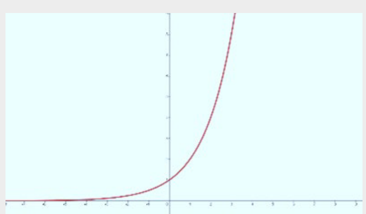
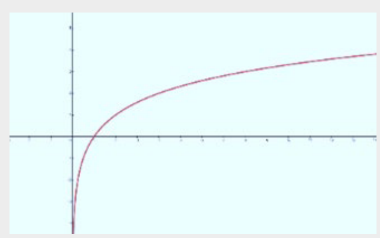
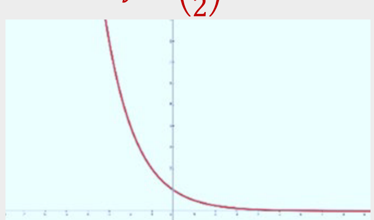
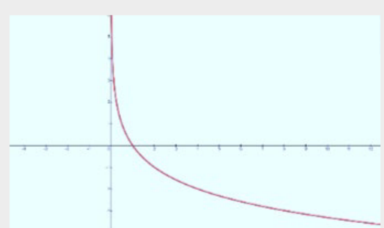
Use the link to try the Desmos activity: https://www.desmos.com/calculator/bvh8odgu6n
How do the exponential and logarithmic graphs change when the base is >1? When the base is between 0 and 1? ____

____

What is the inverse symmetry between the exponential and the logarithmic functions of the same base?

____

____

Comparing the Graphs of $y = b^x$ and $y = \log_b x$ $(b > 0,\ b \neq 1)$		
	$y = b^x$	$y = \log_b x$
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$
x-intercept	None	$(1, 0)$
y-intercept	$(0, 1)$	None
Contains points	$(0,1), (1, b)$	$(1, 0), (b, 1)$
Asymptote	x-axis or horizontal line $y = 0$	y-axis or vertical line $x = 0$
Example $b > 1$ (Always increasing)	$y = 2^x$ 	$y = \log_2 x$ 
Example $0 < b < 1$ (Always decreasing)	$y = \left(\frac{1}{2}\right)^x$ 	$y = \log_{1/2} x$ 

Section 3.3 Properties of Logarithms

Properties of Logarithms

	Properties of Exponents	Corresponding Properties of Logarithms
Product Property	$b^{x+y} = b^x \cdot b^y$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient Property	$b^{x-y} = \frac{b^x}{b^y}$	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property	$(b^x)^n = b^{nx}$	$\log_b(x^n) = n \cdot \log_b x$
	$b^1 = b$	$\log_b(b) = 1$
	$b^0 = 1$	$\log_b(1) = 0$

Exercise 1: Use the properties of logarithm and the values $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to evaluate each of the following. (What is the value of $\log_5 5$?)

$\log_b(b)=1$

$\log_5(5)=1$

$\log(10)=1$

$\ln(e)=1$

a) $\log_5 21 = \log_5(7 \cdot 3) = \log_5(7) + \log_5(3) = \approx 1.209 + .683 \approx 1.892$

b) $\log_5 \frac{3}{7} = \log_5(3) - \log_5(7) \approx .683 - 1.209 \approx -.526$

c) $\log_5 35 = \log_5(7 \cdot 5) = \log_5(7) + \log_5(5) \approx 1.209 + 1 \approx 2.209$

d) $\log_5 \frac{15}{7} = \log_5(15) - \log_5(7) = \log_5(3) + \log_5(5) - \log_5(7)$
$.683 + 1 - 1.209 = .474$

e) $\log_5 49 = \log_5(7) + \log_5(7)$

$\log_5(7^2) = 2\log_5(7) = 2(1.209) \approx 2.418$

The logarithm with base 10 is called the **common logarithm** denoted by $\log_{10}$ or $\log$

$$\log_{10} x = \log x$$

The logarithm with base *e* is called the **natural logarithm** denoted by $\log_e$ or $\ln$

$$\log_e x = \ln x$$

The Number *e*

The **natural base *e*** is a mathematical constant defined as the value $\left(1+\frac{1}{n}\right)^n$ as n approaches ∞. Its value is approximately

$$e \approx 2.71828182845$$

Properties of log and ln

log	ln
$\log(x \cdot y) = \log x + \log y$	$\ln(x \cdot y) = \ln x + \ln y$
$\log\left(\frac{x}{y}\right) = \log x - \log y$	$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
$\log x^n = n \log x$	$\ln x^n = n \ln x$
$\log 10 = 1$	$\ln e = 1$
$\log 1 = 0$	$\ln 1 = 0$

Exercise 2: Expand using the properties of logarithms. Assume x and y are positive.

a) $\log_3\left(\frac{9\sqrt{x}}{y^2}\right) = \log_3(9\sqrt{x}) - \log_3(y^2) = \log_3(9) + \log_3(\sqrt{x}) - \log_3(y^2)$

$= 2 + \log_3(x^{\frac{1}{2}}) - \log_3(y^2)$

$= 2 + \frac{1}{2}\log_3(x) - 2\log_3(y)$

b) $\log\frac{(x+3)}{100} = \log(x+3) - \log_{10}(100)$

$= \log(x+3) - 2$

c) $\ln\left(\frac{6e^4}{4}\right) = \ln(6) + \ln(e^4) - \ln(4)$

$= \ln(6) + 4\ln(e) - \ln(2^2)$

$= \ln(6) + 4 - 2\ln(2)$

$\ln(e) = 1$

$\log_e(e) = 1$

* $e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}$