Exercise 4: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{9\sqrt{7}}{7}$ $\qquad \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

b) $\frac{3}{2\sqrt{x}}$ $\quad \frac{3\sqrt{x}}{2x}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}} \cdot \frac{\sqrt{10}+\sqrt{6}}{\sqrt{10}+\sqrt{6}}$ $\qquad (a-b)(a+b) = a^2-b^2$

$\frac{-12(\sqrt{10}+\sqrt{6})}{(\sqrt{10})^2 - (\sqrt{6})^2} = \frac{-12}{4}(\sqrt{10}+\sqrt{6})$

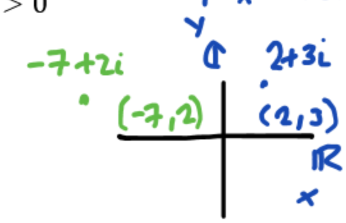
$= -3(\sqrt{10}+\sqrt{6})$

d) $\frac{\sqrt{6}}{5-\sqrt{2}} \cdot \frac{5+\sqrt{2}}{5+\sqrt{2}}$

Complex numbers

We define the imaginary unit $i = \sqrt{-1}$ as the solution to $x^2 = -1$.
The square root of a negative number can be simplified $\sqrt{-b} = \sqrt{-1}\sqrt{b} = i\sqrt{b}$ where $b > 0$

Complex Plane



Exercise 5: Simplify the expressions.

a. $\sqrt{-81}$
$\sqrt{81}\,i$
$9i$

b. $\sqrt{-75}$
$\sqrt{25}\sqrt{3}\,i$
$5i\sqrt{3}$
$5\sqrt{3}\,i$

c. $-\sqrt{-49}$
$-7i$

d. $\sqrt{-15}$
$\sqrt{15}\,i$

Powers of i :

$i^0 = 1$

$i^1 = i$	$i^5 = i^4 i = 1 \cdot i = i$	$i^9 = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$
$i^3 = -i$ $\quad i^3 = i^2 \cdot i = (-1)i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$ $\quad (i^2)^2 = (-1)(-1)$	$i^8 = 1$	$i^{12} = 1$

$i^{13} = i$
$i^{14} = -1$
$i^{15} = -i$
$i^{16} = 1$

What is the pattern for powers of i ? Write a rule for it $i^{4n} = 1$, $i^{4n+1} = i^1 = i$
$i^{4n+2} = i^2 = -1$, $i^{4n+3} = i^3 = -i$

Exercise 6: Simplify the product or quotient in terms of i

a) $(\sqrt{-9})(\sqrt{16}) = 3i \cdot 4$
$= 12i$

b) $\frac{\sqrt{-36}}{\sqrt{-9}} = \frac{6i}{3i} = 2$

c) $(\sqrt{-12})(\sqrt{-6}) = 2\sqrt{3}\,i \cdot \sqrt{2}\sqrt{3}\,i$
$= 2 \cdot 3\sqrt{2}\,i^2$
$= 6\sqrt{2}(-1)$
$= -6\sqrt{2}$

d) $\frac{\sqrt{36}}{\sqrt{-2}} = \frac{6}{\sqrt{2}\,i} \cdot \frac{\sqrt{2}\,i}{\sqrt{2}\,i}$
$= \frac{6\sqrt{2}\,i}{2i^2}$
$= \frac{6\sqrt{2}\,i}{-2}$
$= -3\sqrt{2}\,i$

Caution:

Correct work: $\left(\sqrt{-12}\right)\left(\sqrt{-6}\right) = \left(i\sqrt{12}\right)\left(i\sqrt{6}\right) = i^2\sqrt{72} = -6\sqrt{2}$

Incorrect work: $\left(\sqrt{-12}\right)\left(\sqrt{-6}\right) = \sqrt{72} = 6\sqrt{2}$

Correct work: $\dfrac{\sqrt{36}}{\sqrt{-2}} = \dfrac{\sqrt{36}}{i\sqrt{2}} = \dfrac{\sqrt{36}}{i\sqrt{2}} \cdot \dfrac{i}{i} = \dfrac{i\sqrt{36}}{i^2\sqrt{2}} = \dfrac{i\sqrt{18}}{-1} = -3i\sqrt{2}$

Incorrect work: $\dfrac{\sqrt{36}}{\sqrt{-2}} = \sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$

A complex number is a number of the form $x + yi$ where x and y are real numbers. We say x is the **real part** of the complex number and yi is the **imaginary part** of the complex number.

We can add, subtract, multiply, and divide complex numbers following similar procedures and the operations of radicals if the radicands are positive. If the radicands are negative, we must first convert them to ***i***.

Exercise 7: Perform the indicated operation.

a) $(-5 + 9i) - (-2 + 3i)$

$-5 + 9i + 2 - 3i$

$-3 + 6i$

b) $(2 + 7i)(5 - 3i)$

$10 - 6i + 35i - 21i^2$

$10 - 6i + 35i + 21$

$31 + 29i$

c) $(2 + 3i)(2 - 3i)$

$(a+b)(a-b) = a^2 - b^2$

$(2)^2 - (3i)^2$

$4 - 9i^2$

$4 + 9$

13

$(2)^2 + (3)^2$

$4 + 9$

13

$\rightarrow (a-bi)(a+bi)$

$a^2 - (bi)^2$

$a^2 - b^2i^2$

$a^2 + b^2$

The complex numbers $a + bi$ and $a - bi$ are called conjugates.

The product of complex Conjugates $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$ is a real number.

Exercise 8: Rationalize the denominator.

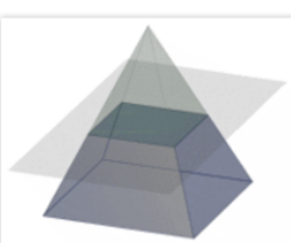
a) $\dfrac{(5+i)}{0+2i} \cdot \dfrac{(0-2i)}{0-2i}$

$\dfrac{-10i - 2i^2}{(0)^2 - (2i)^2} = \dfrac{-10i + 2}{0 - 4i^2} = \dfrac{2 - 10i}{-4i^2} = \dfrac{2 - 10i}{4}$

$= \dfrac{2}{4} - \dfrac{10}{4}i = \dfrac{1}{2} - \dfrac{5}{2}i$

b) $\dfrac{3i}{5-2i} \cdot \dfrac{5+2i}{5+2i} = \dfrac{15i + 6i^2}{(5)^2 + (2)^2} = \dfrac{-6 + 15i}{29}$

$= -\dfrac{6}{29} + \dfrac{15}{29}i$



Complex number or imaginary number concept was first investigated by a mathematician and inventor named Heron (c. 10-70 A.D.) from the city of Alexandria on the coast of the Mediterranean, in Egypt. While trying to find the volume of the frustum of a pyramid (see Figure 1) with a square base of a certain size, Heron of Alexandria first encountered the square root of a negative number (Nahin, 1998).

Section 1.3 Methods for Solving Quadratic Equations

A quadratic function is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$. The roots of the function are solutions to the equation $ax^2 + bx + c = 0$.

The quadratic equation $ax^2 + bx + c = 0$ can be solved in many ways:

1. By factoring and the Zero Factor Theorem
2. By the Square root property
3. By the Quadratic Formula

Method 1: Solving Quadratic Equations by Factoring

Zero Factor Theorem: If $ab = 0$ then $a = 0$ or $b = 0$ or both.

Solving Quadratic Equations by Factoring (Use this method when the equation is factorable)

1. **Put the equation in standard form: $ax^2 + bx + c = 0$.**
2. **Factor completely.**
3. **Use the zero-product rule, set each factor containing the variable equal to zero and solve for x. Note: Do not solve for the constant factor.**

Exercise 1: Solve the equations by factoring

a) Solve: $x^2 - 4x - 32 = 0$

$(x-8)(x+4) = 0$

$x - 8 = 0 \quad | \quad x + 4 = 0$

$x = 8 \quad | \quad x = -4$

$x = -4$ or 8

b) Solve: $36 - 49y^2 = 0$

$(6-7y)(6+7y) = 0$

$6 - 7y = 0 \quad | \quad 6 + 7y = 0$

$-7y = -6$

$y = \frac{6}{7} \quad | \quad y = -\frac{6}{7}$

c) Solve: $a^3 - 6a^2 + 5a = 0$

$a(a^2 - 6a + 5) = 0$

$a(a-5)(a-1) = 0$

$\boxed{a=0} \quad | \quad a - 5 = 0 \quad | \quad a - 1 = 0$

$\boxed{a=5} \quad | \quad \boxed{a=1}$

d) Solve: $2x^3 + 9x^2 = 5x$

$2x^3 + 9x^2 - 5x = 0$

$x(2x^2 + 9x - 5) = 0$

$x(2x-1)(x+5) = 0$

$\boxed{x=0} \quad | \quad 2x - 1 = 0 \quad | \quad x + 5 = 0$

$\boxed{x = \frac{1}{2}} \quad | \quad \boxed{x = -5}$

$a = 2$
$b = 9$
$c = 5$

$b = 9 = \underline{10} + \underline{-1}$

$ac = -10 = \underline{10} \cdot \underline{-1}$

$2x^2 + 10x - 1x - 5$

$(2x^2 + 10x) + (-1x - 5)$

$2x(x+5) + (-1)(x+5)$

$(2x-1)(x+5)$

Method 2: Solving Quadratic Equations by the Square Root Property

The Square Root Property

If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Sometimes it is indicated as $x = \pm\sqrt{a}$

Solving Quadratic Equations by the Square Root Property (Use this method when the equation consists of a square term and a constant term)

- **Isolate the square and then apply the Square Root Property**

Exercise 1: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $x^2 = 100$

$x^2 - 100 = 0$

$(x+10)(x-10) = 0$

$x+10=0$ $\quad$ $x-10=0$

$\boxed{x=-10}$ $\quad$ $\boxed{x=10}$

$x^2 = 100$

$\sqrt{x^2} = \sqrt{100}$

$|x| = 10$

$x = \pm 10$

b) $s^2 - 12 = 0$

$s^2 = 12$

$\sqrt{s^2} = \pm\sqrt{12}$

$s = \pm\sqrt{12}$

$s = \pm\sqrt{4}\sqrt{3}$

$\boxed{s = \pm 2\sqrt{3}}$

c) $s^2 + 27 = 0$

$s^2 = -27$

$\sqrt{s^2} = \pm\sqrt{-27}$

$s = \pm\sqrt{27}\, i$

$s = \pm 3\sqrt{3}\, i$

d) $(y-3)^2 - 98 = 0$

$(y-3)^2 = 98$

$\sqrt{(y-3)^2} = \pm\sqrt{98}$

$y - 3 = \pm 7\sqrt{2}$

$+3 \quad +3$

$y = 3 \pm 7\sqrt{2}$

iv) $2(x+1)^2 + 48 = 0$

$2(x+1)^2 = -48$

$(x+1)^2 = -24$

$\sqrt{(x+1)^2} = \pm\sqrt{-24}$

$x + 1 = \pm 2\sqrt{6}\, i$

$x = -1 \pm 2\sqrt{6}\, i$

Method 3: Solving Quadratic Equations by the Quadratic Formula

The Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving Quadratic Equations by the Quadratic Formula (This method can be used on any quadratic equations; it may not be the best method if the equation can be solved using the other two methods.)

1. **Put the equation in standard form: $ax^2 + bx + c = 0$.**
2. **Substitute, a, b, and c into the quadratic formula and simplify.**

Exercise 1: Solve using the quadratic formula

a) $x^2 - 5x - 8 = 0$

$a=1$

$b=-5$

$c=-8$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4(1)(-8)}}{2(1)} = \frac{5 \pm \sqrt{57}}{2}$$

b) $x^2 + 2x + 6 = 0$

$a=1$

$b=2$

$c=6$

$$x = \frac{-(2) \pm \sqrt{(2)^2-4(1)(6)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-20}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{5}\,i}{2} \qquad x = -1 \pm \sqrt{5}\,i$$

c) $2x^2 + 5 = 4x$

$2x^2 - 4x + 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{-24}}{4} = \frac{4 \pm 2\sqrt{6}\,i}{4} = \frac{4}{4} \pm \frac{2\sqrt{6}}{4}i = \boxed{1 \pm \frac{\sqrt{6}}{2}i}$$

The **discriminant** is the radicand $b^2 - 4ac$ in the quadratic formula.

If the discriminant > 0, then the equation has two real roots.
If the discriminant = 0, then the equation has one real (double) root.
If the discriminant < 0, then the equation has two complex roots.

Exercise 2: Solve $4x^2 - 28x + 49 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 3: Solve $x^2 + 63 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

Exercise 4: Solve $2x^2 - 6x - 9 = 0$. What is the best method for solving this equation? Factoring, the square root property, or the quadratic formula? What type of roots does it have?

SECTION 1.1 SUPPLEMENTARY EXERCISES

1. Factor completely

a) $a^2 - 5a$
b) $25y^3 - 15y^2$
c) $5ab^2 - 15a^3b$
d) $64w^2 - 81$
e) $36x^2 - 25y^2$
f) $16x - 4x^3$
g) $y^3 - 81y$
h) $49 - 9t^2$
i) $36x^2y - 9y$
j) $a^2 - 4a - 12$
k) $x^2 + 9xy - 36y^2$
l) $m^3 - 4m^2 - 21m$
m) $24 + 5n - n^2$
n) $2x^3 - 2x^2y - 12xy^2$
o) $2x^3y + x^2y^2 - 6xy^3$
p) $-6x^2 - 9x + 15$
q) $am - 5a + 2bm - 10b$
r) $15x - 12ax + 10y - 8ay$
s) $7b - 2bd + 21c - 6cd$
t) $16a^5 - 250a^2$

SECTION 1.3 SUPPLEMENTARY EXERCISES

1. Solve the equations by factoring

a) $x^2 + 6x + 8 = 0$
b) $2t^2 + 3t - 14 = 0$
c) $x^2 - 10x + 25 = 0$
d) $w^2 + 14w + 55 = 6$
e) $t^2 - 5t = 24$
f) $2y^3 + 12y^2 - 32y = 0$
g) $k^2 - 25 = 0$
h) $16x^2 = 49$
i) $z^2 - 9z = 0$
j) $-6x^2 - x + 12 = 0$
k) $5m^2 + 20m = 6 - 9m$
l) $(y+5)^2 - 4 = 0$
m) $(n-3)(3n-2) - 8n = 0$
n) $10x^2 = 27x - 18$

2. Solve the equations by the square root property

a) $x^2 = 120$
b) $s^2 - 45 = 0$
c) $(y+2)^2 - 49 = 0$
d) $\frac{1}{2}(x-6)^2 - 16 = 0$
e) $8 = (b-5)^2 - 16$
f) $2k^2 + 80 = 0$
g) $(y-3)^2 + 64 = 0$
h) $y^2 + 88 = 7$
i) $(y+7)^2 + 50 = 0$
j) $(y-1)^2 + 12 = 0$
k) $(2k-1)^2 + 72 = 18$
l) $9(2m-3)^2 + 8 = 449$

3. Solve the equations by the quadratic formula

a) $p^2 + 5p = -2$
b) $a^2 - 2a = 4$
c) $2k^2 = -4k + 3$
d) $12m + 9m^2 = -4$
e) $3x^2 + 6x + 2 = 0$
f) $6x^2 + 2x + 3 = 0$
g) $10x^2 - 13x - 3 = 0$
h) $4x^2 - x + 6 = 0$
i) $m^2 + 2m = -5$
j) $2k^2 + 9k = -7$
k) $(z-2)(z+4) + 6 = 0$
l) $(3x-7)(x+5) = -31$
m) $11n^2 - 4n(n-2) = 6(n+3)$
n) $x(x-3) = -10x - 7$

Module II
Polynomial Functions

Section 2.1 Definition of Functions, Domain, Range, and The Interval Notation

A function f is an assignment (or relation) that assigns to each input x in the domain exactly one output y in the codomain.

The domain of a function f is the set of all possible inputs of f. The codomain of a function is the set of all possible outputs of f. The range of a function f is a subset of codomain and the actual outputs of f.

For example,

$f: \mathbb{R} \to \mathbb{R}$ means the function f has a **domain** of $\mathbb{R}$ (all real numbers) and the **codomain** $\mathbb{R}$ (all real numbers).

If we define f as the assignment $f: x \mapsto x^2$, then f takes an input x and assigns it to the output x^2. We can also write it as

$$f(x) = x^2 \quad \text{or} \quad y = x^2$$

In this case, the **range** of f is the set of nonnegative real numbers, or $[0, \infty)$ using the interval notation.

A function may assign the same output to two different inputs, but it may not assign two outputs to one input. In other words, two different inputs may have the same output, but one input may not have two different outputs.

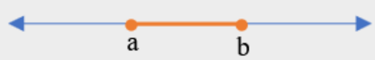
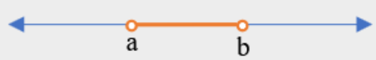
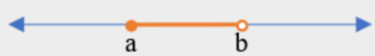
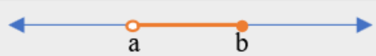
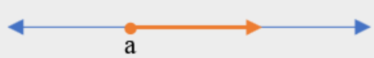
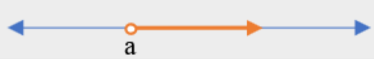
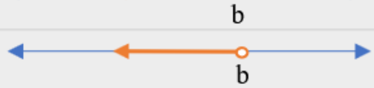
Take the vending machine for example, one code (an input) returns one bottle of beverage. It won't return two bottles of beverages unless the machine malfunctions. On the other hand, it is possible that two different codes (different inputs) give us the same beverage (output).

Exercise 1. Determine if each of the following set forms a function. If it is a function, determine its domain.

	Function? (Yes or No)	Why or Why Not	If a function, what is its domain?
Set of ordered pairs $\{(1, 1), (2, 1), (3, 1)\}$	Yes	Each input has one output	$\{1, 2, 3\}$
Set of ordered pairs $\{(1, 1), (1, 2), (1, 3)\}$	No	1 has multiple output	
Fahrenheit-Celsius conversion formula $F = \frac{9}{5}C + 32$	Yes	Input C results in F	$\mathbb{R}$ all Real # $(-\infty, \infty)$
Equation of a line: $y = mx + b$	Yes if know m&b	x input results in 1 y output	$\mathbb{R}$ all real # $(-\infty, \infty)$
Parabola: $y = ax^2 + bx + c$	Yes if we know a,b,c	x input → 1 y output	$\mathbb{R}$ $(-\infty, \infty)$
Circle: $x^2 + y^2 = 1$	No	Multiple y-output for 1 x-input	

* unless otherwise stated, domain of all polynomial functions is $\mathbb{R}$

Interval on the real number line

Inequality notation	Real number line	Interval notation
$a \leq x \leq b$		$[a, b]$
$a < x < b$		(a, b)
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$x \geq a$		$[a, \infty)$
$x > a$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$
All real numbers		$(-\infty, \infty)$ $\mathbb{R}$

Section 2.2 Quadratic Functions: Graphs and Roots

A **quadratic function** is of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

The graph of a quadratic function is a parabola (U-shaped). A parabola has either a maximum or a minimum point, called the **vertex**. The **axis of symmetry** for the graph of a quadratic function is the vertical line through the vertex.

The vertex can be found by the following methods:

Method 1: The Vertex Form
A quadratic function expressed as $f(x) = a(x-h)^2 + k$ is said to be in the vertex form, where (h, k) is the vertex.

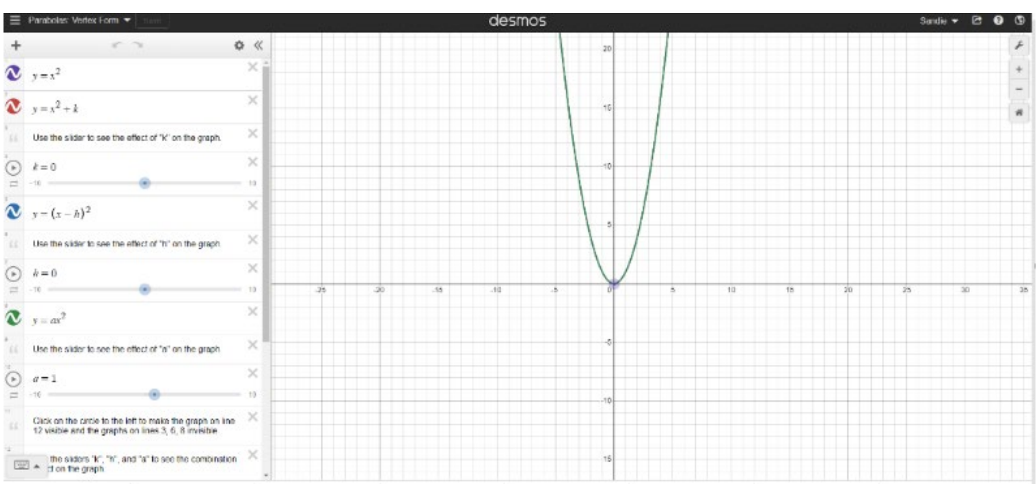
Method 2: The Vertex Formula
If a quadratic function is in standard form $f(x) = ax^2 + bx + c$, the vertex can be computed using the formula:

x-coordinate of the vertex is $-\frac{b}{2a}$

Substitute x-coordinate to find the y-coordinate of the vertex

The **roots** of a quadratic function are those x values such that $f(x) = 0$, or solutions to the equation $ax^2 + bx + c = 0$. They are also called the **zeros** of the function. The quadratic equation $ax^2 + bx + c = 0$ can be solved by factoring, by the square root property, or by the quadratic formula. See section 1.3 for methods of solving quadratic equations.

Use the Desmos link below to see the effect of changing h, k and a on the parabola.
https://www.desmos.com/calculator/0txid19ts5



Parent function : $y=x^2 \to$ Transformed function $y=a(x-h)^2+k$

How does the graph change when you change h? When h is positive? When h is negative? $y=(x-h)^2$

when h is positive graph goes $\rightarrow$ * do not include negative

h is negative graph $\leftarrow$

How does the graph change when you change k? When k is positive? When k is negative? $y=x^2+k$

if k is positive graph $\uparrow$

negative graph $\downarrow$

How does the graph change when you change a? When a is positive? When a is negative? When $|a| > 1$? When $|a| < 1$?

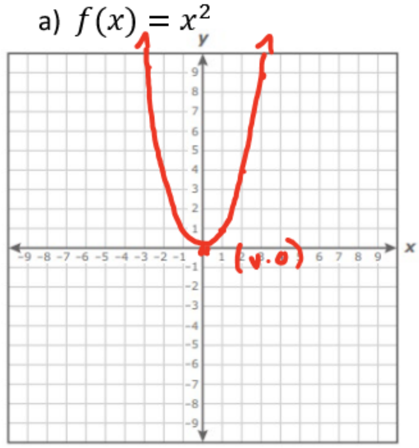
when $|a|>1$ parabola sharpens

$|a|<1$ parabola widens

Exercise 1: For each quadratic function:

i) Identify the vertex.
ii) Sketch the graph.
iii) Determine the domain and the range of the function.
iv) Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = x^2$



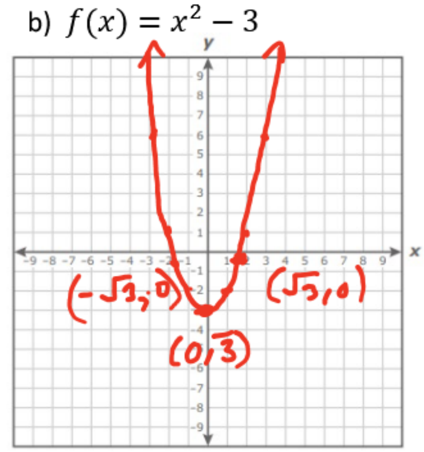
The vertex is $(0,0)$

The Domain is $(-\infty,\infty)$

The Range is $[0,\infty)$

The roots are 0

b) $f(x) = x^2 - 3$



The vertex is $(0,-3)$

The Domain is $(-\infty,\infty)$

The Range is $[-3,\infty)$

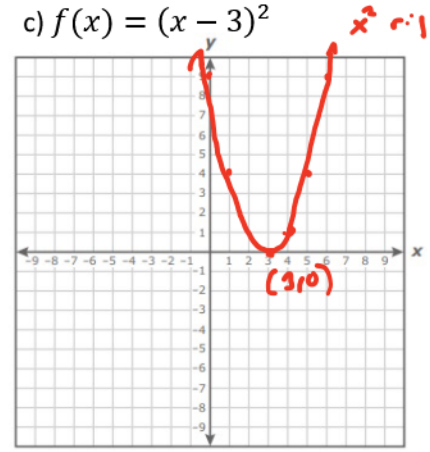
The roots are $\pm\sqrt{3}$

root: $x^2-3=0$

$x^2=3$

$x=\pm\sqrt{3}$

c) $f(x) = (x-3)^2$ x^2 rights 3



The vertex is $(3,0)$

The Domain is $(-\infty,\infty)$

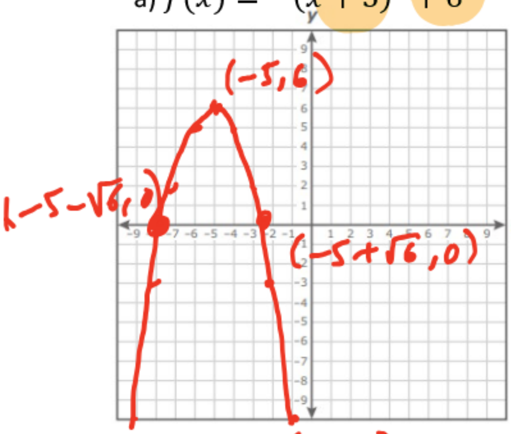
The Range is $[0,\infty)$

The roots are 3

Exercise 2: For each quadratic function,

i. Use the **Vertex Form** to determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.

a) $f(x) = -(x+5)^2 + 6$



The vertex is $(-5,6)$

The Domain is $(-\infty,\infty)$

The Range is $(-\infty,6]$

The roots are $-5\pm\sqrt{6}$

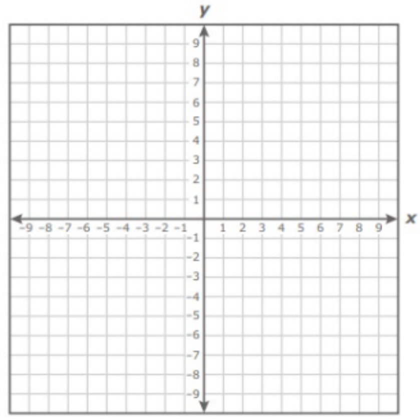
$-(x+5)^2+6=0$

$-(x+5)^2=-6$

$(x+5)^2=6$

$x+5=\pm\sqrt{6}$ $\quad x=-5\pm\sqrt{6}$

b) $f(x) = 2(x+2)^2 - 8$



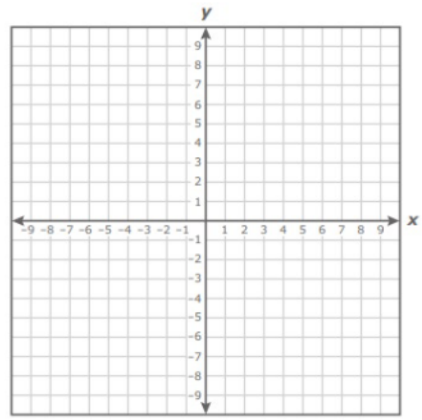
The vertex is ______________

The Domain is ______________

The Range is ______________

The roots are ______________

c) $f(x) = -\frac{1}{2}x^2 - 2$



The vertex is ______________

The Domain is ______________

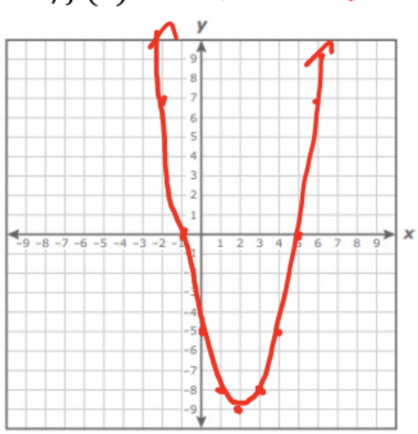
The Range is ______________

The roots are ______________

Exercise 3: For each quadratic function,

i. Use the **Vertex Formula** to determine the vertex.
ii. Sketch the graph.
iii. Determine the domain and the range of the function.
iv. Find the roots. Determine whether the roots are two real roots, one real root, or two complex roots.
v. Determine the interval where the function is positive or negative

a) $f(x) = x^2 + 4x - 5 = (x+1)(x-5)$



The vertex is $(-2,-9)$

The Domain is $(-\infty,\infty)$

The Range is $(-9,\infty)$

The roots are $-1, 5$

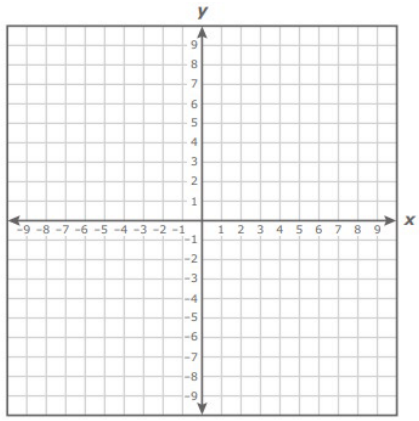
The function is positive on the interval $(-\infty,-1)\cup(5,\infty)$

The function is negative on the interval $(-1,5)$

vertex: $h=-\frac{b}{2a}=-\frac{4}{2(1)}=-2$

$k=(-2)^2+4(-2)-5=-9$

b) $f(x) = -x^2 - 2x - 5$



The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are _____________________________________

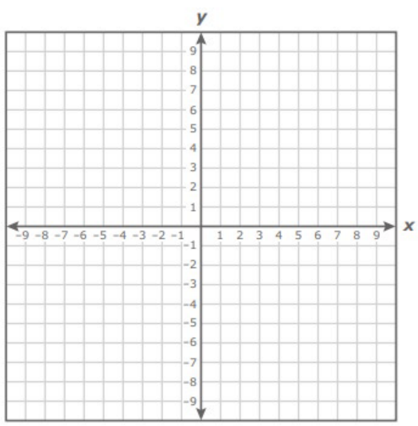
The function is positive on the interval _________________

(use decimal approximation)

The function is negative on the interval _________________

(use decimal approximation)

c) $f(x) = x^2 - 2x + 5$



The vertex is ______________

The Domain is _____________

The Range is _______________

The roots are _____________________________________

The function is positive on the interval _________________

The function is negative on the interval _________________

How do you identify the real roots of the function? ____

____

What do you notice about the roots in exercises 2f and 3c? How are their graphs different from the rest? What can you say about the graph of a quadratic function with complex roots?

____

____

Section 2.3 Polynomial Functions: Graphs and Roots

A polynomial function is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. a_n is the leading coefficient, and a_0 is the constant term, and n is the degree of the polynomial.

The **roots** of a polynomial function are those x values such that $f(x) = 0$. They are also called the **zeros** or **solutions** of the polynomial function.

A polynomial function has the following characteristics:

- A polynomial function with degree n has n roots (including multiplicity), they may be real or complex roots.
- A polynomial function may cross the x-axis at most n times, corresponding to at most n distinct real roots.
- If a polynomial function with real coefficients has complex roots, the complex roots always exist in conjugate pair.
- A polynomial function may have a graph with at most $n - 1$ turning points.

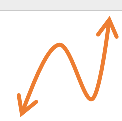
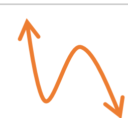
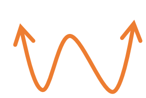
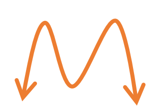
if $a+bi$ is a root
then $a-bi$ is also a root

The END behavior

The END behavior of the graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Is determined by n and a_n, where n is the degree of the polynomial and a_n is the leading coefficient of the polynomial.

Degree of the polynomial	Sign of the Leading Coefficient	Left End Behavior As $x \to -\infty$	Right End Behavior As $x \to \infty$	A maximum or minimum exists?	Domain & Range	Example
n is odd	a_n is positive	$f(x) \to -\infty$ or Down	$f(x) \to \infty$ or Up	No	D = $(-\infty, \infty)$ R = $(-\infty, \infty)$	
n is odd	a_n is negative	$f(x) \to \infty$ or Up	$f(x) \to -\infty$ or Down	No	D = $(-\infty, \infty)$ R = $(-\infty, \infty)$	
n is even	a_n is positive	$f(x) \to \infty$ or Up	$f(x) \to \infty$ or Up	There is a minimum, **m**	D = $(-\infty, \infty)$ R = $[\mathbf{m}, \infty)$	
n is even	a_n is negative	$f(x) \to -\infty$ or Down	$f(x) \to -\infty$ or Down	There is a maximum, **M**	D = $(-\infty, \infty)$ R = $(-\infty, \mathbf{M}]$	

Synthetic Division

The method of Synthetic Division is a quick way to divide a polynomial $f(x)$ by a linear factor of the form $(x-k)$. Instead of using long division, we can simply work with the coefficients of $f(x)$ and k.

Example: If $f(x) = ax^2 + bx + c$. Divide $f(x)$ by $(x-k)$.

Step 1: Set up synthetic division using the coefficients a, b, c and k.

$$\begin{array}{c|ccc} k & a & b & c \\ & & & \\ \hline & & & \end{array}$$

Step 2: Bring down the first coefficient a. Multiply a by k and add to b

$$\begin{array}{c|ccc} k & a & b & c \\ & & ka & \\ \hline & a & b+ka & \end{array}$$

Step 3: Repeat the procedure. Bring down $b + ka$. Multiply $b + ka$ by k, and add to c.

$$\begin{array}{c|ccc} k & a & b & c \\ & & ka & k(b+ka) \\ \hline & a & b+ka & c+k(b+ka) \end{array}$$

Step 4: Repeat the procedure until we finish the last coefficient. The result of the division is a quotient polynomial $q(x) = ax + (b + ka)$ with remainder $c + k(b + ka)$. Note: The degree of the quotient is 1 less than the degree of the original polynomial.

$$\begin{array}{c|ccc} k & a & b & c \\ & & ka & k(b+ka) \\ \hline & a & b+ka & c+k(b+ka) \end{array}$$

Coefficients of $q(x)$ → $a \quad b+ka$

$c + k(b+ka)$ ← Remainder

Exercise 1. If $f(x) = 3x^2 - 8x + 17$ is divided by $x - 4$. Determine the quotient and the remainder.

$x - 4 = 0$

$x = 4$

$$\begin{array}{c|ccc} & x^2 & x & c \\ 4 & 3 & -8 & 17 \\ & \downarrow & 12 & 16 \\ \hline & 3 & 4 & \boxed{33} \end{array}$$

← Remainder

$\times 4$ $\times 4$

$3x + 4 + \dfrac{33}{x-4}$

$f(4) = 33$

Exercise 2. If $f(x) = x^3 - 2x^2 - 5x + 6$.

a) Divide $f(x)$ by $x + 2$ using synthetic division. Determine the quotient and the remainder. What is the significance of the remainder being zero?
b) Set the quotient to zero and solve.
c) Express f in factored form.
d) Find all three roots of f. What is the characteristic of the roots? (Rational, irrational, complex?)
e) Sketch a graph of f, check the graph on Desmos or graphing calculator. How do the roots of the function relate to its graph?

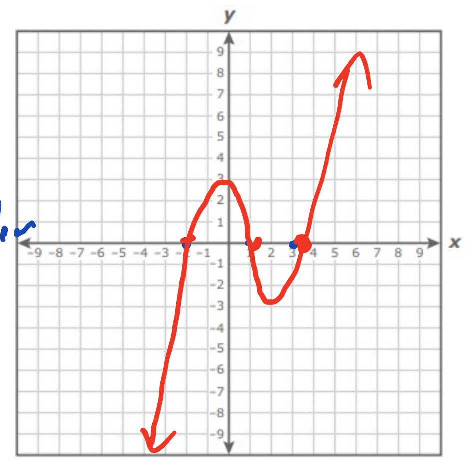
a.) $x+2=0$
$x=-2$

-2	1	-2	-5	6
	↓	-2	8	-6
	1	-4	3	$\boxed{0}$ ← remainder

x^2-4x+3

→ x+2 is a factor of f(x)
-2 is a root of f(x) (-2,0) is x-int

b.) $x^2-4x+3=0$
$(x-1)(x-3)=0$ $x=1,3$



c.) $(x+2)(x-1)(x-3)$
d.) roots: -2, 1, 3
real, rational

The Remainder and Factor Theorems

The Remainder Theorem:
If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $f(k)$.

The Factor Theorem:
If a polynomial $f(x)$ has a factor $x - k$, if and only if $f(k) = 0$.
(i.e., if the remainder is zero, then the polynomial is factorable.)

The following are equivalent:

- r **is a root of** $f(x)$
- r **is a zero of** $f(x)$
- r **is a solution of** $f(x)$
- $x - r$ **is a factor of** $f(x)$
- $f(r) = 0$
- **The remainder of** $f(x)$ **divided by** $x - k$ **is zero**

1/22 ended here.