THIRD EDITION

PRECALCULUS

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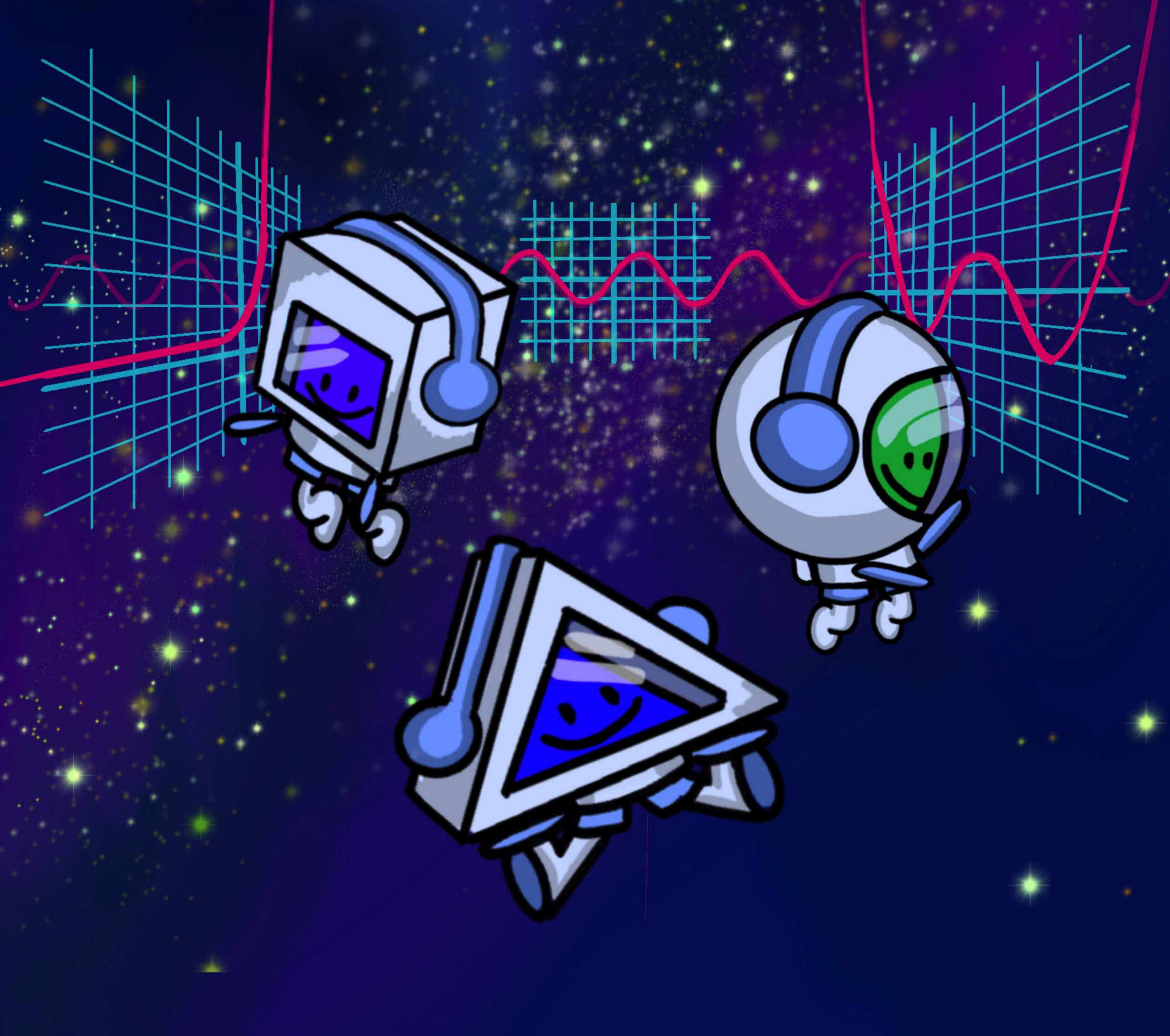

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Preface

This text, our third edition, contains notes for a course in precalculus as it is taught at New York City College of Technology, CUNY, where it is offered under the course number MAT 1375. Our approach is calculator-based. As of the third edition of this text, we use the Desmos graphing calculator to analyze and solve many of the examples studied in this course. An introduction to the Desmos graphing calculator appears in Chapter 4. Moreover, we are very grateful to be able to include in this edition many wonderful illustrations of the cartoon characters featured on the cover of this edition. These characters and images were created by Kate Poirier and provide a charming visualization of the course content.

Our course in precalculus has the overarching theme of "functions." This means that many of the often more algebraic topics studied in prior courses are revisited under this new function theoretic point of view. However, in order to keep this text as self-contained as possible, we will recall all of the results necessary to follow the core of the course even if we assume that the student has familiarity with the formula or topic at hand. Below is an outline of the topics of this course:

Part I: Functions and graphs

Part II: Polynomials and rational functions

Part III: Exponential and logarithmic functions

Part IV: Trigonometric functions

Part V: Vectors, complex numbers, and sequences

After an introduction to the abstract notion of a function and its graph, we study polynomials, rational functions, exponential functions, logarithmic functions, and trigonometric functions. Throughout, we will always assign particular importance to the corresponding graph of the discussed function which will be analyzed with the help of the Desmos graphing calculator. In the fifth and last part of the course, we deviate from the above theme and collect more algebraically oriented topics that will be needed in calculus or other advanced mathematics courses or even other science courses. The fifth part includes a discussion of the 2-dimensional real vector space $\mathbb{R}^2$, the algebra of complex numbers (in particular complex numbers in polar form), and sequences and series with focus on the arithmetic and geometric series. The generalized binomial theorem is discussed in an appendix.

The topics in this book are organized into 25 chapters, each corresponding to one course lesson. Each chapter ends with a list of exercises the student is expected to be able to solve. Answers to all exercises are provided at the end of the book. We cannot overstate the importance of completing these exercises for a successful completion of this course. These 25 lessons, together with four scheduled exams and one review session give a total of 30 class sessions, which is the number of regularly scheduled class meetings in one semester. Each of the five parts also ends with a review of the topics discussed. This may be used as a review for any of the exams during the semester.

We would like to thank our colleagues, students, and friends for their support during the development of this text. In particular, we would like to thank Kate Poirier for creating the wonderful illustrations for the third edition. Moreover, we thank Henry Africk, Laurie Caban, Jean Camilien, Leo Chosid, William Colucci, Samar ElHitti, Johanna Ellner, Natan Ovshey, Satyanand Singh, Johann Thiel, Wendy Wang, Lin Zhou, Archie Worley, Bette Forester, Steven Karaszewski, Josue Enriquez, Mohd Nayum Parvez, Akindiji Fadeyi, Isabel Martinez, Erik Nowak, Sybil Shaver, Faran Hoosain, Kenia Rodriguez, Albert Jaradeh, and Iftekher Hossain for many useful comments that helped to improve this text.

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Contents

Part I

Functions and Graphs

Chapter 1

Intervals and functions

In this chapter, we will give a definition of the main topic of this course, the notion of a function. Before we introduce functions, we review some notation regarding sets of numbers in Section 1.1.

1.1 Review of number sets

We start with a brief review of number systems and intervals.

Review 1.1: Number systems

The **natural numbers** (denoted by $\mathbb{N}$) are the numbers

$$1, 2, 3, 4, 5, \ldots$$

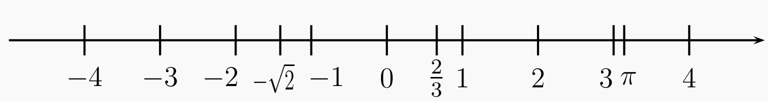
The **integers** (denoted by $\mathbb{Z}$) are the numbers

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

The **rational numbers** (denoted by $\mathbb{Q}$) are the fractions $\frac{a}{b}$ of integers a and b with $b \neq 0$. Here are some examples of rational numbers:

$$\frac{3}{5}, -\frac{2}{6}, 17, 0, \frac{3}{-8}$$

The **real numbers** (denoted by $\mathbb{R}$) are the numbers on the real number line



Here are some examples of real numbers:

$$\sqrt{3}, \pi, -\frac{2}{5}, 18, 0, 6.789$$

A real number that is not a rational number is called an **irrational number**. Here are some examples of irrational numbers:

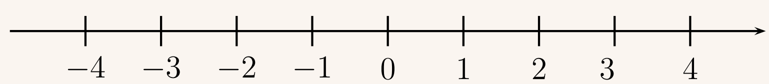
$$\pi, \sqrt{2}, 5^{\frac{2}{3}}, e$$

Recall that there is an order relation on the set of real numbers:

$4 < 9$	reads as	4 is less than 9,
$-3 \leq 2$	reads as	-3 is less than or equal to 2,
$\frac{7}{6} > 1$	reads as	$\frac{7}{6}$ is greater than 1,
$2 \geq -3$	reads as	2 is greater than or equal to -3.

Note 1.2

- Note that $2 < 3$, but $-2 > -3$, which can easily be seen on the number line.

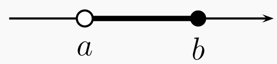
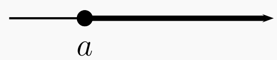
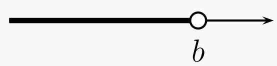


- Note that $5 \leq 5$ and $5 \geq 5$. However the same is not true when using the symbol $<$. We write this as $5 \not< 5$.

We also review some basic notation concerning intervals.

Review 1.3: Intervals

The set of all real numbers x greater than or equal to some number a and/or less than or equal to some number b is a subset of the real numbers, which is an interval. There are several ways to write an interval: in interval notation, graph it on the number line, or write it as an inequality.

Inequality notation	Number line	Interval notation
$a \leq x \leq b$		$[a, b]$
$a < x < b$		(a, b)
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$a \leq x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

Formally, we define the interval $[a, b]$ to be the set of all real numbers x such that $a \leq x \leq b$:

$$[a, b] \quad = \quad \{ \ x \ \mid \ a \leq x \leq b \ \}$$

There are similar definitions for the other intervals shown in the above table.

Be careful!

Be sure to write the smaller number $a < b$ first when writing an interval $[a, b]$. For example, the interval

$$[5, 3] = \{ \ x \ \mid \ 5 \leq x \leq 3 \ \} = \{\}$$

would be the empty set!

Example 1.4

Write the inequality

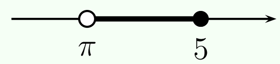
$$\pi < x \leq 5$$

in interval notation and graph it on the number line.

Solution.

Interval notation: $(\pi, 5]$

On the number line:



□

Example 1.5

Write the interval

as an inequality and in interval notation.

Solution.

Inequality notation: $-3 \leq x$

Interval notation: $[-3, \infty)$

□

Example 1.6

Write the interval

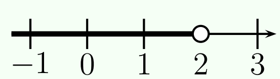
$$(-\infty, 2)$$

as an inequality and graph it on the number line.

Solution.

Inequality notation: $x < 2$

On the number line:



□

Note 1.7: Number line notation

In some texts round and square brackets are also used on the number line to depict an interval. For example, the following depicts the interval $[2,5)$.

1.2 Introduction to functions

We now introduce the notion of a function. An easy and well-known example of a function is given by the equation of a straight line such as, for example, $y = 5x + 4$. Note that for each given x we obtain an induced y. (For example, for $x = 3$, we obtain $y = 5 \cdot 3 + 4 = 19$.)

Definition 1.8: Function, domain, codomain

A **function** f consists: a set D of **inputs** called the **domain**, a set C of possible **outputs** called the **codomain**, and an assignment that assigns to each input x exactly one output y.
A function f with domain D and codomain C is denoted by

$$f : D \to C.$$

If x is in the domain D (an input), then we denote by $f(x) = y$ the output that is assigned by f to x.

Sometimes it is of interest to know the set of all elements in the codomain that actually occur as an output. This set is a subset of the codomain and is called the range. We have:

Definition 1.9: Range

The **range** R of a function f is a subset of the codomain of f, given by all of the outputs of f:

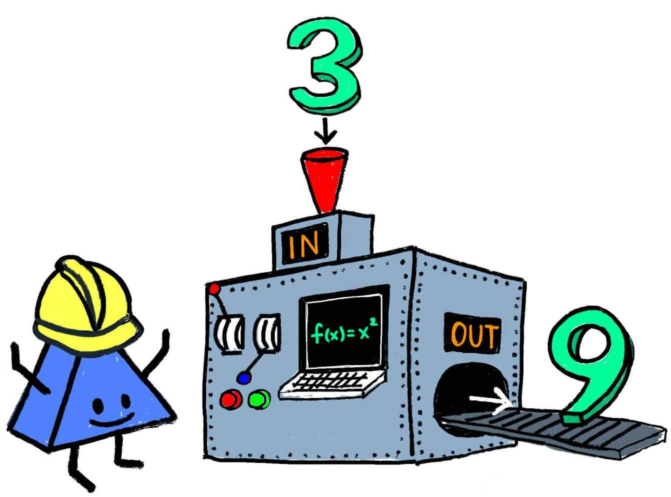
$$R = \{f(x) \,|\, x \text{ is in the domain of } f\}$$

Note 1.10

Some authors use a slightly different convention in that they use the term range for what we called the codomain above.

Since we will be dealing with many functions it is convenient to name various functions (usually with letters f, g, h, etc.). Often we will implicitly assume that a domain and codomain are given without specifying these explicitly. If the range can be determined and the codomain is not given explicitly, then we take the codomain to be the range. If the range cannot easily be determined and the codomain is not explicitly given, then the codomain should be taken to be a set which clearly contains the range. For example, in many instances the codomain can be taken to be the set of all real numbers.

There are many ways that one can describe how a function assigns to an input an output (all of which may not apply to a specific function): via a table of values (listing the input-output pairs); via a formula (with the domain and range explicitly or implicitly given); via a graph (representing input-output pairs on a coordinate plane); or in words, just to name a few. Examples of these ways to represent a function will be given below.



Example 1.11

Define the assignment f by the following table

x	2	5	-3	0	7	4
y	6	8	6	4	-1	8

The assignment f assigns to the input 2 the output 6, which is also written as

$$f(2) = 6.$$

Similarly, f assigns to 5 the number 8, in short $f(5) = 8$, etc.:

$$f(5) = 8, \quad f(-3) = 6, \quad f(0) = 4, \quad f(7) = -1, \quad f(4) = 8.$$

The domain D is the set of all inputs. The domain is therefore

$$D = \{-3, 0, 2, 4, 5, 7\}.$$

The range R is the set of all outputs. The range is therefore

$$R = \{-1, 4, 6, 8\}.$$

The assignment f is indeed a function since each element of the domain gets assigned exactly one element in the range. Note that for an input number that is not in the domain, f does not assign an output to it. For example,

$$f(1) = \text{undefined}.$$

Note also that $f(5) = 8$ and $f(4) = 8$, so that f assigns to the inputs 5 and 4 the same output 8. Similarly, f also assigns the same output to the inputs 2 and -3. Therefore we see that:

- *A function may assign the same output to two different inputs!*

Example 1.12

Consider the assignment f that is given by the following table.

x	2	5	-3	0	5	4
y	6	8	6	4	-1	8

This assignment *does not define* a function! What went wrong? Consider the input value 5. What does f assign to the input 5? The third column states that f assigns to 5 the output 8, whereas the sixth column states that f assigns to 5 the output -1,

$$f(5) = 8, \qquad f(5) = -1.$$

However, by the definition of a function, to each input we have to assign *exactly one* output. So here, to the input 5 we have assigned two outputs 8 and -1. Therefore, f is not a function.

- *A function cannot assign two outputs to one input!*

We repeat the two bullet points from the last two examples, which are crucial for the understanding of a function.

Note 1.13: Same output from inputs versus multiple outputs

- *A function may assign the same output to two different inputs!*

  $f(x_1) = y$ and $f(x_2) = y$ with $x_1 \neq x_2$ is **allowed**!

- *A function cannot assign two outputs to one input!*

  $f(x) = y_1$ and $f(x) = y_2$ with $y_1 \neq y_2$ is **not allowed**!

Example 1.14

A university creates a mentoring program which matches each freshman student with a senior student as his or her mentor. Within this program it is guaranteed that each freshman gets precisely one mentor, however two freshmen may receive the same mentor. Does the assignment of freshmen to mentor, or mentor to freshmen describe a function? If so, what is its domain, what is its range?

Solution.

Since a senior may mentor several freshman, we cannot take a mentor as an "input," as he or she would be assigned to several "output" freshmen students. So freshman is not a function of mentor.

On the other hand, we can assign each freshmen to exactly one mentor, which therefore describes a function.
The domain (the set of all inputs) is given by the set of all freshmen students. The range (the set of all outputs) is given by the set of all senior students that are mentors. The function assigns each "input" freshmen student to his or her unique "output" mentor. □

Example 1.15

The rainfall in a city for each of the 12 months is displayed in the following histogram.

a) Is the rainfall a function of the month?

b) Is the month a function of the rainfall?

Solution.

a) Each month has exactly one amount of rainfall associated to it. Therefore, the assignment that associates to a month its rainfall (in inches) is a function.

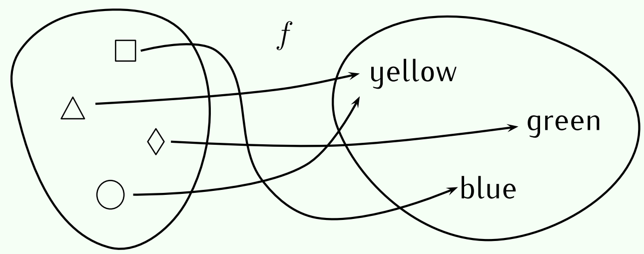
b) If we take a certain rainfall amount as our input data, can we associate a unique month to it? For example, February and March appear to have the same amount of rainfall. If the months February and March do indeed have the same amount of rainfall, then, to one input amount of rainfall we cannot assign a unique month. The month would therefore *not* be a function of the rainfall.

   One could also argue that the rainfall for any two months in the above graph would almost certainly be different if the rainfall is measured to a high enough degree of accuracy, as it is very unlikely that any two months have the exact amount of rainfall. (What does it even mean to have the exact same amount of rainfall?) For this setup, one would conclude that the month *is* a function of the rainfall.

□

Example 1.16

Consider the function f described below.



Here, the function f maps the input symbol $\square$ to the output color blue. Other assignments of f are as follows:

$$f(\square) = \text{blue} \qquad f(\triangle) = \text{yellow}$$
$$f(\Diamond) = \text{green} \qquad f(\bigcirc) = \text{yellow}$$

The domain is the set of symbols $D = \{\square, \triangle, \Diamond, \bigcirc\}$, and the range is the set of colors $R = \{\text{blue, green, yellow}\}$. Notice, in particular, that the inputs $\triangle$ and $\bigcirc$ both have the same output yellow, which is certainly allowed for a function.

Example 1.17

Consider the function $y = 5x+4$ with domain all real numbers and range all real numbers. Note that for each input x, we obtain an exactly one induced output y. For example, for the input $x = 3$ we get the output $y = 5 \cdot 3 + 4 = 19$, etc.

Example 1.18

Consider the function $y = x^2$ with domain all real numbers and range non-negative numbers. The function takes a real number as an input and squares it. For example if $x = -2$ is the input, then $y = 4$ is the output.

Example 1.19

For each real number x, denote by $\lfloor x \rfloor$ the greatest integer that is less or equal to x. We call $\lfloor x \rfloor$ the **floor of** x. For example, to calculate $\lfloor 4.37 \rfloor$, note that all integers $4, 3, 2, \dots$ are less than or equal to 4.37:

$$\dots, -3, -2, -1, 0, 1, 2, 3, 4 \qquad \leq \quad 4.37$$

The greatest of these integers is 4, so that $\lfloor 4.37 \rfloor = 4$. We define the **floor function** as $f(x) = \lfloor x \rfloor$. Here are more examples of function values of the floor function.

$$\lfloor 7.3 \rfloor = 7 \qquad \lfloor \pi \rfloor = 3 \qquad \lfloor -4.65 \rfloor = -5$$
$$\lfloor 12 \rfloor = 12 \qquad \left\lfloor \frac{-26}{3} \right\rfloor = \lfloor -8.667 \rfloor = -9$$

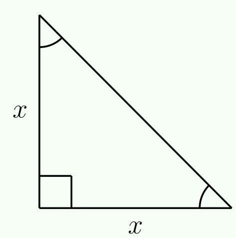
The domain of the floor function is the set of all real numbers, that is $D = \mathbb{R}$. The range is the set of all integers, $R = \mathbb{Z}$.

Example 1.20

Let A be the area of an isosceles right triangle with base side length x. Express A as a function of x.

Solution.

Being an isosceles right triangle means that two side lengths are x, and the angles are $45°$, $45°$, and $90°$ (or in radian measure $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$):



Recall that the area of a triangle is: area $= \frac{1}{2}$ base $\cdot$ height. In this case, we have base$= x$, and height$= x$, so that the area

$$A = \frac{1}{2} x \cdot x = \frac{1}{2} x^2.$$

Therefore, the area $A(x) = \frac{1}{2} \cdot x^2$. □

Example 1.21

Consider the equation $y = x^2 + 3$. This equation associates to each input number a exactly one output number $b = a^2 + 3$. Therefore, the equation defines a function. For example:

To the input 5 we assign the output $5^2 + 3 = 25 + 3 = 28$.

The domain D is all real numbers, $D = \mathbb{R}$. Since x^2 is always ≥ 0, we see that $x^2 + 3 \geq 3$, and vice versa, every number $y \geq 3$ can be written as $y = x^2 + 3$. (To see this, note that the input $x = \sqrt{y-3}$ for $y \geq 3$ gives the output $x^2 - 3 = (\sqrt{y-3})^2 + 3 = y - 3 + 3 = y$.) Therefore, the range is $R = [3, \infty)$.

Example 1.22

Consider the equation $x^2 + y^2 = 25$. Does this equation define y as a function of x? That is, does this equation assign to each input x exactly one output y?
An input number x gets assigned to y with $x^2 + y^2 = 25$. Solving this for y, we obtain

$$y^2 = 25 - x^2 \qquad \Longrightarrow \qquad y = \pm\sqrt{25 - x^2}.$$

Therefore, there are *two* possible outputs associated to the input $x (\neq 5)$:

$$\text{either} \quad y = +\sqrt{25 - x^2} \qquad \text{or} \quad y = -\sqrt{25 - x^2}.$$

For example, the input $x = 0$ has two outputs $y = 5$ and $y = -5$. However, a function cannot assign two outputs to one input x! The conclusion is that $x^2 + y^2 = 25$ does *not* determine y as a function!

Note 1.23: Independent versus dependent variable

Note that if $y = f(x)$ then x is called the **independent variable** and y is called the **dependent variable** (since x can be chosen freely, and y depends on x).
If $x = g(y)$ then y is the independent variable and x is the dependent variable (since now, y can be chosen freely and x depends on y).

1.3 Exercises

Exercise 1.1

Give examples of numbers that are

a) natural numbers
b) integers
c) integers but not natural numbers
d) rational numbers
e) real numbers
f) rational numbers but not integers

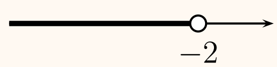
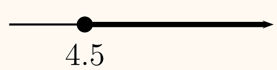
Exercise 1.2

Which of the following numbers are natural numbers, integers, rational numbers, or real numbers? Which of these numbers are irrational?

a) $\frac{7}{3}$ b) -5 c) 0 d) $17,000$ e) $\frac{12}{4}$ f) $\sqrt{7}$ g) $\sqrt{25}$

Exercise 1.3

Complete the table.

Inequality notation	Number line	Interval notation
$2 \leq x < 5$		
$x \leq 3$		
		
		
		$[-2, 6]$
		$(-\infty, 0)$
		
$5 < x \leq \sqrt{30}$		
		$(\frac{13}{7}, \pi)$

Exercise 1.4

The tables below describe assignments between inputs x and outputs y. Determine which of the given tables describe a function. If they do, determine their domain and range. Describe which outputs are assigned to which inputs.

a)

x	-5	3	-1	6	0
y	5	2	8	3	7

b)

x	6	17	4	-2	4
y	8	-2	0	3	-1

c)

x	19	7	6	-2	3	-11
y	3	3	3	3	3	3

d)

x	1	2	3	3	4	5
y	5.33	9	13	13	17	$\sqrt{19}$

e)

x	0	1	2	2	3	4
y	0	1	2	3	3	4

Exercise 1.5

In a store, every item that is for sale has a price.

a) Does the assignment which assigns to an item its price constitute a function (in the sense of Definition 1.8 on page 6)?

b) Does the assignment which assigns to a given price all items with this price constitute a function?

c) In the case where the assignment is a function, what is the domain?

d) In the case where the assignment is a function, what is the range?

Exercise 1.6

A bank offers wealthy customers a certain amount of interest if they keep more than 1 million dollars in their account. The amount is described in the following table.

dollar amount x in the account	interest amount
$x \leq \$1,000,000$	$\$0$
$\$1,000,000 < x \leq \$10,000,000$	2% of x
$\$10,000,000 < x$	1% of x

a) Justify that the assignment cash amount to interest defines a function.

b) Find the interest for an amount of:

i) $\$50,000$ ii) $\$5,000,000$ iii) $\$1,000,000$
iv) $\$30,000,000$ v) $\$10,000,000$ vi) $\$2,000,000$

Exercise 1.7

Find a formula for a function describing the given inputs and outputs.

a) *input*: the radius of a circle
output: the circumference of the circle

b) *input*: the side length in an equilateral triangle
output: the perimeter of the triangle

c) *input*: one side length of a rectangle, with other side length being 3
output: the perimeter of the rectangle

d) *input*: the side length of a cube
output: the volume of the cube

Chapter 2

Functions via formulas

Most of the time we will discuss functions that take some real numbers as inputs, and give real numbers as outputs. Such functions are often described with a formula.

2.1 Functions given by formulas

Here are some examples of functions given by a formula.

Example 2.1

For the given function f, calculate the outputs $f(2)$, $f(-3)$, and $f(-1)$.

a) $f(x) = 3x + 4$

b) $f(x) = \sqrt{x^2 - 3}$

c) $f(x) = \begin{cases} 5x - 6 & \text{, for } -1 \le x \le 1 \\ x^3 + 2x & \text{, for } 1 < x \le 5 \end{cases}$

d) $f(x) = \frac{x+2}{x+3}$

Solution.

a) We substitute the input values into the function and simplify.

$$\begin{aligned} f(2) &= 3 \cdot 2 + 4 = 6 + 4 = 10, \\ f(-3) &= 3 \cdot (-3) + 4 = -9 + 4 = -5, \\ f(-1) &= 3 \cdot (-1) + 4 = -3 + 4 = 1. \end{aligned}$$

b) Similarly, we calculate

$$f(2) = \sqrt{2^2 - 3} = \sqrt{4 - 3} = \sqrt{1} = 1,$$

$$\begin{aligned} f(-3) &= \sqrt{(-3)^2-3} = \sqrt{9-3} = \sqrt{6}, \\ f(-1) &= \sqrt{(-1)^2-3} = \sqrt{1-3} = \sqrt{-2} \text{ is undefined.} \end{aligned}$$

Note that in the last evaluation, we obtained an output of $\sqrt{-2}$. As you are probably aware, $\sqrt{-2}$ is a complex number. However, at this point, we will only allow outputs that are real numbers! Since $\sqrt{-2}$ is *not* a real number (but only a complex number), there is no real output for $f(-1)$, and we say that $f(-1)$ is undefined.

c) The function $f(x) = \begin{cases} 5x-6 & \text{, for } -1 \leq x \leq 1 \\ x^3+2x & \text{, for } 1 < x \leq 5 \end{cases}$ is given as a piecewise defined function. We have to substitute the values into the correct branch:

$$\begin{aligned} f(2) &= 2^3 + 2\cdot 2 = 8+4 = 12, \text{ since } 1 < 2 \leq 5, \\ f(-3) &= \text{undefined, since } -3 \text{ is not in any of the two branches,} \\ f(-1) &= 5\cdot(-1) - 7 = -5-6 = -11, \text{ since } -1 \leq -1 \leq 1. \end{aligned}$$

d) Finally for $f(x) = \frac{x+2}{x+3}$, we have:

$$\begin{aligned} f(2) &= \frac{2+2}{2+3} = \frac{4}{5}, \qquad f(-3) = \frac{-3+2}{-3+3} = \frac{-1}{0} \text{ is undefined,} \\ f(-1) &= \frac{-1+2}{-1+3} = \frac{1}{2}. \end{aligned}$$

□

Example 2.2

Let f be the function given by $f(x) = x^2 + 2x - 3$. Find the following function values.

a) $f(5)$	b) $f(2)$	c) $f(-2)$	d) $f(0)$
e) $f(\sqrt{5})$	f) $f(\sqrt{3}+1)$	g) $f(a)$	h) $f(a)+5$
i) $f(x+h)$	j) $f(x+h) - f(x)$	k) $\frac{f(x+h)-f(x)}{h}$	l) $\frac{f(x)-f(a)}{x-a}$

Solution.

The first four function values ((a)-(d)) can be calculated directly:

$$f(5) = 5^2 + 2\cdot 5 - 3 = 25 + 10 - 3 = 32,$$

$$\begin{aligned} f(2) &= 2^2 + 2\cdot 2 - 3 = 4 + 4 - 3 = 5, \\ f(-2) &= (-2)^2 + 2\cdot(-2) - 3 = 4 + -4 - 3 = -3, \\ f(0) &= 0^2 + 2\cdot 0 - 3 = 0 + 0 - 3 = -3. \end{aligned}$$

The next two values ((e) and (f)) are similar, but the arithmetic is a bit more involved.

$$\begin{aligned} f(\sqrt{5}) &= \sqrt{5}^2 + 2\cdot\sqrt{5} - 3 = 5 + 2\cdot\sqrt{5} - 3 = 2 + 2\cdot\sqrt{5}, \\ f(\sqrt{3}+1) &= (\sqrt{3}+1)^2 + 2\cdot(\sqrt{3}+1) - 3 \\ &= (\sqrt{3}+1)\cdot(\sqrt{3}+1) + 2\cdot(\sqrt{3}+1) - 3 \\ &= \sqrt{3}\cdot\sqrt{3} + 2\cdot\sqrt{3} + 1\cdot 1 + 2\cdot\sqrt{3} + 2 - 3 \\ &= 3 + 2\cdot\sqrt{3} + 1 + 2\cdot\sqrt{3} + 2 - 3 \\ &= 3 + 4\cdot\sqrt{3}. \end{aligned}$$

The last five values ((g)–(l)) are purely algebraic:

$$\begin{aligned} f(a) &= a^2 + 2\cdot a - 3, \\ f(a) + 5 &= a^2 + 2\cdot a - 3 + 5 = a^2 + 2\cdot a + 2, \\ f(x+h) &= (x+h)^2 + 2\cdot(x+h) - 3 \\ &= x^2 + 2xh + h^2 + 2x + 2h - 3, \\ f(x+h) - f(x) &= (x^2 + 2xh + h^2 + 2x + 2h - 3) - (x^2 + 2x - 3) \\ &= x^2 + 2xh + h^2 + 2x + 2h - 3 - x^2 - 2x + 3 \\ &= 2xh + h^2 + 2h, \\ \frac{f(x+h) - f(x)}{h} &= \frac{2xh + h^2 + 2h}{h} \\ &= \frac{h\cdot(2x + h + 2)}{h} = 2x + h + 2, \end{aligned}$$

and

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{(x^2 + 2x - 3) - (a^2 + 2a - 3)}{x - a} \\ &= \frac{x^2 + 2x - 3 - a^2 - 2a + 3}{x - a} = \frac{x^2 - a^2 + 2x - 2a}{x - a} \\ &= \frac{(x+a)(x-a) + 2(x-a)}{x - a} = \frac{(x-a)(x+a+2)}{(x-a)} = x + a + 2. \end{aligned}$$

□

The quotients in the last two examples 2.2(k) and (l) will become particularly important in calculus. They are called difference quotients.

Definition 2.3: Difference quotient

Let $y = f(x)$ be a function. We call the expressions

$$\boxed{\frac{f(x+h)-f(x)}{h}} \quad \text{or} \quad \boxed{\frac{f(x)-f(a)}{x-a}} \tag{2.1}$$

difference quotients for the function f.

We next calculate some more examples of difference quotients.

Example 2.4

Calculate the difference quotient $\frac{f(x+h)-f(x)}{h}$ for

a) $f(x) = x^2 - 4x$ b) $f(x) = 3x^2 + 8x - 5$

Solution.

a) For $f(x) = x^2 - 4x$, we get:

$$\begin{aligned}
f(x+h) &= (x+h)^2 - 4\cdot(x+h) \\
&= x^2 + 2xh + h^2 - 4x - 4h, \\
f(x+h) - f(x) &= (x^2 + 2xh + h^2 - 4x - 4h) - (x^2 - 4x) \\
&= x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x \\
&= 2xh + h^2 - 4h, \\
\frac{f(x+h)-f(x)}{h} &= \frac{2xh + h^2 - 4h}{h} \\
&= \frac{h\cdot(2x + h - 4)}{h} = 2x + h - 4
\end{aligned}$$

b) For $f(x) = 3x^2 + 8x - 5$, we get:

$$\begin{aligned}
f(x+h) &= 3(x+h)^2 + 8\cdot(x+h) - 5 \\
&= 3(x^2 + 2xh + h^2) + 8x + 8h - 5, \\
&= 3x^2 + 6xh + 3h^2 + 8x + 8h - 5,
\end{aligned}$$

$$\begin{aligned} f(x+h)-f(x) &= (3x^2+6xh+3h^2+8x+8h-5) \\ &\qquad -(3x^2+8x-5) \\ &= 3x^2+6xh+3h^2+8x+8h-5-3x^2-8x+5 \\ &= 6xh+3h^2+8h, \\ \frac{f(x+h)-f(x)}{h} &= \frac{6xh+3h^2+8h}{h} \\ &= \frac{h\cdot(6x+3h+8)}{h} = 6x+3h+8 \end{aligned}$$

□

Example 2.5

Calculate the difference quotient $\frac{f(x)-f(a)}{x-a}$ for

a) $f(x)=x^2-7x-2$ b) $f(x)=-2x^2+3x$

Solution.

a) For $f(x)=x^2-7x-2$, we get:

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} &= \frac{(x^2-7x-2)-(a^2-7a-2)}{x-a} \\ &= \frac{x^2-7x-2-a^2+7a+2}{x-a} = \frac{x^2-a^2-7x+7a}{x-a} \\ &= \frac{(x+a)(x-a)-7(x-a)}{x-a} = \frac{(x-a)(x+a-7)}{(x-a)} = x+a-7. \end{aligned}$$

b) For $f(x)=-2x^2+3x$, we get:

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} &= \frac{(-2x^2+3x)-(-2a^2+3a)}{x-a} \\ &= \frac{-2x^2+3x+2a^2-3a}{x-a} = \frac{-2x^2+2a^2+3x-3a}{x-a} \\ &= \frac{-2(x^2-a^2)+3x-3a}{x-a} = \frac{-2(x+a)(x-a)+3(x-a)}{x-a} \\ &= \frac{(x-a)(-2(x+a)+3)}{(x-a)} = -2(x+a)+3 = -2x-2a+3. \end{aligned}$$

□

Here are some difference quotients of a degree 3 polynomial, a rational function, and a square root function.

Example 2.6

Calculate the difference quotient $\frac{f(x+h)-f(x)}{h}$ for

a) $f(x) = x^3 + 2$ b) $f(x) = \dfrac{1}{x}$ c) $f(x) = \sqrt{2x+3}$

Solution.

a) We calculate first the difference quotient step by step.

$$\begin{aligned} f(x+h) &= (x+h)^3 + 2 = (x+h)\cdot(x+h)\cdot(x+h) + 2 \\ &= (x^2 + 2xh + h^2)\cdot(x+h) + 2 \\ &= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 + 2, \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 2. \end{aligned}$$

Subtracting $f(x)$ from $f(x+h)$ gives

$$\begin{aligned} f(x+h) - f(x) &= (x^3 + 3x^2h + 3xh^2 + h^3 + 2) - (x^3 + 2) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 2 - x^3 - 2 \\ &= 3x^2h + 3xh^2 + h^3. \end{aligned}$$

With this we obtain:

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \frac{h\cdot(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2. \end{aligned}$$

b) The computation for (b) is similar.

$$f(x+h) = \frac{1}{x+h},$$

so that

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}.$$

We obtain the solution after simplifying the double fraction:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{(x+h)\cdot x}}{h} = \frac{\frac{x-x-h}{(x+h)\cdot x}}{h} = \frac{\frac{-h}{(x+h)\cdot x}}{h} \\ &= \frac{-h}{(x+h)\cdot x}\cdot\frac{1}{h} = \frac{-1}{(x+h)\cdot x}.\end{aligned}$$

c)

$$\begin{aligned} & f(x+h) &&= \sqrt{2(x+h)+3} = \sqrt{2x+2h+3} \\ \Longrightarrow\ & f(x+h)-f(x) &&= \sqrt{2x+2h+3}-\sqrt{2x+3} \\ \Longrightarrow\ & \frac{f(x+h)-f(x)}{h} &&= \frac{\sqrt{2x+2h+3}-\sqrt{2x+3}}{h}\end{aligned}$$

We can simplify this expression by multiplying both numerator and denominator with $\sqrt{2x+2h+3}+\sqrt{2x+3}$:

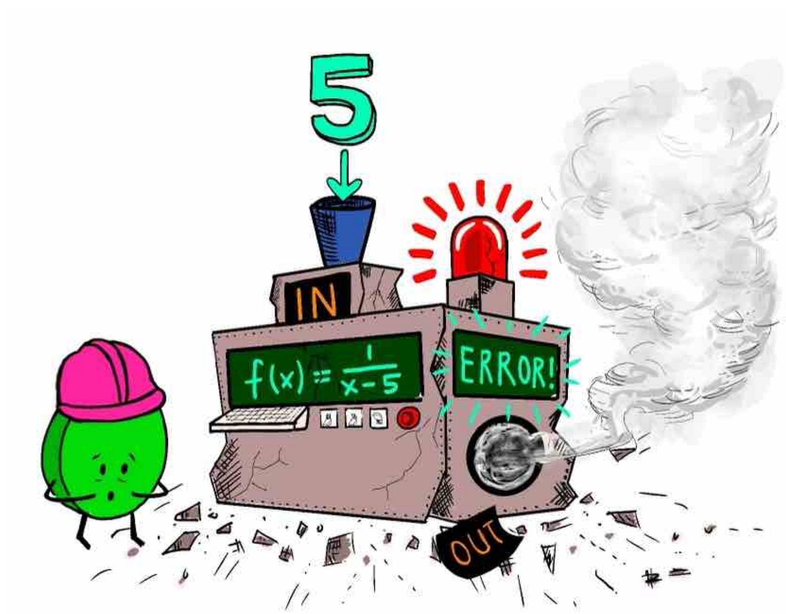
$$\begin{aligned}\Longrightarrow\ \frac{f(x+h)-f(x)}{h} &= \frac{(\sqrt{2x+2h+3}-\sqrt{2x+3})\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{(\sqrt{2x+2h+3})^2-(\sqrt{2x+3})^2}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{(2x+2h+3)-(2x+3)}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2x+2h+3-2x-3}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2h}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2}{\sqrt{2x+2h+3}+\sqrt{2x+3}}\end{aligned}$$

□

So far, we have not mentioned the domain and range of the functions defined above. Indeed, we will often not describe the domain explicitly but use the following convention:

Convention 2.7: Standard convention of the domain

Unless stated otherwise, a function f is assumed to allow any real numbers x as an input for which the output $f(x)$ is a well-defined real number. We refer to this as the **standard convention of the domain**.
In this case, both domain and range are then subsets of the set $\mathbb{R}$ of real numbers. The range is, of course, the set of all outputs obtained by f from the inputs (see also Note 1.10 on page 7).
In particular, under this convention, any polynomial has the domain $\mathbb{R}$ of all real numbers.



Example 2.8

Find the domain of each of the following functions according to the standard convention of the domain.

a) $f(x) = 4x^3 - 2x + 5$
b) $f(x) = |x|$
c) $f(x) = \sqrt{x}$
d) $f(x) = \sqrt{x-3}$
e) $f(x) = \frac{1}{x-5}$
f) $f(x) = \frac{x-2}{x^2+8x+15}$
g) $f(x) = \begin{cases} x+1 & \text{, for } 2 < x \le 4 \\ 2x-1 & \text{, for } 5 \le x \end{cases}$

Solution.

a) There is no problem taking a real number x to the power of any positive integer. Therefore, f is defined for all real numbers x, and the domain is written as $D = \mathbb{R}$.

b) Again, we can take the absolute value for any real number x. The domain is all real numbers, $D = \mathbb{R}$.

c) The square root $\sqrt{x}$ is only defined for $x \geq 0$ (remember we are not using complex numbers yet!). Thus, the domain is $D = [0, \infty)$.

d) Again, the square root is only defined for non-negative numbers. Thus, the argument in the square root has to be greater than or equal to zero: $x - 3 \geq 0$. Solving this for x gives

$$x - 3 \geq 0 \quad \overset{\text{(add 3)}}{\Longrightarrow} \quad x \geq 3.$$

The domain is therefore, $D = [3, \infty)$.

e) A fraction is defined whenever the denominator is not zero, so in this case, $\frac{1}{x-5}$ is defined whenever $x \neq 5$. Therefore, the domain is all real numbers except five, $D = \mathbb{R} - \{5\}$.

f) Again, we need to make sure that the denominator does not become zero, and we disregard the numerator. The denominator is zero exactly when $x^2 + 8x + 15 = 0$. Solving this for x gives:

$$\begin{aligned} x^2 + 8x + 15 = 0 \quad &\Longrightarrow \quad (x+3) \cdot (x+5) = 0 \\ &\Longrightarrow \quad x + 3 = 0 \text{ or } x + 5 = 0 \\ &\Longrightarrow \quad x = -3 \text{ or } x = -5. \end{aligned}$$

The domain is all real numbers except for -3 and -5, that is $D = \mathbb{R} - \{-5, -3\}$.

g) The function is explicitly defined for all $2 < x \leq 4$ and $5 \leq x$. Therefore, the domain is $D = (2, 4] \cup [5, \infty)$.

□

2.2 Exercises

Exercise 2.1

For each of the following functions,

a) $f(x) = 3x + 1$ b) $f(x) = x^2 - x$ c) $f(x) = \sqrt{x^2 - 9}$
d) $f(x) = \frac{1}{x}$ e) $f(x) = \frac{x-5}{x+2}$ f) $f(x) = -x^3$

calculate the function values

i) $f(3)$ ii) $f(5)$ iii) $f(-2)$ iv) $f(0)$ v) $f(\sqrt{13})$
vi) $f(\sqrt{2} + 3)$ vii) $f(-x)$ viii) $f(x + 2)$ ix) $f(x) + h$ x) $f(x + h)$

Exercise 2.2

Let f be the piecewise defined function

$$f(x) = \begin{cases} x - 5 & \text{, for } -4 < x < 3 \\ x^2 & \text{, for } 3 \leq x \leq 6 \end{cases}$$

a) State the domain of the function.
Find the function values

b) $f(2)$ c) $f(5)$ d) $f(-3)$ e) $f(3)$

Exercise 2.3

Let f be the piecewise defined function

$$f(x) = \begin{cases} |x| - x^2 & \text{, for } x < 2 \\ 7 & \text{, for } 2 \leq x < 5 \\ x^2 - 4x + 1 & \text{, for } 5 < x \end{cases}$$

a) State the domain of the function.
Find the function values

b) $f(1)$ c) $f(-2)$ d) $f(3)$
e) $f(2)$ f) $f(5)$ g) $f(7)$

Exercise 2.4

Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the following functions:

a) $f(x) = 5x$ b) $f(x) = 2x - 6$ c) $f(x) = x^2$
d) $f(x) = x^2 + 5x$ e) $f(x) = x^2 - 7$ f) $f(x) = x^2 + 3x + 4$
g) $f(x) = x^2 + 4x - 9$ h) $f(x) = 3x^2 - 2x$ i) $f(x) = 4x^2 + 6x$
j) $f(x) = 2x^2 - 8x - 3$ k) $f(x) = -5x^2 + 3$ l) $f(x) = x^3$

Exercise 2.5

Find the difference quotient $\frac{f(x)-f(a)}{x-a}$ for the following functions:

a) $f(x) = 3x$ b) $f(x) = 4x - 7$ c) $f(x) = x^2 - 3x$
d) $f(x) = x^2 + 4x - 5$ e) $f(x) = 7x^2 + 2x$ f) $f(x) = \frac{1}{x}$

Exercise 2.6

Find the domains of the following functions.

a) $f(x) = x^2 + 3x + 5$ b) $f(x) = |x - 2|$ c) $f(x) = \sqrt{x - 2}$
d) $f(x) = \sqrt{8 - 2x}$ e) $f(x) = \sqrt{|x + 3|}$ f) $f(x) = \frac{1}{x+6}$
g) $f(x) = \frac{x-5}{x-7}$ h) $f(x) = \frac{x+1}{x^2-7x+10}$ i) $f(x) = \frac{x}{|x-2|}$
j) $f(x) = \begin{cases} |x| & \text{for } 1 < x < 2 \\ 2x & \text{for } 3 \leq x \end{cases}$ k) $f(x) = \frac{\sqrt{x}}{x-9}$ l) $f(x) = \frac{5}{\sqrt{x+4}}$

Chapter 3

Functions via graphs

Another way to represent a function is via a graph. Before discussing graphs in general, we first review a familiar kind of graph, namely the graph of a linear function.

3.1 Review of graphs of linear functions

We recall the equation of a linear function.

Review 3.1: Linear function

A **linear function** is a function of the form

$$f(x) = a \cdot x + b$$

for some real numbers a and b where $a \neq 0$. By the standard convention of the domain, the domain of f consists of all real numbers, $D = \mathbb{R}$.

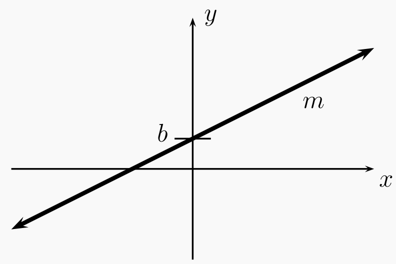
Recall that linear functions can be graphed in the x-y plane as a straight line. In this case, the coefficient a is also denoted by m and has the interpretation of the slope of the line. We review this now.

Review 3.2: Slope-intercept form of the line

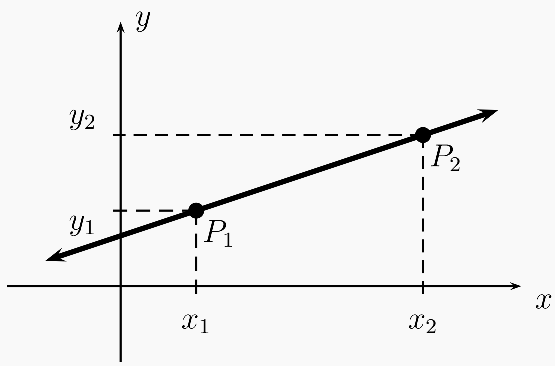
The **slope-intercept form of the line** is the equation

$$\boxed{y = m \cdot x + b} \tag{3.1}$$

Here, m is the **slope** and $(0, b)$ is the y-**intercept** of the line.



Generally, the slope describes how fast the line grows toward the right. For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line L, the slope m is given by the following formula (which is $m = \frac{\text{rise}}{\text{run}}$):

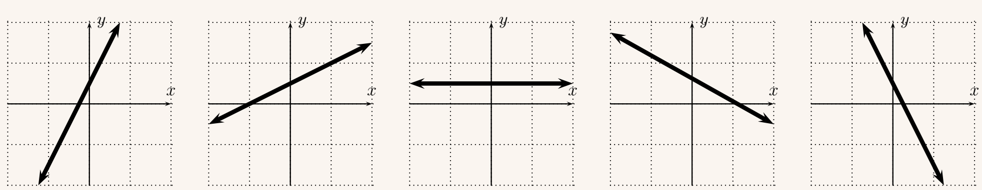


Slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.2}$$

Note 3.3: Sign of the slope

When the slope m is positive, the line rises toward the right. When the slope m is negative, the line declines toward the right.



$m = 2$ $\quad m = \frac{1}{2}$ $\quad m = 0$ $\quad m = -\frac{1}{2}$ $\quad m = -2$

Below is an example of the graph of a line in slope-intercept form.

Example 3.4

Graph the line $y = 2x + 3$.

Solution.

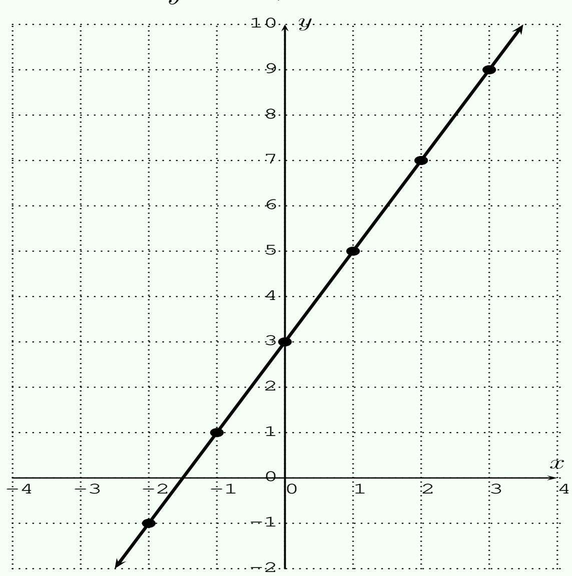
We calculate the output values y for various input values x. For example, when x is $-2, -1, 0, 1, 2$, or 3, we compute

x	-2	-1	0	1	2	3
y	-1	1	3	5	7	9

In the above table each y value is calculated by substituting the corresponding x value into our equation $y = 2x + 3$:

$$\begin{array}{lll} x = -2 & \Longrightarrow & y = 2 \cdot (-2) + 3 = -4 + 3 = -1 \\ x = -1 & \Longrightarrow & y = 2 \cdot (-1) + 3 = -2 + 3 = 1 \\ x = 0 & \Longrightarrow & y = 2 \cdot (0) + 3 = 0 + 3 = 3 \\ x = 1 & \Longrightarrow & y = 2 \cdot (1) + 3 = 2 + 3 = 5 \\ x = 2 & \Longrightarrow & y = 2 \cdot (2) + 3 = 4 + 3 = 7 \\ x = 3 & \Longrightarrow & y = 2 \cdot (3) + 3 = 6 + 3 = 9 \end{array}$$

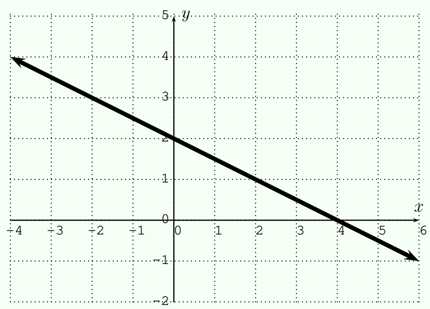
In the above calculation, the values for x were arbitrarily chosen. Since a line is completely determined by knowing two points on it, any two values for x would have worked for the purpose of graphing the line.
Drawing the above points in the coordinate plane and connecting them gives the graph of the line $y = 2x + 3$:



Alternatively, note that the y-intercept is $(0, 3)$ (3 is the additive constant in our initial equation $y = 2x+3$) and the slope $m = 2$ determines the rate at which the line grows: for each step to the right, we have to move two steps up. □

Example 3.5

Find the equation of the line in slope-intercept form.



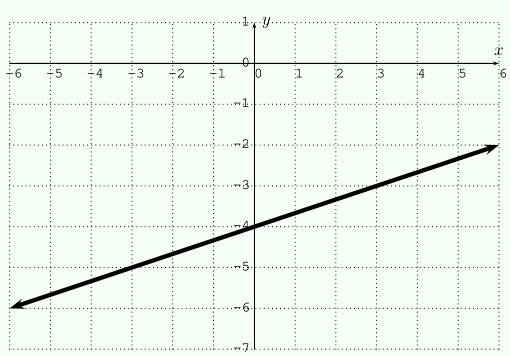
Solution. The y-intercept can be read off the graph giving us that $b = 2$. As for the slope, we use formula (3.2) and the two points on the line $P_1(0, 2)$ and $P_2(4, 0)$. We obtain

$$m = \frac{0-2}{4-0} = \frac{-2}{4} = -\frac{1}{2}.$$

Thus, the line has the slope-intercept form $y = -\frac{1}{2}x + 2$. □

Example 3.6

Find the equation of the line in slope-intercept form.



Solution.
The y-intercept is $b = -4$. To obtain the slope we can again use the y-intercept $P_1(0, -4)$. To use (3.2), we need another point P_2 on the line. We may pick any second point on the line, for example, $P_2(3, -3)$. With this, we obtain

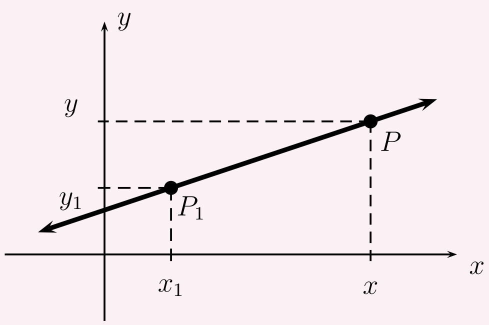
$$m = \frac{(-3) - (-4)}{3 - 0} = \frac{-3 + 4}{3} = \frac{1}{3}.$$

Thus, the line has the slope-intercept form $y = \frac{1}{3}x - 4$. □

There is another important way in which we can write the equation of a line.

Definition 3.7: Point-slope form of the line

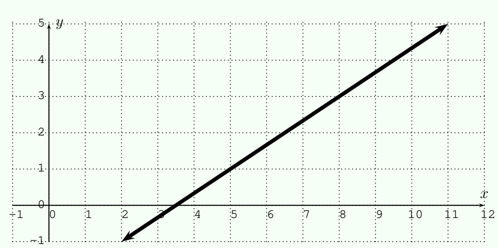
From Equation (3.2), we see that for a given slope m and a point $P_1(x_1, y_1)$ on the line, any other point (x, y) on the line satisfies $m = \frac{y-y_1}{x-x_1}$. Multiplying $(x - x_1)$ on both sides gives what is called the **point-slope form of the line**:



$$\boxed{y - y_1 = m \cdot (x - x_1)} \quad (3.3)$$

Example 3.8

Find the equation of the line in point-slope form (3.3).



Solution.
We need to identify one point (x_1, y_1) on the line together with the slope m of the line so that we can write the line in point-slope form: $y - y_1 = m(x - x_1)$. By direct inspection, we identify the two points $P_1(5, 1)$ and $P_2(8, 3)$ on the line, and with this we calculate the slope as:

$$m = \frac{3-1}{8-5} = \frac{2}{3}$$

Using the point $(5, 1)$ we write the line in point-slope form as follows:

$$y - 1 = \frac{2}{3}(x - 5)$$

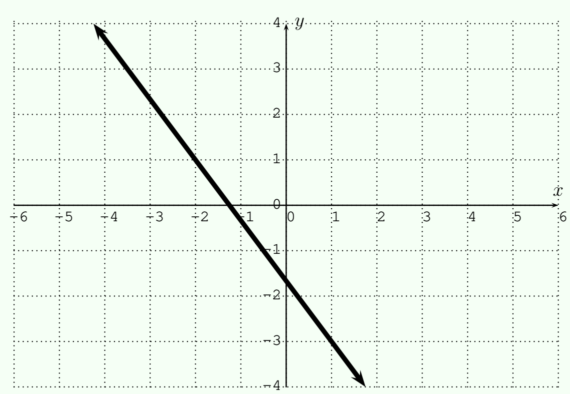
Note that our answer depends on the chosen point $(5, 1)$ on the line. Indeed, if we choose a different point on the line, such as $(8, 3)$, we obtain a different equation, (which nevertheless represents the same line):

$$y - 3 = \frac{2}{3}(x - 8)$$

Note that we do *not* need to solve this for y, since we are looking for an answer in point-slope form. □

Example 3.9

Find the equation of the line in point-slope form (3.3).



Solution.
We identify two points on the line, $P_1(1, -3)$ and $P_2(-2, 1)$. Therefore

the slope is $m = \frac{1-(-3)}{(-2)-1} = \frac{4}{-3} = -\frac{4}{3}$. Using, for example, the point $(1, -3)$, we write the line in point-slope form as follows:

$$y - (-3) = -\frac{4}{3}(x - 1)$$

Alternatively, we can also write this as $y + 3 = -\frac{4}{3}(x - 1)$. □

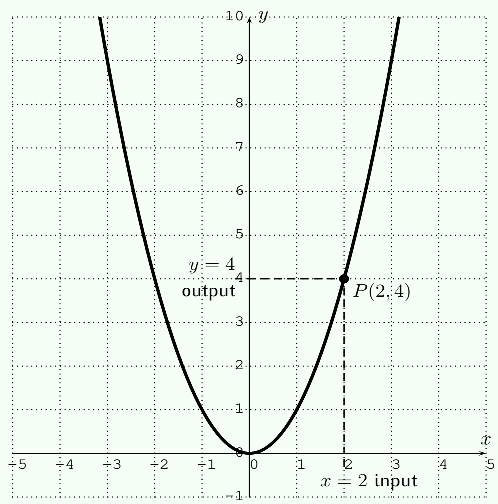
3.2 Functions given by graphs

Next, we study graphs more generally. Recall from the above examples that the graph of a function f is the set of all points (in the coordinate plane) of the form $(x, f(x))$, where x is in the domain of f. Here is another example that shows how we may obtain the graph of a function by computing sample points and plotting and connecting them in the plane.

Example 3.10

Let $y = x^2$ with domain $D = \mathbb{R}$ being the set of all real numbers. We can graph this after calculating a table as follows:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

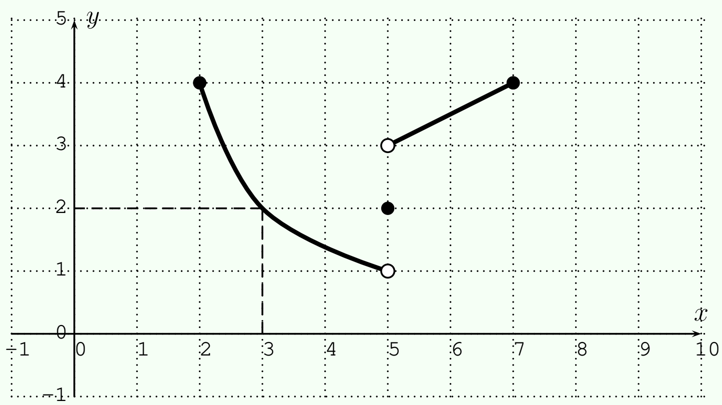


Many function values can be read from this graph. For example, for the input $x = 2$, we obtain the output $y = 4$. This corresponds to the point $P(2, 4)$ on the graph as depicted above.

In general, an input (placed on the x-axis) gets assigned to an output (placed on the y-axis) according to where the vertical line at x intersects with the given graph. This is used in the next example.

Example 3.11

Let f be the function given by the following graph.



Here, the dashed lines show that the input $x = 3$ gives an output of $y = 2$. Similarly, we can obtain other output values from the graph:

$$f(2) = 4, \quad f(3) = 2, \quad f(5) = 2, \quad f(7) = 4.$$

Note that, in the above graph, a closed point means that the point is part of the graph, whereas an open point means that it is not part of the graph.

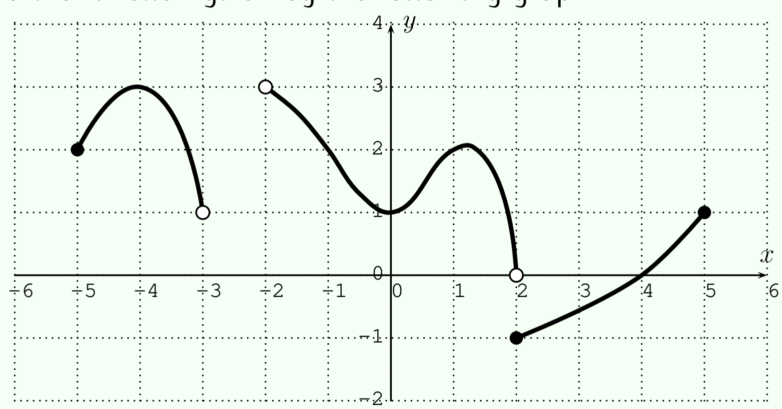
The domain is the set of all possible input values on the x-axis. Since we can take any number $2 \leq x \leq 7$ as an input, the domain is the interval $D = [2, 7]$. The range is the set of all possible output values on the y-axis. Since any number $1 < y \leq 4$ is obtained as an output, the range is $R = (1, 4]$. Note in particular that $y = 1$ is *not* an output, since $f(5) = 2$.

Note 3.12

In the above example we evaluated a function that was given by a graph. Looking at a drawn graph is by its nature an imprecise representation of the function. Indeed, it might be possible that, for example, there are hidden features of the graph that only become apparent after sufficiently zooming in on the graph. So, when studying the above example, we implicitly assumed that there are no hidden features that are not shown in the graph. An accurate evaluation would require more information regarding the function, such as, for example, a precise formula of the function.

Example 3.13

Let f be the function given by the following graph.



Here are some function values that can be read from the graph:

$$f(-5) = 2, \quad f(-4) = 3, \quad f(-3) \text{ and } f(-2) \text{ are undefined,}$$
$$f(-1) = 2, \quad f(0) = 1, \quad f(1) = 2, \quad f(2) = -1, \quad f(4) = 0, \quad f(5) = 1.$$

Note that the output value $f(3)$ is somewhere between -1 and 0, but we can only read off an approximation of $f(3)$ from the graph.
To find the domain of the function, we need to determine all possible x-coordinates (or in other words, we need to project the graph onto the x-axis). The possible x-coordinates are from the interval $[-5, -3)$ together with the intervals $(-2, 2)$ and $[2, 5]$. The last two intervals may be combined. We get the domain:

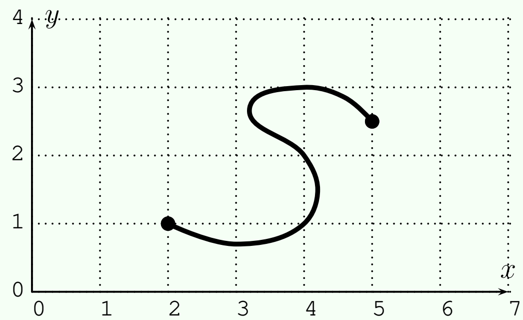
$$D = [-5, -3) \cup (-2, 5].$$

For the range, we look at all possible y-values. These are given by the intervals $(1, 3]$ and $(0, 3)$ and $[-1, 1]$. Combining these three intervals, we obtain the range

$$R = [-1, 3].$$

Example 3.14

Consider the following graph.

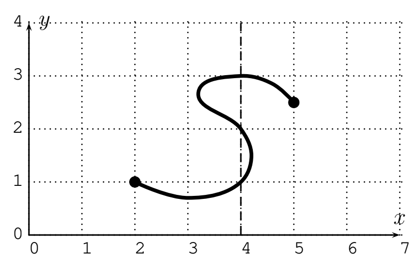


Consider the input $x = 4$. There are several outputs that we get for $x = 4$ from this graph:

$$f(4) = 1, \quad f(4) = 2, \quad f(4) = 3.$$

However, in a function, obtaining more than one output from one input is not allowed! Therefore, this graph is *not* the graph of a function!

The reason why the previous example is not a function is due to some input having more than one output: $f(4) = 1, f(4) = 2, f(4) = 3$.



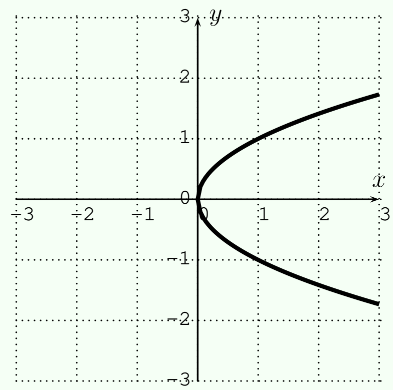
In other words, there is a vertical line ($x = 4$) which intersects the graph in more than one point. This observation is generalized in the following vertical line test.

Observation 3.15: Vertical Line Test

A graph is the graph of a function precisely when every vertical line intersects the graph at most once.

Example 3.16

Consider the graph of the equation $x = y^2$:



This does not pass the vertical line test, so y is not a function of x. However, x is a function of y since, if you consider y to be the input, each input has exactly one output (it passes the 'horizontal line' test).

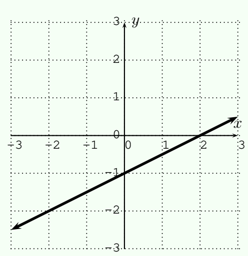
Example 3.17

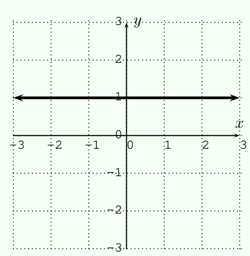
Which of the following equations constitute functions of the form $y = f(x)$?

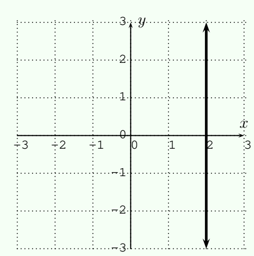
$$\text{a) } y = \frac{1}{2}x - 1 \qquad \text{b) } y = 1 \qquad \text{c) } x = 2$$

Solution.

We graph each of the three equations.

a) 

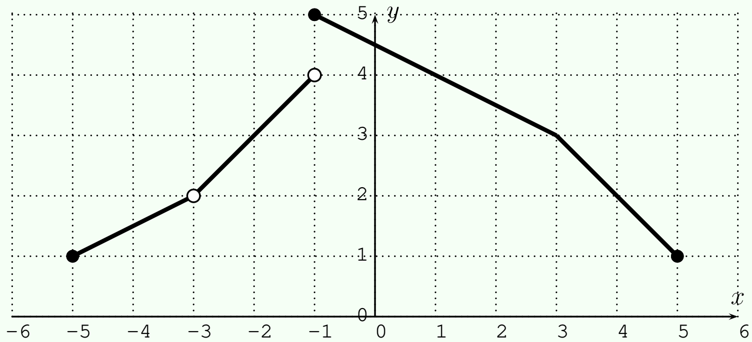
b) 

c) 

From the vertical line test, we see that (a) and (b) are graphs of functions, while (c) is a vertical line, which is not the graph of a function. □

Example 3.18

Let f be the function given by the following graph.



a) What is the domain of f?
b) What is the range of f?
c) For which x is $f(x) = 3$?
d) For which x is $f(x) = 2$?
e) For which x is $f(x) > 2$?
f) For which x is $f(x) \leq 4$?
g) Find $f(1)$ and $f(4)$.
h) Find $f(1) + f(4)$.
i) Find $f(1) + 4$.
j) Find $f(1 + 4)$.

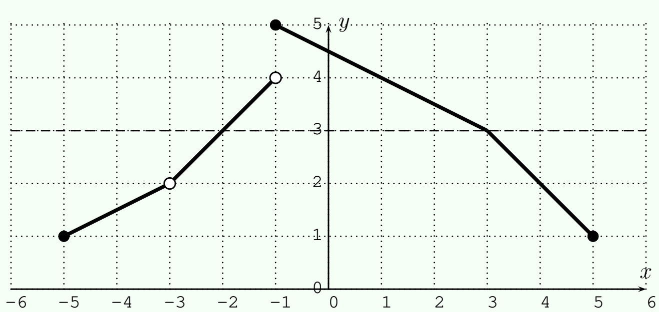
Solution.

Most of the answers can be read immediately from the graph.

a) For the domain, we project the graph to the x-axis. The domain consists of all numbers from -5 to 5 without -3, that is $D = [-5, -3) \cup (-3, 5]$.

b) For the range, we project the graph to the y-axis. The domain consists of all numbers from 1 to 5, that is $R = [1, 5]$.

c) To find x with $f(x) = 3$ we look at the horizontal line at $y = 3$:



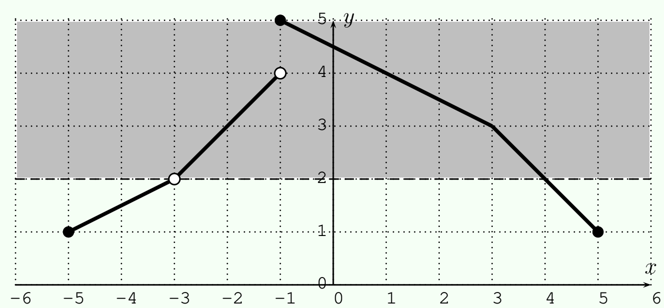
We see that there are two numbers x with $f(x) = 3$:

$$f(-2) = 3, \qquad f(3) = 3.$$

Therefore, the answer is $x = -2$ or $x = 3$.

d) Looking at the horizontal line $y = 2$, we see that there is only one x with $f(x) = 2$; namely $f(4) = 2$. Note that $x = -3$ does not solve the problem, since $f(-3)$ is undefined. The answer is $x = 4$.

e) To find x with $f(x) > 2$, the graph has to lie above the line $y = 2$.



We see that the answer is those numbers x greater than -3 and less than 4. The answer is therefore the interval $(-3, 4)$.

f) For $f(x) \leq 4$, we obtain all numbers x from the domain that are less than -1 or greater than or equal to 1. The answer is therefore,

$$[-5, -3) \cup (-3, -1) \cup [1, 5].$$

Note that -3 is not part of the answer, since $f(-3)$ is undefined.

g) $f(1) = 4$, and $f(4) = 2$

h) $f(1) + f(4) = 4 + 2 = 6$

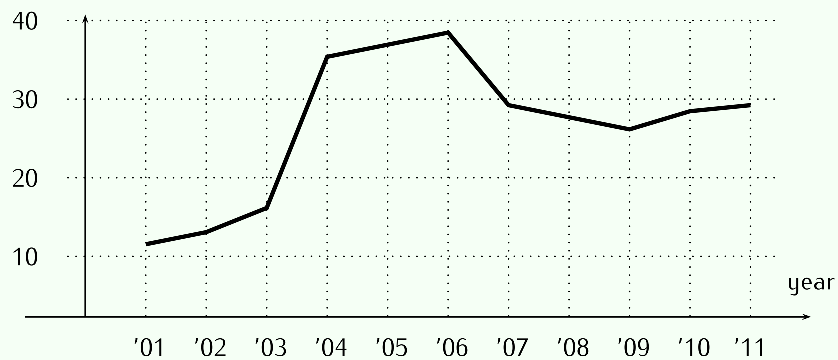
i) $f(1) + 4 = 4 + 4 = 8$

j) $f(1 + 4) = f(5) = 1$

□

Example 3.19

The following graph shows the population size in a small city from the years 2001–2011 in thousands of people.



a) What was the population size in the years 2004 and 2009?
b) In what years did the city have the most population?
c) In what year did the population grow the fastest?
d) In what year did the population decline the fastest?

Solution.
The population size in the year 2004 was approximately $36,000$. In the year 2009, it was approximately $26,000$. The largest population was in the year 2006, where the graph has its maximum. The fastest growth in the population was between the years 2003 and 2004. That is when the graph has the largest slope. Finally, the fastest decline happened from the years 2006–2007. □

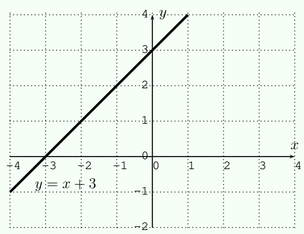
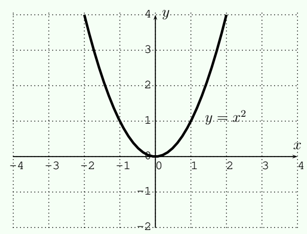
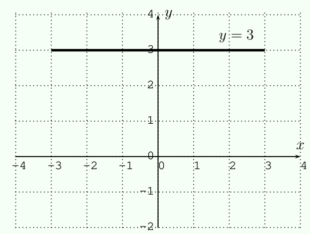
Example 3.20

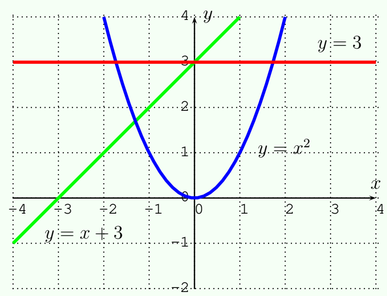
Graph the piecewise defined function described by the following formula:

$$f(x) = \begin{cases} x+3 & \text{, for } -3 \leq x < -1 \\ x^2 & \text{, for } -1 < x < 1 \\ 3 & \text{, for } 2 < x \leq 3 \end{cases}$$

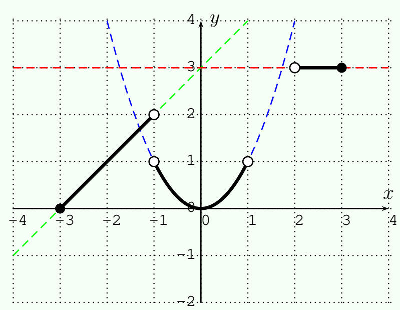
Solution.
We have to graph all three functions $y = x + 3$, $y = x^2$, and $y = 3$, and then restrict them to their respective domain. Graphing the three functions, we obtain the following tables and associated graphs, which

we draw in one x-y plane:

$y = x + 3$	$y = x^2$	$y = 3$
		



However, we need to cut off the functions according to their specific input domain that is given by the original function.

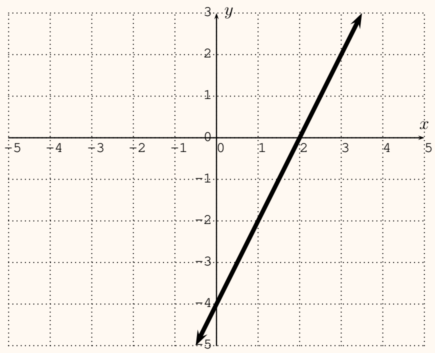


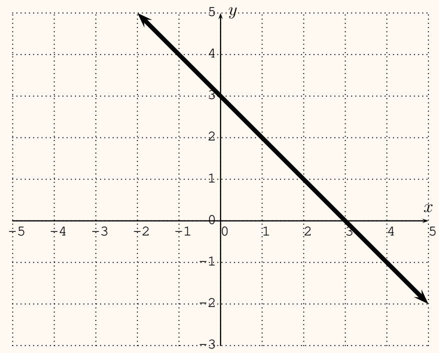
Note that the open and closed circles at the endpoints of each branch correspond to the "$<$" and "$\leq$" rules in the original description of the function. $\square$

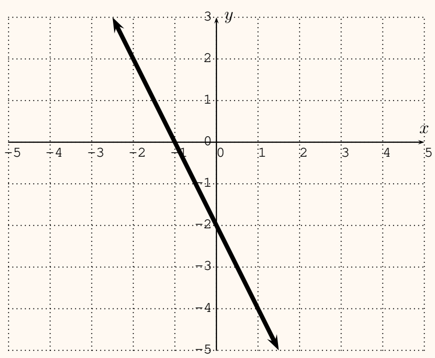
3.3 Exercises

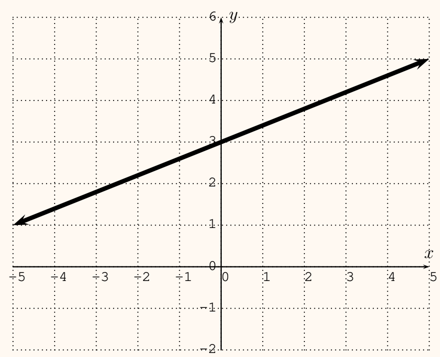
Exercise 3.1

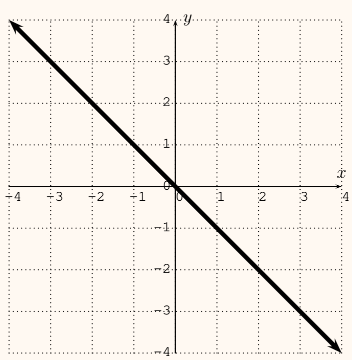
Find the slope and y-intercept of the line with the given data. Using the slope and y-intercept, write the equation of the line in slope-intercept form.

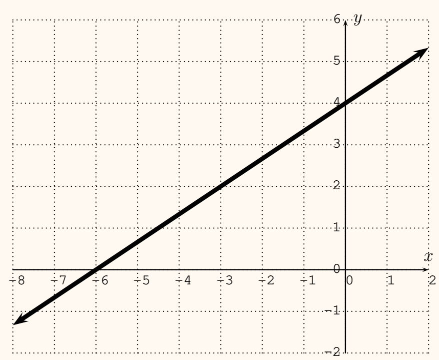
a) 

b) 

c) 

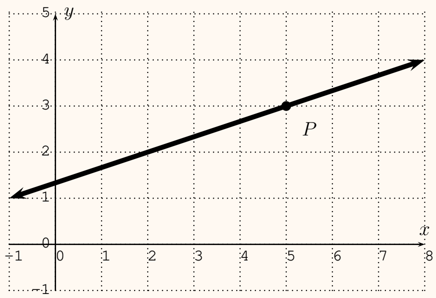
d) 

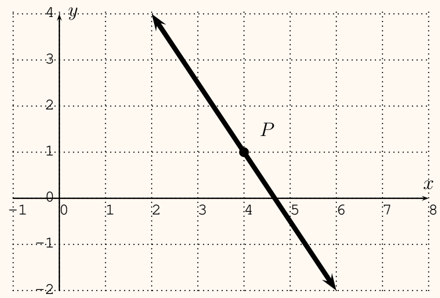
e) 

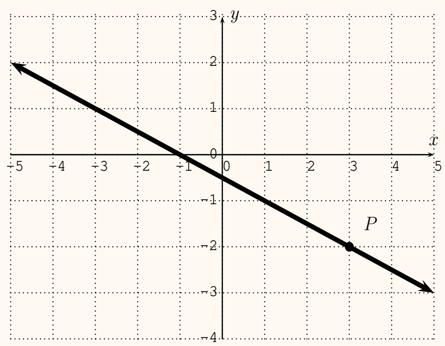
f) 

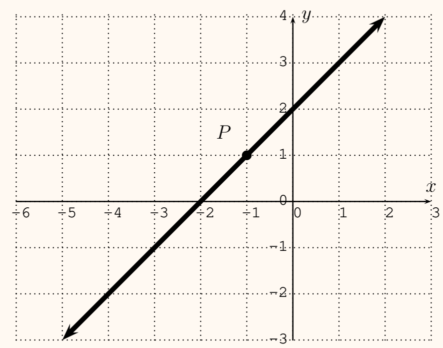
Exercise 3.2

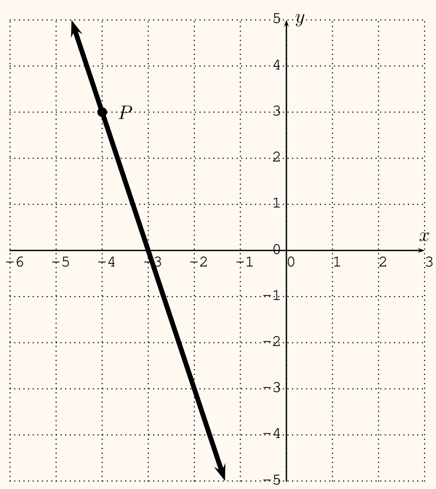
Find the equation of the line in point-slope form (3.3) using the indicated point P.

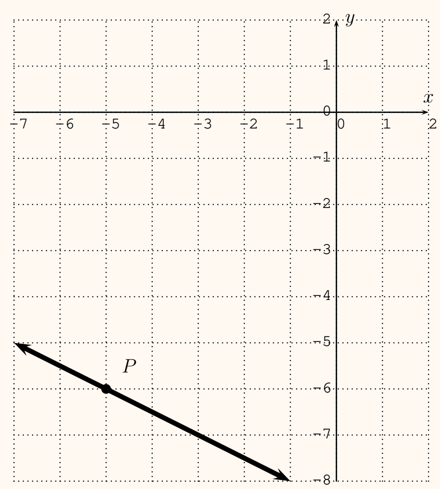
a) 

b) 

c) 

d) 

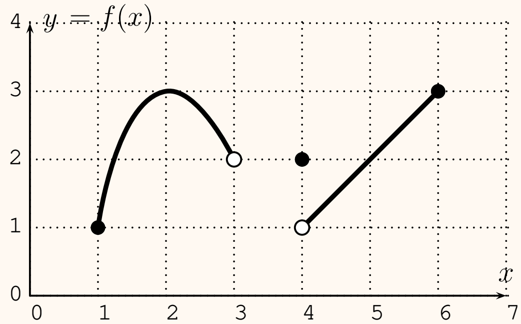
e) 

f) 

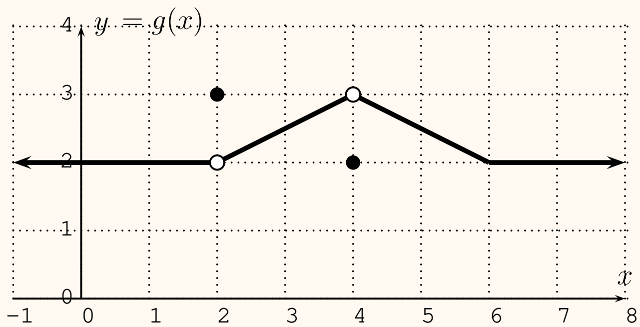
Exercise 3.3

Below are three graphs for the functions f, g, and h.

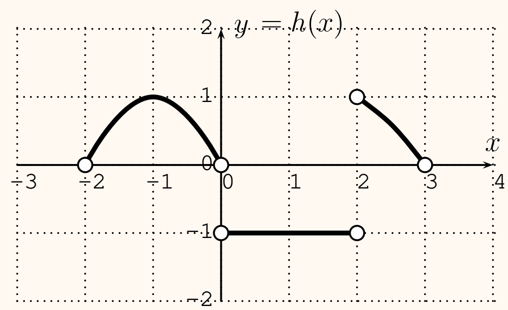
function f :



function g :



function h :



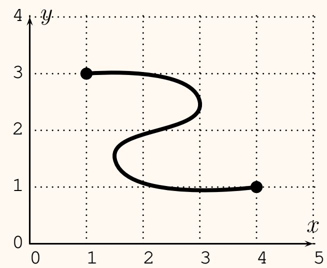
a) Find the domain and range of f.
b) Find the domain and range of g.
c) Find the domain and range of h.

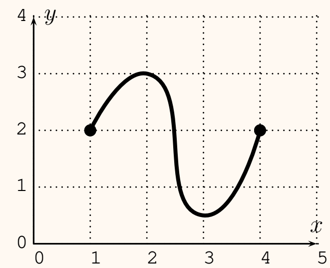
Find the following function values:

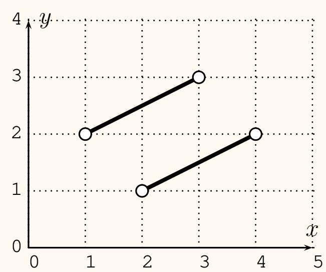
d) $f(1)$	e) $f(2)$	f) $f(3)$	g) $f(4)$	h) $f(5)$	i) $f(6)$	j) $f(7)$
k) $g(0)$	l) $g(1)$	m) $g(2)$	n) $g(3)$	o) $g(4)$	p) $g(6)$	q) $g(13.2)$
r) $h(-2)$	s) $h(-1)$	t) $h(0)$	u) $h(1)$	v) $h(2)$	w) $h(3)$	x) $h(\sqrt{2})$

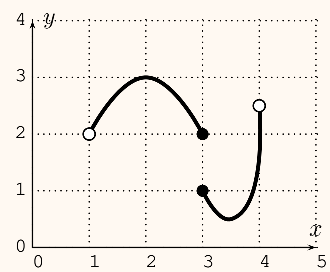
Exercise 3.4

Use the vertical line test to determine which of the following graphs are the graphs of functions.

a) 

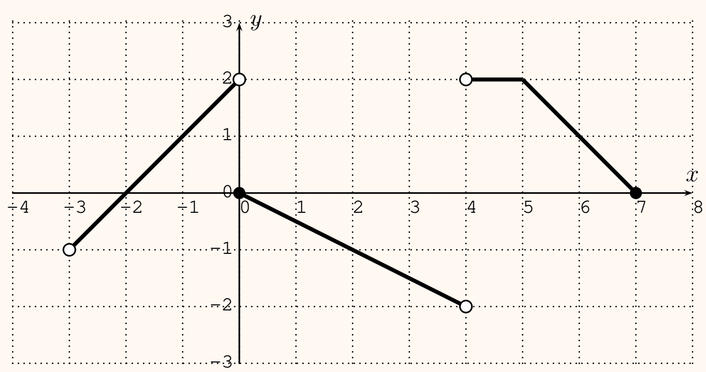
b) 

c) 

d) 

Exercise 3.5

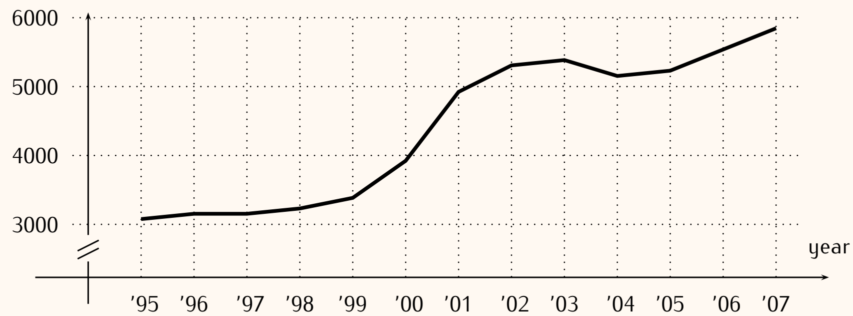
Let f be the function given by the following graph.



a) What is the domain of f?
b) What is the range of f?
c) For which x is $f(x) = 0$?
d) For which x is $f(x) = 2$?
e) For which x is $f(x) \leq 1$?
f) For which x is $f(x) > 0$?
g) Find $f(2)$ and $f(5)$.
h) Find $f(2) + f(5)$.
i) Find $f(2) + 5$.
j) Find $f(2 + 5)$.

Exercise 3.6

The graph below displays the number of students admitted to a college during the years 1995 to 2007.



a) How many students were admitted in the year 2000?
b) In what years were the most students admitted?
c) In what years did the number of admitted students rise fastest?
d) In what year(s) did the number of admitted students decline?

Exercise 3.7

Consider the function described by the following formula:

$$f(x) = \begin{cases} x^2+1 & \text{, for } -2 < x \leq 0 \\ x-1 & \text{, for } 0 < x \leq 2 \\ -x+4 & \text{, for } 2 < x \leq 5 \end{cases}$$

What is the domain of the function f? Graph the function f.

Chapter 4

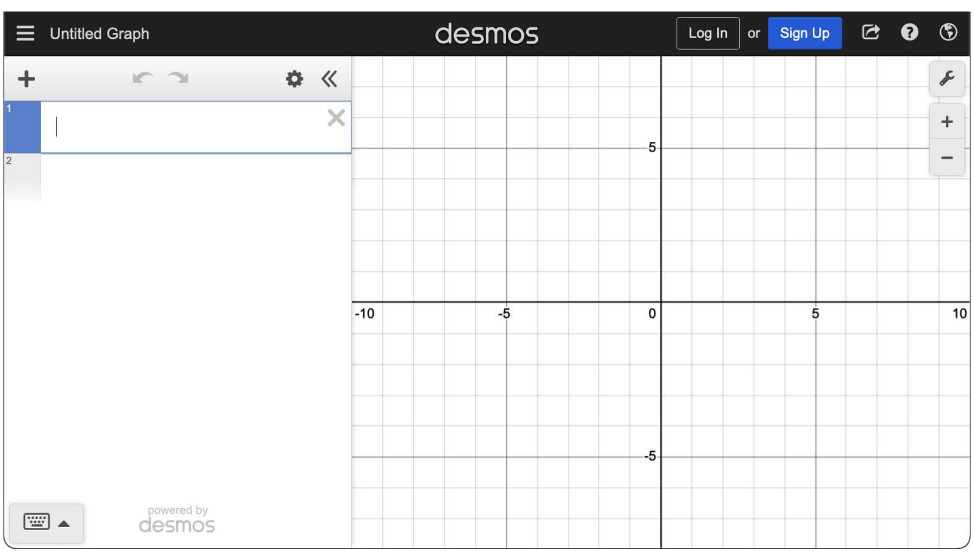
Basic functions and transformations

We now give an introduction to the Desmos graphing calculator that can be used to graph functions given by formulas. We also graph a list of basic functions and observe how these graphs change when adding or multiplying constants to their inputs or outputs.

4.1 Basics of the Desmos graphing calculator

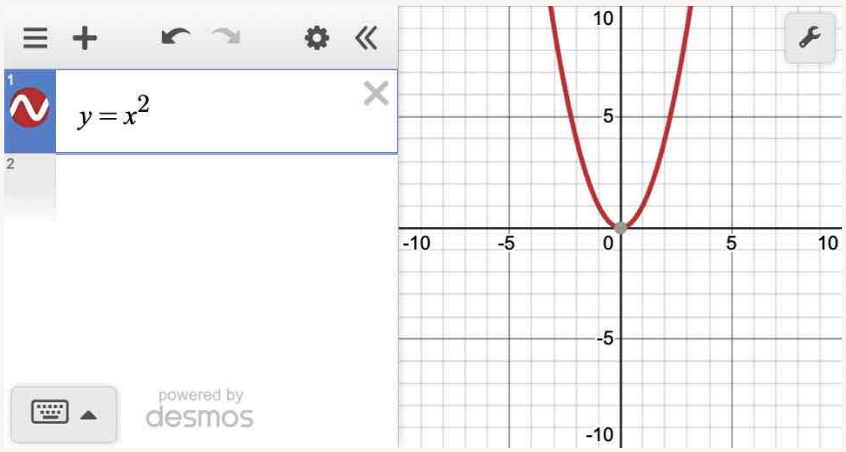
To get started, visit the Desmos graphing calculator at the URL

https://www.desmos.com/calculator



Note 4.1

Graphing a function is straightforward in Desmos, say $y = x^2$. Simply type $y = x^2$ into the input field on the top left.



Moreover, we can easily locate *local maxima* and *local minima* of a function (the peaks and valleys of its graph), as well as its x- and y-intercepts by simply clicking on the graph.
The x-intercepts are also commonly called **zeros** or **roots** of the function. In other words, a root or a zero of a function f is a number c for which $f(c) = 0$.

We demonstrate this in the next example.

Example 4.2

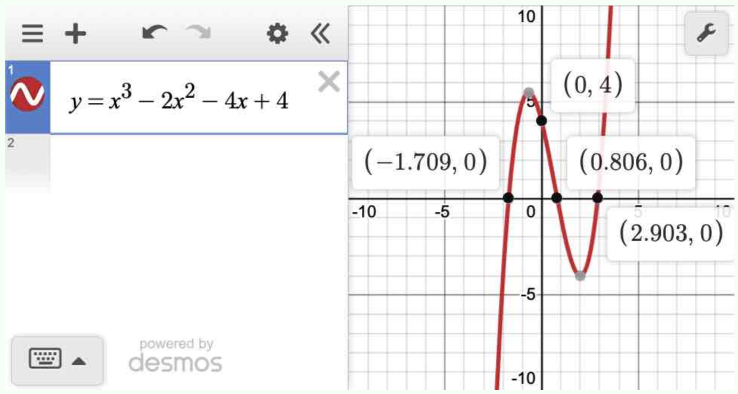
Graph the function $y = x^3 - 2x^2 - 4x + 4$.

a) Approximate the x-intercepts and the y-intercept of the function.

b) Approximate the (local) maximum and minimum. A (local) maximum or minimum is also called a **(local) extremum**.

Solution.

a) Enter the function in Desmos and click on the x-intercepts (points where the graph intersects with the x-axis) and the y-intercept (the point where the graph intersects with the y-axis).

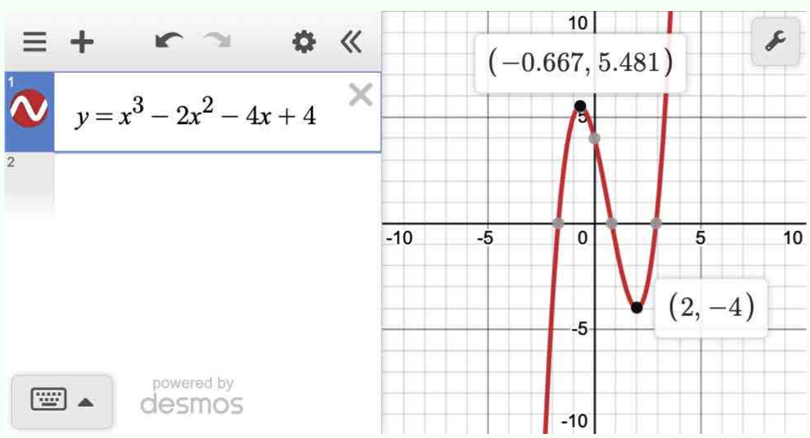


There are three x-intercepts and one y-intercept:

$$x\text{-intercepts: } (x,y)\approx(-1.709,0), \quad (x,y)\approx(0.806,0)$$
$$(x,y)\approx(2.903,0)$$
$$y\text{-intercept: } (x,y)=(0,4)$$

Note that the y-intercept is precisely at $(0,4)$, whereas the x-intercepts are only approximated.

b) Similarly, we obtain the local maximum and the local minimum.



$$\text{local max: } (x,y)\approx(-0.667,5.481), \qquad \text{local min: } (x,y)=(2,-4)$$

One can check (using methods from calculus) that the local minimum is precisely at $(2,-4)$, whereas the local maximum is only approximated.

□

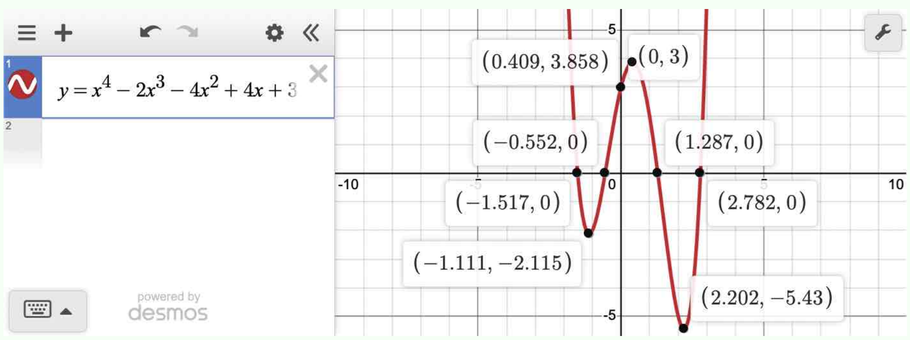
Example 4.3

For the two functions below, find all intercepts and all extrema. Approximate your answer to the nearest thousandth.

a) $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 3$ b) $f(x) = x^3 - 9x^2 - x$

Solution.

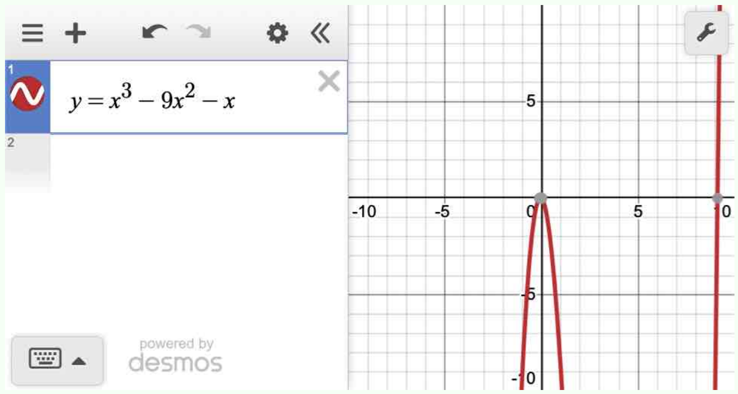
a) Graphing the function in Desmos, we can read off the coordinates of the wanted points (the intercepts and extrema) by clicking on them.



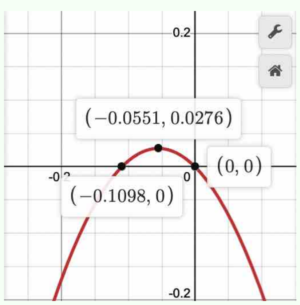
$$
\begin{aligned}
x\text{-intercepts:}\ & (x,y) \approx (-1.517, 0), \quad (x,y) \approx (-0.552, 0), \\
& (x,y) \approx (1.287, 0), \quad (x,y) \approx (2.782, 0) \\
y\text{-intercept:}\ & (x,y) = (0, 3) \\
\text{local maximum:}\ & (x,y) \approx (0.409, 3.858) \\
\text{local mininima:}\ & (x,y) \approx (-1.111, -2.115), \quad (x,y) \approx (2.202, -5.430)
\end{aligned}
$$

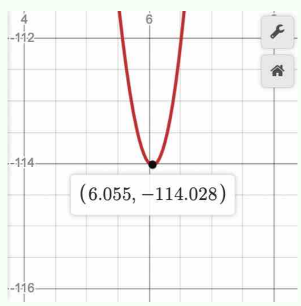
Here, Desmos already rounded coordinates to the nearest thousandth. For example, the maximum with one more digit is $(x,y) \approx (0.4088, 3.8580)$, which rounds to $(0.409, 3.858)$.

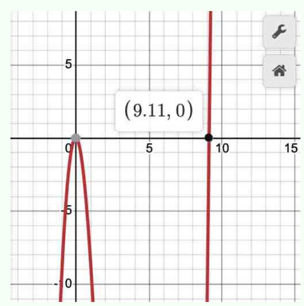
b) Graphing $f(x) = x^3 - 9x^2 - x$ shows that we don't have a complete view of all the points of interest in the initial viewing window.



Zooming in and out (by using the plus and minus symbols on the graph or by scrolling on the screen), we can get a closer view of the wanted points.







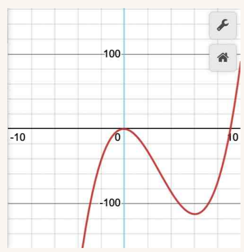
Note that by zooming in, Desmos may display more than three digits after the decimal point. Rounding to the nearest thousandth gives the following answers.

$$\begin{aligned}
x\text{-intercepts: } & (x,y)\approx(-0.110,0), \quad (x,y)=(0,0),\\
& (x,y)\approx(9.110,0)\\
y\text{-intercept: } & (x,y)=(0,0)\\
\text{local maximum: } & (x,y)\approx(-0.055,0.028)\\
\text{local mininimum: } & (x,y)\approx(6.055,-114.028)
\end{aligned}$$

□

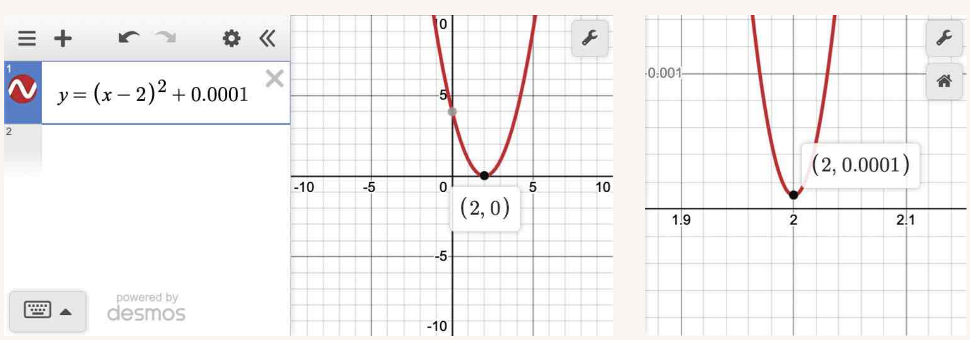
Note 4.4: Zooming

Besides zooming in and out, the display window can also be set manually via the Graph Settings menu (click on the wrench symbol ). The home button  resets the window to a size in which the x is approximately between -10 and 10, and with a matching scale for y.
There is also a possibility to rescale each axis individually. To this end, hover the pointer over the axis that needs to be rescaled and press and hold the shift key. The axis will appear in blue, and can then be rescaled (click and move the pointer in the wanted direction). Below is the rescaled graph for $y = x^3 - 9x^2 - x$.



Note 4.5

Desmos only approximates its answers, such as intercepts, maxima, and minima. It is *our* task to correctly interpret and confirm any answers inferred from Desmos.
For example, graphing $y = (x-2)^2 + 0.0001$ appears to show a root at $(2, 0)$. Nevertheless, a closer look reveals that this function does *not* have a root at $(2, 0)$.



We next show how to graph piecewise defined functions with Desmos.

Example 4.6

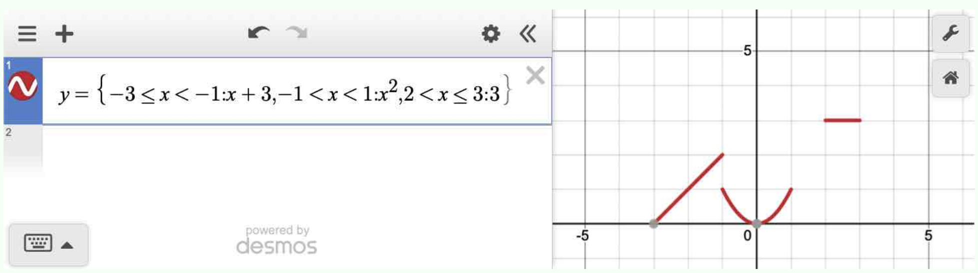
Graph the piecewise defined function from Example 3.20 with Desmos.

$$y = \begin{cases} x+3 & \text{, for } \quad -3 \le x < -1 \\ x^2 & \text{, for } \quad -1 < x < 1 \\ 3 & \text{, for } \quad 2 < x \le 3 \end{cases}$$

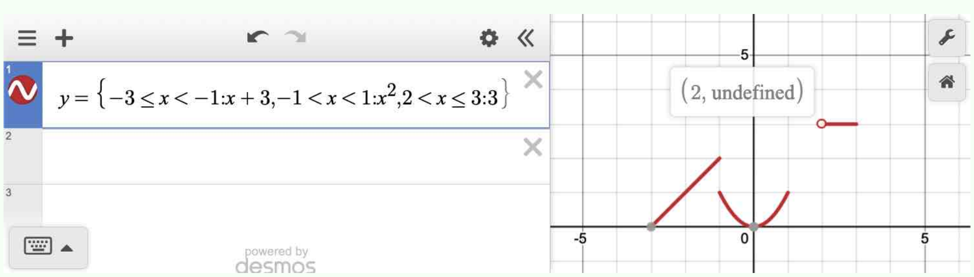
Solution.

A piecewise defined function is entered in Desmos with a set bracket $\{\}$, separating each branch with a comma. Each branch is entered as "condition:function value"; for example, the top branch in our example is entered as $-3 \le x < -1 : x+3$. Combining the three branches, we obtain:

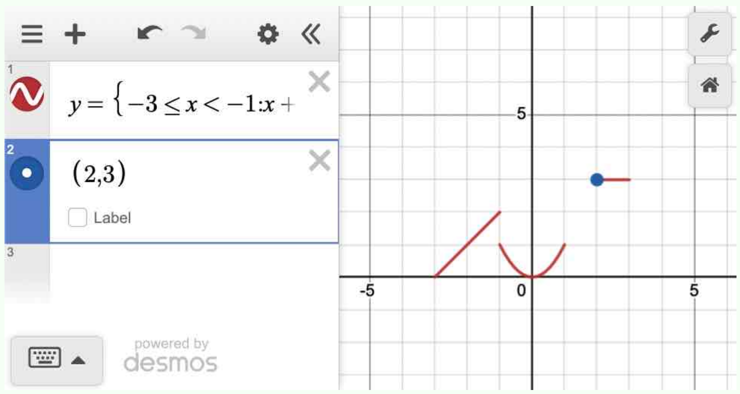
$$y = \{ \quad -3 \le x < -1 : x+3 \quad , \quad -1 < x < 1 : x^2 \quad , \quad 2 < x \le 3 : 3 \quad \}$$



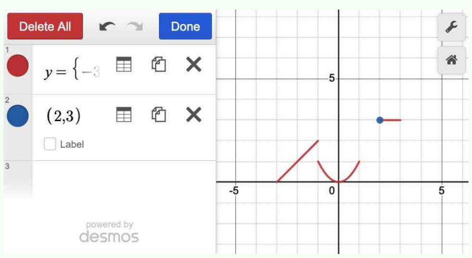
Although there are no open or closed circles at the endpoints of the line segments, Desmos does interpret these endpoints correctly. This can be seen by clicking on the endpoint of a branch.

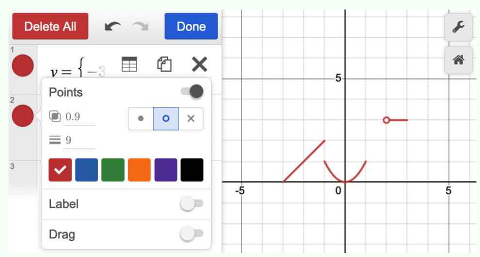


We can add the missing endpoints manually by entering the coordinates for each point.

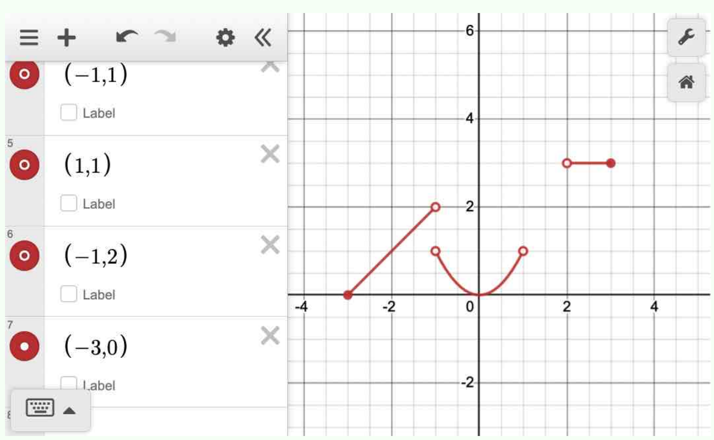


To adjust properties of the point such as its color or it being depicted as open or closed, we click the Edit List button on top of the panel, then click on the big (in this case blue) circle to the left of the entered point. We can then set the style and color of the point.





Performing this for each endpoint, we obtain the following graph.



□

Our last example in this section shows how to easily compute function values and how to create tables in Desmos.

Example 4.7

Consider the functions

$$f(x) = x^3 - 4x^2 + 5$$

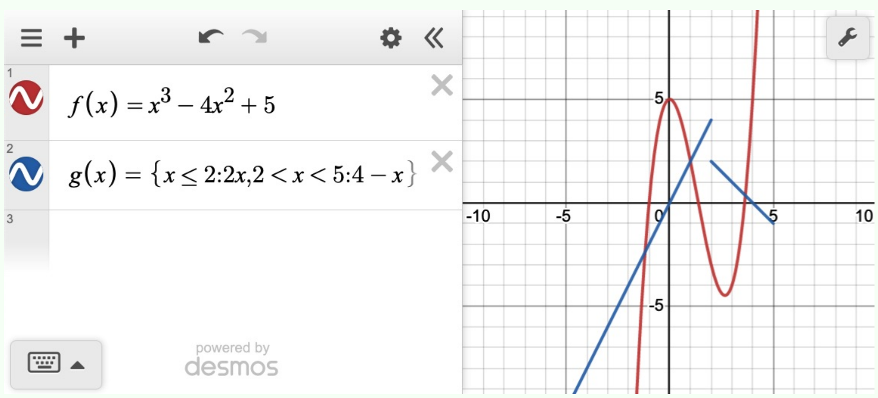
$$\text{and} \quad g(x) = \begin{cases} 2x & \text{, for } \quad x \leq 2 \\ 4 - x & \text{, for } \quad 2 < x < 5 \end{cases}$$

Compute the function values

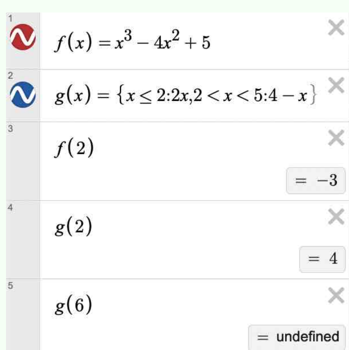
$$f(2), \qquad f(4), \qquad g(1), \qquad g(2), \qquad g(6).$$

Solution.

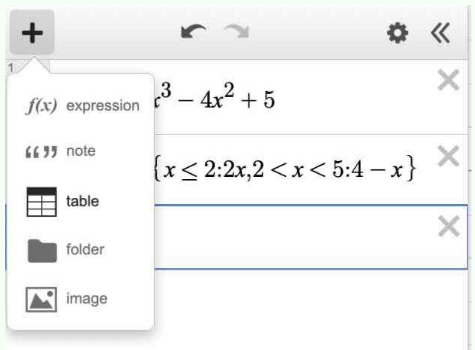
First, graph both functions f and g.



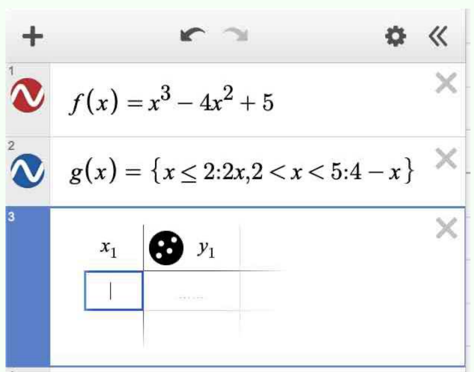
A direct way of computing function values is given by simply entering the wanted expression. Note that undefined function values (such as $g(6)$) are indeed stated as undefined.

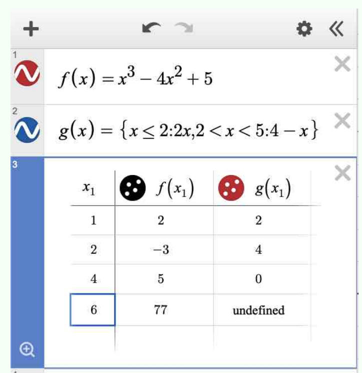


Another way to calculate function values comes from generating a table. Press the "Add Item" button + on top, and click on "table".



Modify the table by replacing y_1 with $f(x_1)$. We can also compute multiple function values, such as $f(x_1)$ and $g(x_1)$, by putting $g(x_1)$ next to $f(x_1)$. Below x_1, we enter the desired inputs.





Thus, $f(2) = -3$, $f(4) = 5$, $g(1) = 2$, $g(2) = 4$, and $g(6)$ is undefined. □

4.2 *Optional section:* Exploring Desmos further

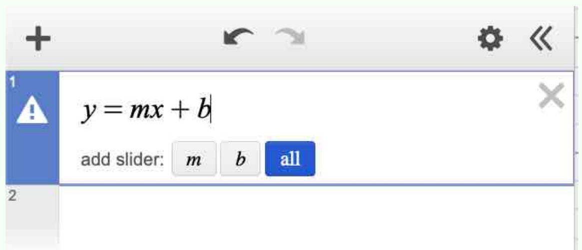
We now explore sliders in Desmos and we revisit the domain and range of a function, as well as the vertical line test. We also give an example of finding intersection points of two graphs.

Example 4.8

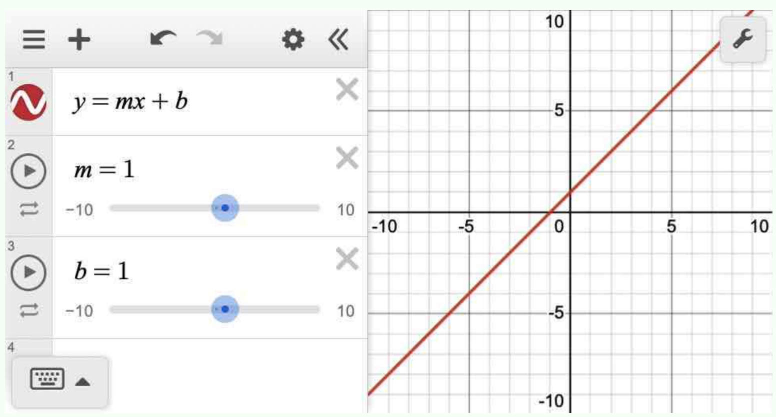
Explore the equation of a line $y = m \cdot x + b$ for various values of m and b using sliders.

Solution.

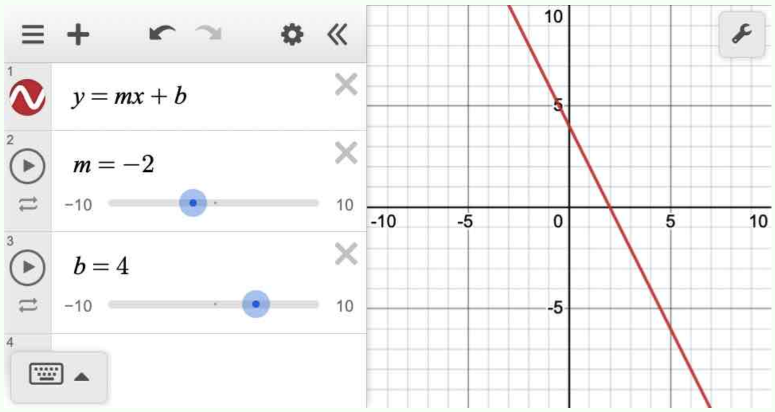
We enter $y = mx + b$ into Desmos.



We want m and b to be interpreted as constants, but these constants can be adjusted. This is precisely what sliders provide in Desmos. We therefore add the sliders m and b.



Changing the values for m and b, we instantly see the effect on its graph.



□

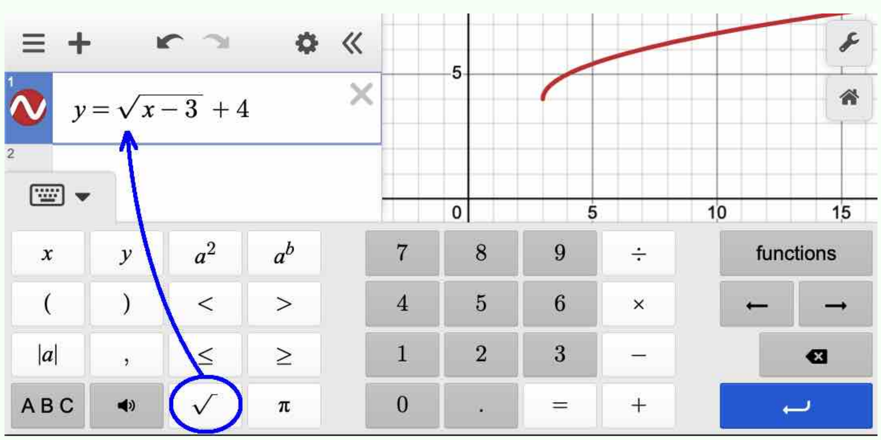
Example 4.9

Find the (approximate) domain and range of the following functions.

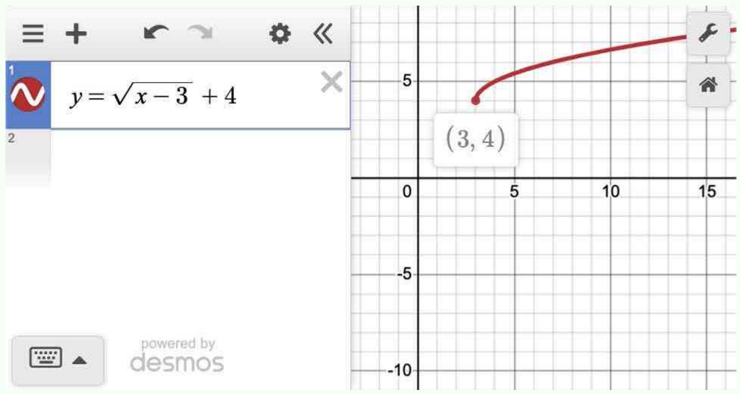
a) $f(x) = \sqrt{x-3} + 4$ b) $f(x) = x^2 + 8x - 7$

Solution.

a) Enter $y = \sqrt{x-3}+4$ into Desmos. Note that the square-root symbol can be entered by typing the letters $sqrt$, or alternatively, first show the keyboard by clicking  and then clicking the square-root symbol.

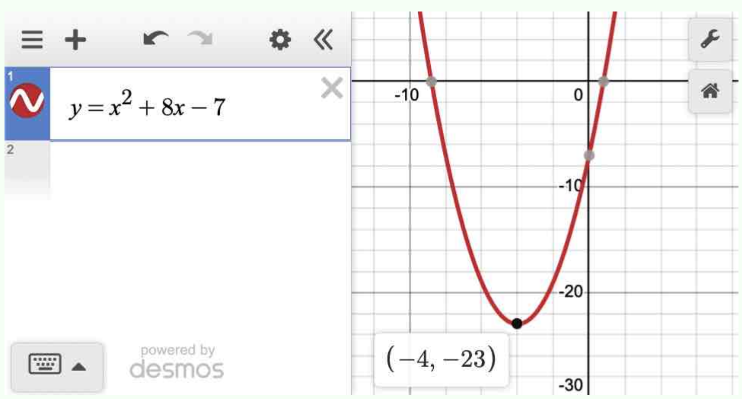


We see that the function starts from some vertex, and then increases as x increases. To find the vertex, click on it.



Thus, the domain and range appear to be $D = [3, \infty)$ and $R = [4, \infty)$.

b) Similarly, we can graph the function $y = x^2 + 8x - 7$ and read off its domain and range. Note that there is no restriction on the inputs x, so that the domain is all real numbers.



From the graph, the domain and range are therefore $D = \mathbb{R}$ and $R = [-23, \infty)$.

□

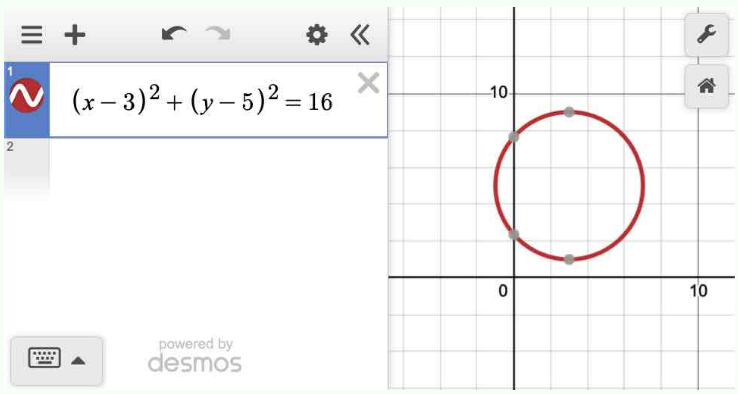
Example 4.10

Graph each of the following equations. Determine whether the graph is the graph of a function or not.

a) $(x-3)^2 + (y-5)^2 = 16$ b) $3x^2 + y^3 + 5xy = 7$

Solution.

a) Graphing $(x-3)^2 + (y-5)^2 = 16$ shows that we obtain a circle.

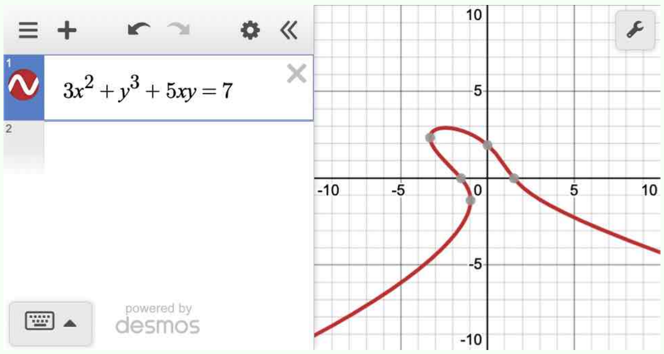


To see if this is the graph of a function, we can use the vertical line test (from Observation 3.15). The y-axis (which is the vertical line at $x = 0$) intersects the circle at two points. This shows that the circle is not the graph of a function. Indeed, if we solve the equation for y, we get:

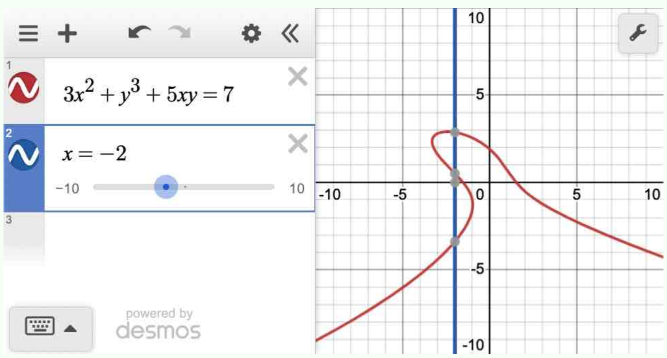
$$\begin{aligned}(x-3)^2 + (y-5)^2 = 16 \quad &\Longrightarrow \quad (y-5)^2 = 16 - (x-3)^2 \\ &\Longrightarrow \quad y - 5 = \pm\sqrt{16-(x-3)^2} \\ &\Longrightarrow \quad y = 5 \pm \sqrt{16-(x-3)^2}\end{aligned}$$

This shows that the circle is made up of two parts, the upper half-circle $y = 5 + \sqrt{16-(x-3)^2}$ and the lower half-circle $y = 5 - \sqrt{16-(x-3)^2}$, each of which *is* the graph of a function.

b) We can easily enter more complicated equations into Desmos.



The vertical line at $x = -2$ intersects the graph at three points, which shows that it is not the graph of a function.



□

Note 4.11: Equation of the circle

We recall that the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

always forms a circle in the plane. Indeed, this equation describes a circle with center $C(h, k)$ and radius r.
This can easily be explored in Desmos using sliders; see Exercise 4.6 below.

In the last example of this section we solve an equation by determining the intersection of two graphs.

Example 4.12

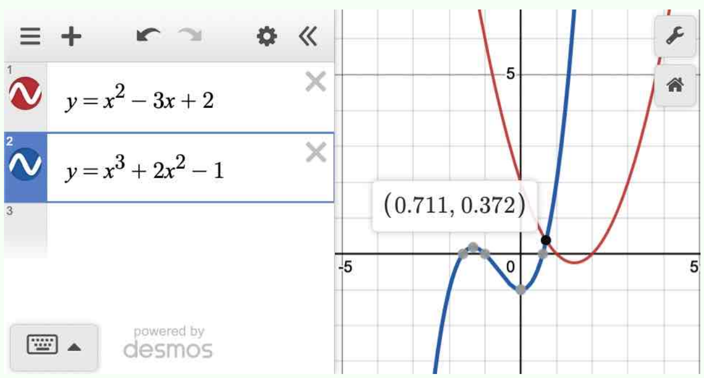
Solve the equation

$$x^2 - 3x + 2 = x^3 + 2x^2 - 1$$

Approximate your answer to the nearest thousandth.

Solution.

We can solve the equation by graphing the left-hand side $y = x^2 - 3x + 2$ and the right-hand side $y = x^3 + 2x^2 - 1$, and by determining those values of x where both sides are equal. This occurs precisely at the intersection of the two graphs. Graphing both functions and clicking on the intersection, we obtain:



The intersection is at $(x, y) \approx (0.711, 0.372)$. Therefore, the two sides of the equation are equal for $x \approx 0.711$ (in which case both the left-hand side and right-hand side are approximately 0.372). Therefore, $x \approx 0.711$ is the approximate solution. □

4.3 Graphs of basic functions and transformations

It will be useful to study the shape of graphs of some basic functions, which may then be taken as building blocks for more advanced and complicated functions.

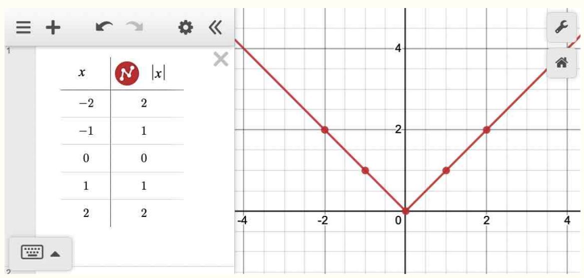
Observation 4.13: Basic function

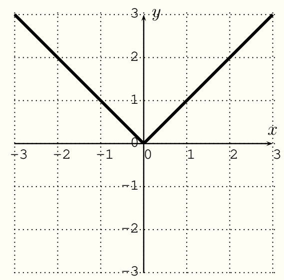
We start by examining the following functions, which we will sometimes refer to as *basic functions*:

$$y = |x|, \quad y = x^2, \quad y = x^3, \quad y = \sqrt{x}, \quad y = \frac{1}{x}$$

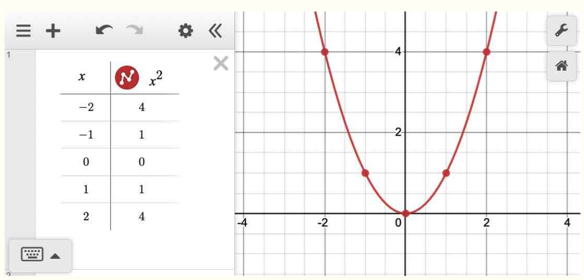
We can either graph these functions by hand by calculating a table, or by using the graphing calculator.

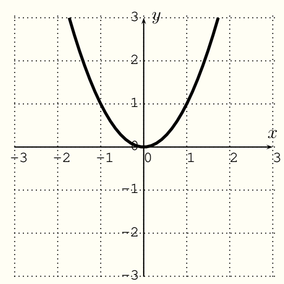
- We begin with the absolute value function $y = |x|$. The domain of $y = |x|$ is all real numbers, $D = \mathbb{R}$.



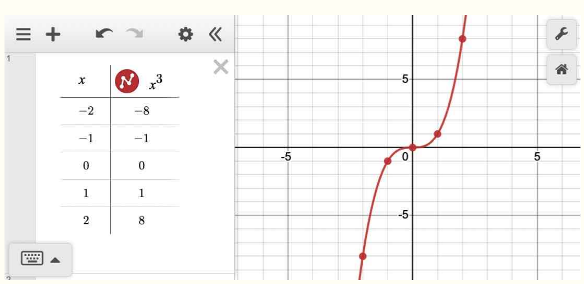


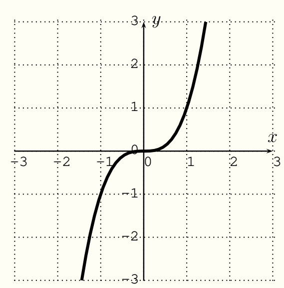
- Similarly, we obtain the graph for $y = x^2$, which is a parabola. The domain of the function $y = x^2$ is $D = \mathbb{R}$.



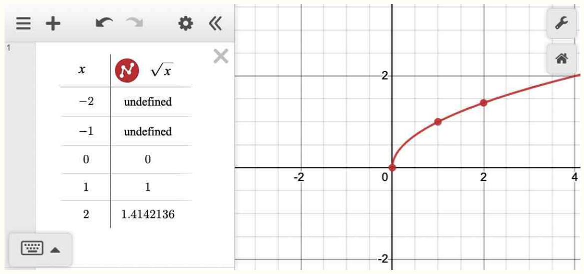


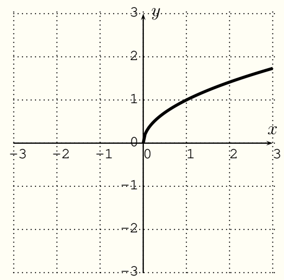
- Here is the graph for $y = x^3$. The domain is $D = \mathbb{R}$.



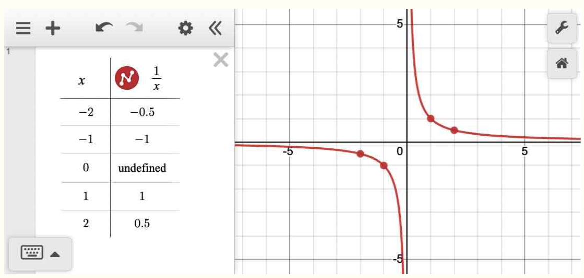


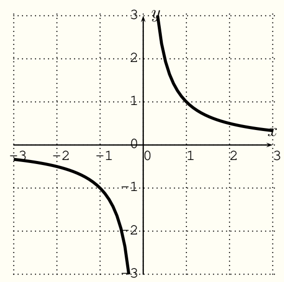
- Next we graph $y = \sqrt{x}$. The domain is $D = [0, \infty)$.



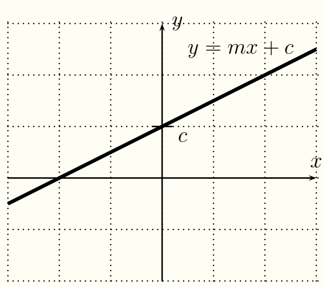


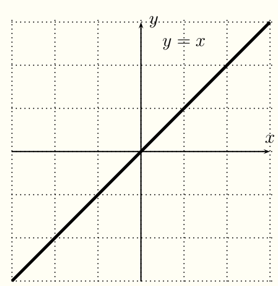
- Finally, here is the graph for $y = \frac{1}{x}$. The domain is $D = \mathbb{R} - \{0\}$.





These graphs together with the line $y = mx + b$ studied in Section 3.1 are our basic building blocks for more complicated graphs in the next sections. Note in particular, that the graph of $y = x$ is the diagonal line.

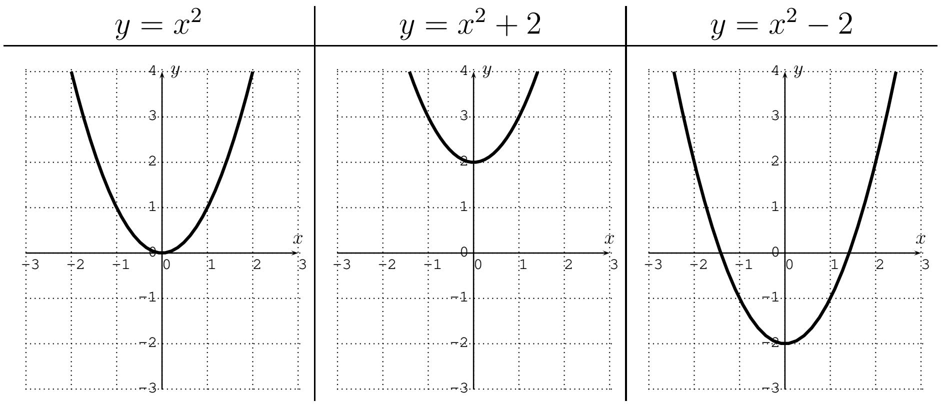




For a given function (such as one of the basic functions above), we now study how the graph of the function changes when performing elementary operations, such as adding, subtracting, or multiplying a constant number to the input or output. We will study the behavior in five specific transformations.

Adding or subtracting a constant to the output

Consider the following graphs:



We see that the function $y = x^2$ is shifted up by 2 units, respectively down by 2 units. In general, we have:

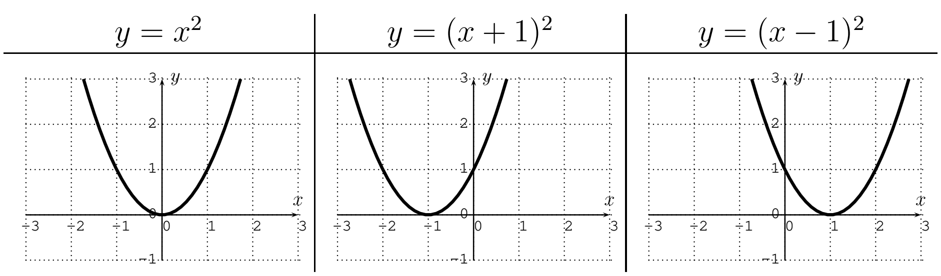
Observation 4.14: Adding or subtracting a constant to the output

Consider the graph of a function $y = f(x)$. Then, the graph of $y = f(x) + c$ is that of $y = f(x)$ shifted up or down by c.

- If c is positive, the graph is shifted up; if c is negative, the graph is shifted down.

Adding or subtracting a constant to the input

Next, we consider the transformation of $y = x^2$ given by adding or subtracting a constant to the input x.



Now we see that the function is shifted to the left or right. Note that $y = (x+1)^2$ shifts the function to the left, which can be seen to be correct, since the input $x = -1$ gives the output $y = ((-1)+1)^2 = 0^2 = 0$.

Observation 4.15: Adding or subtracting a constant to the input

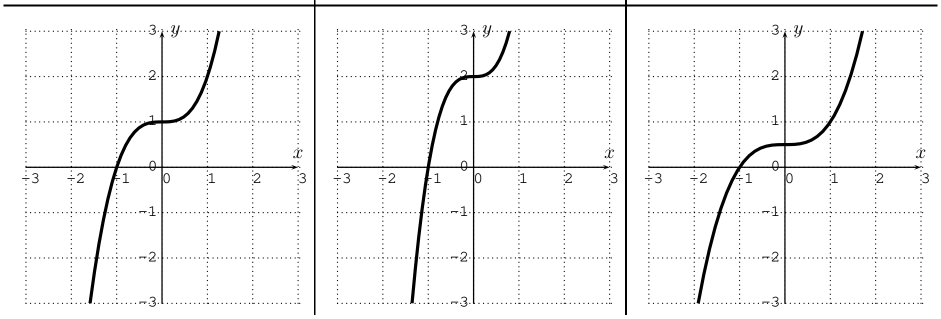
Consider the graph of a function $y = f(x)$. Then, the graph of $y = f(x+c)$ is that of $y = f(x)$ shifted to the left or right by c.

- If c is positive, the graph is shifted to the left; if c is negative, the graph is shifted to the right.

Multiplying a positive constant to the output

Another transformation is given by multiplying the function by a fixed positive factor.

$y = x^3 + 1$	$y = 2 \cdot (x^3 + 1)$	$y = \frac{1}{2} \cdot (x^3 + 1)$



This time, the function is stretched away or compressed toward the x-axis.

Observation 4.16: Multiplying a positive constant to the output

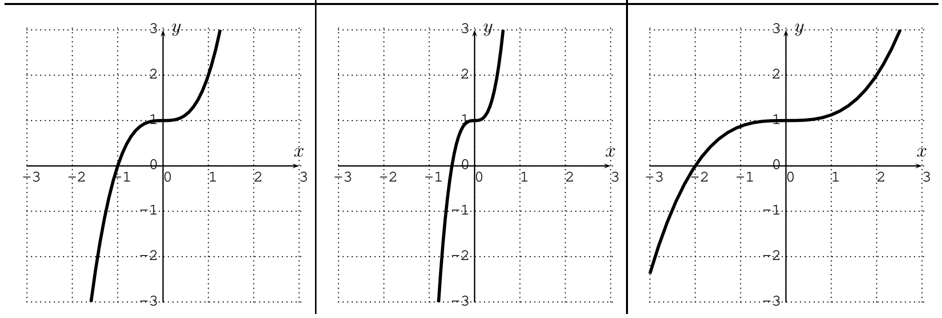
Consider the graph of a function $y = f(x)$ and let $c > 0$. Then, the graph of $y = c \cdot f(x)$ is that of $y = f(x)$ stretched away or compressed toward the x-axis by a factor c.

- If $c > 1$, the graph is stretched away from the x-axis; if $0 < c < 1$, the graph is compressed toward the x-axis.

Multiplying a positive constant to the input

Similarly, we can multiply the input by a positive factor.

$y = x^3 + 1$	$y = (2 \cdot x)^3 + 1$	$y = (\frac{1}{2} \cdot x)^3 + 1$



This time, the function is stretched away or compressed toward the y-axis.

Observation 4.17: Multiplying a positive constant to the input

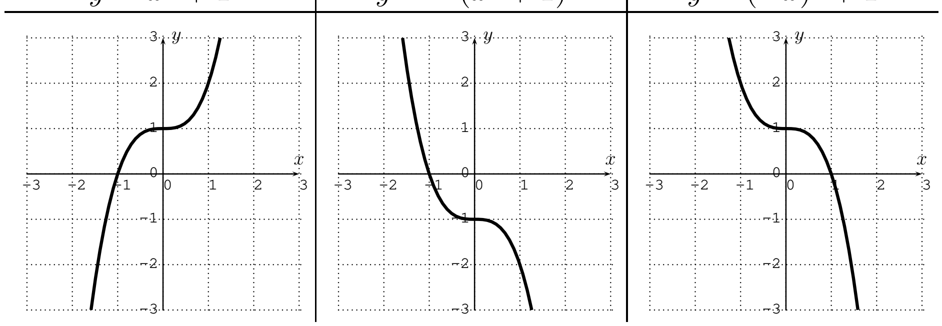
Consider the graph of a function $y = f(x)$ and let $c > 0$. Then, the graph of $y = f(c \cdot x)$ is that of $y = f(x)$ stretched away or compressed toward the y-axis by a factor c.

- If $c > 1$, the graph is compressed toward the y-axis; if $0 < c < 1$, the graph is stretched away from the y-axis.

Multiplying (-1) to the input or output

The last transformation is given by multiplying (-1) to the input or output, as displayed in the following chart.

$y = x^3 + 1$	$y = -(x^3 + 1)$	$y = (-x)^3 + 1$



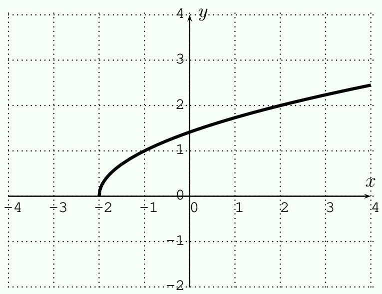
Here, the function is reflected either about the x-axis or about the y-axis.

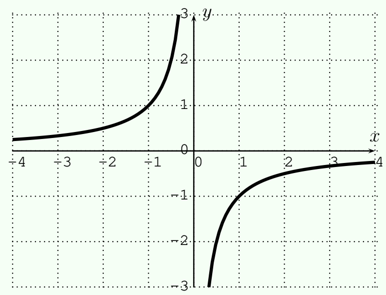
Observation 4.18: Multiplying (-1) to the input or output

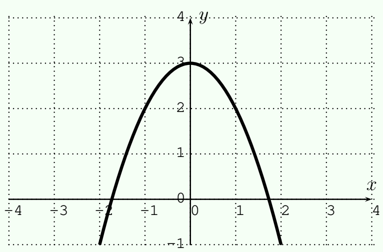
Consider the graph of a function $y = f(x)$. Then, the graph of $y = -f(x)$ is that of $y = f(x)$ reflected about the x-axis. Furthermore, the graph of $y = f(-x)$ is that of $y = f(x)$ reflected about the y-axis.

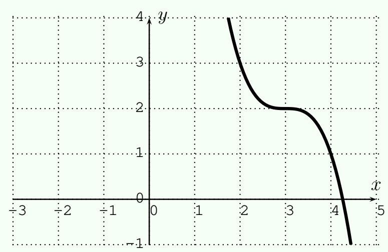
Example 4.19

Guess the formula for the function based on the basic graphs in Section 4.3 and the transformations described above.

a) 

b) 

c) 

d) 

Solution.

a) This is the square-root function shifted to the left by 2. Thus, by Observation 4.14, this is the function $f(x) = \sqrt{x+2}$.

b) This is the graph of $y = \frac{1}{x}$ reflected about the x-axis (or also $y = \frac{1}{x}$ reflected about the y-axis). In either case, we obtain the rule $y = -\frac{1}{x}$.

c) This is a parabola reflected about the x-axis and then shifted up by 3. Thus, we get:

	$y = x^2$
reflecting about the x-axis gives	$y = -x^2$
shifting the graph up by 3 gives	$y = -x^2 + 3$

d) Starting from the graph of the cubic equation $y = x^3$, we need to reflect about the x-axis (or also y-axis), then shift up by 2 and to the right by 3. These transformations affect the formula as follows:

	$y = x^3$
reflecting about the x-axis gives	$y = -x^3$
shifting up by 2 gives	$y = -x^3 + 2$
shifting the the right by 3 gives	$y = -(x-3)^3 + 2$

All of the above answers can be checked by graphing the function with the graphing calculator. □

Example 4.20

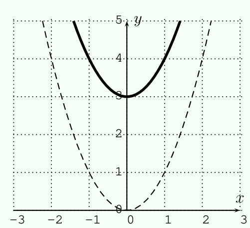
Sketch the graph of the function based on the basic graphs in Section 4.3 and the transformations described above.

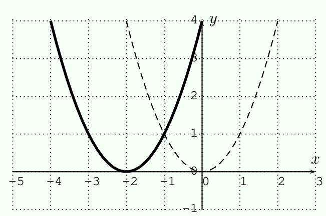
a) $y = x^2 + 3$ b) $y = (x+2)^2$ c) $y = |x-3| - 2$
d) $y = 2 \cdot \sqrt{x+1}$ e) $y = -\left(\frac{1}{x} + 2\right)$ f) $y = (-x+1)^3$

Solution.

a) This is the parabola $y = x^2$ shifted up by 3. The graph is shown below.

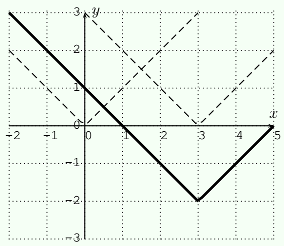
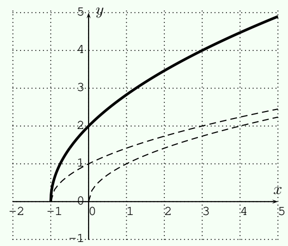
b) $y = (x+2)^2$ is the parabola $y = x^2$ shifted 2 units to the left.

a) 

b) 

c) The graph of the function $f(x) = |x-3| - 2$ is the absolute value shifted to the right by 3 and down by 2. (Alternatively, we can first shift down by 2 and then to the right by 3.)

d) Similarly, to get from the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{x+1}$, we shift the graph to the left, and then for $y = 2 \cdot \sqrt{x+1}$, we need to stretch the graph by a factor 2 away from the x-axis. (Alternatively, we could first stretch the the graph away from the x-axis, then shift the graph by 1 to the left.)

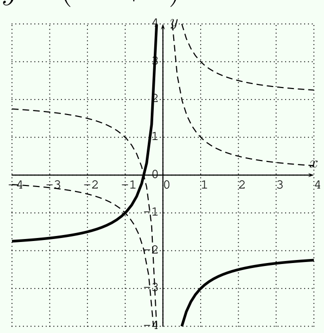
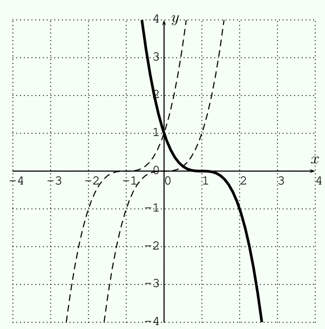
c)  d) 

e) For $y = -(\frac{1}{x} + 2)$, we start with $y = \frac{1}{x}$ and add 2, giving $y = \frac{1}{x} + 2$, which shifts the graph up by 2. Then, taking the negative gives $y = -(\frac{1}{x} + 2)$, which corresponds to reflecting the graph about the x-axis.

Note that in this case, we cannot perform these transformations in the opposite order, since the negative of $y = \frac{1}{x}$ gives $y = -\frac{1}{x}$, and adding 2 gives $y = -\frac{1}{x} + 2$, which is *not* equal to $-(\frac{1}{x} + 2)$.

f) We start with $y = x^3$. Adding 1 to the argument, $y = (x+1)^3$, shifts its graph to the left by 1. Then, applying a minus to the argument gives $y = (-x+1)^3$, which reflects the graph about the y-axis.

Here, the order in which we perform these transformations is again important. In fact, if we first take the negative of the argument, we obtain $y = (-x)^3$. Then, adding one to the argument would give $y = (-(x+1))^3 = (-x-1)^3$, which is different than our given function $y = (-x+1)^3$.

e)  f) 

All these solutions may also be checked easily by using the graphing calculator. □

Example 4.21

a) The graph of $f(x) = |x^3 - 5|$ is stretched away from the y-axis by a factor of 3. What is the formula for the new function?

b) The graph of $f(x) = \sqrt{6x^2 + 3}$ is shifted up 5 units, and then reflected about the x-axis. What is the formula for the new function?

c) How are the graphs of $y = 2x^3 + 5x - 9$ and $y = 2(x-2)^3 + 5(x-2) - 9$ related?

d) How are the graphs of $y = (x-2)^2$ and $y = (-x+3)^2$ related?

Solution.

a) Based on Observation 4.17 on page 68, we have to multiply the argument by $\frac{1}{3}$. The new function is therefore:

$$f\left(\frac{1}{3} \cdot x\right) = \left|\left(\frac{1}{3} \cdot x\right)^3 - 5\right| = \left|\frac{1}{27} \cdot x^3 - 5\right|$$

b) After the shift, we have the graph of a new function $y = \sqrt{6x^2 + 3} + 5$. Then, a reflection about the x-axis gives the graph of the function $y = -(\sqrt{6x^2 + 3} + 5)$.

c) Based on Observation 4.15 on page 67, we see that we need to shift the graph of $y = 2x^3 + 5x - 9$ by 2 units to the right.

d) The formulas can be transformed into each other as follows:

We begin with	$y = (x-2)^2$.
Replacing x by $x+5$ gives	$y = ((x+5)-2)^2 = (x+3)^2$.
Replacing x by $-x$ gives	$y = ((-x)+3)^2 = (-x+3)^2$.

Therefore, we have performed a shift to the left by 5, and then a reflection about the y-axis.

We want to point out that there is a second solution for this problem:

We begin with $y = (x-2)^2$.
Replacing x by $-x$ gives
$y = ((-x)-2)^2 = (-x-2)^2$.
Replacing x by $x-5$ gives
$y = (-(x-5)-2)^2 = (-x+5-2)^2 = (-x+3)^2$.

Therefore, we could also first perform a reflection about the y-axis, and then shift the graph to the right by 5.

□

Some of the above functions have special symmetries, which we investigate now.

Definition 4.22: Even function, odd function

A function f is called **even** if $f(-x) = f(x)$ for all x.
A function f is called **odd** if $f(-x) = -f(x)$ for all x.

Example 4.23

Determine if the following functions are even, odd, or neither:

$$f(x) = x^2, \qquad g(x) = x^3, \qquad h(x) = x^4, \qquad k(x) = x^5,$$
$$l(x) = 4x^5 + 7x^3 - 2x, \qquad m(x) = x^2 + 5x$$

Solution.

The function $f(x) = x^2$ is even, since $f(-x) = (-x)^2 = x^2$. Similarly, $g(x) = x^3$ is odd, $h(x) = x^4$ is even, and $k(x) = x^5$ is odd, since

$$\begin{aligned} g(-x) &= (-x)^3 = -x^3 = -g(x) \\ h(-x) &= (-x)^4 = x^4 = h(x) \\ k(-x) &= (-x)^5 = -x^5 = -k(x) \end{aligned}$$

Indeed, we see that a function $y = x^n$ is even precisely when n is even, and $y = x^n$ is odd precisely when n is odd. (These examples are in fact the motivation behind defining even and odd functions as in Definition 4.22 above.)
Next, in order to determine if the function l is even or odd, we calculate $l(-x)$ and compare it with $l(x)$.

$$\begin{aligned} l(-x) &= 4(-x)^5 + 7(-x)^3 - 2(-x) = -4x^5 - 7x^3 + 2x \\ &= -(4x^5 + 7x^3 - 2x) = -l(x) \end{aligned}$$

Therefore, l is an odd function.

Finally, for $m(x) = x^2 + 5x$, we calculate $m(-x)$ as follows:

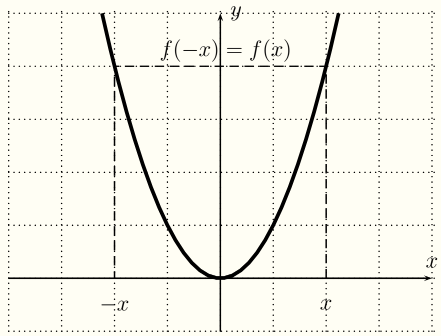
$$m(-x) = (-x)^2 + 5(-x) = x^2 - 5x$$

Note that m is not an even function, since $x^2 - 5x \neq x^2 + 5x$. Furthermore, m is also not an odd function, since $x^2 - 5x \neq -(x^2 + 5x)$. Therefore, m is a function that is *neither* even *nor* odd. □

Observation 4.24: Graph of even or odd function

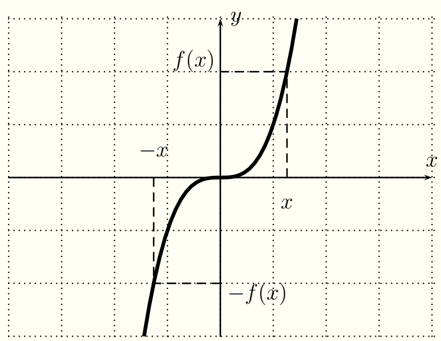
An even function f is symmetric with respect to the $y-$axis (if you reflect the graph of f about the $y-$axis, you get the same graph back), since even functions satisfy $f(-x) = f(x)$:

Example $y = x^2$:



An odd function f is symmetric with respect to the origin (if you reflect the graph of f about the $y-$axis and then about the $x-$axis, you get the same graph back), since odd functions satisfy $f(-x) = -f(x)$:

Example $y = x^3$:



4.4 Exercises

Exercise 4.1

Graph the function in Desmos.

a) $y = 3x - 5$ b) $y = x^2 - 3x - 2$ c) $y = x^4 - 3x^3 + 2x - 1$
d) $y = \sqrt{x^2 - 4}$ e) $y = \frac{4x+3}{2x+5}$ f) $y = |x + 3|$

Exercise 4.2

For each of the functions below, use Desmos to find all roots, all local maxima, all local minima, and the y-intercept.

a) $f(x) = x^3 + 4x^2 - 2x - 9$ b) $f(x) = x^3 - 6x^2 + 7x + 4$
c) $f(x) = -4x^3 + 3x^2 + 7x + 1$ d) $f(x) = 5x^3 + 2x^2$
e) $f(x) = x^4 - x^3 - 4x^2 + 1$ f) $f(x) = -x^4 + 5x^3 - 4x + 3$
g) $f(x) = x^5 + 2x^4 - x^3 - 3x^2 - x$ h) $f(x) = \sqrt{|2^x - 3|} - 2x + 3$

Exercise 4.3

Determine the domain and range using Desmos.

a) $y = |x - 2| + 5$ b) $y = -2x + 7$
c) $y = x^2 - 6x + 4$ d) $y = -x^2 - 8x + 3$
e) $y = 3 + \sqrt{x + 5}$ f) $y = 6 - x + \sqrt{4 - x}$
g) $y = x^4 - 8x^2 + 9$ h) $y = \frac{x-2}{x-3}$

Exercise 4.4

Use Desmos to determine whether the equation describes a function or not.

a) $x^2 + 2y - 3x = 7$ b) $x^2 + 2y^2 - 3x = 7$
c) $y^2 + 8y + 15 = x$ d) $y^3 + x^2 + y + x = 1$
e) $y = \frac{2x-5}{x-3}$ f) $x^2 + \left(y - \sqrt{|x|}\right)^2 = 1$

Exercise 4.5

Solve the equation for y and graph all branches in Desmos in the same window.

a) $x^2 + y^2 = 4$ b) $(x+5)^2 + y^2 = 15$
c) $(x-1)^2 + (y-2)^2 = 9$ d) $y^2 = x^2 + 3$

Exercise 4.6

Set up the general equation of a circle in Desmos, where the center and the radius can be changed using sliders. If a circle of radius 3 with center at the origin $(0,0)$ is shifted 4 units to the right and shifted 2 units down, then what is its equation?

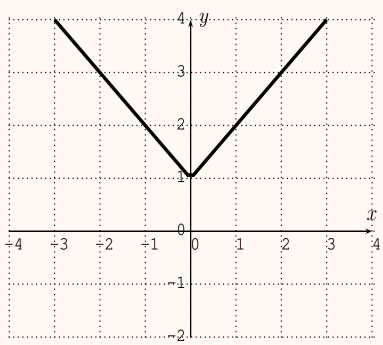
Exercise 4.7

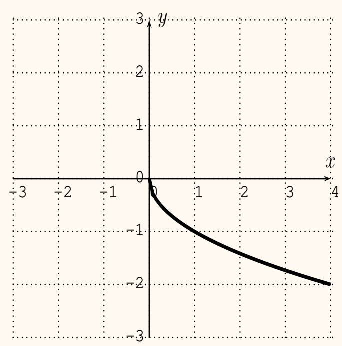
Use Desmos to find all solutions of the equation. Round your answer to the nearest thousandth.

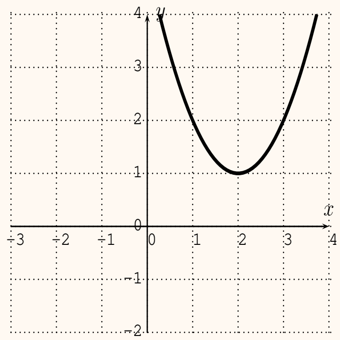
a) $x^3 + 3 = x^5 + 7$ b) $4x^3 + 6x^2 - 3x - 2 = 0$
c) $\frac{2x}{x-3} = \frac{x^2+2}{x+1}$ d) $5^{3x+1} = x^5 + 6$
e) $x^3 + x^2 = x^4 - x^2 + x$ f) $3x^2 = x^3 - x^2 + 3x$

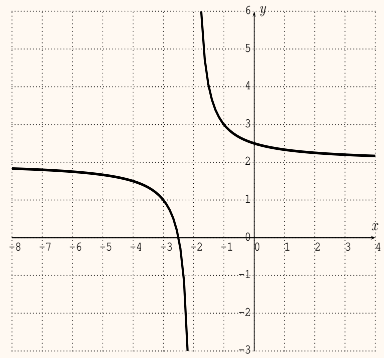
Exercise 4.8

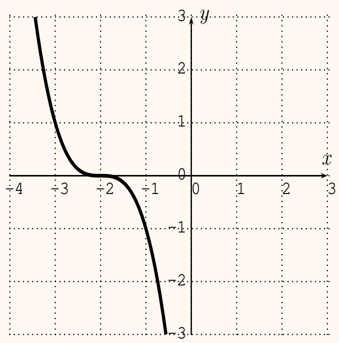
Find a possible formula of the graph displayed below.

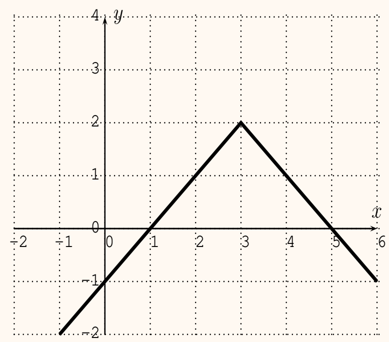
a) 

b) 

c) 

d) 

e) 

f) 

Exercise 4.9

Sketch the graph of the function based on the basic graphs and their transformation described in Section 4.3. Confirm your answer by graphing the function with the graphing calculator.

a) $f(x) = \|x\| - 3$	b) $f(x) = \frac{1}{x+2}$
c) $f(x) = -x^2$	d) $f(x) = (x-1)^3$
e) $f(x) = \sqrt{-x}$	f) $f(x) = 4 \cdot \|x-3\|$
g) $f(x) = -\sqrt{x} + 1$	h) $f(x) = (\frac{1}{2} \cdot x)^2 + 3$

Exercise 4.10

Consider the graph of $f(x) = x^2 - 7x + 1$. Find the formula of the function that is given by performing the following transformations on the graph.

a) Shift the graph of f down by 4.
b) Shift the graph of f to the left by 3 units.
c) Reflect the graph of f about the x-axis.
d) Reflect the graph of f about the y-axis.
e) Stretch the graph of f away from the y-axis by a factor 3.
f) Compress the graph of f toward the y-axis by a factor 2.

Exercise 4.11

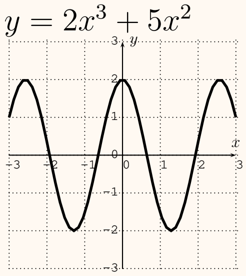
How are the graphs of f and g related?

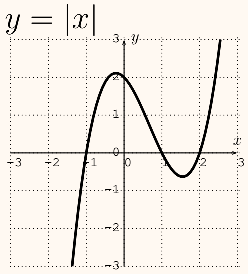
a)	$f(x) = \sqrt{x}$,	$g(x) = \sqrt{x-5}$
b)	$f(x) = \|x\|$,	$g(x) = 2 \cdot \|x\|$
c)	$f(x) = (x+1)^3$,	$g(x) = (x-3)^3$
d)	$f(x) = x^2 + 3x + 5$,	$g(x) = (2x)^2 + 3(2x)^2 + 5$
e)	$f(x) = \frac{1}{x+3}$,	$g(x) = -\frac{1}{x}$
f)	$f(x) = 2 \cdot \|x\|$,	$g(x) = \|x+1\| + 1$

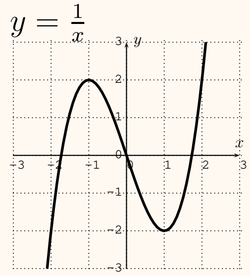
Exercise 4.12

Determine if the function is even, odd, or neither.

a) $y = 2x^3$ b) $y = 5x^2$ c) $y = 3x^4 - 4x^2 + 5$
d) $y = 2x^3 + 5x^2$ e) $y = |x|$ f) $y = \frac{1}{x}$

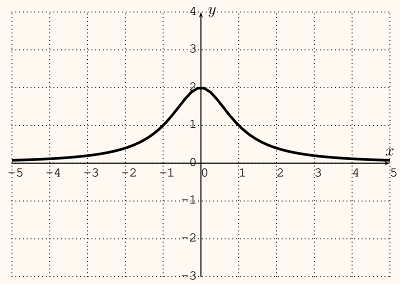
g) 

h) 

i) 

Exercise 4.13

The graph of the function $y = f(x)$ is displayed below.



Sketch the graph of the following functions.

a) $y = f(x) + 1$ b) $y = f(x - 3)$ c) $y = -f(x)$
d) $y = 2f(x)$ e) $y = f(2x)$ f) $y = f(\frac{1}{2}x)$

Chapter 5

Operations on functions

We can combine two functions in many different ways, for example, by combining their output values (such as adding or multiplying them), or by composing the functions (that is, using the output of one as the input of the next function). In this chapter, we define and study these operations.

5.1 Operations on functions given by formulas

In the first example of this section, we show how to add, subtract, multiply, and divide functions that are given by a formula.

Example 5.1

Let $f(x) = x^2 + 5x$ and $g(x) = 7x - 3$. Find the following functions, and state their domain.

$$(f+g)(x), (f-g)(x), (f\cdot g)(x), \text{ and } \left(\frac{f}{g}\right)(x).$$

Solution.
The functions are calculated by adding them (or subtracting, multiplying, or dividing them).

$$\begin{aligned}
(f+g)(x) &= (x^2+5x)+(7x-3) = x^2+12x-3,\\
(f-g)(x) &= (x^2+5x)-(7x-3)\\
&= x^2+5x-7x+3 = x^2-2x+3,
\end{aligned}$$

$$\begin{aligned}(f\cdot g)(x) &= (x^2+5x)\cdot(7x-3)\\ &= 7x^3-3x^2+35x^2-15x = 7x^3+32x^2-15x,\\ \left(\frac{f}{g}\right)(x) &= \frac{x^2+5x}{7x-3}.\end{aligned}$$

The calculation of these functions was straightforward. To state their domain is also straightforward, except for the domain of the quotient $\frac{f}{g}$. Note that $f+g$, $f-g$, and $f\cdot g$ are all polynomials. According to the standard convention (Convention 2.7 on page 24), all these functions have the domain $\mathbb{R}$; that is, their domain is all real numbers.

Now, for the domain of $\frac{f}{g}$, we have to be a bit more careful, since the denominator of a fraction cannot be zero. The denominator of $\frac{f}{g}(x) = \frac{x^2+5x}{7x-3}$ is zero, exactly when

$$7x-3=0 \quad\Longrightarrow\quad 7x=3 \quad\Longrightarrow\quad x=\frac{3}{7}.$$

We have to exclude $\frac{3}{7}$ from the domain. The domain of the quotient $\frac{f}{g}$ is therefore $\mathbb{R}-\{\frac{3}{7}\}$. □

We can formally state the observation we made in the previous example.

Observation 5.2: Domain when adding, multiplying, dividing

Let f be a function with domain D_f, and let g be a function with domain D_g. A value x can be used as an input of $f+g$, $f-g$, and $f\cdot g$, exactly when x is an input of both f and g. Therefore, the domains of the combined functions are the intersection of the domains D_f and D_g:

$$\begin{aligned}D_{f+g} &= D_f\cap D_g = \{x \mid x\in D_f \text{ and } x\in D_g\},\\ D_{f-g} &= D_f\cap D_g,\\ D_{f\cdot g} &= D_f\cap D_g.\end{aligned}$$

For the quotient $\frac{f}{g}$, we also have to make sure that the denominator $g(x)$ is not zero.

$$D_{\frac{f}{g}} = \{x \mid x\in D_f, x\in D_g, \text{ and } g(x)\neq 0\}.$$

Example 5.3

Let $f(x) = \sqrt{x+2}$, and let $g(x) = x^2 - 5x + 4$. Find the functions $\frac{f}{g}$ and $\frac{g}{f}$ and state their domains.

Solution.
First, the domain of f consists of those numbers x for which the square root is defined. In other words, we need $x + 2 \geq 0$, which means that $x \geq -2$, and so the domain of f is $D_f = [-2, \infty)$. On the other hand, the domain of g is all real numbers, $D_g = \mathbb{R}$. Now, we have the quotients

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+2}}{x^2 - 5x + 4} \qquad \text{and} \qquad \left(\frac{g}{f}\right)(x) = \frac{x^2 - 5x + 4}{\sqrt{x+2}}.$$

For the domain of $\frac{f}{g}$, we need to exclude those numbers x for which $x^2 - 5x + 4 = 0$. Thus, factoring $x^2 - 5x + 4 = 0$ gives

$$(x-1)(x-4) = 0 \quad \Longrightarrow \quad x = 1 \text{ or } x = 4$$

We obtain the domain for $\frac{f}{g}$ as the combined domain for f and g, and exclude 1 and 4. Therefore, $D_{\frac{f}{g}} = [-2, \infty) - \{1, 4\}$.

Now, for $\frac{g}{f}(x) = \frac{x^2-5x+4}{\sqrt{x+2}}$, the denominator becomes zero exactly when

$$x + 2 = 0 \quad \Longrightarrow \quad x = -2$$

Therefore, we need to exclude -2 from the domain, that is

$$D_{\frac{g}{f}} = [-2, \infty) - \{-2\} = (-2, \infty).$$

□

Note 5.4

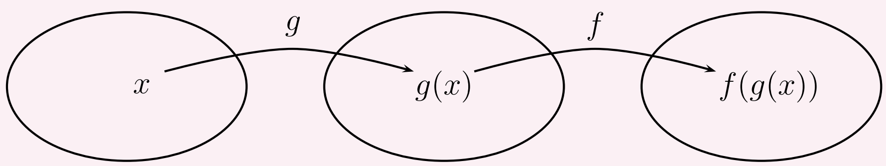
To form the quotient $\frac{f}{g}(x)$, where $f(x) = x^2 - 1$ and $g(x) = x + 1$, we write $\frac{f}{g}(x) = \frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x - 1$. One might be tempted to say that the domain is all real numbers. But it is not! The domain is all real numbers except -1, and the last step of the simplification performed above is only valid for $x \neq -1$.

Another operation we can perform is the composition of two functions.

Definition 5.5: Composition of functions

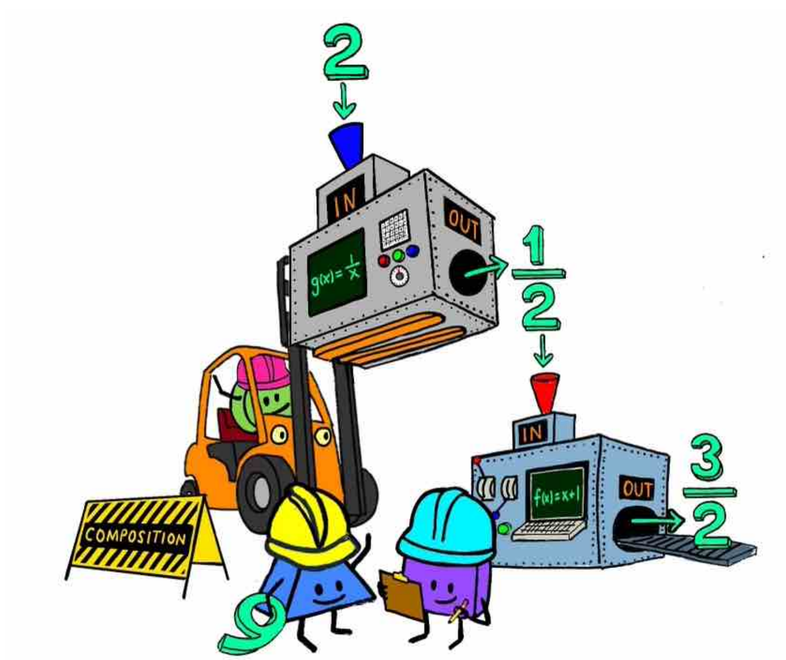
Let f and g be functions, and assume that $g(x)$ is in the domain of f. Then define the **composition of** f **and** g at x to be

$$(f \circ g)(x) := f(g(x)).$$



We can take any x as an input of $f \circ g$ which is an input of g and for which $g(x)$ is an input of f. Therefore, if D_f is the domain of f and D_g is the domain of g, the domain of $D_{f \circ g}$ is

$$D_{f \circ g} = \{\ x \mid x \in D_g,\ g(x) \in D_f\ \}.$$



Example 5.6

Let $f(x) = 2x^2 + 5x$ and $g(x) = 2 - x$. Find the following compositions:

a) $f(g(3))$ b) $g(f(3))$ c) $f(f(1))$ d) $f(2 \cdot g(5))$ e) $g(g(4) + 5)$

Solution.

We evaluate the expressions as follows:

$$
\begin{array}{lrcl}
\text{a)} & f(g(3)) & = & f(2-3) = f(-1) = 2 \cdot (-1)^2 + 5 \cdot (-1) \\
 & & = & 2 - 5 = -3, \\
\text{b)} & g(f(3)) & = & g(2 \cdot 3^2 + 5 \cdot 3) = g(18 + 15) = g(33) \\
 & & = & 2 - 33 = -31, \\
\text{c)} & f(f(1)) & = & f(2 \cdot 1^2 + 5 \cdot 1) = f(2+5) = f(7) \\
 & & = & 2 \cdot 7^2 + 5 \cdot 7 = 98 + 35 = 133, \\
\text{d)} & f(2 \cdot g(5)) & = & f(2 \cdot (2-5)) = f(2 \cdot (-3)) = f(-6) \\
 & & = & 2 \cdot (-6)^2 + 5 \cdot (-6) = 72 - 30 = 42, \\
\text{e)} & g(g(4)+5) & = & g((2-4)+5) = g((-2)+5) \\
 & & = & g(3) = 2 - 3 = -1.
\end{array}
$$

□

We can also calculate composite functions for arbitrary x in the domain.

Example 5.7

Let $f(x) = x^2 + 1$ and $g(x) = x + 3$. Find the following compositions:

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$

Solution.

a) There are essentially two ways to evaluate $(f \circ g)(x) = f(g(x))$. We can either first use the explicit formula for $f(x)$ and then the one for $g(x)$, or vice versa. We will evaluate $f(g(x))$ by substituting $g(x)$ into the formula for $f(x)$:

$$
\begin{array}{rcl}
(f \circ g)(x) & = & f(g(x)) = (g(x))^2 + 1 = (x+3)^2 + 1 \\
 & = & x^2 + 6x + 9 + 1 = x^2 + 6x + 10.
\end{array}
$$

Similarly, we evaluate the other expressions (b)-(d):

b) $(g \circ f)(x) = g(f(x)) = f(x) + 3 = x^2 + 1 + 3 = x^2 + 4$

c) $(f \circ f)(x) = f(f(x)) = (f(x))^2 + 1 = (x^2+1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2$

d) $(g \circ g)(x) = g(g(x)) = g(x) + 3 = x + 3 + 3 = x + 6$

□

Example 5.8

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the following functions, and state their domains.

a) $f(x) = \frac{3}{x+2}$ and $g(x) = x^2 - 3x$
b) $f(x) = |3x - 2| - 6x + 4$ and $g(x) = 5x + 1$
c) $f(x) = \sqrt{\frac{1}{2} \cdot (x - 4)}$ and $g(x) = 2x^2 + 4$

Solution.

a) Composing $f \circ g$, we obtain

$$(f \circ g)(x) = f(g(x)) = \frac{3}{g(x) + 2} = \frac{3}{x^2 - 3x + 2}.$$

The domain is the set of numbers x for which the denominator is non-zero.

$$\begin{aligned} x^2 - 3x + 2 = 0 \quad &\Longrightarrow \quad (x-2)(x-1) = 0 \\ &\Longrightarrow \quad x = 2 \text{ or } x = 1 \\ &\Longrightarrow \quad D_{f \circ g} = \mathbb{R} - \{1, 2\}. \end{aligned}$$

Similarly,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = f(x)^2 - 3f(x) = \left(\frac{3}{x+2}\right)^2 - 3\frac{3}{x+2} \\ &= \frac{9}{(x+2)^2} - \frac{9}{x+2} = \frac{9 - 9(x+2)}{(x+2)^2} \\ &= \frac{9 - 9x - 18}{(x+2)^2} = \frac{-9x - 9}{(x+2)^2} = \frac{-9 \cdot (x+1)}{(x+2)^2} \end{aligned}$$

The domain is all real numbers except $x = -2$, that is $D_{g \circ f} = \mathbb{R} - \{-2\}$.

b) We calculate the compositions as follows:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = |3g(x) - 2| - 6g(x) + 4 \\
&= |3(5x + 1) - 2| - 6(5x + 1) + 4 \\
&= |15x + 1| - 30x - 2 \\
(g \circ f)(x) &= g(f(x)) = 5f(x) + 1 = 5 \cdot (|3x - 2| - 6x + 4) + 1 \\
&= 5 \cdot |3x - 2| - 30x + 20 + 1 = 5 \cdot |3x - 2| - 30x + 21
\end{aligned}$$

Since the domains of f and g are all real numbers, so are also the domains for both $f \circ g$ and $g \circ f$.

c) Again we calculate the compositions.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = \sqrt{\frac{1}{2} \cdot (g(x) - 4)} = \sqrt{\frac{1}{2} \cdot (2x^2 + 4 - 4)} \\
&= \sqrt{\frac{1}{2} \cdot 2x^2} = \sqrt{x^2} = |x|.
\end{aligned}$$

The domain of g is all real numbers, and the outputs $g(x) = 2x^2 + 4$ are all ≥ 4, (since $2x^2 \geq 0$). Therefore, $g(x)$ is in the domain of f, and we have a combined domain of $f \circ g$ of $D_{f \circ g} = \mathbb{R}$. On the other hand,

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) = 2(f(x))^2 + 4 = 2 \cdot \left(\sqrt{\frac{1}{2} \cdot (x - 4)} \right)^2 + 4 \\
&= 2 \cdot \left(\frac{1}{2} \cdot (x - 4) \right) + 4 = (x - 4) + 4 = x.
\end{aligned}$$

The domain of $g \circ f$ consists of all numbers x which are in the domain of f and for which $f(x)$ is in the domain of g. Now, the domain of f consists of all real numbers x that give a non-negative argument in the square-root, that is: $\frac{1}{2}(x - 4) \geq 0$. Therefore, we must have $x - 4 \geq 0$, so that $x \geq 4$, and we obtain the domain $D_f = [4, \infty)$. Since the domain $D_g = \mathbb{R}$, the composition $g \circ f$ has the same domain as f:

$$D_{g \circ f} = D_f = [4, \infty).$$

We remark that at a first glance, we might have expected that $(g \circ f) = x$ has a domain of all real numbers. However, the composition $g(f(x))$ can only have those inputs that are also allowed inputs of f. We see that the domain of a composition is sometimes smaller than the domain that we use via the standard convention (Convention 2.7).

□

5.2 Operations on functions given by tables

We now show how to combine two functions that are given by tables.

Example 5.9

Let f and g be the functions defined by the following table.

x	1	2	3	4	5	6	7
$f(x)$	6	3	1	4	0	7	6
$g(x)$	4	0	2	5	-2	3	1

Describe the following functions via a table:

a) $2 \cdot f(x) + 3$ b) $f(x) - g(x)$ c) $f(x+2)$ d) $g(-x)$

Solution.
For (a) and (b) we obtain by immediate calculation:

x	1	2	3	4	5	6	7
$2 \cdot f(x) + 3$	15	9	5	11	3	17	15
$f(x) - g(x)$	2	3	-1	-1	2	4	5

For example, for $x = 3$, we obtain $2 \cdot f(x) + 3 = 2 \cdot f(3) + 3 = 2 \cdot 1 + 3 = 5$ and $f(x) - g(x) = f(3) - g(3) = 1 - 2 = -1$.
For part (c), we have a similar calculation of $f(x+2)$. For example, for $x = 1$, we get $f(1+2) = f(1+2) = f(3) = 1$.

x	1	2	3	4	5	6	7	-1	0
$f(x+2)$	1	4	0	7	6	undef.	undef.	6	3

Note that for the last two inputs $x = 6$ and $x = 7$ the expression $f(x+2)$ is undefined, since, for example for $x = 6$, we have $f(x+2) = f(6+2) = f(8)$ which is undefined. However, for $x = -1$, we obtain $f(x+2) = f(-1+2) = f(1) = 6$. If we define $h(x) = f(x+2)$, then the domain of h is therefore $D_h = \{-1, 0, 1, 2, 3, 4, 5\}$.
Finally, for part (d), we need to take x as inputs, for which $g(-x)$ is defined via the table for g. We obtain the following answer:

x	-1	-2	-3	-4	-5	-6	-7
$g(-x)$	4	0	2	5	-2	3	1

□

Example 5.10

Let f and g be the functions defined by the following table.

x	1	3	5	7	9	11
$f(x)$	3	5	11	4	9	7
$g(x)$	7	-6	9	11	9	5

Describe the following functions via a table:

a) $f \circ g$ b) $g \circ f$ c) $f \circ f$ d) $g \circ g$

Solution.
The compositions are calculated by repeated evaluation. For example,

$$(f \circ g)(1) = f(g(1)) = f(7) = 4.$$

The complete answer is displayed below.

x	1	3	5	7	9	11
$(f \circ g)(x)$	4	undef.	9	7	9	11
$(g \circ f)(x)$	-6	9	5	undef.	9	11
$(f \circ f)(x)$	5	11	7	undef.	9	4
$(g \circ g)(x)$	11	undef.	9	5	9	9

□

5.3 Exercises

Exercise 5.1

Find $f+g$, $f-g$, $f \cdot g$ for the functions below. State their domain.

a) $f(x) = x^2 + 6x$ and $g(x) = 3x - 5$
b) $f(x) = x^3 + 5$ and $g(x) = 5x^2 + 7$
c) $f(x) = 3x + 7\sqrt{x}$ and $g(x) = 2x^2 + 5\sqrt{x}$
d) $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{5x}{x+2}$
e) $f(x) = \sqrt{x-3}$ and $g(x) = 2\sqrt{x-3}$
f) $f(x) = x^2 + 2x + 5$ and $g(x) = 3x - 6$
g) $f(x) = x^2 + 3x$ and $g(x) = 2x^2 + 3x + 4$

Exercise 5.2

Find $\frac{f}{g}$, and $\frac{g}{f}$ for the functions below. State their domain.

a) $f(x) = 3x + 6$ and $g(x) = 2x - 8$
b) $f(x) = x + 2$ and $g(x) = x^2 - 5x + 4$
c) $f(x) = \frac{1}{x-5}$ and $g(x) = \frac{x-2}{x+3}$
d) $f(x) = \sqrt{x+6}$ and $g(x) = 2x + 5$
e) $f(x) = x^2 + 8x - 33$ and $g(x) = \sqrt{x}$

Exercise 5.3

Let $f(x) = 2x - 3$ and $g(x) = 3x^2 + 4x$. Find the following compositions:

a) $f(g(2))$
b) $g(f(2))$
c) $f(f(5))$
d) $f(5g(-3))$
e) $g(f(2) - 2)$
f) $f(f(3) + g(3))$
g) $g(f(2 + x))$
h) $f(f(-x))$
i) $f(f(-3) - 3g(2))$
j) $f(f(f(2)))$
k) $f(x + h)$
l) $g(x + h)$

Exercise 5.4

Find the composition $(f \circ g)(x)$ for the following functions:

a) $f(x) = 3x - 5$ and $g(x) = 2x + 3$
b) $f(x) = x^2 + 2$ and $g(x) = x + 3$
c) $f(x) = x^2 - 3x + 2$ and $g(x) = 2x + 1$
d) $f(x) = x^2 + \sqrt{x+3}$ and $g(x) = x^2 + 2x$
e) $f(x) = \frac{2}{x+4}$ and $g(x) = x + h$
f) $f(x) = x^2 + 4x + 3$ and $g(x) = x + h$

Exercise 5.5

Find the compositions

$$(f \circ g)(x), \quad (g \circ f)(x), \quad (f \circ f)(x), \quad (g \circ g)(x)$$

for the following functions:

a) $f(x) = 2x + 4$ and $g(x) = x - 5$
b) $f(x) = x + 3$ and $g(x) = x^2 - 2x$
c) $f(x) = 2x^2 - x - 6$ and $g(x) = \sqrt{3x+2}$
d) $f(x) = \frac{1}{x+3}$ and $g(x) = \frac{1}{x} - 3$
e) $f(x) = (2x-7)^2$ and $g(x) = \frac{\sqrt{x}+7}{2}$

Exercise 5.6

Let f and g be the functions defined by the table below. Complete the table by performing the indicated operations.

x	1	2	3	4	5	6	7
$f(x)$	4	5	7	0	-2	6	4
$g(x)$	6	-8	5	2	9	11	2
$f(x)+3$							
$4g(x)+5$							
$g(x)-2f(x)$							
$f(x+3)$							

Exercise 5.7

Let f and g be the functions defined by the table below. Complete the table by composing the given functions.

x	1	2	3	4	5	6
$f(x)$	3	1	2	5	6	3
$g(x)$	5	2	6	1	2	4
$(g \circ f)(x)$						
$(f \circ g)(x)$						
$(f \circ f)(x)$						
$(g \circ g)(x)$						

Exercise 5.8

Let f and g be the functions defined by the table below. Complete the table by composing the given functions.

x	0	2	4	6	8	10	12
$f(x)$	4	8	5	6	12	-1	10
$g(x)$	10	2	0	-6	7	2	8
$(g \circ f)(x)$							
$(f \circ g)(x)$							
$(f \circ f)(x)$							
$(g \circ g)(x)$							

Chapter 6

The inverse of a function

For some functions, we can reverse the meaning of input and output. We can do this when each output of the function comes from exactly one input (the function is one-to-one). The function resulting from switching inputs and outputs is called the inverse of the function.

6.1 One-to-one functions

We have seen that some functions f may have the *same* outputs for *different* inputs. For example for $f(x) = x^2$, the inputs $x = 2$ and $x = -2$ have the same output $f(2) = 4$ and $f(-2) = 4$. A function is one-to-one, precisely when this is *not* the case.

Definition 6.1: One-to-one function

A function f is called **one-to-one** (or **injective**), if any two different inputs $x_1 \neq x_2$ always have different outputs $f(x_1) \neq f(x_2)$.

We now give a graphical interpretation for when a function is one-to-one.

Note 6.2

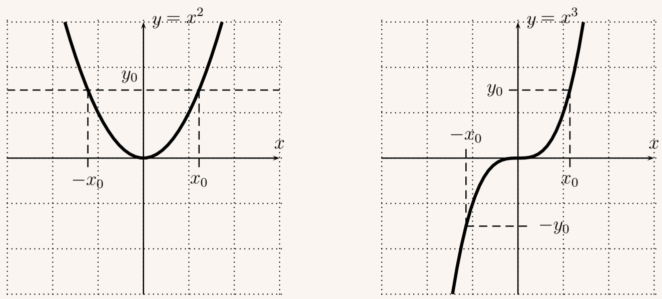
As was noted above, the function $f(x) = x^2$ is not one-to-one, because, for example, for inputs 2 and -2, we have the same output

$$f(-2) = (-2)^2 = 4, \qquad f(2) = 2^2 = 4.$$

On the other hand, $g(x) = x^3$ is one-to-one, since, for example, for inputs -2 and 2, we have different outputs:

$$g(-2) = (-2)^3 = -8, \qquad g(2) = 2^3 = 8.$$

The difference between the functions f and g can be seen from their graphs.

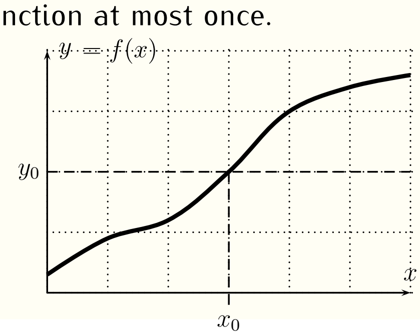


The graph of $f(x) = x^2$ on the left has for different inputs (x_0 and $-x_0$) the same output ($y_0 = (x_0)^2 = (-x_0)^2$). This is shown in the graph since the horizontal line at y_0 intersects the graph at two different points. In general, two inputs that have the same output y_0 give two points on the graph which also lie on the horizontal line at y_0.
Now, the graph of $g(x) = x^3$ on the right intersects with a horizontal line at some y_0 only once. This shows that for two different inputs, we can never have the same output y_0, so that the function g is one-to-one.

We summarize the above in the following observation.

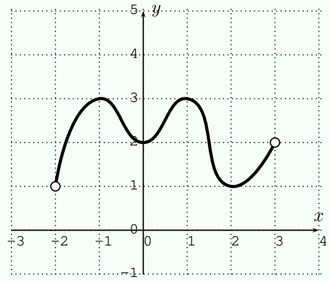
Observation 6.3: Horizontal Line Test

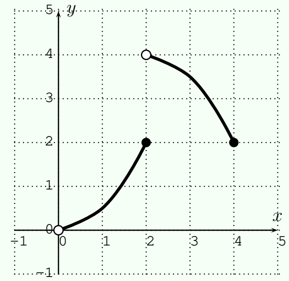
A function is one-to-one exactly when every horizontal line intersects the graph of the function at most once.



Example 6.4

Which of the following are, or represent, one-to-one functions?

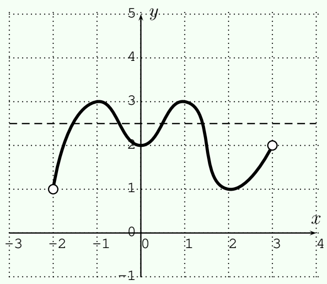
a) 

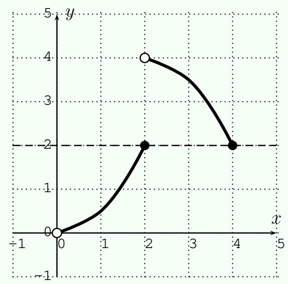
b) 

c) $f(x) = -x^3 + 6x^2 - 13x + 12$ d) $f(x) = x^3 - 2x^2 + 3$

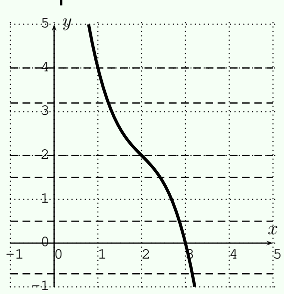
Solution.

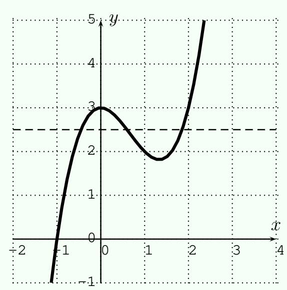
We use the horizontal line test to see which functions are one-to-one. For (a) and (b), we see that the functions are *not* one-to-one since there is a horizontal line that intersects with the graph more than once:

a) 

b) 

For (c), using the graphing calculator to graph the function $f(x) = -x^3 + 6x^2 - 13x + 12$, we see that all horizontal lines intersect the graph exactly once. Therefore, the function in part (c) is one-to-one. The function in part (d) however has a graph that intersects some horizontal line in several points. Therefore, $f(x) = x^3 - 2x^2 + 3$ is not one-to-one:

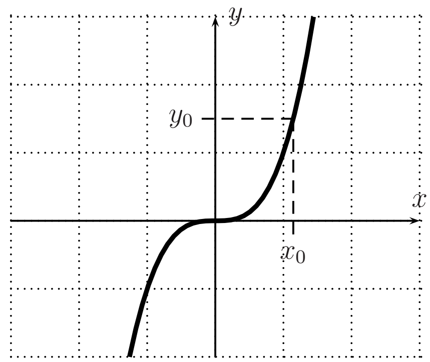
c) 

d) 

□

6.2 Inverse function

A function is one-to-one when each output is determined by exactly one input. Therefore, we can construct a new function called the inverse function, in which we reverse the roles of inputs and outputs. For example, when $y = x^3$, each y_0 comes from exactly one x_0 as shown in the picture below:

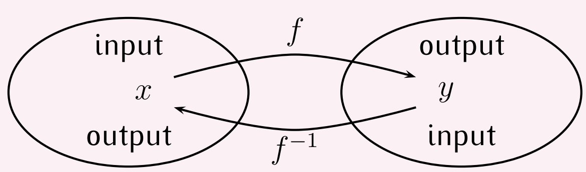


The inverse function assigns to the *input* y_0 the *output* x_0.

Definition 6.5: Inverse function

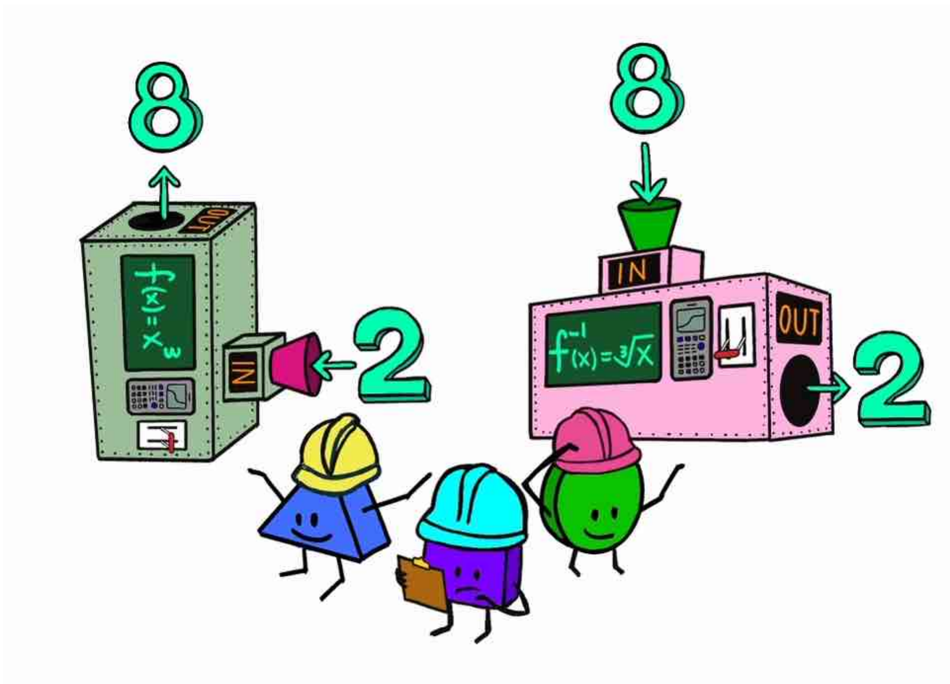
Let f be a function with domain D_f and range R_f, and assume that f is one-to-one. The **inverse** of f is the function f^{-1}, determined by:

$$f(x) = y \quad \text{means precisely that} \quad f^{-1}(y) = x.$$



Here the outputs of f are the inputs of f^{-1}, and the inputs of f are the outputs of f^{-1}. Therefore, the inverse function f^{-1} has a domain equal to the range of f, $D_{f^{-1}} = R_f$; and f^{-1} has a range equal to the domain of f, $R_{f^{-1}} = D_f$. In short, when f is a function $f : D_f \to R_f$, then the inverse function f^{-1} is a function $f^{-1} : R_f \to D_f$.

The inverse function reverses the roles of inputs and outputs.



Example 6.6

Find the inverse of the following functions.

a) $f(x) = 2x - 7$
b) $g(x) = \sqrt{x+2}$
c) $h(x) = \frac{1}{x+4}$
d) $j(x) = \frac{x+1}{x+2}$
e) $k(x) = (x-2)^2 + 3$ for $x \geq 2$

Solution.

a) First, reverse the role of input and output in $y = 2x-7$ by exchanging the variables x and y. That is, we write $x = 2y - 7$. We need to solve this for y:

$$\overset{\text{(add 7)}}{\Longrightarrow} \quad x + 7 = 2y \quad \Longrightarrow \quad y = \frac{x+7}{2}$$

Therefore, we obtain that the inverse of f is $f^{-1}(x) = \frac{x+7}{2}$.

For the other parts, we always exchange x and y and solve for y:

b) Write $y = \sqrt{x+2}$ and exchange x and y:

$$x = \sqrt{y+2} \implies x^2 = y+2 \implies y = x^2 - 2$$
$$\implies g^{-1}(x) = x^2 - 2$$

c) Write $y = \frac{1}{x+4}$ and exchange x and y:

$$x = \frac{1}{y+4} \implies y+4 = \frac{1}{x} \implies y = \frac{1}{x} - 4$$
$$\implies h^{-1}(x) = \frac{1}{x} - 4$$

d) Write $y = \frac{x+1}{x+2}$ and exchange x and y:

$$x = \frac{y+1}{y+2} \overset{\times(y+2)}{\implies} x(y+2) = y+1 \implies xy + 2x = y+1$$
$$\implies xy - y = 1 - 2x \implies y(x-1) = 1 - 2x$$
$$\implies y = \frac{1-2x}{x-1} \implies j^{-1}(x) = \frac{1-2x}{x-1}$$

e) Write $y = (x-2)^2 + 3$ and exchange x and y:

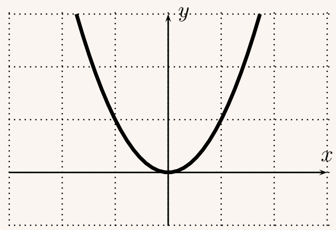
$$x = (y-2)^2 + 3 \implies x - 3 = (y-2)^2 \implies \sqrt{x-3} = y - 2$$
$$\implies y = 2 + \sqrt{x-3} \implies k^{-1}(x) = 2 + \sqrt{x-3}$$

□

The function in the last example is not one-to-one when allowing x to be any real number. This is why we had to restrict the example to the inputs $x \geq 2$. We exemplify the situation in the following note.

Note 6.7

Note that the function $y = x^2$ can be restricted to a one-to-one function by choosing the domain to be all non-negative numbers $[0, \infty)$, or by choosing the domain to be all non-positive numbers $(-\infty, 0]$.



Let $f : [0, \infty) \to [0, \infty)$ be the function $f(x) = x^2$, so that f has a domain of all non-negative numbers. Then, the inverse is the function $f^{-1}(x) = \sqrt{x}$.
On the other hand, we can take $g(x) = x^2$, whose domain consists of all non-positive numbers $(-\infty, 0]$, that is $g : (-\infty, 0] \to [0, \infty)$. Then, the inverse function g^{-1} must reverse domain and range, that is $g^{-1} : [0, \infty) \to (-\infty, 0]$. The inverse is obtained by exchanging x and y in $y = x^2$ as follows:

$$x = y^2 \quad \Longrightarrow y = \pm\sqrt{x} \quad \Longrightarrow g^{-1}(x) = -\sqrt{x}.$$

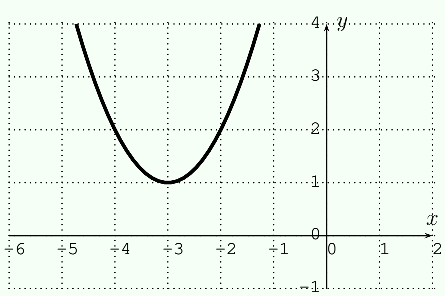
Example 6.8

Restrict the function to a one-to-one function. Find the inverse function, if possible.

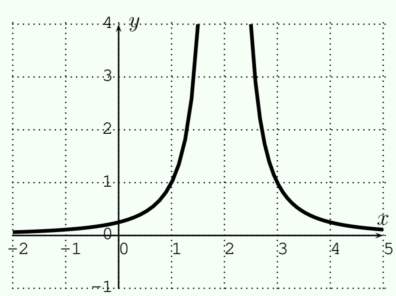
a) $f(x) = (x+3)^2 + 1$ b) $g(x) = \dfrac{1}{(x-2)^2}$ c) $h(x) = x^3 - 3x^2$

Solution.

The graphs of f and g are displayed below.



a)



b)

a) The graph shows that f is one-to-one when restricted to all numbers $x \geq -3$, which is the choice we make to find an inverse function. Next, we replace x and y in $y = (x+3)^2 + 1$ to give $x = (y+3)^2 + 1$. When solving this for y, we must now remember that our choice of $x \geq -3$ becomes $y \geq -3$, after replacing x with y. We now solve for y.

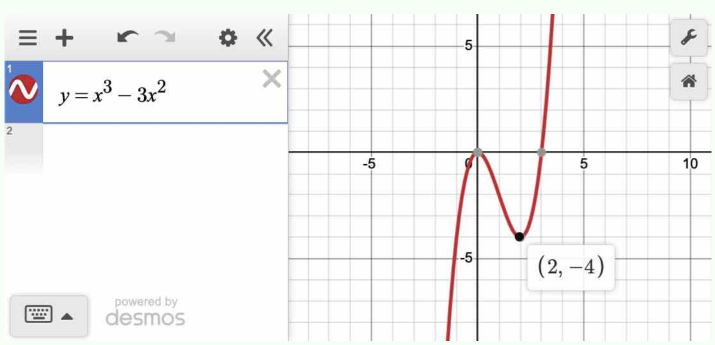
$$x = (y+3)^2 + 1 \implies x - 1 = (y+3)^2 \implies y + 3 = \pm\sqrt{x-1} \implies y = -3 \pm \sqrt{x-1}$$

Since we have chosen the restriction of $y \geq -3$, we use the expression with the positive sign, $y = -3 + \sqrt{x-1}$, so that the inverse function is $f^{-1}(x) = -3 + \sqrt{x-1}$.

b) For the function g, the graph shows that we can restrict g to $x > 2$ to obtain a one-to-one function. The inverse for this choice is given as follows. Replacing x and y in $y = \frac{1}{(x-2)^2}$ gives $x = \frac{1}{(y-2)^2}$, which we solve for y under the condition $y > 2$.

$$x = \frac{1}{(y-2)^2} \implies (y-2)^2 = \frac{1}{x} \implies y - 2 = \pm\frac{1}{\sqrt{x}} \implies y = 2 \pm \frac{1}{\sqrt{x}} \implies g^{-1}(x) = 2 + \frac{1}{\sqrt{x}}$$

c) Finally, $h(x) = x^3 - 3x^2$ can be graphed as follows:

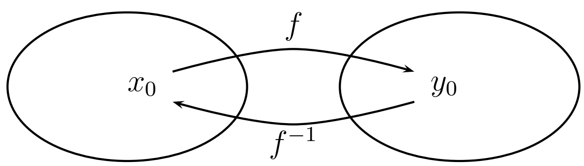


The above picture shows that the local minimum is at (approximately) $x = 2$. Therefore, if we restrict h to all $x \geq 2$, we obtain a one-to-one function. We replace x and y in $y = x^3 - 3x^2$, so that the inverse function is obtained by solving the equation $x = y^3 - 3y^2$ for y. However, this equation is quite complicated and solving it is beyond our capabilities at this time. Therefore, we simply say that $h^{-1}(x)$ is that $y \geq 2$ for which $y^3 - 3y^2 = x$, and leave the example with this.

□

Let f be a one-to-one function. If f maps x_0 to $y_0 = f(x_0)$, then f^{-1} maps y_0 to x_0. In other words, the inverse function is precisely the function for which

$$\boxed{f^{-1}(f(x_0)) = f^{-1}(y_0) = x_0} \quad \text{and} \quad \boxed{f(f^{-1}(y_0)) = f(x_0) = y_0}.$$



We therefore have the following observation.

Observation 6.9: Inverses compose to the identity

Let f and g be two one-to-one functions. Then f and g are inverses of each other exactly when

$$\boxed{f(g(x)) = x} \quad \text{and} \quad \boxed{g(f(x)) = x} \quad \text{for all } x. \qquad (6.1)$$

In this case we write that $g = f^{-1}$ and $f = g^{-1}$.

Example 6.10

Are the following functions inverse to each other?

a) $f(x) = 5x + 7$, $\quad g(x) = \frac{x-7}{5}$
b) $f(x) = \frac{3}{x-6}$, $\quad g(x) = \frac{3}{x} + 6$
c) $f(x) = \sqrt{x} - 3$, $\quad g(x) = x^2 + 3$

Solution.

We calculate the compositions $f(g(x))$ and $g(f(x))$.

$$\text{a)} \quad f(g(x)) = f(\frac{x-7}{5}) = 5 \cdot \frac{x-7}{5} + 7 = (x-7) + 7 = x$$

$$g(f(x)) = g(5x+7) = \frac{(5x+7)-7}{5} = \frac{5x}{5} = x$$

b) $$f(g(x)) = f(\frac{3}{x}+6) = \frac{3}{(\frac{3}{x}+6)-6} = \frac{3}{\frac{3}{x}} = 3\cdot\frac{x}{3} = x$$

$$g(f(x)) = g(\frac{3}{x-6}) = \frac{3}{\frac{3}{x-6}}+6 = 3\cdot\frac{x-6}{3}+6$$
$$= (x-6)+6 = x$$

Using the Observation 6.9, we see that in both part (a) and (b) the functions are inverse to each other. For part (c), we calculate for a general x in the domain of g:

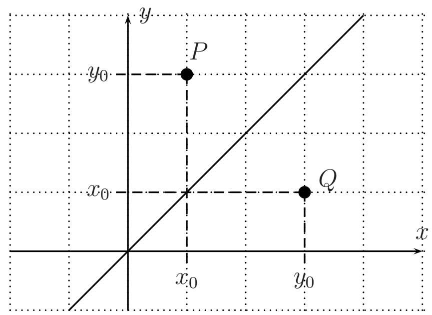
$$f(g(x)) = \sqrt{x^2+3}-3 \neq x.$$

It is enough to show that for one composition $(f \circ g)(x)$ does not equal x to conclude that f and g are not inverses. (It is not necessary to also calculate the other composition $g(f(x))$.) □

Be careful!

If f and g are functions such that the range of f is the domain of g, and the range of g is the domain of f, then one of the two equations in (6.1) also implies the validity of the other equation in (6.1). In other words, if, for example, we know that $f(g(x)) = x$ is true, then $g(f(x)) = x$ is also true. Nevertheless, we recommend to always check *both* equations. The reason for this is that it is easy to mistake one of the relations when we are not careful about the domain and range.
For example, let $f(x) = x^2$ and $g(x) = -\sqrt{x}$. Then, naively, we would calculate $f(g(x)) = (-\sqrt{x})^2 = x$ and $g(f(x)) = -\sqrt{x^2} = -|x|$, so that the first equation would say f and g are inverses, whereas the second equation may lead us to think they are not inverses.
We can resolve this apparent contradiction by being precise about the domain that we consider for f. Note that we can only find an inverse for f if we choose a domain that makes f into a one-to-one function. For example, if we take the domain of f to be all positive numbers and zero, $D_f = [0,\infty)$, then $f(g(x)) = f(-\sqrt{x})$ which is undefined, since f only takes non-negative inputs. Also, we have $g(f(x)) = -\sqrt{x^2} = -x$. Therefore, neither $f(g(x))$ equals x, nor $(g(f(x))$ equals x. The functions f and g *are not* inverse to each other!
On the other hand, if we restrict the function $f(x) = x^2$ to all negative numbers and zero, $D_f = (-\infty, 0]$, then $f(g(x)) = (-\sqrt{x})^2 = x$, since now f is defined for the negative input $-\sqrt{x}$. Also, for a negative number $x < 0$, we have $g(f(x)) = -\sqrt{x^2} = -|x| = x$. So, in this case, f and g *are* inverse to each other!

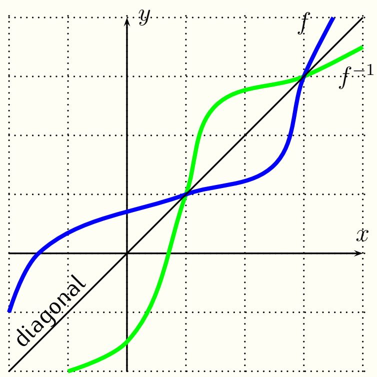
Our last observation in this chapter concerns the graph of inverse functions. If $f(x_0) = y_0$ then $f^{-1}(y_0) = x_0$, and the point $P(x_0, y_0)$ is on the graph of f, whereas the point $Q(y_0, x_0)$ is on the graph of f^{-1}.



We see that Q is the reflection of P along the diagonal $y = x$. Since this is true for any point on the graph of f and f^{-1}, we have the following general observation.

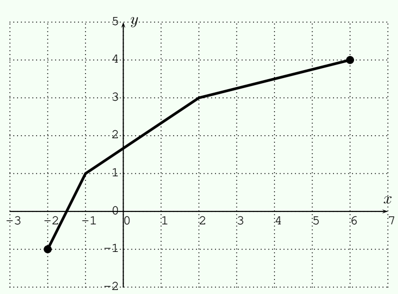
Observation 6.11: Graph of an inverse function

The graph of f^{-1} is the graph of f reflected about the diagonal.



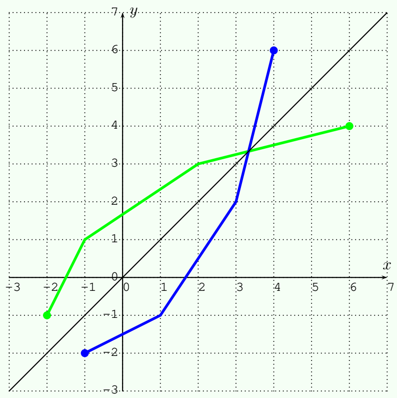
Example 6.12

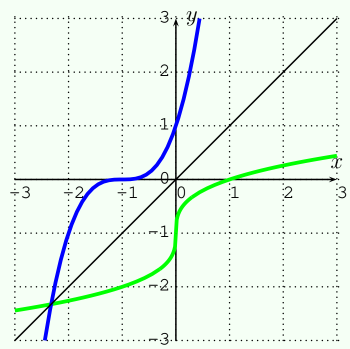
Find the graph of the inverse function of the function given below.

a) 

b) $f(x) = (x + 1)^3$

Solution.
Carefully reflecting the graphs given in part (a) and (b) gives the following solution. The function $f(x) = (x+1)^3$ in part (b) can be graphed with a graphing calculator first.

a) 

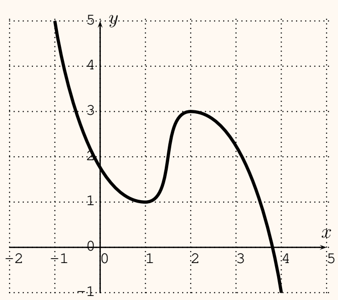
b) 

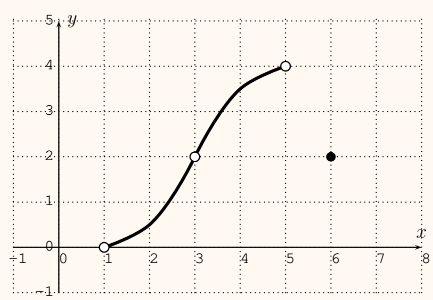
□

6.3 Exercises

Exercise 6.1

Use the horizontal line test to determine whether the function is one-to-one.

a) 

b) 

c) $f(x) = x^2 + 2x + 5$
d) $f(x) = x^2 - 14x + 29$
e) $f(x) = x^3 - 5x^2$
f) $f(x) = \frac{x^2}{x^2-3}$
g) $f(x) = \sqrt{x+2}$
h) $f(x) = \sqrt{|x+2|}$

Exercise 6.2

Find the inverse of the function f and check your solution.

a) $f(x) = 4x + 9$
b) $f(x) = -8x - 3$
c) $f(x) = \sqrt{x+8}$
d) $f(x) = \sqrt{3x+7}$
e) $f(x) = 6 \cdot \sqrt{-x-2}$
f) $f(x) = x^3$
g) $f(x) = (2x+5)^3$
h) $f(x) = 2 \cdot x^3 + 5$

i) $f(x) = \frac{1}{x}$
j) $f(x) = \frac{1}{x-1}$
k) $f(x) = \frac{1}{\sqrt{x-2}}$
l) $f(x) = \frac{-5}{4-x}$
m) $f(x) = \frac{x}{x+2}$
n) $f(x) = \frac{3x}{x-6}$
o) $f(x) = \frac{x+2}{x+3}$
p) $f(x) = \frac{7-x}{x-5}$
q) f given by the table below:

x	2	4	6	8	10	12
$f(x)$	3	7	1	8	5	2

Exercise 6.3

Restrict the domain of the function f in such a way that f becomes a one-to-one function. Find the inverse of f with the restricted domain.

a) $f(x) = x^2$
b) $f(x) = (x+5)^2 + 1$
c) $f(x) = |x|$
d) $f(x) = |x-4| - 2$
e) $f(x) = \frac{1}{x^2}$
f) $f(x) = \frac{-3}{(x+7)^2}$
g) $f(x) = x^4$
h) $f(x) = \frac{(x-3)^4}{10}$

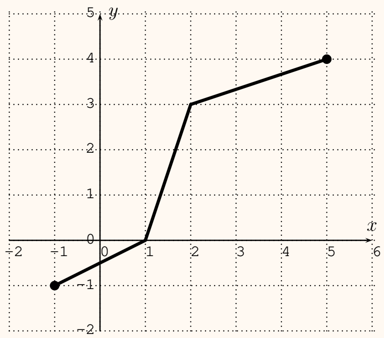
Exercise 6.4

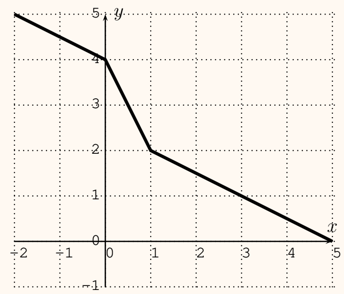
Determine whether the following functions f and g are inverse to each other.

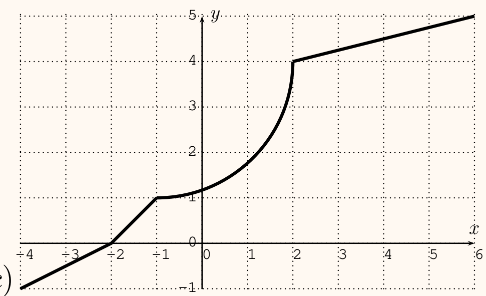
a) $f(x) = x + 3$ and $g(x) = x - 3$
b) $f(x) = -x - 4$ and $g(x) = 4 - x$
c) $f(x) = 2x + 3$ and $g(x) = x - \frac{3}{2}$
d) $f(x) = 6x - 1$ and $g(x) = \frac{x+1}{6}$
e) $f(x) = x^3 - 5$ and $g(x) = 5 + \sqrt[3]{x}$
f) $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{1}{x} + 2$

Exercise 6.5

Draw the graph of the inverse of the function given below.

a) 

b) 

c) 

d) $f(x) = \sqrt{x}$
e) $f(x) = x^3 - 4$
f) $f(x) = 2x - 4$
g) $f(x) = 2^x$
h) $f(x) = \frac{1}{x-2}$ for $x > 2$
i) $f(x) = \frac{1}{x-2}$ for $x < 2$.

Review of functions and graphs

Exercise I.1

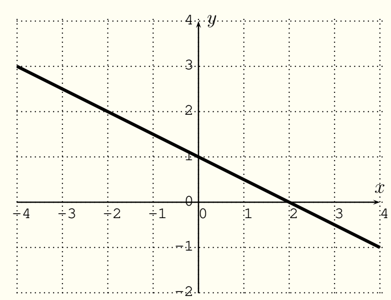
Let f be the piecewise defined function:

$$f(x) = \begin{cases} 2x+3 & \text{, for } -8 < x \leq -4 \\ \frac{x}{2} & \text{, for } -3 < x < 2 \\ x^2+5x & \text{, for } 4 \leq x \end{cases}$$

Find $f(6)$, $f(2)$, $f(-6)$. State the domain of f.

Exercise I.2

Find the equation of the line displayed below.



Exercise I.3

Find all solutions of the equation with the graphing calculator:

$$x^3 - 4x^2 + 2x + 2 = 0$$

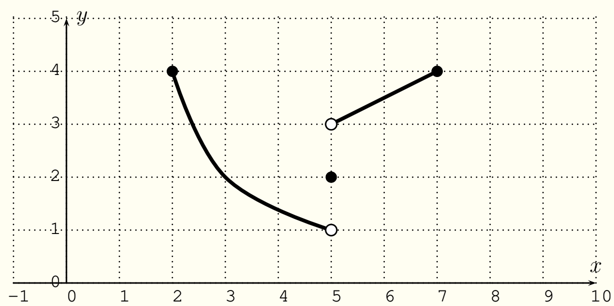
Approximate your answer to the nearest thousandth.

Exercise I.4

Let $f(x) = x^2 - 2x + 5$. Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ as much as possible.

Exercise I.5

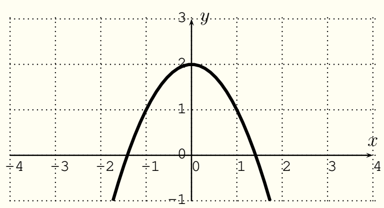
Consider the following graph of a function f.



Find: domain of f, range of f, $f(3)$, $f(5)$, $f(7)$, $f(9)$.

Exercise I.6

Find the possible formula for the graph displayed below.



Exercise I.7

Let $f(x) = 5x + 4$ and $g(x) = x^2 + 8x + 7$. Find the quotient $\left(\frac{f}{g}\right)(x)$ and state its domain.

Exercise I.8

Let $f(x) = x^2 + \sqrt{x-3}$ and $g(x) = 2x - 3$. Find the composition $(f \circ g)(x)$ and state its domain.

Exercise I.9

Consider the assignments for f and g given by the table below.

x	2	3	4	5	6
$f(x)$	5	0	2	4	2
$g(x)$	6	2	3	4	1

Is f a function? Is g a function? Write the composed assignment for $(f \circ g)(x)$ as a table.

Exercise I.10

Find the inverse of the function $f(x) = \frac{1}{2x+5}$.

Part II

Polynomials and Rational Functions

Chapter 7

Dividing polynomials

We now start our discussion of specific classes of examples of functions. The first class of functions that we discuss are polynomials and rational functions. In this section we discuss an important tool for analyzing these functions, which consists of dividing two polynomials, also known as long division. Before we get to this, let us first recall the definition of polynomials and rational functions.

Definition 7.1: Monomial, polynomial

A **monomial** is a number, a variable, or a product of numbers and variables. A **polynomial** is a sum (or difference) of monomials.

Example 7.2

The following are examples of monomials:

$$5, \quad x, \quad 7x^2y, \quad -12x^3y^2z^4, \quad \sqrt{2}\cdot a^3n^2xy$$

The following are examples of polynomials:

$$x^2+3x-7, \quad 4x^2y^3+2x+z^3+4mn^2, \quad -5x^3-x^2-4x-9, \quad 5x^2y^4$$

In particular, every monomial is also a polynomial.

We are mainly interested in polynomials in one variable x, and consider these as functions. For example, $f(x)=x^2+3x-7$ is such a function.

Definition 7.3: Polynomial in one variable

A **polynomial in one variable** is a function f of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for some constants $a_0, a_1, \ldots, a_n$, where $a_n \neq 0$ and n is a non-negative integer. The numbers $a_0, a_1, \ldots, a_n$ are called **coefficients**. For each k, the number a_k is the **coefficient of** x^k. The number a_n is called the **leading coefficient** and n is the **degree** of the polynomial.
We usually consider polynomials f with real coefficients. In this case, the domain of a f is all real numbers (see our standard convention 2.7).
A **root** or **zero** of a polynomial $f(x)$ is a number c so that $f(c) = 0$.

Definition 7.4: Rational function

A **rational function** is a fraction of two polynomials $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are both polynomials, and $h \neq 0$ is not the zero function. The domain of f is all real numbers for which the denominator $h(x)$ is not zero:

$$D_f \;=\; \{\ x \mid h(x) \neq 0\ \}.$$

Example 7.5

The following are examples of rational functions:

$$f(x) = \frac{-3x^2 + 7x - 5}{2x^3 + 4x^2 + 3x + 1}, \qquad f(x) = \frac{1}{x}, \qquad f(x) = -x^2 + 3x + 5$$

7.1 Long division

An important tool for analyzing polynomials consists of dividing two polynomials. The method of dividing polynomials that we use is that of *long division*, which is similar to the long division of natural numbers. Our first example shows the procedure in detail.

Example 7.6

Divide the following fractions via long division:

$$\text{a)}\ \frac{3571}{11} \qquad \text{b)}\ \frac{x^3+5x^2+4x+2}{x+3}$$

Solution.

a) Recall the procedure for long division of natural numbers:

$$\begin{array}{r|r}
 & 324 \\ \hline
11 & 3571 \\
 & -33 \\ \hline
 & 271 \\
 & -22 \\ \hline
 & 51 \\
 & -44 \\ \hline
 & 7 \quad = \text{remainder}
\end{array}$$

The steps above are performed as follows. First, we find the largest multiple of 11 less or equal to 35. The answer 3 is written as the first digit on the top line. Multiply 3 times 11, and subtract the answer 33 from the first two digits 35 of the dividend. The remaining digits 71 are copied below to give 271. Now we repeat the procedure, until we arrive at the remainder 7. In short, what we have shown is that:

$$3571 = 324 \cdot 11 + 7 \quad \text{or alternatively,} \quad \frac{3571}{11} = 324 + \frac{7}{11}.$$

b) We repeat the steps from part (a) as follows. First, write the dividend and divisor as in the format above:

$$x+3 \;\big|\; x^3 \quad +5x^2 \quad +4x \quad +2$$

Next, consider the highest term x^3 of the dividend and the highest term x of the divisor. Since $\frac{x^3}{x} = x^2$, we start with the first term x^2 of the quotient:

Step 1:

$$\begin{array}{r|llll}
 & x^2 & & & \\ \hline
x+3 & x^3 & +5x^2 & +4x & +2
\end{array}$$

Multiply x^2 by the divisor $x+3$ and write it below the dividend:

Step 2:
$$\begin{array}{r|llll} & x^2 & & & \\ \cline{2-5} x+3 & x^3 & +5x^2 & +4x & +2 \\ & x^3 & +3x^2 & & \end{array}$$

Since we need to subtract x^3+3x^2, so we equivalently add its negative (don't forget to distribute the negative):

Step 3:
$$\begin{array}{r|rrrr} & & x^2 & & \\ \cline{2-5} x+3 & x^3 & +5x^2 & +4x & +2 \\ & -(x^3 & +3x^2) & & \\ \cline{2-3} & & 2x^2 & & \end{array}$$

Now, carry down the remaining terms of the dividend:

Step 4:
$$\begin{array}{r|rrrr} & & x^2 & & \\ \cline{2-5} x+3 & x^3 & +5x^2 & +4x & +2 \\ & -(x^3 & +3x^2) & & \\ \cline{2-3} & & 2x^2 & +4x & +2 \end{array}$$

Now, repeat steps 1–4 for the remaining polynomial $2x^2+4x+2$. The outcome after going through steps 1–4 is the following:

$$\begin{array}{r|rrrrl} & & x^2 & +2x & & \text{(add } \frac{2x^2}{x} = 2x\text{)} \\ \cline{2-5} x+3 & x^3 & +5x^2 & +4x & +2 & \\ & -(x^3 & +3x^2) & & & \\ \cline{2-3} & & 2x^2 & +4x & +2 & \\ & & -(2x^2 & +6x) & & \text{(multiply } 2x \text{ by } (x+3)\text{)} \\ \cline{3-4} & & & -2x & +2 & \text{(subtract from the above)} \end{array}$$

Since x can be divided into $-2x$, we can proceed with the above steps 1–4 one more time. The outcome is this:

$$
\begin{array}{r|llll}
 & & x^2 & +2x & -2 \\
\hline
x+3 & x^3 & +5x^2 & +4x & +2 \\
 & -(x^3 & +3x^2) & & \\
\hline
 & & 2x^2 & +4x & +2 \\
 & & -(2x^2 & +6x) & \\
\hline
 & & & -2x & +2 \\
 & & & -(-2x & -6) \\
\hline
 & & & & 8 \quad = \text{remainder}
\end{array}
$$

Note now that x cannot be divided into 8 so we stop here. The final term 8 is called the remainder. The term $x^2 + 2x - 2$ is called the quotient. In analogy with our result in part (a), we can write our conclusion as:

$$x^3 + 5x^2 + 4x + 2 = (x^2 + 2x - 2) \cdot (x + 3) + 8.$$

Alternatively, we could also divide this by $(x + 3)$ and write it as:

$$\frac{x^3 + 5x^2 + 4x + 2}{x + 3} = x^2 + 2x - 2 + \frac{8}{x + 3}.$$

□

Note 7.7: Dividend, divisor, quotient, remainder

Just as with a division operation involving numbers, when dividing $\frac{f(x)}{g(x)}$, $f(x)$ is called the **dividend** and $g(x)$ is called the **divisor**. As a result of dividing $f(x)$ by $g(x)$ via long division with **quotient** $q(x)$ and **remainder** $r(x)$, we can write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}. \tag{7.1}$$

If we multiply this equation by $g(x)$, we obtain the following alternative version:

$$f(x) = q(x) \cdot g(x) + r(x) \tag{7.2}$$

Example 7.8

Divide the following fractions via long division.

a) $\frac{x^2+4x+5}{x-4}$ b) $\frac{x^4+3x^3-5x+1}{x+1}$

c) $\frac{4x^3+2x^2+6x+18}{2x+3}$ d) $\frac{x^3+x^2+2x+1}{x^2+3x+1}$

Solution.

a) We calculate:

$$
\begin{array}{r|rrr}
 & & x & +8 \\ \cline{2-4}
x-4 & x^2 & +4x & +5 \\
 & -(x^2 & -4x) & \\ \cline{2-3}
 & & 8x & +5 \\
 & & -(8x & -32) \\ \cline{3-4}
 & & & 37
\end{array}
$$

Therefore, $x^2+4x+5 = (x+8)\cdot(x-4)+37$.

b) Note that there is no x^2 term in the dividend. This can be resolved by adding $+0\,x^2$ to the dividend:

$$
\begin{array}{r|rrrrr}
 & & x^3 & +2x^2 & -2x & -3 \\ \cline{2-6}
x+1 & x^4 & +3x^3 & +0x^2 & -5x & +1 \\
 & -(x^4 & +x^3) & & & \\ \cline{2-3}
 & & 2x^3 & +0x^2 & -5x & +1 \\
 & & -(2x^3 & +2x^2) & & \\ \cline{3-4}
 & & & -2x^2 & -5x & +1 \\
 & & & -(-2x^2 & -2x) & \\ \cline{4-5}
 & & & & -3x & +1 \\
 & & & & -(-3x & -3) \\ \cline{5-6}
 & & & & & 4
\end{array}
$$

Therefore, we showed:

$$\frac{x^4+3x^3-5x+1}{x+1} = x^3+2x^2-2x-3+\frac{4}{x+1}.$$

c)

$$
\begin{array}{r|rrrr}
 & & 2x^2 & -2x & +6 \\ \cline{2-5}
2x+3 & 4x^3 & +2x^2 & +6x & +18 \\
 & -(4x^3 & +6x^2) & & \\ \cline{2-3}
 & & -4x^2 & +6x & +18 \\
 & & -(-4x^2 & -6x) & \\ \cline{3-4}
 & & & 12x & +18 \\
 & & & -(12x & +18) \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Since the remainder is zero, we succeeded in factoring $4x^3 + 2x^2 + 6x + 18$:

$$4x^3 + 2x^2 + 6x + 18 = (2x^2 - 2x + 6) \cdot (2x + 3)$$

d) The last example has a divisor that is a polynomial of degree 2. Therefore, the remainder is not a number, but a polynomial of degree 1.

$$
\begin{array}{r|rrrr}
 & & & x & -2 \\ \cline{2-5}
x^2+3x+1 & x^3 & +x^2 & +2x & +1 \\
 & -(x^3 & +3x^2 & +x) & \\ \cline{2-4}
 & & -2x^2 & +x & +1 \\
 & & -(-2x^2 & -6x & -2) \\ \cline{3-5}
 & & & 7x & +3
\end{array}
$$

Here, the remainder is $r(x) = 7x + 3$.

$$\frac{x^3 + x^2 + 2x + 1}{x^2 + 3x + 1} = x - 2 + \frac{7x + 3}{x^2 + 3x + 1}$$

□

Note 7.9: Factoring and zero remainder

The divisor $g(x)$ is a factor of $f(x)$ exactly when the remainder $r(x)$ is zero, that is:

$$f(x) = q(x) \cdot g(x) \quad \iff \quad r(x) = 0.$$

For example, in the above Example 7.8, only part (c) results in a factorization of the dividend, since this is the only part with remainder zero.

7.2 Dividing by $(x-c)$

We now restrict our attention to the case in which the divisor is $g(x)=x-c$ for some real number c. In this case, the remainder r of the division $f(x)$ by $g(x)$ is a real number. We make the following observations.

Observation 7.10: Remainder theorem, factor theorem

Assume that $g(x)=x-c$, and the long division of $f(x)$ by $g(x)$ has remainder r, that is,

$$\text{Assumption: } f(x)=q(x)\cdot(x-c)+r.$$

When we evaluate both sides in the above equation at $x=c$, we see that $f(c)=q(c)\cdot(c-c)+r=q(c)\cdot 0+r=r$. In short:

$$\boxed{\text{The remainder when dividing } f(x) \text{ by } (x-c) \text{ is } \quad r=f(c).} \tag{7.3}$$

In particular:

$$\boxed{f(c)=0 \quad\Longleftrightarrow\quad g(x)=x-c \text{ is a factor of } f(x).} \tag{7.4}$$

The above statement (7.3) is called the **remainder theorem**, and (7.4) is called the **factor theorem**.

Example 7.11

Find the remainder of dividing $f(x)=x^2+3x+2$ by

a) $x-3$ b) $x+4$ c) $x+1$ d) $x-\frac{1}{2}$

Solution.

a) By Observation 7.10, we know that the remainder r of the division by $x-c$ is $f(c)$. Thus, the remainder for part (a), when dividing by

$x - 3$ is

$$r = f(3) = 3^2 + 3 \cdot 3 + 2 = 9 + 9 + 2 = 20.$$

b) For (b), note that $g(x) = x + 4 = x - (-4)$, so that taking $c = -4$ for our input yields a remainder of $r = f(-4) = (-4)^2 + 3 \cdot (-4) + 2 = 16 - 12 + 2 = 6$. Similarly, the other remainders are:

c) $r = f(-1) = (-1)^2 + 3 \cdot (-1) + 2 = 1 - 3 + 2 = 0,$

d) $r = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \frac{1}{2} \cdot 3 + 2 = \frac{1}{4} + \frac{3}{2} + 2 = \frac{1+6+8}{4} = \frac{15}{4}.$

Note that in part (c), we found a remainder 0, so that $(x + 1)$ is a factor of $f(x)$.

□

Example 7.12

Determine whether $g(x)$ is a factor of $f(x)$.

a) $f(x) = x^3 + 2x^2 + 5x + 1,\quad g(x) = x - 2$
b) $f(x) = x^4 + 4x^3 + x^2 + 18,\quad g(x) = x + 3$
c) $f(x) = x^5 + 3x^2 + 7,\quad g(x) = x + 1$

Solution.

a) We need to determine whether 2 is a root of $f(x) = x^3 + 2x^2 + 5x + 1$, that is, whether $f(2)$ is zero.

$$f(2) = 2^3 + 2 \cdot 2^2 + 5 \cdot 2 + 1 = 8 + 8 + 10 + 1 = 27.$$

Since $f(2) = 27 \neq 0$, we see that $g(x) = x - 2$ is not a factor of $f(x)$.

b) Now, $g(x) = x + 3 = x - (-3)$, so that we calculate:

$$f(-3) = (-3)^4 + 4 \cdot (-3)^3 + (-3)^2 - 18 = 81 - 108 + 9 + 18 = 0.$$

Since the remainder is zero, we see that $x + 3$ is a factor of $x^4 + 4 \cdot x^3 + x^2 + 18$. Therefore, if we wanted to find the other factor, we could use long division to obtain the quotient.

c) Finally, we have:

$$f(-1) = (-1)^5 + 3 \cdot (-1)^2 + 7 = -1 + 3 + 7 = 9.$$

$g(x) = x + 1$ is not a factor of $f(x) = x^5 + 3x^2 + 7$.

□

Example 7.13

a) Show that -2 is a root of $f(x) = x^5 - 3x^3 + 5x^2 - 12$, and use this to factor f.

b) Show that 5 is a root of $f(x) = x^3 - 19x - 30$, and use this to factor f *completely*.

Solution.

a) First, we calculate that -2 is a root.

$$f(-2) = (-2)^5 - 3 \cdot (-2)^3 + 5 \cdot (-2)^2 + 12 = -32 + 24 + 20 - 12 = 0.$$

So we can divide $f(x)$ by $g(x) = x - (-2) = x + 2$:

$$\begin{array}{r|rrrrrr}
 & & x^4 & -2x^3 & +x^2 & +3x & -6 \\ \hline
x+2 & x^5 & +0x^4 & -3x^3 & +5x^2 & +0x & -12 \\
 & -(x^5 & +2x^4) & & & & \\ \cline{2-3}
 & & -2x^4 & -3x^3 & +5x^2 & +0x & -12 \\
 & & -(-2x^4 & -4x^3) & & & \\ \cline{3-4}
 & & & x^3 & +5x^2 & +0x & -12 \\
 & & & -(x^3 & +2x^2) & & \\ \cline{4-5}
 & & & & 3x^2 & +0x & -12 \\
 & & & & -(3x^2 & +6x) & \\ \cline{5-6}
 & & & & & -6x & -12 \\
 & & & & & -(-6x & -12) \\ \cline{6-7}
 & & & & & & 0
\end{array}$$

So we factored $f(x)$ as

$$f(x) = (x^4 - 2x^3 + x^2 + 3x - 6) \cdot (x + 2).$$

b) Again, we start by calculating $f(5) = 5^3 - 19 \cdot 5 - 30 = 125 - 95 - 30 = 0$. Long division by $g(x) = x - 5$ gives:

$$
\begin{array}{r|rrrr}
 & & x^2 & +5x & +6 \\
\hline
x-5 & x^3 & +0x^2 & -19x & -30 \\
 & -(x^3 & -5x^2) & & \\
\cline{2-3}
 & & 5x^2 & -19x & -30 \\
 & & -(5x^2 & -25x) & \\
\cline{3-4}
 & & & 6x & -30 \\
 & & & -(6x & -30) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Thus, $x^3 - 19x - 30 = (x^2 + 5x + 6) \cdot (x - 5)$. To factor f completely, we also factor $x^2 + 5x + 6$.

$$f(x) = (x^2 + 5x + 6) \cdot (x - 5) = (x + 2) \cdot (x + 3) \cdot (x - 5).$$

□

7.3 *Optional section:* Synthetic division

When dividing a polynomial $f(x)$ by $g(x) = x - c$, the actual calculation of the long division has a lot of unnecessary repetitions, and we may want to reduce this redundancy as much as possible. In fact, we can extract the essential part of the long division, the result of which is called **synthetic division**.

Example 7.14

Our first example is the long division of $\frac{5x^3+7x^2+x+4}{x+2}$.

$$\begin{array}{r|rrrr} & & 5x^2 & -3x & +7 \\ \hline x+2 & 5x^3 & +7x^2 & +x & +4 \\ & -(5x^3 & +10x^2) & & \\ \cline{2-3} & & -3x^2 & +x & +4 \\ & & -(-3x^2 & -6x) & \\ \cline{3-4} & & & 7x & +4 \\ & & & -(7x & +14) \\ \cline{4-5} & & & & -10 \end{array}$$

Here, the first term $5x^2$ of the quotient is just copied from the first term of the dividend. We record this together with the coefficients of the dividend $5x^3+7x^2+x+4$ and of the divisor $x+2 = x-(-2)$ as follows:

$$\begin{array}{r|rrrr} & 5 & 7 & 1 & 4 \\ -2 & & & & \\ \hline & 5 & & & \end{array} \qquad \begin{array}{l} \text{(dividend } (5x^3+7x^2+x+4)) \\ \text{(divisor } (x-(-2))) \\ \text{(quotient)} \end{array}$$

The first actual calculation is performed when multiplying the $5x^2$ term with 2, and subtracting it from $7x^2$. We record this as follows:

$$\begin{array}{r|rrrr} & 5 & 7 & 1 & 4 \\ -2 & & -10 & & \\ \hline & 5 & -3 & & \end{array} \qquad \begin{array}{l} (-10 \text{ is the product of } -2 \cdot 5) \\ (-3 \text{ is the sum } 7+(-10)) \end{array}$$

Similarly, we obtain the next step by multiplying the $2x$ by (-3) and subtracting it from $1x$. Therefore, we get

$$\begin{array}{r|rrrr} & 5 & 7 & 1 & 4 \\ -2 & & -10 & 6 & \\ \hline & 5 & -3 & 7 & \end{array} \qquad \begin{array}{l} (6 \text{ is the product of } -2 \cdot (-3)) \\ (7 \text{ is the sum } 1+(6)) \end{array}$$

The last step multiplies 7 times 2 and subtracts this from 4. In short, we write:

$$\begin{array}{r|rrrr} & 5 & 7 & 1 & 4 \\ -2 & & -10 & 6 & -14 \\ \hline & 5 & -3 & 7 & -10 \end{array} \qquad \begin{array}{l} (-14 \text{ is the product of } -2 \cdot (7)) \\ (-10 \text{ is the sum } 4+(-14)) \end{array}$$

The answer can be determined from these coefficients. The quotient is $5x^2 - 3x + 7$, and the remainder is -10.

Example 7.15

Find the following quotients via synthetic division.

$$\text{a) } \frac{4x^3 - 7x^2 + 4x - 8}{x - 4} \qquad \text{b) } \frac{x^4 - x^2 + 5}{x + 3}$$

Solution.

a) We need to perform the synthetic division.

$$\begin{array}{r|rrrr} & 4 & -7 & 4 & -8 \\ 4 & & 16 & 36 & 160 \\ \hline & 4 & 9 & 40 & 152 \end{array}$$

Therefore we have

$$\frac{4x^3 - 7x^2 + 4x - 8}{x - 4} = 4x^2 + 9x + 40 + \frac{152}{x - 4}.$$

b) Similarly, we calculate part (b). Note that some of the coefficients are now zero.

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & 0 & 5 \\ -3 & & -3 & 9 & -24 & 72 \\ \hline & 1 & -3 & 8 & -24 & 77 \end{array}$$

We obtain the following result.

$$\frac{x^4 - x^2 + 5}{x + 3} = x^3 - 3x^2 + 8x - 24 + \frac{77}{x + 5}$$

□

Note 7.16

We have only considered synthetic division when dividing by a polynomial of the form $x - c$. The method for dividing by polynomials such as $3x + 7$ or $x^2 + 5x - 4$ is more elaborate.

7.4 Exercises

Exercise 7.1

Divide by long division.

a) $\frac{x^3-4x^2+2x+1}{x-2}$ b) $\frac{x^3+6x^2+7x-2}{x+3}$ c) $\frac{x^2+7x-4}{x+1}$

d) $\frac{x^3+3x^2+2x+5}{x+2}$ e) $\frac{2x^3+x^2+3x+5}{x-1}$ f) $\frac{2x^4+7x^3+x+3}{x+5}$

g) $\frac{2x^4-31x^2-13}{x-4}$ h) $\frac{x^3+27}{x+3}$ i) $\frac{3x^4+7x^3+5x^2+7x+4}{3x+1}$

j) $\frac{8x^3+18x^2+21x+18}{2x+3}$ k) $\frac{x^3+3x^2-4x-5}{x^2+2x+1}$ l) $\frac{x^5+3x^4-20}{x^2+3}$

Exercise 7.2

Find the remainder when dividing $f(x)$ by $g(x)$.

a) $f(x) = x^3 + 2x^2 + x - 3$, $g(x) = x - 2$
b) $f(x) = x^3 - 5x + 8$, $g(x) = x - 3$
c) $f(x) = x^5 - 1$, $g(x) = x + 1$
d) $f(x) = x^5 + 5x^2 - 7x + 10$, $g(x) = x + 2$

Exercise 7.3

Determine whether the given $g(x)$ is a factor of $f(x)$. If so, name the corresponding root of $f(x)$.

a) $f(x) = x^2 + 5x + 6$, $g(x) = x + 3$
b) $f(x) = x^3 - x^2 - 3x + 8$, $g(x) = x - 4$
c) $f(x) = x^4 + 7x^3 + 3x^2 + 29x + 56$, $g(x) = x + 7$
d) $f(x) = x^{999} + 1$, $g(x) = x + 1$

Exercise 7.4

Check that the given numbers for x are roots of $f(x)$ (see Observation 7.10). If the numbers x are indeed roots, then use this information to factor $f(x)$ as much as possible.

a) $f(x) = x^3 - 2x^2 - x + 2$, $x = 1$
b) $f(x) = x^3 - 6x^2 + 11x - 6$, $x = 1, x = 2, x = 3$
c) $f(x) = x^3 - 3x^2 + x - 3$, $x = 3$
d) $f(x) = x^3 + 6x^2 + 12x + 8$, $x = -2$
e) $f(x) = x^3 + 13x^2 + 50x + 56$, $x = -2, x = -4$
f) $f(x) = x^3 + 3x^2 - 16x - 48$, $x = 2, x = -4$
g) $f(x) = x^5 + 5x^4 - 5x^3 - 25x^2 + 4x + 20$, $x = 1, x = -1,$ $x = 2, x = -2$

Exercise 7.5

Divide by using synthetic division.

a) $\frac{2x^3+3x^2-5x+7}{x-2}$ b) $\frac{4x^3+3x^2-15x+18}{x+3}$ c) $\frac{x^3+4x^2-3x+1}{x+2}$

d) $\frac{x^4+x^3+1}{x-1}$ e) $\frac{x^5+32}{x+2}$ f) $\frac{x^3+5x^2-3x-10}{x+5}$

Chapter 8

Graphing polynomials

We now discuss the features of graphs of polynomial functions.

8.1 Graphs of polynomials

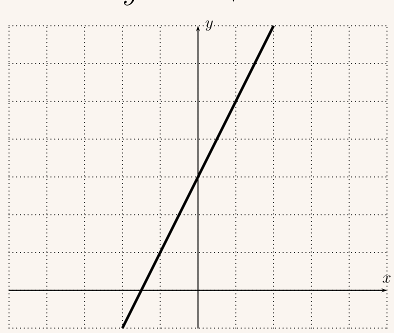
We study graphs of polynomials of various degrees. Recall from definition 7.3 that a polynomial function f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad \text{with } a_n \neq 0.$$

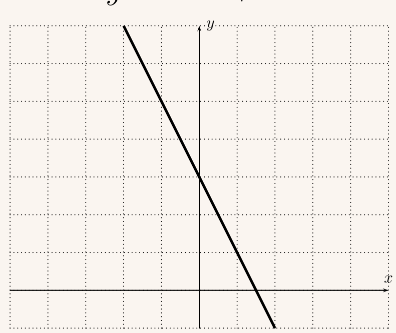
Note 8.1: Polynomial of degree 1

We already know from Section 3.1 that the graphs of polynomials of degree 1, that is, $f(x) = ax + b$, are straight lines.

$y = 2x + 3$



$y = -2x + 3$

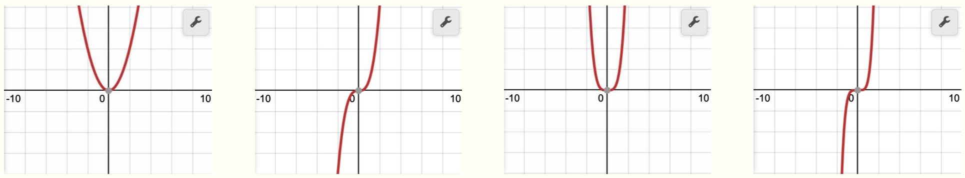


Polynomials of degree 1 have only one root.

We can also easily sketch the graphs of the functions $f(x) = x^n$.

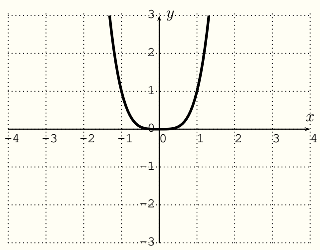
Observation 8.2: $f(x) = x^n$

Graphing $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, we obtain:

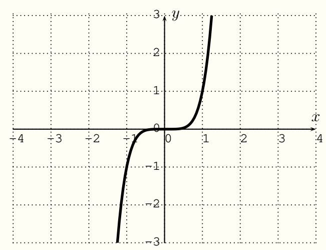


$y = x^2$ $\qquad$ $y = x^3$ $\qquad$ $y = x^4$ $\qquad$ $y = x^5$

From this, we see that the shape of the graph of $f(x) = x^n$ depends on n being even or odd.



$y = x^n$, for n even
If x approaches $\pm\infty$,
$\Longrightarrow$ y approaches $+\infty$.



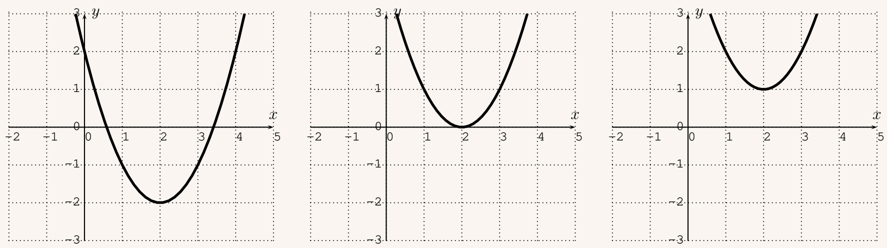
$y = x^n$, for n odd
If x approaches $\pm\infty$,
$\Longrightarrow$ y approaches $\pm\infty$.

Next, we look at graphs of general polynomials of degrees 2, 3, 4, 5, and more generally, of any degree n. In particular, we will be interested in the number of real roots (which are shown at the x-intercepts in the graph of f) and the number of extrema (that is the number of maxima or minima) of a polynomial f.

Note 8.3: Polynomial of degree 2

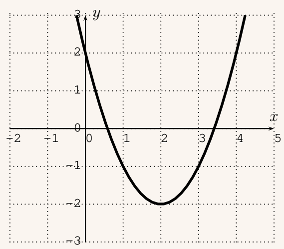
Let $f(x) = ax^2 + bx + c$ be a polynomial of degree 2. The graph of f is a parabola.

- f has at most 2 real roots (displayed at the x-intercepts). f has one extremum (that is one maximum or minimum).

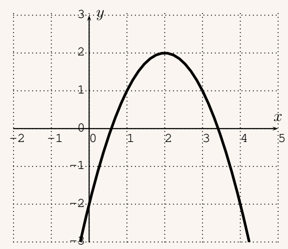


- If $a > 0$ then f opens upward; if $a < 0$ then f opens downward.

$y = x^2 - 4x + 2$



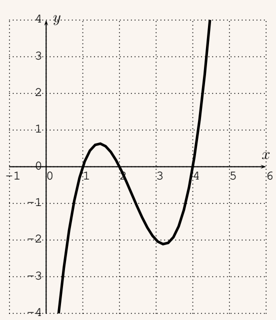
$y = -x^2 + 4x - 2$



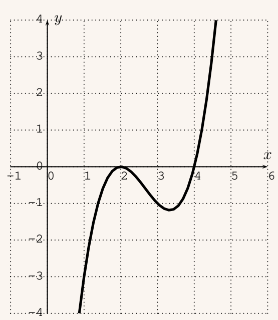
Note 8.4: Polynomial of degree 3

Let $f(x) = ax^3 + bx^2 + cx + d$ be a polynomial of degree 3. The graph may change its direction at most twice.

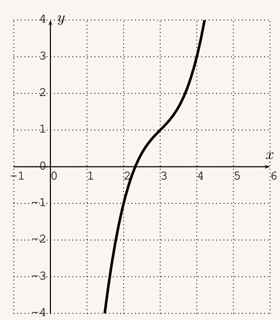
- f has at most 3 real roots. f has at most 2 extrema.



3 roots, 2 extrema



2 roots, 2 extrema

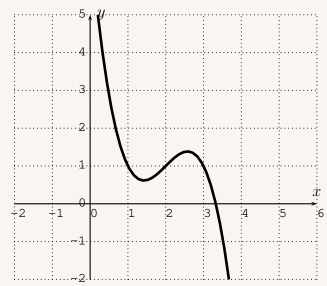


1 root, 0 extrema

- If $a > 0$ then $f(x)$ approaches $+\infty$ when x approaches $+\infty$ (that is, $f(x)$ gets large when x gets large), and $f(x)$ approaches $-\infty$ when x approaches $-\infty$. If $a < 0$ then $f(x)$ approaches $-\infty$ when x approaches $+\infty$, and $f(x)$ approaches $+\infty$ when x approaches $-\infty$.

$y = x^3 - 6x^2 + 11x - 4$

$y = -x^3 + 6x^2 - 11x + 7$

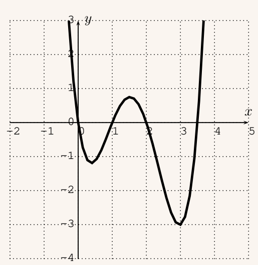
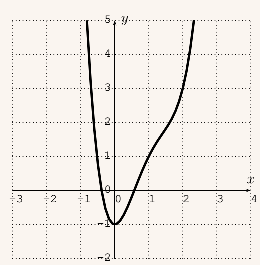
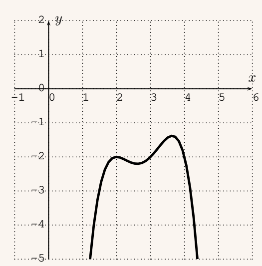


Above, we have an instance of a polynomial of degree n which "changes its direction" one more time than a polynomial of one lesser degree $n-1$. This phenomenon happens for higher degrees as well.

Note 8.5: Polynomial of degree 4

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ be a polynomial of degree 4.

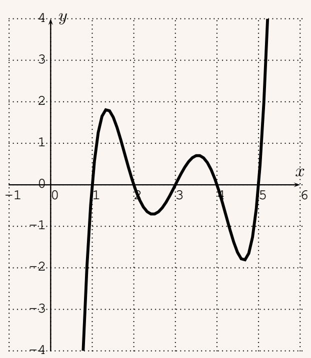
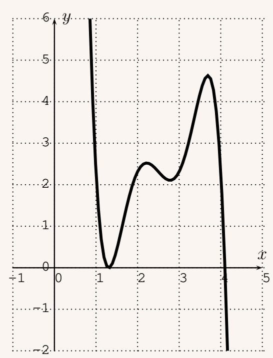
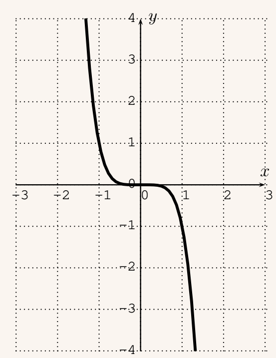
- f has at most 4 real roots. f has at most 3 extrema. If $a > 0$ then f opens upward, if $a < 0$ then f opens downward.

$y = x^4 - 6.5x^3 + 12.5x^2 - 7x$	$y = x^4 - 4x^3 + 5x^2 - 1$	$y = -x^4 + 11x^3 - 44x^2 + 76x - 50$
		
4 roots, 3 extrema	2 roots, 1 extremum	0 roots, 3 extrema

Note 8.6: Polynomial of degree 5

Let f be a polynomial of degree 5.

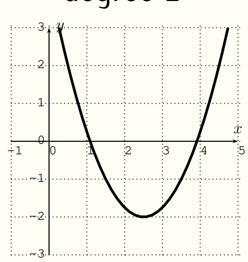
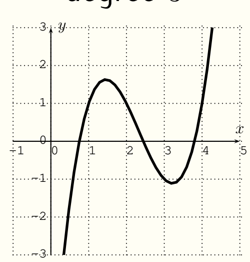
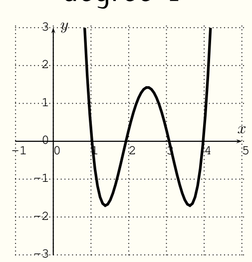
- f has at most 5 real roots. f has at most 4 extrema. If $a > 0$ then $f(x)$ approaches $+\infty$ when x approaches $+\infty$, and $f(x)$ approaches $-\infty$ when x approaches $-\infty$. If $a < 0$ then $f(x)$ approaches $-\infty$ when x approaches $+\infty$, and $f(x)$ approaches $+\infty$ when x approaches $-\infty$.

		$y = -x^5$
		
5 roots, 4 extrema	2 roots, 4 extrema	1 root, 0 extrema

We summarize our findings in the following observation.

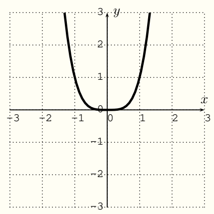
Observation 8.7: Graphs of polynomials

- Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial of degree n. Then f has at most n real roots, and at most $n-1$ extrema.

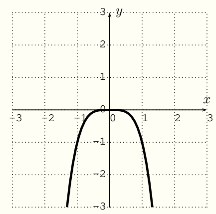
degree 2	degree 3	degree 4
		

- Assume the degree of f is even, $n = 2, 4, 6, \ldots$. If $a_n > 0$, then the polynomial opens upward. If $a_n < 0$ then the polynomial opens downward.

$f(x) = x^4$

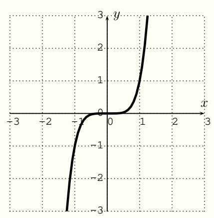


$f(x) = -x^4$

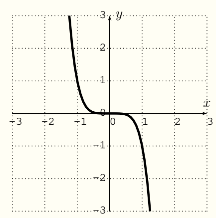


- Assume the degree of f is odd, $n = 1, 3, 5, \ldots$. If $a_n > 0$, then $f(x)$ approaches $+\infty$ when x approaches $+\infty$, and $f(x)$ approaches $-\infty$ as x approaches $-\infty$. If $a_n < 0$, then $f(x)$ approaches $-\infty$ when x approaches $+\infty$, and $f(x)$ approaches $+\infty$ as x approaches $-\infty$.

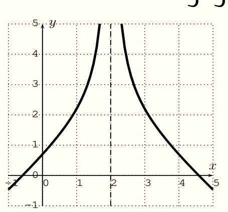
$f(x) = x^5$



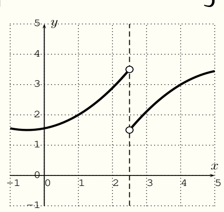
$f(x) = -x^5$



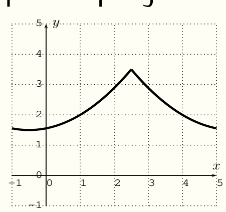
- The domain of a polynomial f is all real numbers, and f is continuous for all real numbers (there are no jumps in the graph). The graph of f has no horizontal or vertical asymptotes, no discontinuities (jumps in the graph), and no corners. Furthermore, $f(x)$ approaches $\pm\infty$ when x approaches $\pm\infty$. Therefore, the following graphs ***cannot*** be graphs of polynomials.



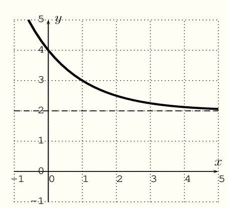
vertical asymptote / pole



discontinuity



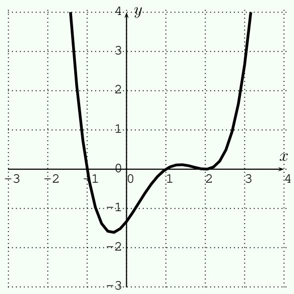
corner

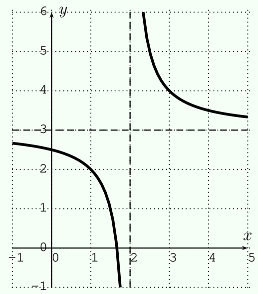


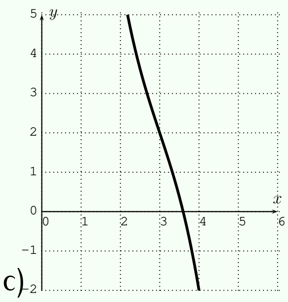
horizontal asymptote

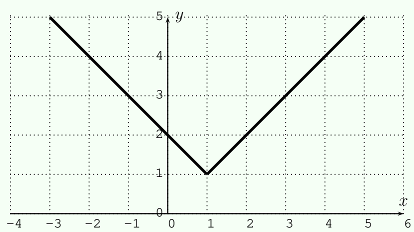
Example 8.8

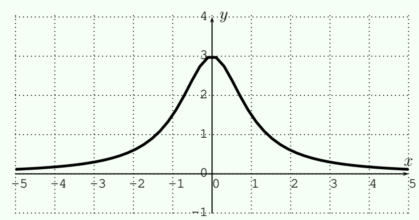
Which of the following graphs could be the graphs of a polynomial? If the graph could indeed be a graph of a polynomial, then determine a possible degree of the polynomial.

a) 

b) 

c) 

d) 

e) 

Solution.

a) Yes, this could be a polynomial. The degree could be, for example, 4.

b) No, since the graph has a pole.

c) Yes, this could be a polynomial. A possible degree would be degree 3.

d) No, since the graph has a corner.

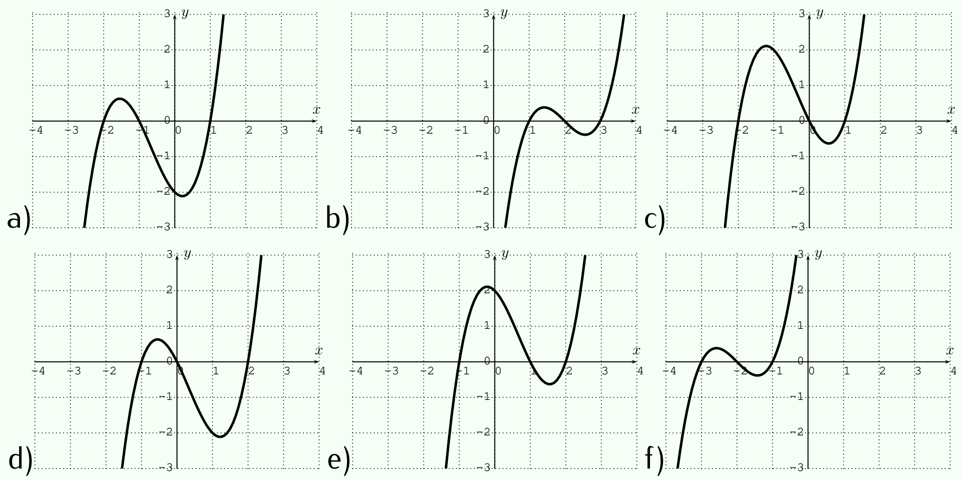
e) No, since $f(x)$ does not approach ∞ or $-\infty$ as x approaches ∞. (In fact, $f(x)$ approaches 0 as x approaches $\pm\infty$ and we say that the function (or graph) has a horizontal asymptote $y = 0$.)

□

Example 8.9

Identify the graphs of the polynomials f, g, h, and k.

$$\begin{aligned} f(x) &= (x-1)\cdot(x+1)\cdot(x-2) \\ g(x) &= (x+1)\cdot(x+2)\cdot(x+3) \\ h(x) &= (x-1)\cdot(x-2)\cdot(x-3) \\ k(x) &= x\cdot(x+1)\cdot(x-2) \end{aligned}$$



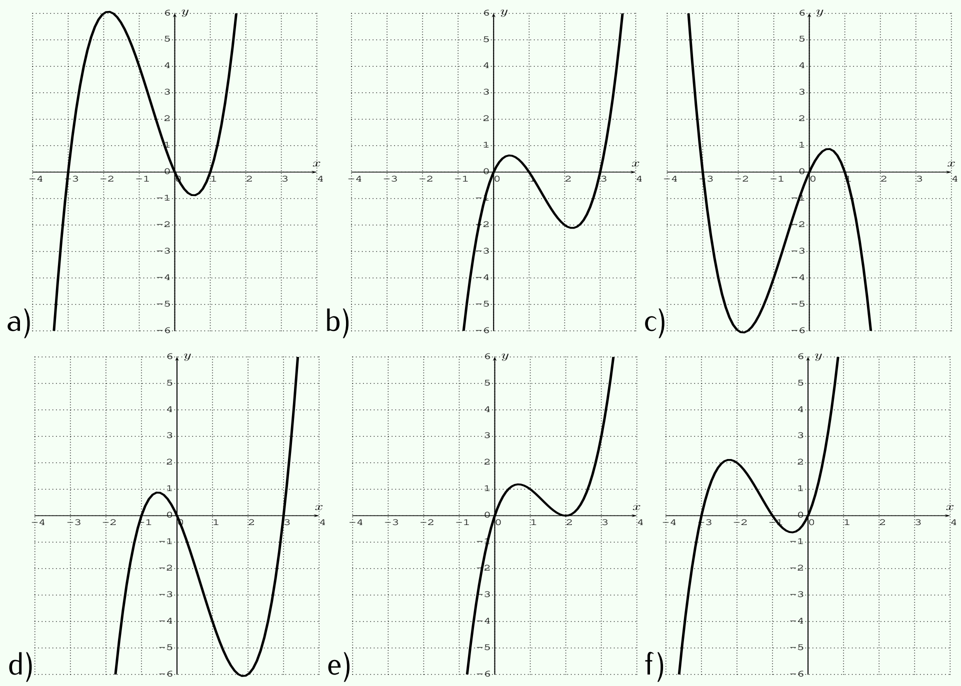
Solution.

Since $(x-1)$ is a factor of f, the factor theorem (7.4) tells us that $f(1)=0$, that is, 1 is a root of f. Similarly, we see that the function f has roots at 1, -1, and 2. The only graph with roots 1, -1, and 2 is graph (e), so that the graph of f is (e). Similarly, the roots of g are -1, -2, -3, so that it's graph is (f). This should not be confused with the function h, which has roots at 1, 2, 3, and thus has graph (b). To identify the function k, note that the factor x can be expressed as $(x-0)$, so that k can also be written as $k(x)=(x-0)\cdot(x+1)\cdot(x-2)$. The roots of k are 0, -1, 2, and so k has graph (d). □

Example 8.10

Identify the graphs of the polynomials f, g, h, and k.

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 3x \\ g(x) &= x^3 - 4x^2 + 4x \\ h(x) &= x^3 - 4x^2 + 3x \\ k(x) &= -x^3 - 2x^2 + 3x \end{aligned}$$



Solution.

Factoring first x, and then factoring again, the functions can be written as

$$\begin{aligned} f(x) &= x(x^2 - 2x - 3) = x(x+1)(x-3) \\ g(x) &= x(x^2 - 4x + 4) = x(x-2)(x-2) = x \cdot (x-2)^2 \\ h(x) &= x(x^2 - 4x + 3) = x(x-1)(x-3) \\ k(x) &= -x(x^2 + 2x - 3) = -x(x-1)(x+3) \end{aligned}$$

Note that $x = (x - 0)$, so that a factor x gives a root at 0. Therefore, f has roots 0, -1, 3, and thus has graph (d). For g, note that we have roots 0 and 2, and thus g has graph (e). Note that the factor $(x - 2)$

appears twice in the factored form for g. The number of times a root appears in the factored expression is called the *multiplicity* of the root. Thus, the multiplicity of the root 2 of g is 2. This higher multiplicity can also be observed in the graph of g: the graph of g does not cut through the x-axis at 2, but only touches the x-axis at 2. In fact, the graph resembles a parabola close to the root 2; of course it looks very different than a parabola further away from 2. Next, the function h has roots 0, 1, 3, and thus has graph (b). Finally, the function k has roots 0, 1, -3. There are two graphs with these roots, namely, graph (a) and graph (c). Since the first coefficient is negative, the correct graph has to be (c); see Note 8.4. □

When graphing a function, we want to make sure to draw the function in a window that shows all the interesting properties of the graph.

Note 8.11: A complete graph

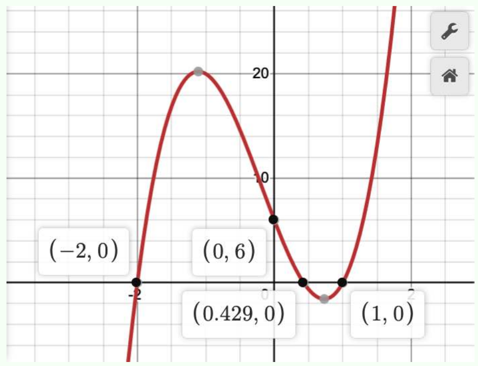
Generally, we would like to graph a function in a way that includes all essential parts of the function, such as all *intercepts* (both x-intercepts and y-intercept), all *roots*, all *asymptotes* (as discussed in the following chapters), and the *long-range behavior* of the function (that is how the function behaves when x approaches $\pm\infty$). Moreover, if possible, we also want to include all *extrema* (that is all maxima and minima) of the function. Such a graph is called a **complete graph**.
Note that we have a certain amount of choice when graphing a complete graph, as we want to pick a "reasonable" viewing window that displays the wanted features. Depending on the graph, it may be sometimes difficult or even impossible to make a good choice. Moreover, it may not be clear if all of the wanted features (such as all maxima, minima, etc.) have been displayed in the graph. In fact, some of the tools that will be developed in a course in calculus may be needed to ensure that this has indeed been achieved.

Example 8.12

Draw a complete graph of the function below. Label all intercepts and roots.

$$f(x) = 7x^3 + 4x^2 - 17x + 6$$

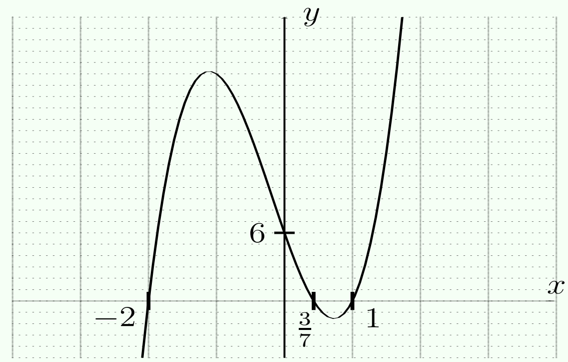
Solution.
We use the graphing calculator to graph $y = f(x)$.



By clicking on the intercepts, we see their approximate values. From the graph, it appears that there are roots at $x = -2$ and $x = 1$, and there is another root that is not an integer (≈ 0.429). Now to confirm, for example, that $x = -2$ is a root, we could check directly that $f(-2) = 0$. However, to find the other roots, we will need to use the factor theorem (7.4) and divide $f(x)$ by $(x+2)$. So, if we perform the long division and we obtain a remainder of 0, then this also confirms that -2 is indeed a root.

$$
\begin{array}{r|cccc}
 & & 7x^2 & -10x & +3 \\ \hline
x+2 & 7x^3 & +4x^2 & -17x & +6 \\
 & -(7x^3 & +14^2) & & \\ \hline
 & & -10x^2 & -17x & +6 \\
 & & -(-10x^2 & -20x) & \\ \hline
 & & & 3x & +6 \\
 & & & -(3x & +6) \\ \hline
 & & & & 0
\end{array}
$$

Thus, $f(x) = (x+2)\cdot(7x^2 - 10x + 3)$. To find the other roots of f, we factor the quotient as $7x^2 - 10x + 3 = (x-1)\cdot(7x-3)$, and we get that $f(x) = (x+2)\cdot(x-1)\cdot(7x-3)$. (Note that 1 is a root of f, so it is not surprising that $(x-1)$ appears as a factor of f.) The third root is where $7x - 3 = 0$, i.e., $7x = 3$, or $x = \frac{3}{7}$, which is approximately 0.429. We can now draw a complete graph of f, using a graphing window similar to the one above, labeling all roots and intercepts. We can calculate the y-intercept to be at $y = f(0) = 6$.



□

8.2 Roots and factors of a polynomial

We have seen that the roots are an important feature of a polynomial. Recall that the roots of the polynomial f are those x for which $f(x) = 0$. These are, of course, precisely the x-intercepts of the graph. By the factor theorem (7.4), this is precisely the information needed to factor a polynomial.

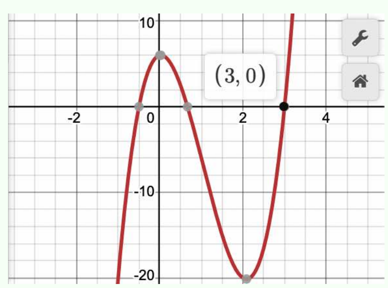
Example 8.13

Find the roots of the polynomial and factor the polynomial completely.

a) $f(x) = 6x^3 - 19x^2 + x + 6$
b) $g(x) = -x^3 - 5x^2 - 3x + 9$
c) $h(x) = 2x^3 + 11x^2 + 11x - 4$

Solution.

a) We start by graphing the polynomial $f(x) = 6x^3 - 19x^2 + x + 6$.



The graph suggests a root at $x = 3$, so that we divide $f(x)$ by $x - 3$.

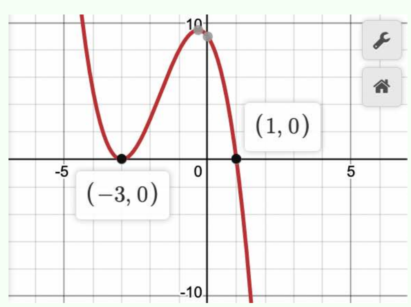
$$
\begin{array}{r|rrrr}
 & & 6x^2 & -x & -2 \\ \hline
x-3 & 6x^3 & -19x^2 & +x & +6 \\
 & -(6x^3 & -18^2) & & \\ \hline
 & & -x^2 & +x & +6 \\
 & & -(-x^2 & +3x) & \\ \hline
 & & & -2x & +6 \\
 & & & -(-2x & +6) \\ \hline
 & & & & 0
\end{array}
$$

We therefore obtain $f(x) = (x - 3) \cdot (6x^2 - x - 2)$. Continuing to factor, we obtain $f(x) = (x - 3) \cdot (3x - 2) \cdot (2x + 1)$. Note that we can factor 3 from $(3x - 2) = 3 \cdot (x - \frac{2}{3})$ and we can factor 2 from $2x + 1 = 2 \cdot (x + \frac{1}{2})$, so that the final factored expression for $f(x)$ is

$$f(x) = (x-3)\cdot 3\cdot \left(x-\frac{2}{3}\right)\cdot 2\cdot \left(x+\frac{1}{2}\right) = 6\cdot (x-3)\cdot \left(x-\frac{2}{3}\right)\cdot \left(x+\frac{1}{2}\right)$$

The roots of f are therefore 3, $\frac{2}{3}$, and $-\frac{1}{2}$.

b) From the graph we see that the roots of $g(x) = -x^3 - 5x^2 - 3x + 9$ appear to be -3 and 1.



Dividing by $x-1$, we obtain

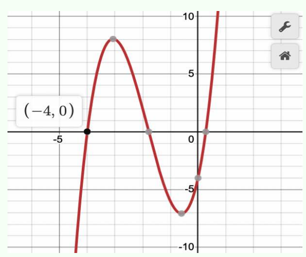
$$
\begin{array}{r|rrrr}
 & & -x^2 & -6x & -9 \\
\hline
x-1 & -x^3 & -5x^2 & -3x & +9 \\
 & -(-x^3 & +x^2) & & \\
\hline
 & & -6x^2 & -3x & +9 \\
 & & -(-6x^2 & +6x) & \\
\hline
 & & & -9x & +9 \\
 & & & -(-9x & +9) \\
\hline
 & & & & 0
\end{array}
$$

Therefore, $g(x)=(x-1)(-x^2-6x-9)$, and factoring $-x^2-6x-9=-(x^2+6x+9)=-(x+3)(x+3)$, we obtain

$$g(x)=-(x-1)\cdot(x+3)^2$$

The roots are indeed 1 and -3. Note that -3 is a root of multiplicity 2, as it appears twice in the factored expression for g. The graph of g does not cut the x-axis at -3, but only touches the x-axis.

c) The graph of $h(x)=2x^3+11x^2+11x-4$ displays an integer root at -4.



Factoring by $x+4$, we get

$$
\begin{array}{r|rrrr}
 & & 2x^2 & +3x & -1 \\
\hline
x+4 & 2x^3 & +11x^2 & +11x & -4 \\
 & -(2x^3 & +8^2) & & \\
\hline
 & & 3x^2 & +11x & -4 \\
 & & -(3x^2 & +12x) & \\
\hline
 & & & -x & -4 \\
 & & & -(-x & -4) \\
\hline
 & & & & 0
\end{array}
$$

Therefore, $h(x) = (x+4) \cdot (2x^2 + 3x - 1)$. There does not seem to be an immediate way to factor $2x^2 + 3x - 1$. However, we may use the quadratic formula (reviewed in Proposition 8.14 below) to find the roots of $2x^2 + 3x - 1$. Setting $2x^2 + 3x - 1 = 0$, we get

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9+8}}{4} = \frac{-3 \pm \sqrt{17}}{4}$$

These are indeed the remaining two roots of h, and we can write

$$h(x) = 2 \cdot (x+4) \cdot \left(x - \frac{-3+\sqrt{17}}{4}\right) \cdot \left(x - \frac{-3-\sqrt{17}}{4}\right)$$

Note that there is an overall coefficient 2, which has to appear to obtain the correct leading coefficient for $h(x) = 2x^3 + 11x^2 + 11x - 4$.

□

In the last example we found the roots and factors of a quadratic polynomial via the quadratic formula. We now recall the well-known quadratic formula and state how it can be used to factor any quadratic polynomial.

Proposition 8.14: The quadratic formula

The solutions of the equation $ax^2 + bx + c = 0$ for some real numbers a, b, and c are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We may combine the two solutions x_1 and x_2 and simply write this as:

$$\boxed{x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{8.1}$$

Since we have an explicit formula for the roots of a quadratic polynomial, it is always possible to give an explicit formula of a quadratic polynomial in factored form. We record this in the following note.

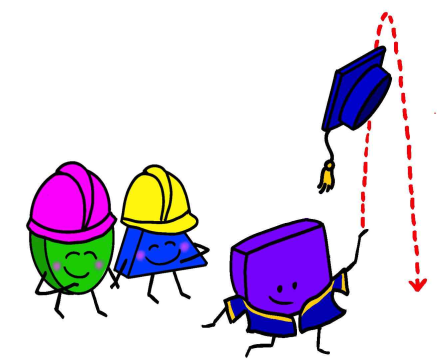
Note 8.15: Factoring a quadratic polynomial

We may always use the roots x_1 and x_2 of a quadratic polynomial $f(x) = ax^2 + bx + c$ from the quadratic formula and rewrite the polynomial as

$$\boxed{ax^2 + bx + c = a \cdot \left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)}$$

Application: Vertical position of an object in gravity

An an application, we note that the height $h = h(t)$ of an object thrown into the air as a function of time t will follow a quadratic function. Here, for simplicity, we only consider the effect of the gravitational force and ignore issues such as air resistance and friction, etc.



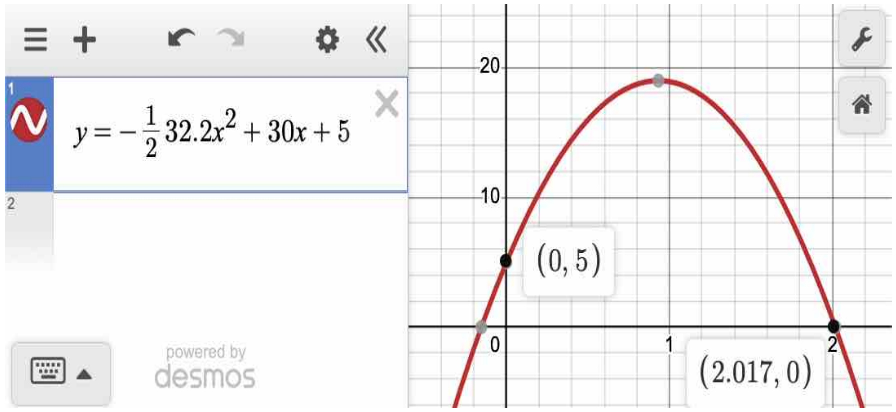
In fact, the vertical position $h(t)$ of an object is a quadratic function in time:[1]

$$\boxed{h(t) = -\frac{1}{2}gt^2 + v_0t + h_0} \tag{8.2}$$

Here, v_0 is the initial velocity, h_0 is the initial height, and $g = 32.2\frac{\text{ft}}{\text{sec}^2}$ is the acceleration due to the gravitational pull from the Earth.
Therefore, if an object is thrown from an initial height of $h_0 = 5\text{ft}$ with an initial velocity of $v_0 = 30\frac{\text{ft}}{\text{sec}}$, then $h(t)$ follows the formula:

$$h(t) = -\tfrac{1}{2} \cdot 32.2 \cdot t^2 + 30 \cdot t + 5$$

[1] For more information, see https://openstax.org/books/college-physics-2e/pages/3-4-projectile-motion



In the above graph the horizontal axis describes time t in seconds, and the vertical axis describes height $h(t)$ in feet. The graph shows that, in particular, it would take about 2 seconds until the object hits the ground.

8.3 *Optional section:* Graphing polynomials by hand

In this section we will show how to sketch the graph of a factored polynomial without the use of a calculator.

Example 8.16

Sketch the graph of the following polynomial without using the calculator:

$$p(x) = -2(x+10)^3(x+9)x^2(x-8)$$

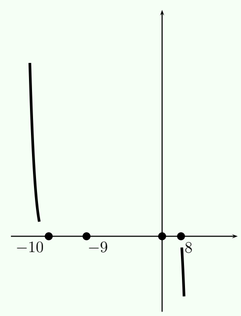
Solution.

Note that on the calculator it is impossible to get a window which will give all of the features of the graph (by focusing on a window view that captures the maximum, other features will become invisible). We will sketch the graph by hand so that some of the main features are visible. This will only be a sketch and not the actual graph up to scale. Again, the graph cannot be drawn to scale while being able to see the features.

We first start by putting the x-intercepts on the graph in the right order, but not necessarily to scale. Then note that

$$p(x) = -2x^7 + \dots \text{(lower terms)} \quad \approx -2x^7 \qquad \text{for large } |x|.$$

This is the leading term of the polynomial (if you expand p it is the term with the largest power) and therefore dominates the polynomial for large $|x|$. So the graph of our polynomial should look something like the graph of $y = -2x^7$ on the extreme left and right side.

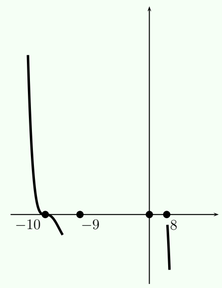


Now we look at what is going on at the roots. Near each root the factor corresponding to that root dominates. So we have

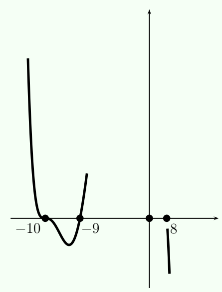
for $x \approx$	$p(x) \approx$	
-10	$C_1(x+10)^3$	cubic
-9	$C_2(x+9)$	line
0	$C_3\ x^2$	parabola
8	$C_4(x-8)$	line

where $C_1, C_2, C_3,$ and C_4 are constants which can, but need not, be calculated. For example, whether or not the parabola near 0 opens up or down will depend on whether the constant $C_3 = -2{\cdot}(0{+}10)^3(0{+}9)(0{-}8)$ is negative or positive. In this case C_3 is positive, so it opens upward, but we will not use this fact to graph. We will see this independently which is a good check of our work.
Starting from the left of our graph where we had determined the behavior for large negative x, we move toward the left-most zero, -10. Near -10 the graph looks cubic, so we imitate a cubic curve as we pass through

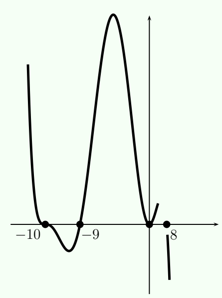
$(-10, 0)$.



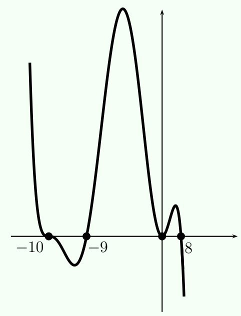
Now we turn and head toward the next zero, -9. Here the graph looks like a line, so we pass through the point $(-9, 0)$ as a line would.



Now we turn and head toward the root 0. Here the graph should look like a parabola. So we form a parabola there. (Note that, as we had said before, the parabola should be opening upward here—and we see that it is).



Now we turn toward the final zero 8. We pass through the point $(8, 0)$ like a line and we join (perhaps with the use of an eraser) to the large x part of the graph. If this does not join nicely (if the graph is going in the wrong direction) then there has been a mistake. This is a check of our work. Here is the final sketch.

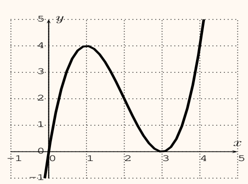


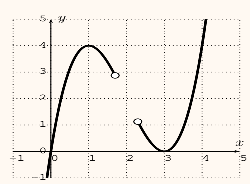
What can be understood from this sketch? Questions like "when is $p(x) > 0$?" can be answered by looking at the sketch. Further, the general shape of the curve is correct so that other properties can be concluded. For example, p has a local minimum between $x = -10$ and $x = -9$ and a local maximum between $x = -9$ and $x = 0$, and between $x = 0$ and $x = 8$. The exact point where the function reaches its maximum or minimum cannot be decided by looking at this sketch. But it will help to decide on an appropriate window so that the minimum or maximum finder on the calculator can be used. □

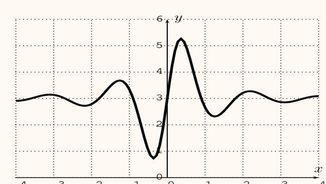
8.4 Exercises

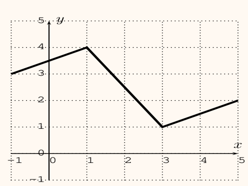
Exercise 8.1

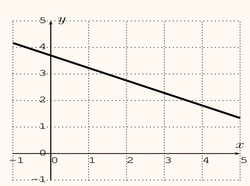
Assuming the graphs below are complete graphs, which of the graphs could be the graphs of a polynomial?

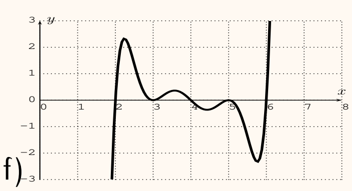
a) 

b) 

c) 

d) 

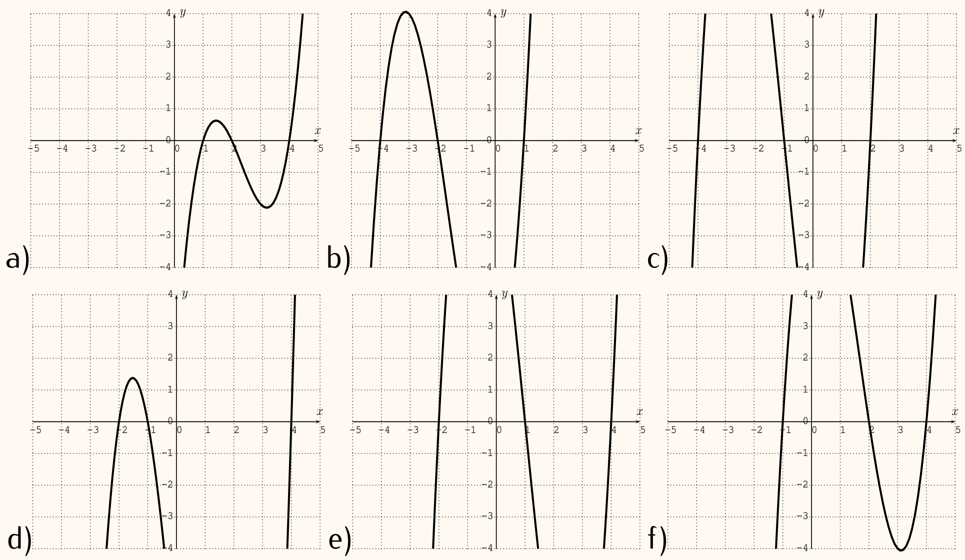
e) 

f) 

Exercise 8.2

For each of the polynomials f, g, h, and k, find the corresponding graph from (a)–(f) below.

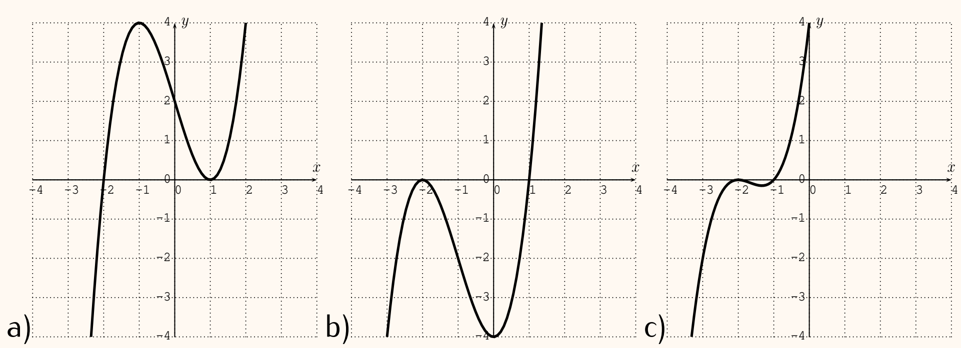
$$f(x) = (x-1)\cdot(x+2)\cdot(x-4) \qquad g(x) = (x+1)\cdot(x-2)\cdot(x+4)$$
$$h(x) = (x-1)\cdot(x-2)\cdot(x-4) \qquad k(x) = (x+1)\cdot(x-2)\cdot(x-4)$$

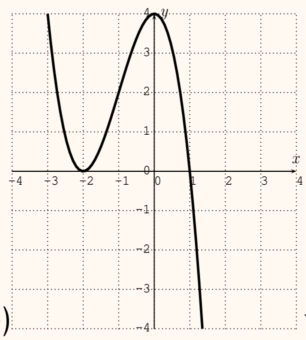
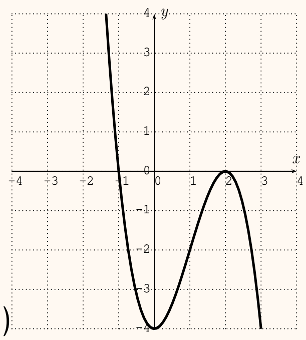


Exercise 8.3

For each of the polynomials f, g, h, and k, find the corresponding graph from (a)–(f) below.

$$f(x) = (x+1)\cdot(x+2)^2 \qquad g(x) = -(x+1)\cdot(x-2)^2$$
$$h(x) = -(x-1)^2\cdot(x+2) \qquad k(x) = (x-1)\cdot(x+2)^2$$

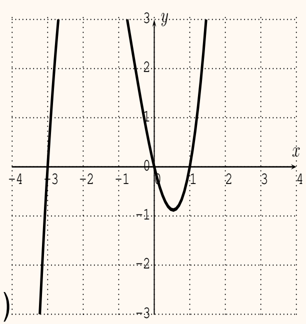
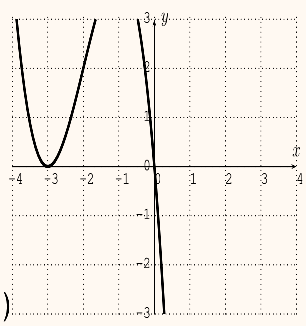
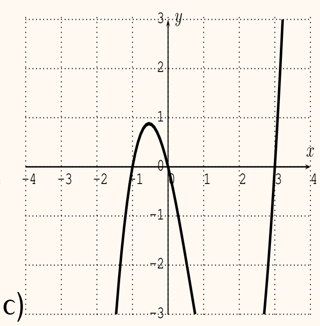


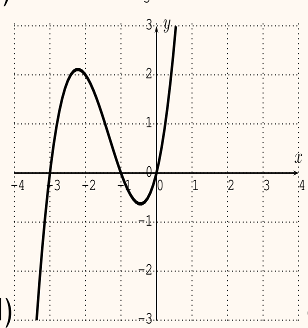
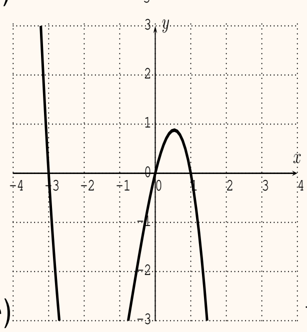
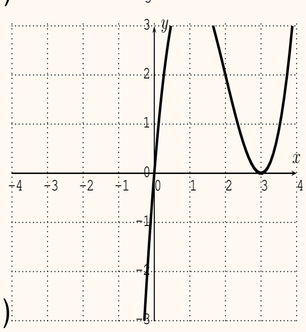
d)  e)  f) 

Exercise 8.4

For each of the polynomials f, g, h, and k, find the corresponding graph from (a)–(f) below.

$$f(x) = x^3 + 4x^2 + 3x \qquad g(x) = -x^3 - 2x^2 + 3x$$
$$h(x) = x^2 - 2x^2 - 3x \qquad k(x) = -x^3 - 6x^2 - 9x$$

a)  b)  c) 

d)  e)  f) 

Exercise 8.5

Sketch a complete the graph of the function. Label all intercepts of the graph.

a) $f(x) = x^3 + 4x^2 + x - 6$
b) $f(x) = 2x^3 - 15x^2 + 34x - 24$
c) $f(x) = x^3 - 16x - 21$
d) $f(x) = -2x^3 - 5x^2 - 2x + 1$
e) $f(x) = x^4 - 7x^3 + 15x^2 - 7x - 6$
f) $f(x) = 3x^4 + 11x^3 - x^2 - 19x + 6$

Exercise 8.6

Find the exact value of at least one root of the given polynomial.

a) $f(x) = x^3 - 10x^2 + 31x - 30$
b) $f(x) = -x^3 - x^2 + 8x + 8$
c) $f(x) = x^3 - 11x^2 - 3x + 33$
d) $f(x) = x^4 + 9x^3 - 6x^2 - 136x - 192$
e) $f(x) = x^2 + 6x + 3$
f) $f(x) = x^4 - 6x^3 + 3x^2 + 5x$

Exercise 8.7

Find all roots and factor the polynomial completely.

a) $f(x) = x^3 - 5x^2 + 2x + 8$
b) $f(x) = x^3 + 7x^2 + 7x - 15$
c) $f(x) = x^3 + 9x^2 + 26x + 24$
d) $f(x) = x^3 + 4x^2 - 11x + 6$
e) $f(x) = 3x^3 + 13x^2 - 52x + 28$
f) $f(x) = 6x^3 - 5x^2 - 13x - 2$
g) $f(x) = 6x^3 - x^2 - 31x - 10$
h) $f(x) = x^3 - 7x^2 + 13x - 3$
i) $f(x) = x^3 + 2x^2 - 11x + 8$
j) $f(x) = 2x^3 + 7x^2 + 5x - 2$
k) $f(x) = 3x^3 - 10x^2 - 4x + 21$

Exercise 8.8

Graph the following polynomials without using the calculator.

a) $f(x) = (x+4)^2(x-5)$
b) $f(x) = -3(x+2)^3x^2(x-4)^5$
c) $f(x) = 2(x-3)^2(x-5)^3(x-7)$
d) $f(x) = -(x+4)(x+3)(x+2)^2(x+1)(x-2)^2$

Chapter 9

Roots of polynomials

We have seen in Observation 7.10 on page 117 that every root c of a polynomial $f(x)$ gives a factor $(x - c)$ of $f(x)$. As we would like to use this to factor polynomials, it will be helpful to know more about the nature of roots of polynomials. In Section 9.1, we will discuss a statement concerning roots that are rational numbers (the rational root theorem), while in Section 9.2 we give a general statement about the existence of roots (the fundamental theorem of algebra).

9.1 *Optional section:* The rational root theorem

Our first comment concerns rational roots for a polynomial with integer coefficients.

Note 9.1

Consider, for example, the equation $10x^3 - 6x^2 + 5x - 3 = 0$. Let x be a rational solution of this equation, that is $x = \frac{p}{q}$ is a rational number such that

$$10 \cdot \left(\frac{p}{q}\right)^3 - 6 \cdot \left(\frac{p}{q}\right)^2 + 5 \cdot \frac{p}{q} - 3 = 0.$$

We assume that $x = \frac{p}{q}$ is *completely reduced*, that is, p and q have no common factors that can be used to cancel the numerator and denominator of the fraction $\frac{p}{q}$. Now, simplifying the above equation, and combining terms, we obtain:

$$
\begin{aligned}
& & 10\cdot\frac{p^3}{q^3} - 6\cdot\frac{p^2}{q^2} + 5\cdot\frac{p}{q} - 3 = 0 \\
\text{(multiply by } q^3\text{)} & \implies & 10p^3 - 6p^2q + 5pq^2 - 3q^3 = 0 \\
\text{(add } 3q^3\text{)} & \implies & 10p^3 - 6p^2q + 5pq^2 = 3q^3 \\
\text{(factor } p \text{ on the left)} & \implies & p\cdot(10p^2 - 6pq + 5q^2) = 3q^3.
\end{aligned}
$$

Therefore, p is a factor of $3q^3$ (with the other factor being $(10p^2 - 6pq + 5q^2)$). Since p and q have no common factors, p must be a factor of 3. That is, p is one of the following integers: $p = +1, +3, -1, -3$.
Similarly, starting from $10p^3 - 6p^2q + 5pq^2 - 3q^3 = 0$, we can write

$$
\begin{aligned}
\text{(add } +6p^2q - 5pq^2 + 3q^3\text{)} & \implies & 10p^3 = 6p^2q - 5pq^2 + 3q^3 \\
\text{(factor } q \text{ on the right)} & \implies & 10p^3 = (6p^2 - 5pq + 3q^2)\cdot q.
\end{aligned}
$$

Now, q must be a factor of $10p^3$. Since q and p have no common factors, q must be a factor of 10. In other words, q is one of the following numbers: $q = \pm1, \pm2, \pm5, \pm10$. Putting this together with the possibilities for $p = \pm1, \pm3$, we see that all possible rational roots are the following:

$$\pm\frac{1}{1},\quad \pm\frac{1}{2},\quad \pm\frac{1}{5},\quad \pm\frac{1}{10},\quad \pm\frac{3}{1},\quad \pm\frac{3}{2},\quad \pm\frac{3}{5},\quad \pm\frac{3}{10}.$$

The observation in the previous example holds for a general polynomial equation with integer coefficients.

Observation 9.2: Rational root theorem

Consider the equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0, \tag{9.1}$$

where every coefficient $a_n, a_{n-1}, \ldots, a_0$ is an integer and $a_0 \neq 0$, $a_n \neq 0$. Assume that $x = \frac{p}{q}$ is a rational solution of (9.1) and the fraction $x = \frac{p}{q}$ is completely reduced. Then a_0 is an integer multiple of p, and a_n is an integer multiple of q. In particular, if x is an integer root of (9.1), then a_0 is an integer multiple of x (which follows if we apply the above to the case $x = \frac{p}{1}$).

In other words:

- Any rational solution of (9.1) can be written as a fraction $x = \frac{p}{q}$ where p is a factor of a_0 and q is a factor of a_n.
- Any integer solution x of (9.1) is a factor of a_0.

We can use this observation to find good candidates for the roots of a given polynomial.

Example 9.3

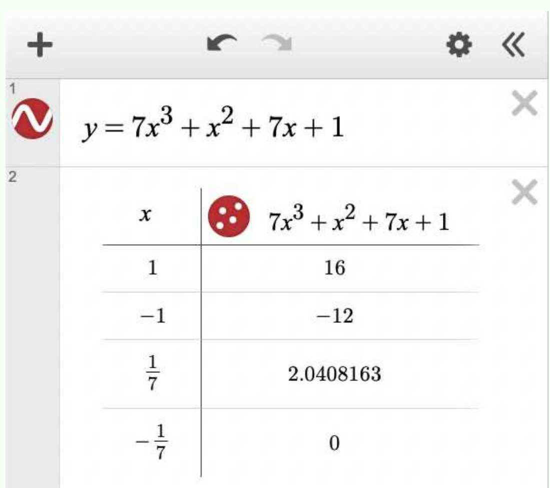
a) Find all *rational* roots of $f(x) = 7x^3 + x^2 + 7x + 1$.

b) Find all *real* roots of $f(x) = 2x^3 + 11x^2 - 2x - 2$.

c) Find all *real* roots of $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$.

Solution.

a) If $x = \frac{p}{q}$ is a rational root, then p is a factor of 1, that is $p = \pm 1$; and q is a factor of 7, that is $q = \pm 1, \pm 7$. The candidates for rational roots are therefore $x = \pm\frac{1}{1}, \pm\frac{1}{7}$. To see which of these candidates are indeed roots of f we plug these numbers into f via the table function on the graphing calculator (see Example 4.7). We obtain the following:



x	$7x^3 + x^2 + 7x + 1$
1	16
-1	-12
$\frac{1}{7}$	2.0408163
$-\frac{1}{7}$	0

The only root among $\pm 1, \pm\frac{1}{7}$ is $x = -\frac{1}{7}$.

b) We need to identify all real roots of $f(x) = 2x^3 + 11x^2 - 2x - 2$. In general, it is a quite difficult task to find a root of a polynomial of degree 3, so that it will be helpful if we can find the rational roots first. If $x = \frac{p}{q}$ is a rational root, then p is a factor of -2, that is $p = \pm 1, \pm 2$; and q is a factor of 2, that is $q = \pm 1, \pm 2$. The possible rational roots $x = \frac{p}{q}$ of f are:

$$\pm 1, \quad \pm 2, \quad \pm\frac{1}{2}$$

Using the calculator, we see that the only rational root is $x = \frac{1}{2}$.

x	$2x^3 + 11x^2 - 2x - 2$
1	9
−1	9
2	54
−2	30
$\frac{1}{2}$	0
$-\frac{1}{2}$	1.5

Therefore, by the factor theorem (Observation 7.10), we see that $(x - \frac{1}{2})$ is a factor of f, that is $f(x) = q(x) \cdot (x - \frac{1}{2})$. To avoid fractions in the long division, we rewrite this as

$$f(x) = q(x) \cdot (x - \frac{1}{2}) = q(x) \cdot \frac{2x - 1}{2} = \frac{q(x)}{2} \cdot (2x - 1),$$

so that we may divide $f(x)$ by $(2x - 1)$ instead of $(x - \frac{1}{2})$ (note that this cannot be done with synthetic division). We obtain the following quotient:

$$\begin{array}{r|rrrr} & & x^2 & +6x & +2 \\ \hline 2x-1 & 2x^3 & +11x^2 & -2x & -2 \\ & -(2x^3 & -x^2) & & \\ \hline & & 12x^2 & -2x & -2 \\ & & -(12x^2 & -6x) & \\ \hline & & & 4x & -2 \\ & & & -(4x & -2) \\ \hline & & & & 0 \end{array}$$

Therefore, $f(x) = (x^2 + 6x + 2)(2x - 1)$, and any root of f is either a root of $x^2 + 6x + 2$ or of $2x - 1$. We know that the root of $2x - 1$ is $x = \frac{1}{2}$, and that x^2+6x+2 has no other rational roots. Nevertheless, we can identify all other real roots of $x^2 + 6x + 2$ via the quadratic formula, (see Proposition 8.14).

$$\begin{aligned} & x^2 + 6x + 2 = 0 \\ \implies \quad x_{1/2} &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 2}}{2} \\ &= \frac{-6 \pm \sqrt{36 - 8}}{2} = \frac{-6 \pm \sqrt{28}}{2} \\ &= \frac{-6 \pm \sqrt{4 \cdot 7}}{2} = \frac{-6 \pm 2\sqrt{7}}{2} \\ &= -3 \pm \sqrt{7} \end{aligned}$$

Therefore, the roots of f are precisely the following:

$$x_1 = -3 + \sqrt{7}, \quad x_2 = -3 - \sqrt{7}, \quad x_3 = \frac{1}{2}.$$

c) First we find the rational roots $x = \frac{p}{q}$ of $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$. Since p is a factor of -6 it must be $p = \pm1, \pm2, \pm3, \pm6$, and since q is a factor of 4 it must be $q = \pm1, \pm2, \pm4$. All candidates for rational roots $x = \frac{p}{q}$ are the following (where we excluded repeated ways of writing x):

$$\pm1, \quad \pm2, \quad \pm3, \quad \pm6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{1}{4}, \quad \pm\frac{3}{4}$$

Checking all these candidates with the calculator produces exactly two rational roots: $x = 6$ and $x = -\frac{1}{4}$. Therefore, we may divide $f(x)$ by both $(x-6)$ and by $(x+\frac{1}{4})$ without remainder. To avoid fractions, we use the term $4 \cdot (x+\frac{1}{4}) = (4x+1)$ instead of $(x+\frac{1}{4})$ for our factor of f. Therefore, $f(x) = q(x) \cdot (x-6) \cdot (4x+1)$. The quotient $q(x)$ is determined by performing a long division by $(x-6)$ and then another long division by $(4x+1)$, or alternatively by only one long division by

$$(x-6) \cdot (4x+1) = 4x^2 + x - 24x - 6 = 4x^2 - 23x - 6.$$

Dividing $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$ by $4x^2 - 23x - 6$ produces the quotient $q(x)$:

$$\begin{array}{r|rrrrr}
 & & & x^2 & & +1 \\ \hline
4x^2-23x-6 & 4x^4 & -23x^3 & -2x^2 & -23x & -6 \\
 & -(4x^4 & -23x^3 & -6x^2) & & \\ \hline
 & & & 4x^2 & -23x & -6 \\
 & & & -(4x^2 & -23x & -6) \\ \hline
 & & & & & 0
\end{array}$$

We obtain the factored expression for $f(x)$ as $f(x) = (x^2+1)(4x+1)(x-6)$. The only remaining real roots we need to find are those of x^2+1. However,

$$x^2 + 1 = 0 \quad \Longrightarrow \quad x^2 = -1$$

has no real solution. In other words, there are only complex solutions of $x^2 = -1$, which are $x = i$ and $x = -i$ (we will discuss complex solutions in more detail in the next section). Since the problem requires us to find the *real* roots of f, our answer is that the only real roots are $x_1 = 6$ and $x_2 = -\frac{1}{4}$.

□

9.2 The fundamental theorem of algebra

There is a general theorem which tells us when a polynomial has a root. This theorem is called the *fundamental theorem of algebra*. Since complex numbers play a crucial role in this theorem, we briefly recall the basic notations concerning complex numbers. A more thorough discussion of complex numbers will be given in Chapter 23.

Review 9.4: Complex numbers

There is no real number whose square is minus 1, that is, there is no x with $x^2 = -1$. So we *denote* by i a solution of this equation. This i is not a real number but a new kind of number called a *complex* number. We can think of i as $i = \sqrt{-1}$.
We can then consider numbers of the form $a+bi$ where a and b are real numbers. Numbers of this form constitute the set of complex numbers, denoted by $\mathbb{C}$. a is called the *real part* and b is called the *imaginary part* of the complex number $a + bi$.
We can add two complex numbers by adding their real and imaginary parts to form the real and imaginary parts of the sum. We can multiply two complex numbers by ordinary distribution (FOIL) then use the property that $i^2 = -1$.

Example 9.5

Here is an example for the subtraction and multiplication of two complex numbers.

$$(2-3i)-(4+3i) = (2-4)+(-3-3)i = -2-6i,$$
$$(2-3i)\cdot(4+3i) = 8+6i-12i-9i^2 = 8-6i-9(-1) = 17-6i.$$

We can see that these numbers arise naturally as roots of quadratic equations, such as, for example $x^2+6 = 0$, which can be written as $x^2 = -6$ and has a solution given by $x = \sqrt{-6} = \sqrt{-1}\cdot\sqrt{6} = i\sqrt{6}$. The following fundamental theorem of algebra guarantees the existence of a root of *any* polynomial of degree ≥ 1, as long as we allow complex numbers for our roots.

Theorem 9.6: Fundamental theorem of algebra

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree ≥ 1. Then there exists a complex number c which is a root of f.

Let us make two remarks about the fundamental theorem of algebra to clarify the statement of the theorem.

Note 9.7

- In the above Theorem 9.6, we did not specify what kind of coefficients $a_0, \dots a_n$ are allowed for the theorem to hold. In fact, to be precise, the fundamental theorem of algebra states that for any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree ≥ 1 where $a_0, \dots a_n$ are *complex* numbers, the polynomial f has a root c (which is also a complex number).

- The theorem states that a polynomial f of degree ≥ 1 always has a *complex* root c, but, in general, f may not have any *real* roots. For example, consider $f(x) = x^2 + 1$, and consider a root c of f, that is $c^2 + 1 = 0$. Since, for any real number c, we always have $c^2 \geq 0$, so that $f(c) = c^2 + 1 \geq 1$, this shows that there cannot be a real root c of f. However, we can easily check that the complex number i is a root of f:

  $$f(i) = i^2 + 1 = -1 + 1 = 0$$

  Indeed $f(x)$ has the roots i and $-i$, and can be factored as

  $$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1.$$

Now, while the fundamental theorem of algebra guarantees a root c of a polynomial f, we can use the remainder theorem from Observation 7.10 together with the calculator (and also the rational root theorem) to check possible candidates c for the roots. Once we found a root, we can use the factor theorem (also from Observation 7.10) to factor $f(x) = q(x) \cdot (x - c)$.

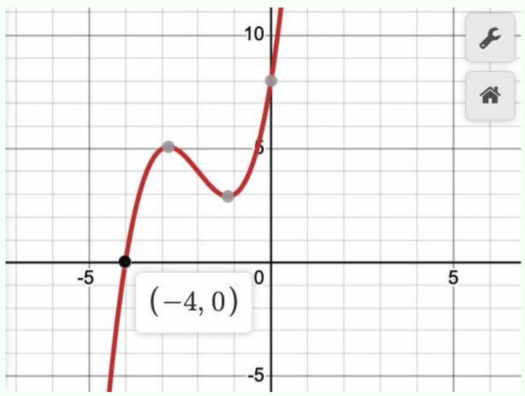
Example 9.8

Find all (real and complex) roots of the polynomial. Sketch a complete graph and label all roots.

a) $f(x) = x^3 + 6x^2 + 10x + 8$
b) $g(x) = x^3 - 6x^2 + 10x - 4$
c) $h(x) = x^4 + 2x^3 - 6x^2 - 3x + 18$

Solution.

a) In order to find a root of f, we use the graph to make a guess for one of the roots.



The graph suggests that the root may be at $x = -4$, which is also easily confirmed by plugging -4 into the function:

$$\begin{aligned} f(-4) &= (-4)^3 + 6 \cdot (-4)^2 + 10 \cdot (-4) + 8 \\ &= -64 + 96 - 40 + 8 = 0 \end{aligned}$$

Next, we divide $f(x) = x^3 + 6x^2 + 10x + 8$ by $(x + 4)$.

$$\begin{array}{r|rrrr} & x^2 & +2x & +2 & \\ \hline x+4 & x^3 & +6x^2 & +10x & +8 \\ & -(x^3 & +4x^2) & & \\ \hline & & 2x^2 & +10x & +8 \\ & & -(2x^2 & +8x) & \\ \hline & & & 2x & +8 \\ & & & -(2x & +8) \\ \hline & & & & 0 \end{array}$$

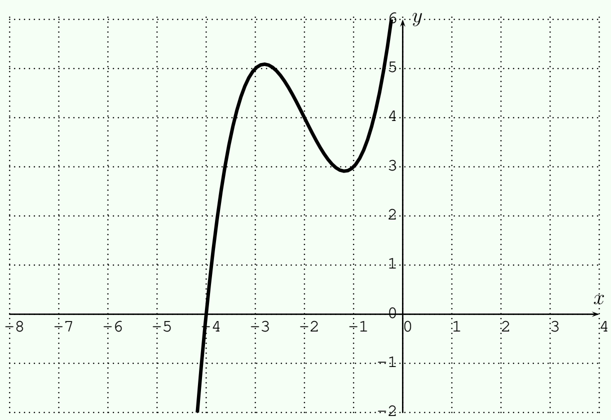
Therefore, $f(x) = (x+4)(x^2+2x+2)$. To find the remaining roots of f, we use the quadratic formula for the second polynomial x^2+2x+2:

$$x^2+2x+2 = 0 \implies x = \frac{-2\pm\sqrt{2^2-4\cdot1\cdot2}}{2} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2}$$
$$= \frac{-2\pm\sqrt{-1}\sqrt{4}}{2} = \frac{-2\pm i\cdot 2}{2} = \frac{2(-1\pm i)}{2} = -1 \pm i$$

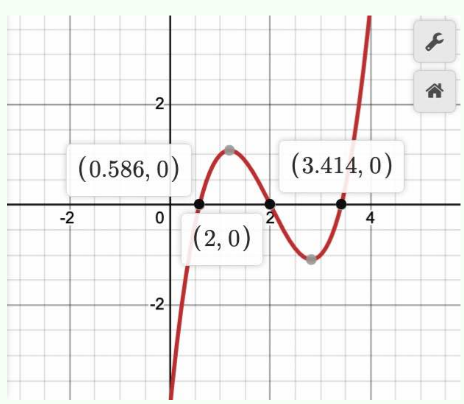
Therefore, there is only one real root -4, and two complex roots $-1+i$ and $-1-i$. The polynomial can be factored as

$$f(x) = (x+4)\cdot(x-(-1+i))\cdot(x-(-1-i))$$

The complete graph is displayed below. The only real root is shown at -4.



b) We first check the graph of $g(x) = x^3 - 6x^2 + 10x - 4$.



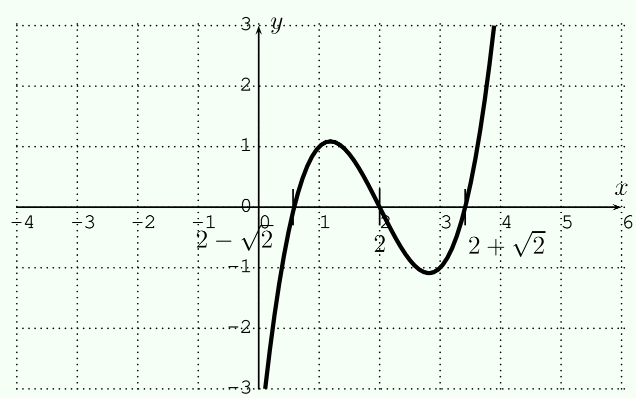
From the graph we suspect that $x = 2$ is a root, while there are two more real roots which are not at integer values. We confirm the root at 2 by direct computation, or by performing a long division by $x - 2$.

$$
\begin{array}{r|rrrr}
 & x^2 & -4x & +2 & \\
\hline
x-2 & x^3 & -6x^2 & +10x & -4 \\
 & -(x^3 & -2x^2) & & \\
\cline{2-3}
 & & -4x^2 & +10x & -4 \\
 & & -(-4x^2 & +8x) & \\
\cline{3-4}
 & & & 2x & -4 \\
 & & & -(2x & -4) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

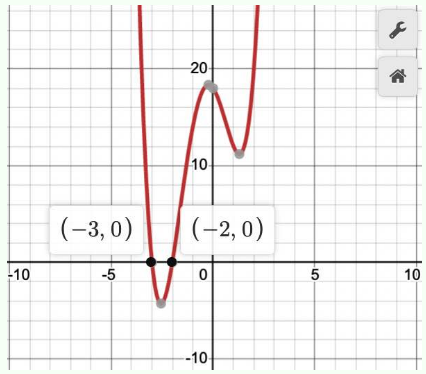
We find the remaining roots via the quadratic formula. Setting $x^2 - 4x + 2 = 0$ gives

$$
\begin{aligned}
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} \\
&= \frac{4 \pm \sqrt{4}\sqrt{2}}{2} = \frac{4 \pm 2 \cdot \sqrt{2}}{2} = \frac{2 \cdot (2 \pm \sqrt{2})}{2} = 2 \pm \sqrt{2}
\end{aligned}
$$

Therefore, $g(x) = (x-2) \cdot (x - (2 + \sqrt{2})) \cdot (x - (2 - \sqrt{2}))$. The roots of g are 2, $2 + \sqrt{2}$, $2 - \sqrt{2}$. The complete graph of g is drawn below.



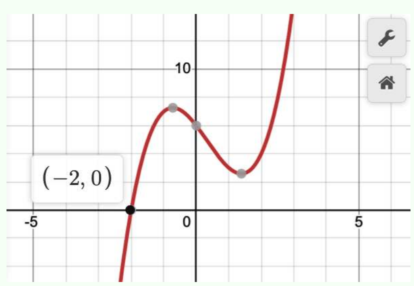
c) We first graph $h(x) = x^4 + 2x^3 - 6x^2 - 3x + 18$. Note that if we want to see a complete graph, we need to zoom out vertically.



Two integer roots of h appear to be at $x = -2$ and $x = -3$. Dividing by, say, $x + 3$, gives

$$
\begin{array}{r|rrrrr}
 & & x^3 & -x^2 & -3x & +6 \\
\hline
x+3 & x^4 & +2x^3 & -6x^2 & -3x & +18 \\
 & -(x^4 & +3x^3) & & & \\
\hline
 & & -x^3 & -6x^2 & -3x & +18 \\
 & & -(-x^3 & -3x^2) & & \\
\hline
 & & & -3x^2 & -3x & +18 \\
 & & & -(-3x^2 & -9x) & \\
\hline
 & & & & 6x & +18 \\
 & & & & -(6x & +18) \\
\hline
 & & & & & 0
\end{array}
$$

Therefore, $h(x) = (x+3)(x^3 - x^2 - 3x + 6)$. To factor $x^3 - x^2 - 3x + 6$, we graph it to find possible roots.



There is a root at $x = -2$, as we might have already suspected, since we noted that -2 is a root of h. Next, we performing another long division.

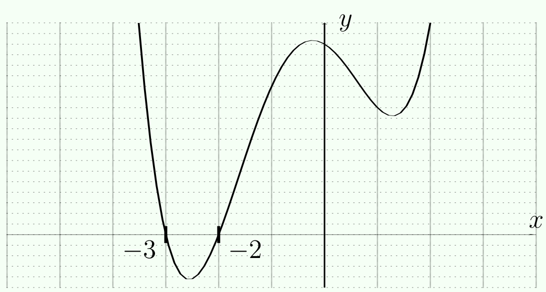
$$
\begin{array}{r|rrrr}
 & & x^2 & -3x & +3 \\
\hline
x+2 & x^3 & -x^2 & -3x & +6 \\
 & -(x^3 & +2x^2) & & \\
\cline{2-3}
 & & -3x^2 & -3x & +6 \\
 & & -(-3x^2 & -6x) & \\
\cline{3-4}
 & & & 3x & +6 \\
 & & & -(3x & +6) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

With this, we have $h(x) = (x+3)(x+2)(x^2 - 3x + 3)$. To find the roots of the last factor, apply the quadratic formula to $x^2 - 3x + 3 = 0$.

$$
\begin{aligned}
x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{3 \pm \sqrt{9-12}}{2} \\
&= \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm \sqrt{-1}\sqrt{3}}{2} = \frac{3 \pm i \cdot \sqrt{3}}{2}
\end{aligned}
$$

Thus, $h(x) = (x+3) \cdot (x+2) \cdot \left(x - \frac{3 + i \cdot \sqrt{3}}{2}\right) \cdot \left(x - \frac{3 - i \cdot \sqrt{3}}{2}\right)$.

The complete graph is shown below.



$h(x)$ has roots at -3, -2, $\frac{3+i\cdot\sqrt{3}}{2}$, $\frac{3-i\cdot\sqrt{3}}{2}$.

□

As we have seen in the last example, we can use the roots to factor a polynomial completely so that all factors are polynomials of degree 1. Furthermore, in this example, we had a complex root, $a + ib$, and its complex conjugate $a - ib$ was also a root. These observations hold more generally, as we state now.

Observation 9.9: Factors and roots of polynomials

(1) Every polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree n can be factored as

$$f(x) = a_n \cdot (x - c_1) \cdot (x - c_2) \cdot \cdots \cdot (x - c_n). \quad (9.2)$$

This follows, since we can find a root c_1 of f (as guaranteed by the fundamental theorem of algebra), and use it to factor $f(x) = (x - c_1) \cdot g(x)$. We do the same for $g(x)$ and repeat until we arrive at (9.2).

(2) In particular, every polynomial of degree n has at most n roots. (However, these roots may be real or complex.)

(3) The factor $(x - c)$ for a root c could appear multiple times in (9.2), that is, we may have $(x - c)^k$ as a factor of f. The **multiplicity** of a root c is the number of times k that a root appears in the factored expression for f, as in (9.2).

(4) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has only *real* coefficients $a_0, \ldots, a_n$, and $c = a + bi$ is a *complex* root of f, then the complex conjugate $\bar{c} = a - bi$ is also a root of f.

Proof. If x is any root, then $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Applying the complex conjugate to this and using that $\overline{u \cdot v} = \bar{u} \cdot \bar{v}$ gives $\overline{a_n}\bar{x}^n + \overline{a_{n-1}}\bar{x}^{n-1} + \cdots + \overline{a_1}\bar{x} + \overline{a_0} = 0$. Since the coefficients a_j are real, we have that $\overline{a_j} = a_j$, so that $a_n \bar{x}^n + a_{n-1}\bar{x}^{n-1} + \cdots + a_1 \bar{x} + a_0 = 0$. This shows that the complex conjugate $\bar{x}$ is a root of f as well. □

Example 9.10

For a chosen real number C, let f be the function (dependent on the C):

$$f(x) = 4x^3 - 16x^2 + 9x + C$$

a) Find the number C so that the polynomial $f(x)$ has a root at 3.

b) Find all remaining roots of $f(x)$ and write them in simplest radical form.

Solution.

a) For 3 to be a root of f, we know that $x-3$ has to be a factor of $f(x)$. We therefore perform a long division by $f(x) \div (x-3)$.

$$
\begin{array}{r|rrrr}
 & & 4x^2 & -4x & -3 \\
\hline
x-3 & 4x^3 & -16x^2 & +9x & +C \\
 & -(4x^3 & -12x^2) & & \\
\cline{2-3}
 & & -4x^2 & +9x & +C \\
 & & -(-4x^2 & +12x) & \\
\cline{3-4}
 & & & -3x & +C \\
 & & & -(-3x & +9) \\
\cline{4-5}
 & & & & C-9
\end{array}
$$

Thus, $x-3$ is a factor of $f(x)$ exactly when the remainder $C-9$ is zero, that is, $C=9$. We thus have that

$$f(x) = 4x^3 - 16x^2 + 9x + 9$$

b) From (a), we know that f factors as $f(x) = (x-3)(4x^2-4x-3)$. We can use the quadratic formula to find the remaining roots of f by setting $4x^2-4x-3=0$.

$$
\begin{aligned}
\implies \quad x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 4\cdot(-3)}}{2\cdot 4} \\
&= \frac{4 \pm \sqrt{16+48}}{8} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} \\
\implies x_1 &= \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2}, \quad x_2 = \frac{4-8}{8} = \frac{-4}{8} = -\frac{1}{2}
\end{aligned}
$$

We get that the roots of f are 3, $\frac{3}{2}$ and $-\frac{1}{2}$.

Note that, alternatively, we could have factored $4x^2-4x-3 = (2x-3)(2x+1) = 4(x-\frac{3}{2})(x+\frac{1}{2})$, resulting in the same roots $x_1 = \frac{3}{2}$ and $x_2 = -\frac{1}{2}$.

□

Example 9.11

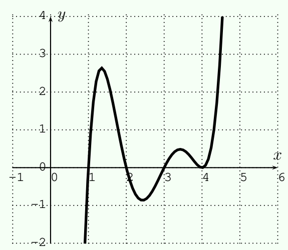
Find a polynomial f with the following properties:

a) f has degree 3; the roots of f are precisely 4, 5, 6; and the leading coefficient of f is 7

b) f has degree 3 with real coefficients; f has roots $3i$, -5 (and possibly other roots as well); and $f(0) = 90$

c) f has degree 4 with complex coefficients; f has roots $i+1$, $2i$, 3

d) f has roots that are determined by the following graph of f:



Solution.

a) In general a polynomial f of degree 3 is of the form $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3)$. Identifying the roots and the leading coefficient, we obtain the polynomial

$$f(x) = 7 \cdot (x-4) \cdot (x-5) \cdot (x-6)$$

b) A polynomial f of degree 3 is of the form $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3)$. Roots of f are $3i$ and -5, and since the coefficients of f are real, it follows from Observation 9.9(4) that the complex conjugate $-3i$ is also a root of f. Therefore, $f(x) = m \cdot (x+5) \cdot (x-3i) \cdot (x+3i)$. To identify m, we use the last condition $f(0) = 90$.

$$90 = m \cdot (0+5) \cdot (0-3i) \cdot (0+3i) = m \cdot 5 \cdot (-9)i^2 = m \cdot 5 \cdot 9 = 45m$$

Dividing by 45, we obtain $m = 2$, so that

$$f(x) = 2 \cdot (x+5) \cdot (x-3i) \cdot (x+3i) = 2 \cdot (x+5) \cdot (x^2+9),$$

which clearly has real coefficients.

c) Since f is of degree 4, it can be written as $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3) \cdot (x - c_4)$. Three of the roots are identified as $i + 1$, $2i$, and 3:

$$f(x) = m \cdot (x - (1 + i)) \cdot (x - 2i) \cdot (x - 3) \cdot (x - c_4)$$

However, we have no further information on the fourth root c_4 or the leading coefficient m. (Note that Observation 9.9(4) cannot be used here, since we are ***not*** assuming that the polynomial has ***real*** coefficients. Indeed f ***cannot*** have real coefficients since then, besides the complex roots $1 + i$ and $2i$, their complex conjugates $1 - i$ and $-2i$ would also be roots of f, giving us 5 roots of f. However, a polynomial of degree 4 cannot have 5 roots.) We can therefore *choose any number* for these remaining variables. For example, a possible solution to the problem is given by choosing $m = 3$ and $c_4 = 2$, for which we obtain:

$$f(x) = 3 \cdot (x - (1 + i)) \cdot (x - 2i) \cdot (x - 3) \cdot (x - 2)$$

d) f is of degree 5, and we know that the leading coefficient is 1. The graph is zero at $x = 1, 2, 3$, and 4, so that the roots are 1, 2, 3, and 4. Moreover, since the graph just touches the root $x = 4$, this must be a multiple root, that is, it must occur more than once (see Section 8.3 for a discussion of multiple roots and their graphical consequences). We obtain the following solution:

$$f(x) = (x - 1)(x - 2)(x - 3)(x - 4)^2$$

Note that the root $x = 4$ is a root of multiplicity 2.

□

Note 9.12

By Observation 9.9(4), polynomials with real coefficients have complex roots that come in complex conjugate pairs. To find the product of the corresponding factors, an appropriate grouping may help to simplify the computation.

For example, when multiplying $(x-(2+3i))(x-(2-3i))$, we can group the x and 2, and then use the binomial formula $(a+b)(a-b)=a^2-b^2$ to evaluate:

$$\begin{aligned}(x-(2+3i))(x-(2-3i)) &= ((x-2)-3i)((x-2)+3i)\\ &= (x-2)^2-9i^2=(x-2)^2+9\end{aligned}$$

9.3 Exercises

Exercise 9.1

a) Find all rational roots of $f(x)=2x^3-3x^2-3x+2$.
b) Find all rational roots of $f(x)=3x^3-x^2+15x-5$.
c) Find all rational roots of $f(x)=6x^3+7x^2-11x-12$.
d) Find all real roots of $f(x)=6x^4+25x^3+8x^2-7x-2$.
e) Find all real roots of $f(x)=4x^3+9x^2+26x+6$.

Exercise 9.2

Find a root of the polynomial by guessing possible candidates of the root.

a) $f(x)=x^5-1$
b) $f(x)=x^4-1$
c) $f(x)=x^3-27$
d) $f(x)=x^3+1000$
e) $f(x)=x^4-81$
f) $f(x)=x^3-125$
g) $f(x)=x^5+32$
h) $f(x)=x^{777}-1$
i) $f(x)=x^2+64$

Exercise 9.3

Find the roots of the polynomial and use it to factor the polynomial completely.

a) $f(x)=x^3-7x+6$
b) $f(x)=x^3-x^2-16x-20$
c) $f(x)=x^3-7x^2+17x-20$
d) $f(x)=x^3+x^2-5x-2$
e) $f(x)=2x^3+x^2-7x-6$
f) $f(x)=12x^3+49x^2-2x-24$
g) $f(x)=x^3-3x^2+9x+13$
h) $f(x)=x^4-5x^2+4$
i) $f(x)=x^4-1$
j) $f(x)=x^5-6x^4+8x^3+6x^2-9x$
k) $f(x)=x^3-27$
l) $f(x)=x^4+2x^2-15$

Exercise 9.4

Find the exact roots of the polynomial; write the roots in simplest radical form, if necessary. Sketch a graph of the polynomial with all roots clearly marked.

a) $f(x) = x^3 - 2x^2 - 5x + 6$
b) $f(x) = x^3 + 5x^2 + 3x - 4$
c) $f(x) = -x^3 + 5x^2 + 7x - 35$
d) $f(x) = x^3 + 7x^2 + 13x + 7$
e) $f(x) = 2x^3 - 8x^2 - 18x - 36$
f) $f(x) = x^4 - 4x^2 + 3$
g) $f(x) = -x^4 + x^3 + 24x^2 - 4x - 80$
h) $f(x) = 7x^3 - 11x^2 - 10x + 8$
i) $f(x) = -15x^3 + 41x^2 + 15x - 9$
j) $f(x) = x^4 - 6x^3 + 6x^2 + 4x$

Exercise 9.5

Find a real number C so that the polynomial has a root as indicated. Then, for this choice of C, find all remaining roots of the polynomial.

a) $f(x) = x^3 + 6x^2 + 5x + C$ has root at $x = 1$
b) $f(x) = x^3 - 4x^2 - 2x + C$ has root at $x = -2$
c) $f(x) = x^3 - x^2 - 9x + C$ has root at $x = 3$
d) $f(x) = x^3 + 8x^2 + 5x + C$ has root at $x = -1$
e) $f(x) = x^3 - 5x^2 + 15x + C$ has root at $x = 2$

Exercise 9.6

Find a polynomial f that fits the given data.

a) f has degree 3. The roots of f are precisely 2, 3, 4. The leading coefficient of f is 2.

b) f has degree 4. The roots of f are precisely -1, 2, 0, -3. The leading coefficient of f is -1.

c) f has degree 3. f has roots -2, -1, 2, and $f(0) = 10$.

d) f has degree 4. f has roots 0, 2, -1, -4, and $f(1) = 20$.

e) f has degree 3. The coefficients of f are all real. The roots of f are precisely $2 + 5i$, $2 - 5i$, 7. The leading coefficient of f is 3.

f) f has degree 3. The coefficients of f are all real. f has roots i, 3, and $f(0) = 6$.

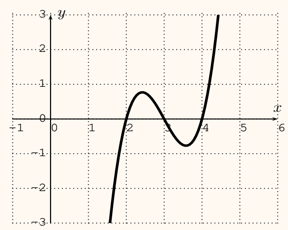
g) f has degree 4. The coefficients of f are all real. f has roots $5+i$ and $5-i$ of multiplicity 1, the root 3 of multiplicity 2, and $f(5)=7$.

h) f has degree 4. The coefficients of f are all real. f has roots i and $3+2i$.

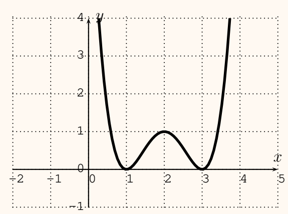
i) f has degree 6. f has complex coefficients. f has roots $1+i$, $2+i$, $4-3i$ of multiplicity 1 and the root -2 of multiplicity 3.

j) f has degree 5. f has complex coefficients. f has roots i, 3, -7 (and possibly other roots).

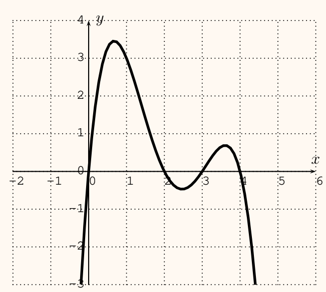
k) f has degree 3. The roots of f are determined by its graph:



l) f has degree 4. The coefficients of f are all real. The leading coefficient of f is 1. The roots of f are determined by its graph:



m) f has degree 4. The coefficients of f are all real. f has the following graph:



Chapter 10

Rational functions

Recall that a rational function is a fraction of polynomials:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

In this chapter, we will study some of the characteristics of graphs of rational functions.

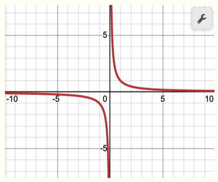
10.1 Graphs of rational functions

The graph of rational functions can have new features that were not present in the graph of polynomials, such as, for example, *asymptotes*. An *asymptote* is a line that is approached by the graph of a function: $x = a$ is a *vertical asymptote* if $f(x)$ approaches $\pm\infty$ as x approaches a from either the left or from the right, and $y = b$ is a *horizontal asymptote* if $f(x)$ approaches b as x approaches ∞ or $-\infty$.

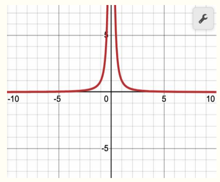
In order to get an idea of some of the features of graphs of rational functions, we look at various sample graphs. First, we graph the basic functions $y = \frac{1}{x^n}$.

Observation 10.1: $f(x) = \frac{1}{x^n}$

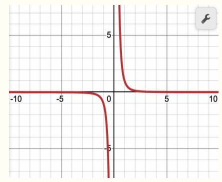
Graphing $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^4}$, we obtain:



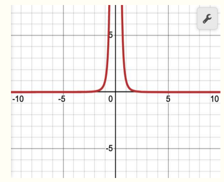
$y = \frac{1}{x}$



$y = \frac{1}{x^2}$

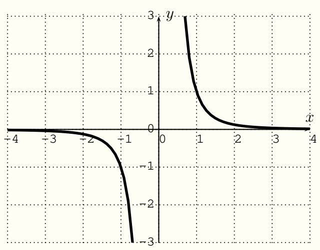


$y = \frac{1}{x^3}$

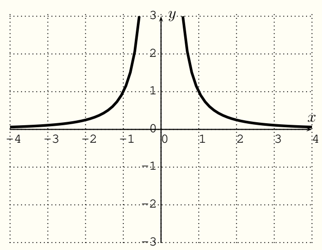


$y = \frac{1}{x^4}$

In general, we see that $x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote. The shape of $y = \frac{1}{x^n}$ depends on n being even or odd. We have:



$y = \frac{1}{x^n}$, n odd

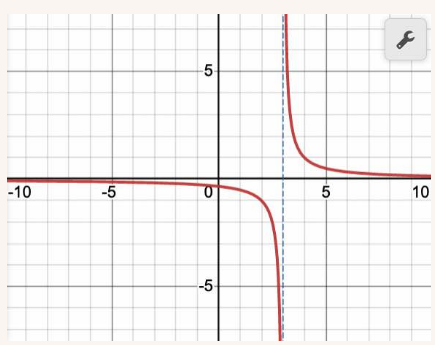


$y = \frac{1}{x^n}$, n even

In the next note, we graph four sample rational functions. These examples will help us understand many general aspects of graphs of rational functions.

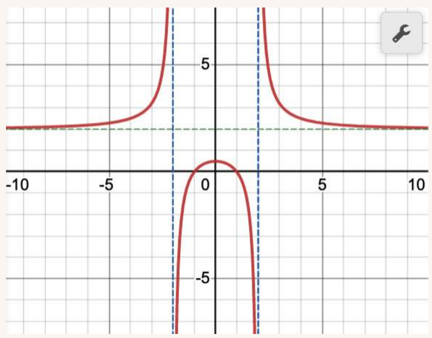
Note 10.2

(1) Our first graph is $f(x) = \frac{1}{x-3}$.



Here, the domain is all numbers where the denominator is not zero, that is $D = \mathbb{R} - \{3\}$. There is a vertical asymptote, $x = 3$. Furthermore, the graph approaches 0 as x approaches $\pm\infty$. Therefore, f has a horizontal asymptote, $y = 0$. Indeed, whenever the denominator has a higher degree than the numerator, the line $y = 0$ will be the horizontal asymptote.

(2) Next, we graph $f(x) = \frac{8x^2-8}{4x^2-16}$.



Here, the domain is all x for which $4x^2 - 16 \neq 0$. To see where this occurs, calculate

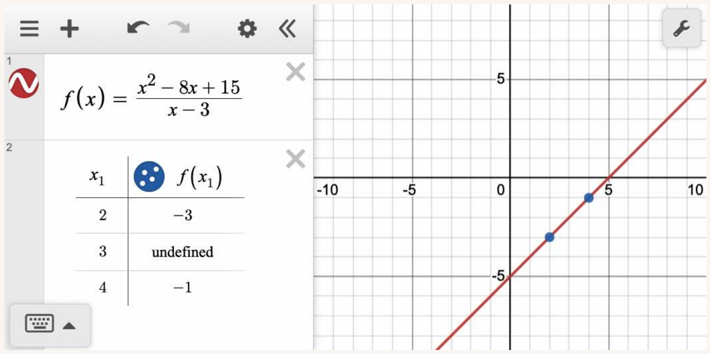
$$4x^2 - 16 = 0 \implies 4x^2 = 16 \implies x^2 = 4 \implies x = \pm 2.$$

Therefore, the domain is $D = \mathbb{R} - \{-2, 2\}$. As before, we see from the graph that the domain reveals the vertical asymptotes $x = 2$ and $x = -2$ (the vertical dashed lines). To find the horizontal asymptote (the horizontal dashed line), we note that, when x becomes very large, the highest terms of both numerator and denominator dominate the function value, so that

$$\text{for } |x| \text{ very large} \implies f(x) = \frac{8x^2 - 8}{4x^2 - 16} \approx \frac{8x^2}{4x^2} = 2$$

Therefore, when x approaches $\pm\infty$, the function value $f(x)$ approaches 2, and therefore the horizontal asymptote is at $y = 2$ (the horizontal dashed line).

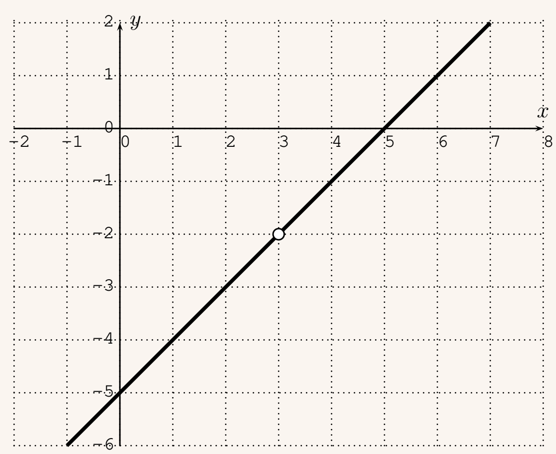
(3) Our next graph is $f(x) = \frac{x^2-8x+15}{x-3}$.



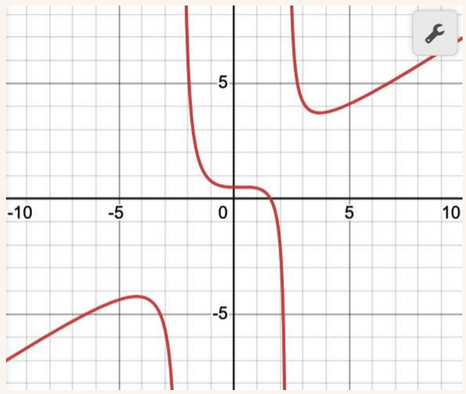
We see that there does not appear to be any vertical asymptote, despite the fact that 3 is not in the domain. The reason for this is that we can "remove the singularity" by canceling the troubling term $x-3$ as follows:

$$f(x) = \frac{x^2-8x+15}{x-3} = \frac{(x-3)(x-5)}{(x-3)} = \frac{x-5}{1} = x-5, \quad x \neq 3$$

Therefore, the function f reduces to $x-5$ for all values where it is defined. However, note that $f(x) = \frac{x^2-8x+15}{x-3}$ is not defined at $x=3$. We denote this in the graph by an open circle at $x=3$, and call this a removable singularity (or a hole).



(4) Our fourth and last graph before stating the rules in full generality is $f(x) = \frac{2x^3-8}{3x^2-16}$.



The graph indicates that there is no horizontal asymptote, as the graph appears to increase toward ∞ and decrease toward $-\infty$. To make this observation precise, we calculate the behavior when x approaches $\pm\infty$ by ignoring the lower terms in the numerator and denominator

$$\text{for } |x| \text{ very large} \quad \Longrightarrow \quad f(x) = \frac{2x^3-8}{3x^2-16} \approx \frac{2x^3}{3x^2} = \frac{2x}{3}$$

Therefore, when x becomes very large, $f(x)$ behaves like $\frac{2}{3}x$, which approaches ∞ when x approaches ∞, and approaches $-\infty$ when x approaches $-\infty$. (In fact, after performing a long division we obtain $\frac{2x^3-8}{3x^2-16} = \frac{2}{3} \cdot x + \frac{r(x)}{3x^2-16}$, which would give rise to what is called a *slant asymptote* $y = \frac{2}{3} \cdot x$; see also Observation 11.4 below.) Indeed, whenever the degree of the numerator is greater than the degree of the denominator, we find that there is no horizontal asymptote, but the graph blows up to $\pm\infty$. (Compare this also with example (c) above).

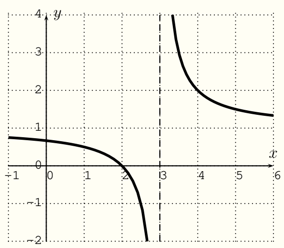
We summarize the observations from the above examples in the following observation.

Observation 10.3: Rational functions

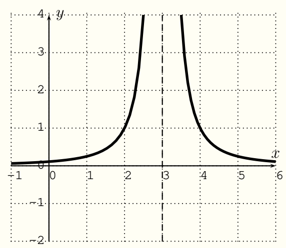
Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function with polynomials $p(x)$ and $q(x)$ in the numerator and denominator, respectively.

- The **domain** of f is all real numbers x for which the denominator is not zero,
$$D = \{ \quad x \in \mathbb{R} \quad | \quad q(x) \neq 0 \quad \}$$
- Assume that $q(x_0) = 0$, so that f is not defined at x_0. If x_0 is not a root of $p(x)$, or if x_0 is a root of $p(x)$ but of a lesser multiplicity than the root in $q(x)$, then f has a **vertical asymptote** $x = x_0$.

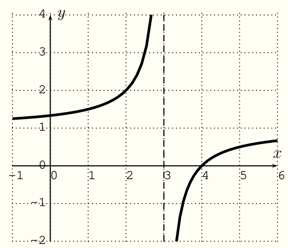
$f(x) = \frac{x-2}{x-3}$



$f(x) = \frac{1}{(x-3)^2}$

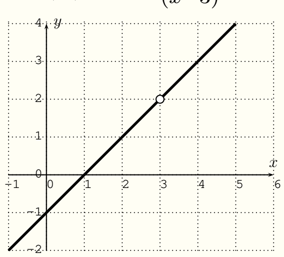


$f(x) = \frac{(x-3)(x-4)}{(x-3)^2}$

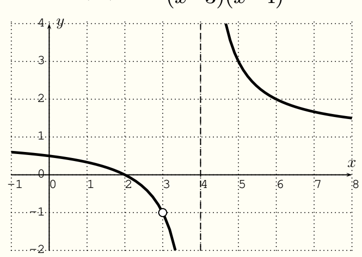


- If $p(x_0) = 0$ and $q(x_0) = 0$, and the multiplicity of the root x_0 in $p(x)$ is at least the multiplicity of the root in $q(x)$, then these roots can be canceled, and we say that there is a **removable discontinuity** (also called a **hole**) at $x = x_0$.

$f(x) = \frac{(x-1)(x-3)}{(x-3)}$



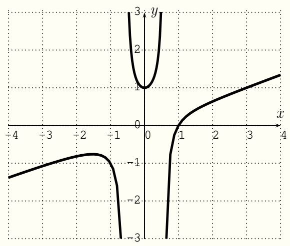
$f(x) = \frac{(x-2)(x-3)}{(x-3)(x-4)}$



- To find the **horizontal asymptotes**, we need to distinguish the cases where the degree of $p(x)$ is less than, equal to, or greater than $q(x)$.

deg$(p) >$deg(q)

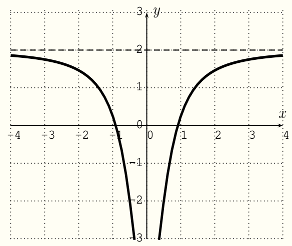
$f(x) = \frac{x^3-1}{3x^2-1}$



no horizontal asymptote

deg$(p) =$deg(q)

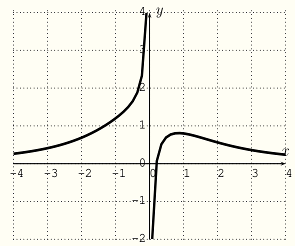
$f(x) = \frac{6x^2-5}{3x^2+1}$



asymptote: $y = \frac{\text{highest coeff. of } p}{\text{highest coeff. of } q}$

deg$(p) <$deg(q)

$f(x) = \frac{5x-1}{x^3+4x}$

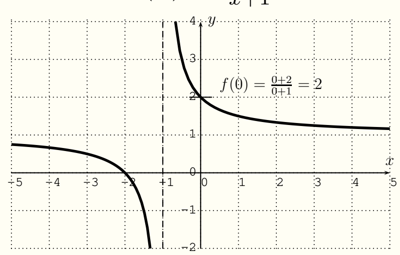


asymptote: $y = 0$

In addition, it is also useful to determine the x- and y-intercepts.

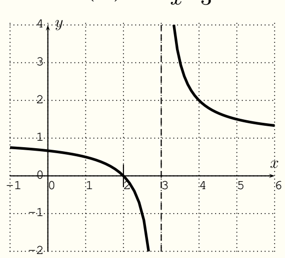
- If 0 is in the domain of f, then the y**-intercept** is $(0, f(0))$.

$f(x) = \frac{x+2}{x+1}$

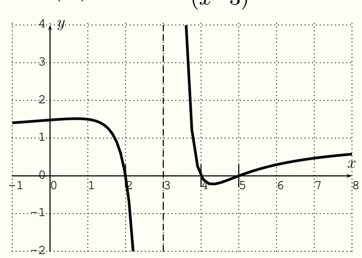


- If $p(x_0) = 0$ but $q(x_0) \neq 0$, then $f(x_0) = \frac{p(x_0)}{q(x_0)} = \frac{0}{q(x_0)} = 0$, so that $(x_0, 0)$ is an x**-intercept**, that is, the graph intersects with the x-axis at x_0.

$f(x) = \frac{x-2}{x-3}$



$f(x) = \frac{(x-2)(x-4)(x-5)}{(x-3)^3}$



Example 10.4

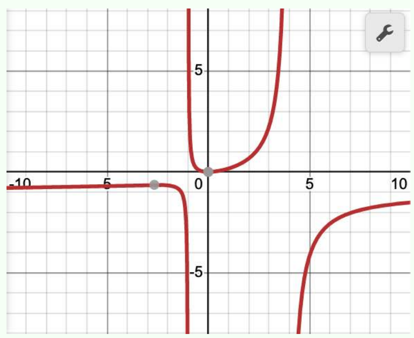
Find the domain, all horizontal asymptotes, vertical asymptotes, removable singularities, and x- and y-intercepts. Use this information together with the graph of the calculator to sketch the graph of f.

a) $f(x) = \frac{-x^2}{x^2-3x-4}$ b) $f(x) = \frac{5x}{x^2-2x}$ c) $f(x) = \frac{x^3-9x^2+26x-24}{x^2-x-2}$

d) $f(x) = \frac{x-4}{(x-2)^2}$ e) $f(x) = \frac{3x^2-12}{2x^2+1}$

Solution.

a) We combine our knowledge of rational functions and its algebra with the particular graph of the function. The graphing calculator shows the following graph:



To find the domain of f we only need to exclude from the real numbers those x that make the denominator zero. Since $x^2-3x-4=0$ exactly when $(x+1)(x-4)=0$, which gives $x=-1$ or $x=4$, we have the domain:

$$\text{domain } D = \mathbb{R} - \{-1, 4\}$$

The numerator has a root exactly when $-x^2 = 0$, that is $x = 0$. Therefore, $x=-1$ and $x=4$ are vertical asymptotes, and since we cannot cancel terms in the fraction, there is no removable singularity. Furthermore, since $f(x)=0$ exactly when the numerator is zero, the only x-intercept is $(0,0)$.

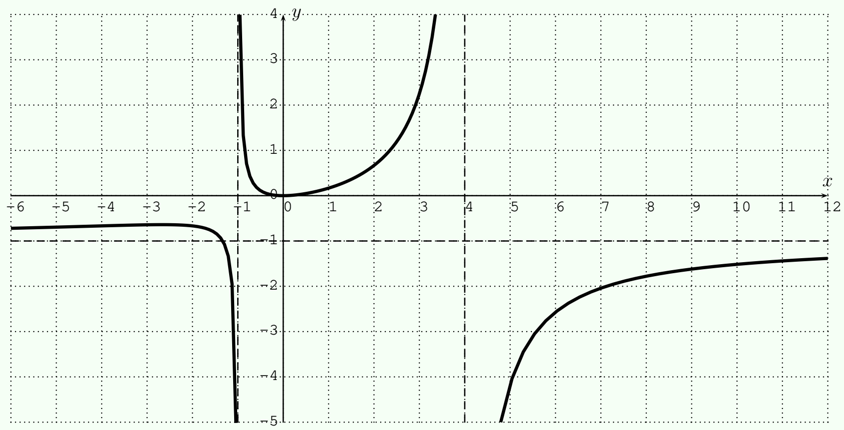
To find the horizontal asymptote, we consider $f(x)$ for large values of x by ignoring the lower order terms in numerator and denominator,

$$|x| \text{ large} \implies f(x) \approx \frac{-x^2}{x^2} = -1$$

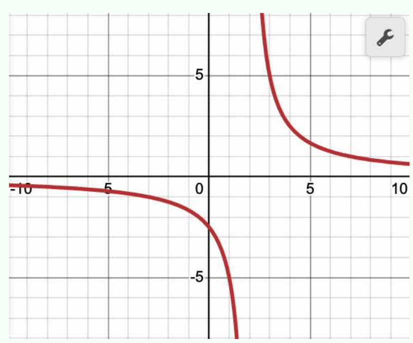
We see that the horizontal asymptote is $y = -1$. Finally, for the y-intercept, we calculate $f(0)$:

$$f(0) = \frac{-0^2}{0^2 - 3 \cdot 0 - 4} = \frac{0}{-4} = 0.$$

Therefore, the y-intercept is $(0, 0)$. The function is then graphed as follows:



b) The graph of $f(x) = \frac{5x}{x^2 - 2x}$ as drawn with the graphing calculator is shown below.



For the domain, we find the roots of the denominator,

$$x^2 - 2x = 0 \implies \quad x(x-2) = 0 \implies \quad x = 0 \text{ or } x = 2.$$

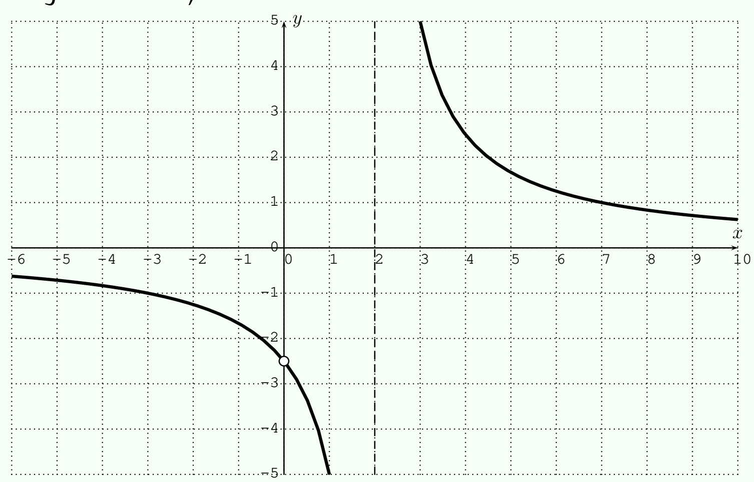
The domain is $D = \mathbb{R} - \{0, 2\}$. For the vertical asymptotes and removable singularities, we calculate the roots of the numerator,

$$5x = 0 \implies \quad x = 0.$$

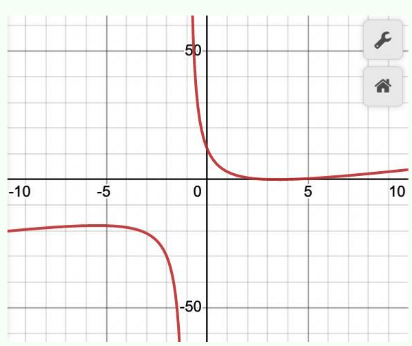
Therefore, $x = 2$ is a vertical asymptote, and $x = 0$ is a removable singularity. Furthermore, the denominator has a higher degree than the numerator, so that $y = 0$ is the horizontal asymptote. For the y-intercept, we calculate $f(0)$ by evaluating the fraction $f(x)$ at 0

$$\frac{5 \cdot 0}{0^2 - 2 \cdot 0} = \frac{0}{0},$$

which is undefined. Therefore, there is no y-intercept (we, of course, already noted that there is a removable singularity when $x = 0$). Finally, for the x-intercept, we need to analyze where $f(x) = 0$, that is where $5x = 0$. The only candidate is $x = 0$ for which f is undefined. Again, we see that there is no x-intercept. The function is then graphed as follows. (Notice in particular the removable singularity at $x = 0$.)



c) We start again by graphing the function $f(x) = \frac{x^3-9x^2+26x-24}{x^2-x-2}$ with the calculator. After zooming to an appropriate window, we get:



To find the domain of f, we find the zeros of the denominator

$$x^2 - x - 2 = 0 \implies (x+1)(x-2) = 0 \implies x = -1 \text{ or } x = 2.$$

The domain is $D = \mathbb{R} - \{-1, 2\}$. The graph suggests that there is a vertical asymptote $x = -1$. However the $x = 2$ appears not to be a vertical asymptote. This would happen when $x = 2$ is a removable singularity, that is, $x = 2$ is a root of both numerator and denominator of $f(x) = \frac{p(x)}{q(x)}$. To confirm this, we calculate the numerator $p(x)$ at $x = 2$:

$$p(2) = 2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 8 - 36 + 52 - 24 = 0$$

Therefore, $x = 2$ is indeed a removable singularity. To analyze f further, we also factor the numerator. Using the factor theorem, we know that $x-2$ is a factor of the numerator. Its quotient is calculated via long division.

$$\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & -7x & +12 & \\
\cline{2-5}
x-2 & x^3 & -9x^2 & +26x & -24 \\
 & -(x^3 & -2x^2) & & \\
\cline{2-3}
 & & -7x^2 & +26x & -24 \\
 & & -(-7x^2 & +14x) & \\
\cline{3-4}
 & & & 12x & -24 \\
 & & & -(12x & -24) \\
\cline{4-5}
 & & & & 0
\end{array}$$

With this, we obtain:

$$f(x) = \frac{(x-2)(x^2 - 7x + 12)}{x^2 - x - 2} = \frac{(x-2)(x-3)(x-4)}{(x+1)(x-2)}$$

Therefore, we conclude that $x = -1$ is a vertical asymptote and $x = 2$ is a removable singularity. We also see that the x-intercepts are $(3, 0)$ and $(4, 0)$ (that is $x-$ values where the numerator is zero).

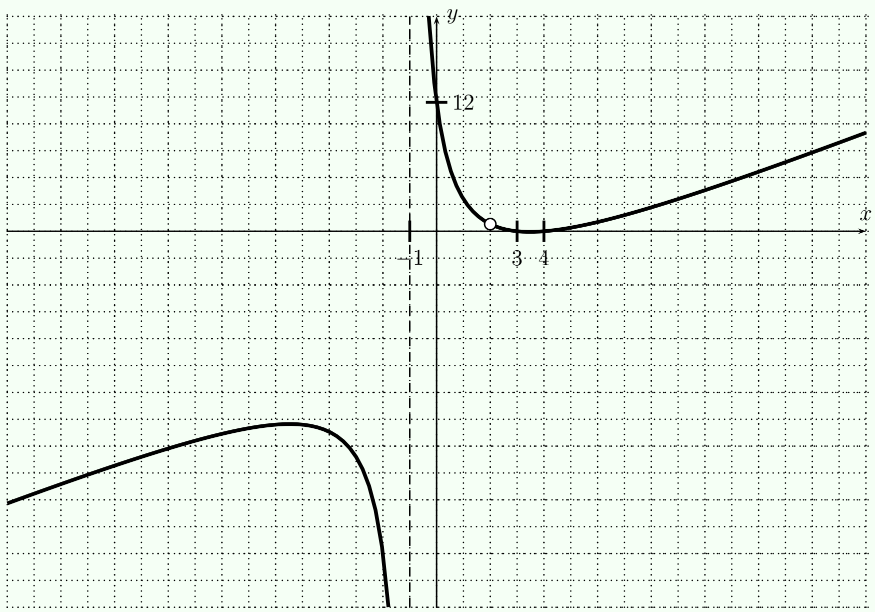
Now, the long range behavior is determined by ignoring the lower terms in the fraction,

$$|x| \text{ large} \implies f(x) \approx \frac{x^3}{x^2} = x \implies \text{no horizontal asymptote}$$

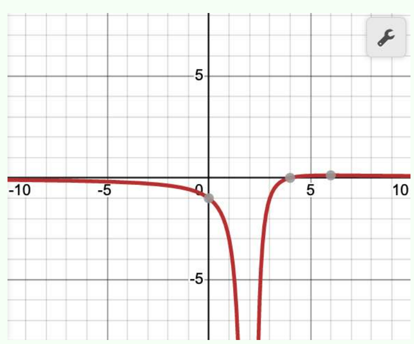
Finally, the y-coordinate of the y-intercept is given by

$$y = f(0) = \frac{0^3 - 9 \cdot 0^2 + 26 \cdot 0 - 24}{0^2 - 0 - 2} = \frac{-24}{-2} = 12.$$

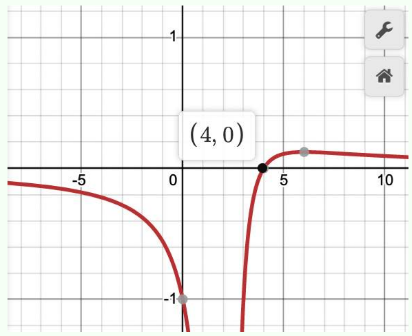
We draw the graph as follows:



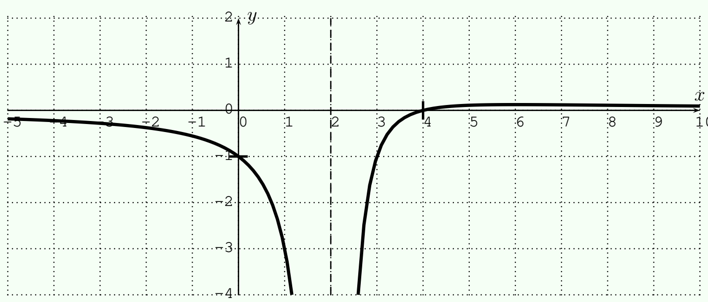
d) We first graph $f(x) = \frac{x-4}{(x-2)^2}$.



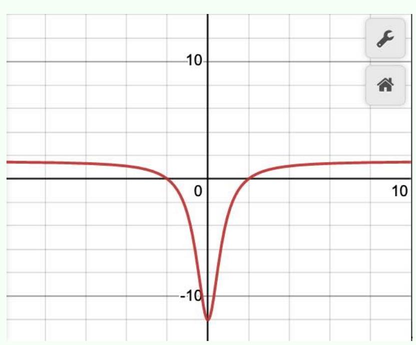
The domain is all real numbers except where the denominator becomes zero, that is, $D = \mathbb{R} - \{2\}$. The graph has a vertical asymptote $x = 2$ and no hole. The horizontal asymptote is at $y = 0$, since the denominator has a higher degree than the numerator. The y-intercept is at $y_0 = f(0) = \frac{0-4}{(0-2)^2} = \frac{-4}{4} = -1$. The x-intercept is where the numerator is zero, $x - 4 = 0$, that is at $x = 4$. Since the above graph did not show the x-intercept clearly, we can observe it better by zooming into the graph vertically (see Note 4.4):



Note in particular that the graph intersects the x-axis at $x = 4$ and then changes its direction to approach the x-axis from above. A graph of the function f which includes all these features is displayed below.



e) We graph $f(x) = \frac{3x^2-12}{2x^2+1}$.



For the domain, we determine the zeros of the denominator.

$$2x^2 + 1 = 0 \implies 2x^2 = -1 \implies x^2 = -\frac{1}{2}.$$

The only solutions of this equation are given by complex numbers, but not by any real numbers. In particular, for any real number x, the denominator of $f(x)$ is not zero. The domain of f is all real numbers, $D = \mathbb{R}$. This implies in turn that there are no vertical asymptotes, and no removable singularities.

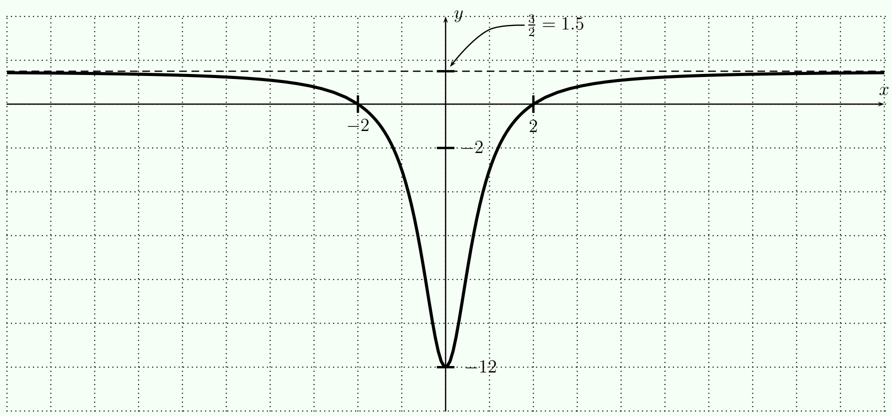
The x-intercepts are determined by $f(x) = 0$, that is where the numerator is zero,

$$3x^2 - 12 = 0 \implies 3x^2 = 12 \implies x^2 = 4 \implies x = \pm 2.$$

The horizontal asymptote is given by $f(x) \approx \frac{3x^2}{2x^2} = \frac{3}{2}$, that is, it is at $y = \frac{3}{2} = 1.5$. The y-intercept is at

$$y = f(0) = \frac{3 \cdot 0^2 - 12}{2 \cdot 0^2 + 1} = \frac{-12}{1} = -12.$$

We sketch the graph as follows:



Since the graph is symmetric with respect to the y-axis, we can make one more observation, namely that the function f is even (see Observation 4.24 on page 74):

$$f(-x) = \frac{3(-x)^2 - 12}{2(-x)^2 + 1} = \frac{3x^2 - 12}{2x^2 + 1} = f(x)$$

□

10.2 *Optional section:* Graphing rational functions by hand

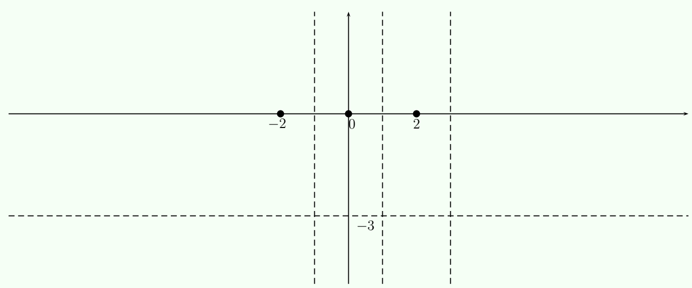
In this section we will show how to sketch the graph of a factored rational function without the use of a calculator. It will be helpful to the reader to have read Section 8.3 on graphing a polynomial by hand before continuing in this section. In addition to having the same difficulties as polynomials, calculators often have difficulty graphing rational functions near an asymptote.

Example 10.5

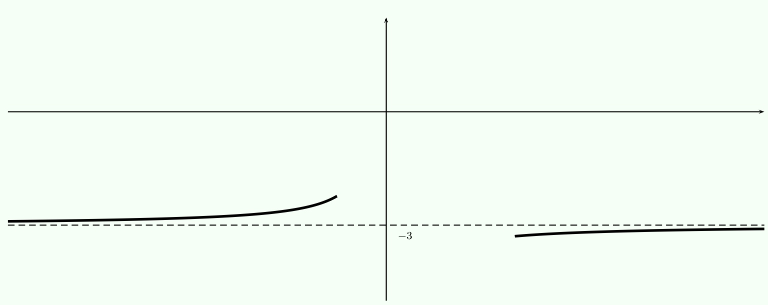
Graph the function $p(x) = \frac{-3x^2(x-2)^3(x+2)}{(x-1)(x+1)^2(x-3)^3}$.

Solution.

We can see that p has zeros at $x = 0, 2,$ and -2 and vertical asymptotes $x = 1$, $x = -1$ and $x = 3$. Also note that for large $|x|$, $p(x) \approx -3$. So there is a horizontal asymptote $y = -3$. We indicate each of these facts on the graph:



We can in fact get a more precise statement by performing a long division and writing $p(x) = \frac{n(x)}{d(x)} = -3 + \frac{r(x)}{d(x)}$. If we drop all but the leading order terms in the numerator and the denominator of the second term, we see that $p(x) \approx -3 - \frac{12}{x}$, whose graph for large $|x|$ looks like

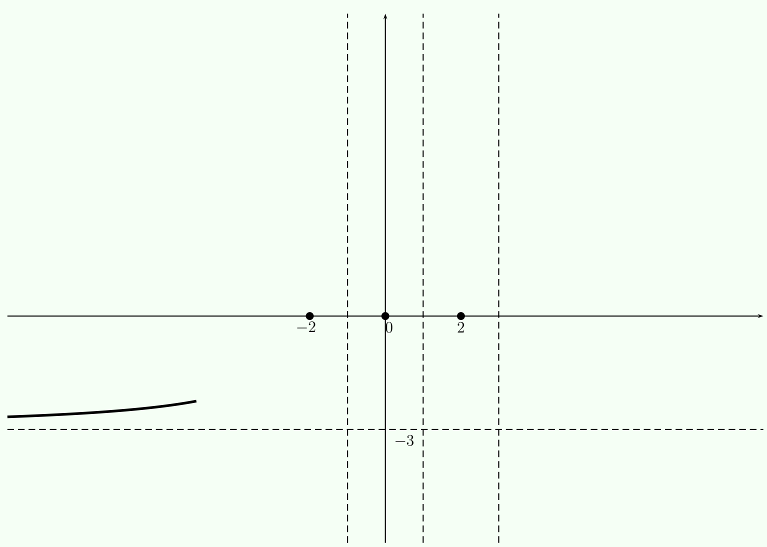


This sort of reasoning can make the graph a little more accurate but is not necessary for a sketch.

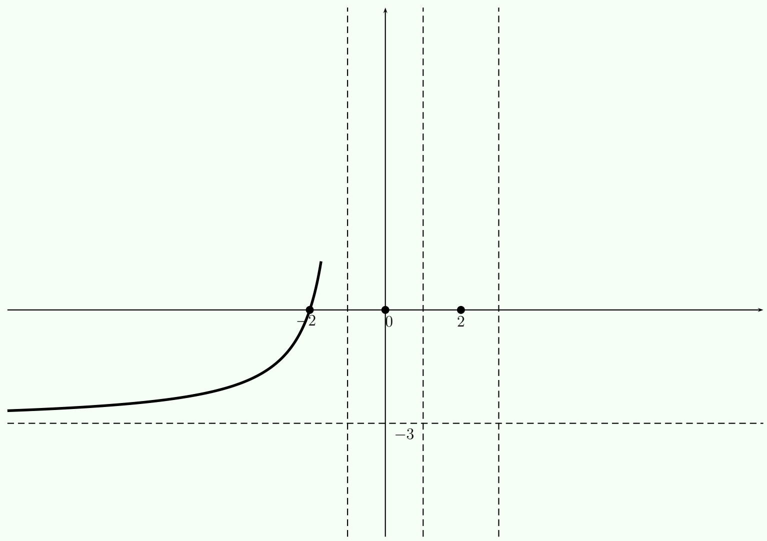
We also have the following table:

for a	near a, $p(x) \approx$	type	sign change at a
-2	$C_1(x+2)$	linear	changes
-1	$C_2/(x+1)^2$	asymptote	does not change
0	$C_3\ x^2$	parabola	does not change
1	$C_4/(x-1)$	asymptote	changes
2	$C_5(x-2)^3$	cubic	changes
3	$C_6/(x-3)^3$	asymptote	changes

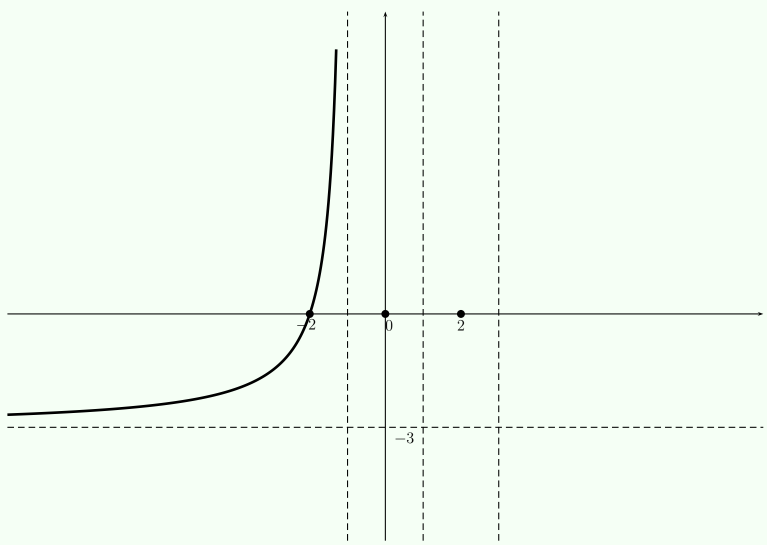
Note that, if the power appearing in the second column is even, then the function does not change from one side of a to the other. If the power is odd, the sign changes (either from positive to negative or from negative to positive).
Now we move from large negative x values toward the right, taking into account the above table. For large negative x, we start our sketch as follows:



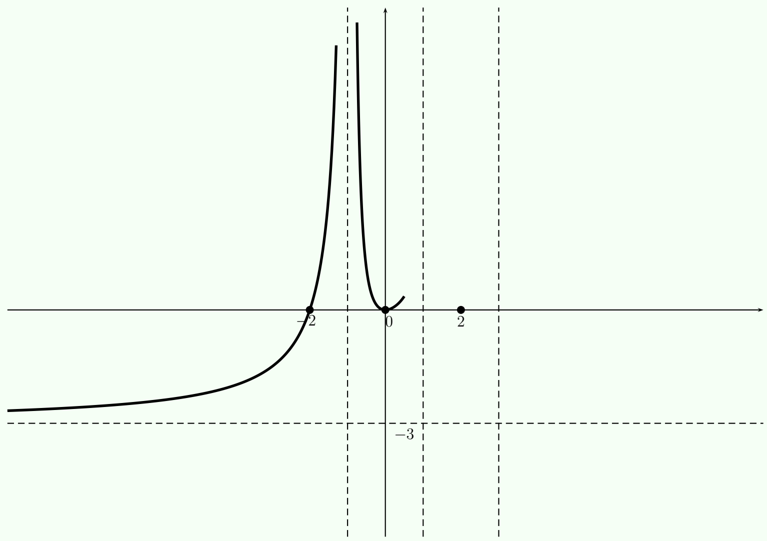
And noting that near $x = -2$ the function $p(x)$ is approximately linear, we have



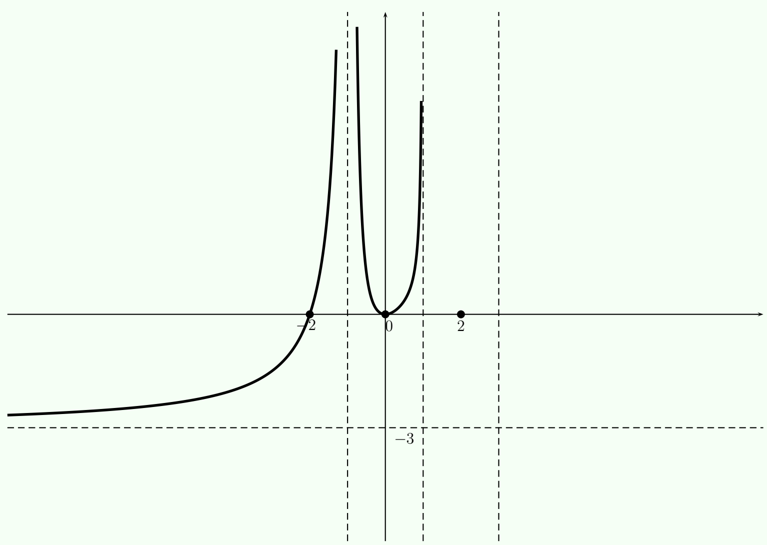
Then noting that we have an asymptote (noting that we cannot cross the x-axis without creating an x-intercept) we have



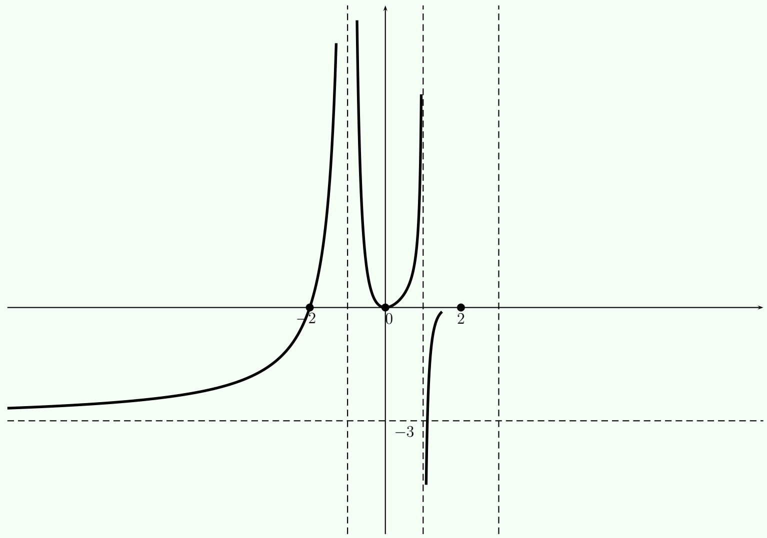
Now, from the table we see that there is no sign change at -1 so we have



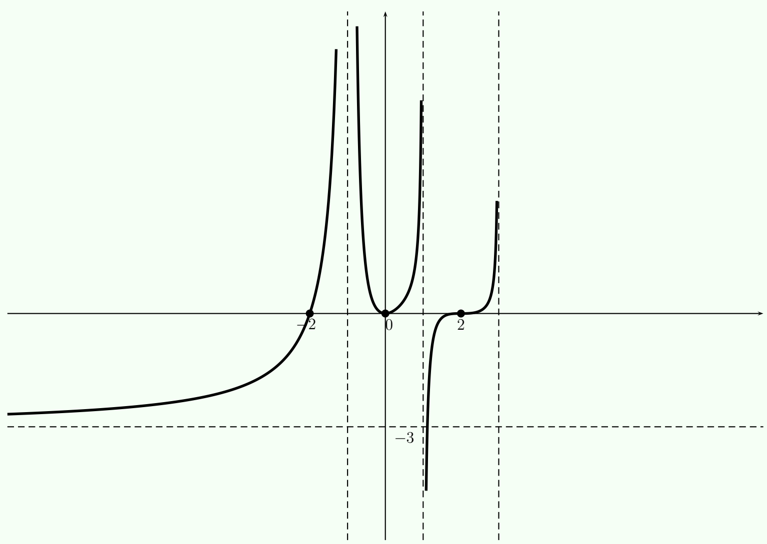
and from the table we see that near $x = 0$ the function $p(x)$ is approximately quadratic and therefore the graph looks like a parabola. This together with the fact that there is an asymptote at $x = 1$ gives



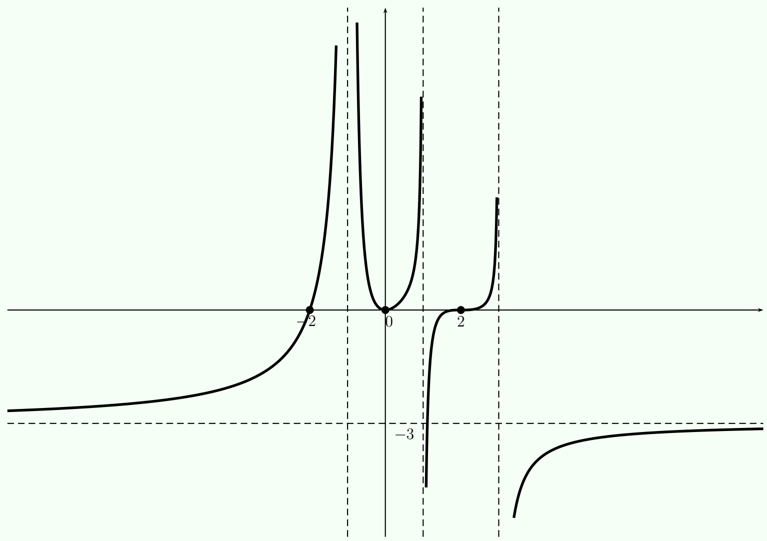
Now, from the table we see that the function changes sign at the asymptote, so while the graph "hugs" the top of the asymptote on the left-hand side, it "hugs" the bottom on the right-hand side giving



Now, from the table we see that near $x = 2$, $p(x)$ is approximately cubic. Also, there is an asymptote $x = 3$ so we get



Finally, we see from the table that $p(x)$ changes sign at the asymptote $x = 3$ and has a horizontal asymptote $y = -3$, so we complete the sketch:



Note that if we had made a mistake somewhere there is a good chance that we would have not been able to get to the horizontal asymptote on the right side without creating an additional x-intercept.
What can we conclude from this sketch? This sketch exhibits only the general shape which can help decide on an appropriate window size if we want to investigate details using technology. Furthermore, we can infer where $p(x)$ is positive and where $p(x)$ is negative. However, it is important to notice that there may be wiggles in the graph that we have not included in our sketch. □

We now give one more example of graphing a rational function where the horizontal asymptote is $y = 0$.

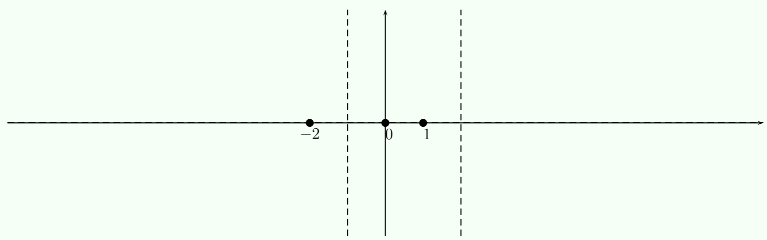
Example 10.6

Sketch the graph of

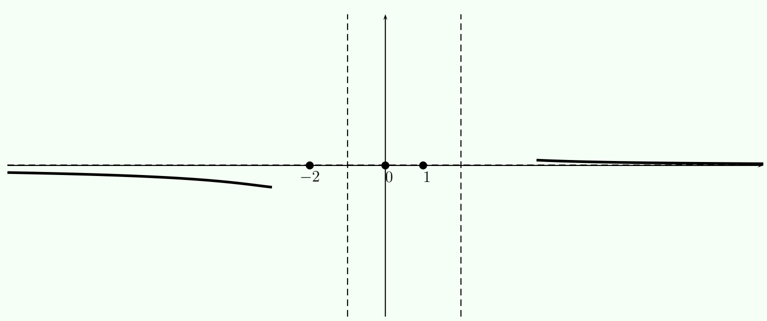
$$r(x) = \frac{2x^2(x-1)^3(x+2)}{(x+1)^4(x-2)^3}.$$

Solution.

Here we see that there are x-intercepts at $(0,0)$, $(0,1)$, and $(0,-2)$. There are two vertical asymptotes: $x = -1$ and $x = 2$. In addition, there is a horizontal asymptote at $y = 0$. (Why?) Putting this information on the graph gives



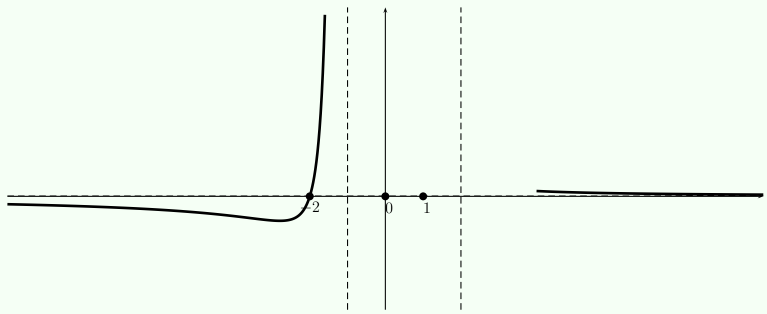
In this case, it is easy to get more information for large $|x|$ that will be helpful in sketching the function. Indeed, when $|x|$ is large, we can approximate $r(x)$ by dropping all but the highest order term in the numerator and denominator which gives $r(x) \approx \frac{2x^6}{x^7} = \frac{2}{x}$. So for large $|x|$, the graph of r looks like



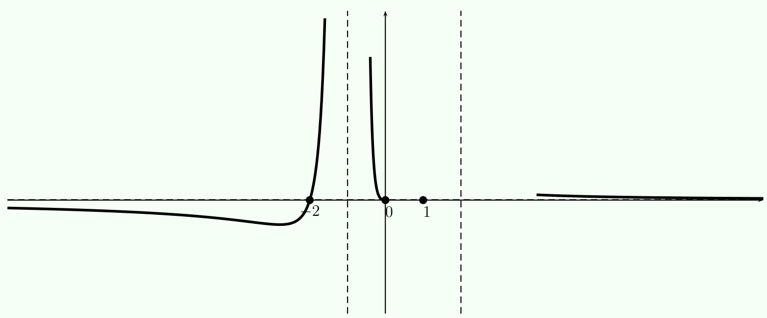
The function gives the following table:

for a	near a, $p(x) \approx$	type	sign change at a
-2	$C_1(x+2)$	linear	changes
-1	$C_2/(x+1)^4$	asymptote	does not change
0	$C_3\ x^2$	parabola	does not change
1	$C_4(x-1)^3$	cubic	changes
2	$C_5/(x-2)^3$	asymptote	changes

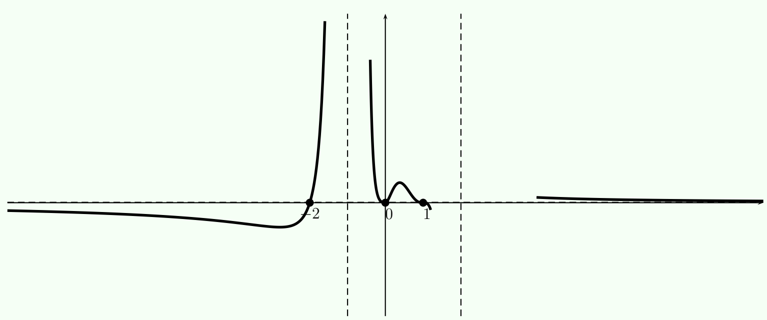
Looking at the table for this function, we see that the graph should look like a line near the zero $(0, -2)$ and since it has an asymptote $x = -1$, the graph looks something like:



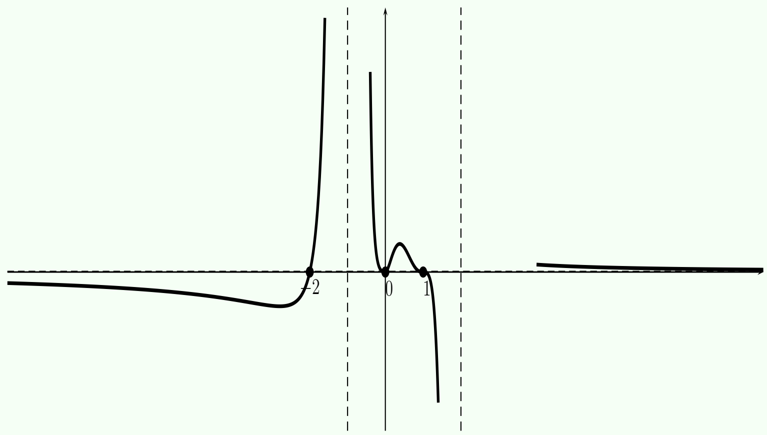
Then, looking at the table, we see that $r(x)$ does not change its sign near $x = -1$, so that we obtain:



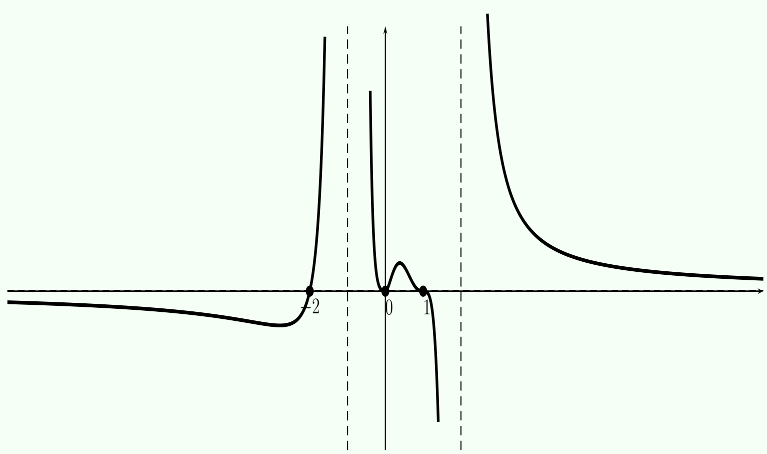
Now, the function is approximately quadratic near $x = 0$, so the graph looks like:



Next, looking at the root at 1, we note that the function is approximately cubic there, and that there is an asymptote at $x = 2$. We draw this as follows:



Finally, we see that the function changes sign at $x = 2$ (see the table). So, since the graph "hugs" the asymptote near the bottom of the graph on the left side of the asymptote, it will "hug" the asymptote near the top on the right side. So this, together with the fact that $y = 0$ is an asymptote, gives the sketch (perhaps using an eraser to match the part of the graph on the right that uses the large x):



Note that if the graph could not be matched at the end without creating an extra x-intercept, then a mistake has been made. □

10.3 Exercises

Exercise 10.1

Find the domain, the vertical asymptotes, and removable discontinuities of the functions:

a) $f(x) = \frac{2}{x-2}$ b) $f(x) = \frac{x^2+2}{x^2-6x+8}$

c) $f(x) = \frac{3x+6}{x^3-4x}$ d) $f(x) = \frac{(x-2)(x+3)(x+4)}{(x-2)^2(x+3)(x-5)}$

e) $f(x) = \frac{x-1}{x^3-1}$ f) $f(x) = \frac{2}{x^3-2x^2-x+2}$

Exercise 10.2

Find the horizontal asymptotes of the functions:

a) $f(x) = \frac{8x^2+2x+1}{2x^2+3x-2}$ b) $f(x) = \frac{1}{(x-3)^2}$

c) $f(x) = \frac{x^2+3x+2}{x-1}$ d) $f(x) = \frac{12x^3-4x+2}{-3x^3+2x^2+1}$

Exercise 10.3

Find the x- and y-intercepts of the functions:

a) $f(x) = \frac{x-3}{x-1}$ b) $f(x) = \frac{x^3-4x}{x^2-8x+15}$

c) $f(x) = \frac{(x-3)(x-1)(x+4)}{(x-2)(x-5)}$ d) $f(x) = \frac{x^2+5x+6}{x^2+2x}$

Exercise 10.4

Sketch a complete graph of the function f. To this end, calculate the domain of f, the horizontal and vertical asymptotes, the removable singularities, the x- and y-intercepts of the function, and graph the function with the graphing calculator.

a) $f(x) = \frac{7x+2}{3x-5}$ b) $f(x) = \frac{x^2-x-2}{x^2+2x-3}$

c) $f(x) = \frac{3x^2-7x+2}{x^2-3x-10}$ d) $f(x) = \frac{x^2+7x+12}{x^2+6x+8}$

e) $f(x) = \frac{x-3}{x^3-3x^2-6x+8}$ f) $f(x) = \frac{x^3-3x^2-x+3}{x^3-2x^2}$

Exercise 10.5

Find a rational function f that satisfies all the given properties.

a) vertical asymptote at $x = 4$ and horizontal asymptote at $y = 0$

b) vertical asymptotes at $x = 2$ and $x = 3$ and horizontal asymptote at $y = 5$

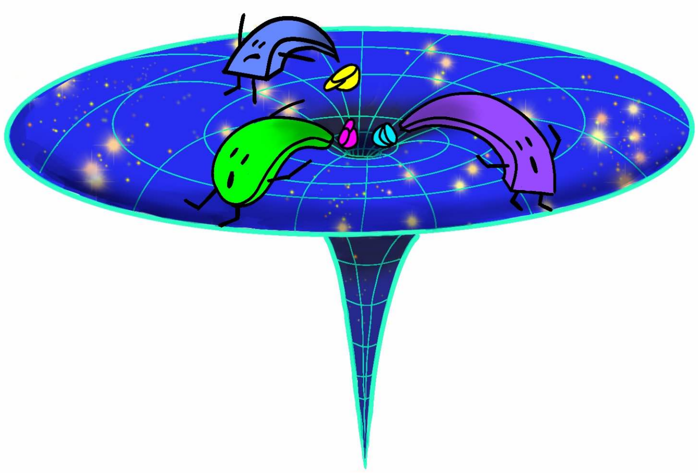
c) removable singularity at $x = 1$ and no horizontal asymptote

Chapter 11

Exploring discontinuities and asymptotes

We have seen that rational functions have certain features that were not present in polynomials, such as discontinuities. These discontinuities can be removable ("holes"), or non-removable (at vertical asymptotes), where the function can become arbitrarily large, and thus approaches infinity.

It may be perplexing to think about functions approaching an infinite value, as we do not experience infinities in everyday life. Indeed, a quantity that approaches an infinite value would probably come with some strange side effects. For example, it is theorized that gravity approaches an infinite value at the center of a black hole (often called the singularity), and we definitely do not recommend to get anywhere near such an object!



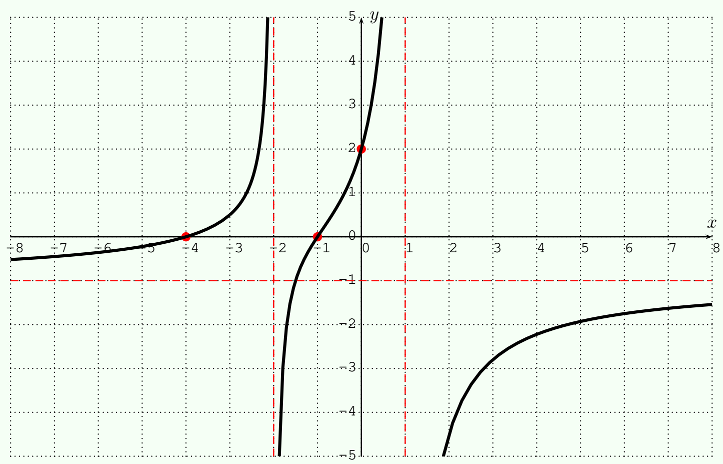
In this chapter, we will explore the behavior of functions near an input x_0. In Section 11.1 we look at asymptotes and removable singularities for rational functions, while we look at the general case in Section 11.2.

11.1 More on rational functions

We now explore the asymptotic behavior of rational functions near a discontinuity, as well as the behavior at infinity in more detail. First, we review how one can recover the formula of a rational function $f(x) = \frac{p(x)}{q(x)}$ from its asymptotes and roots.

Example 11.1

The graph of the rational function $f(x) = \frac{p(x)}{q(x)}$ is displayed below, where p and q are polynomials of degree 2. Assuming that all intercepts and asymptotes are at integer values as indicated (in red), find these intercepts and asymptotes. Use this information to find a formula for $f(x)$.



Solution.

The intercepts and asymptotes can be read off from the graph:

x-intercepts :	$(x, y) = (-4, 0)$ and $(x, y) = (-1, 0)$
y-intercept :	$(x, y) = (0, 2)$
vertical asymptotes :	$x = -2$ and $x = 1$
horizontal asymptote :	$y = -1$

Since the x-intercepts determine the roots of the numerator of f, and the vertical asymptotes determine the roots of the denominator of f, we see that f must be of the form

$$f(x) = a \cdot \frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$$

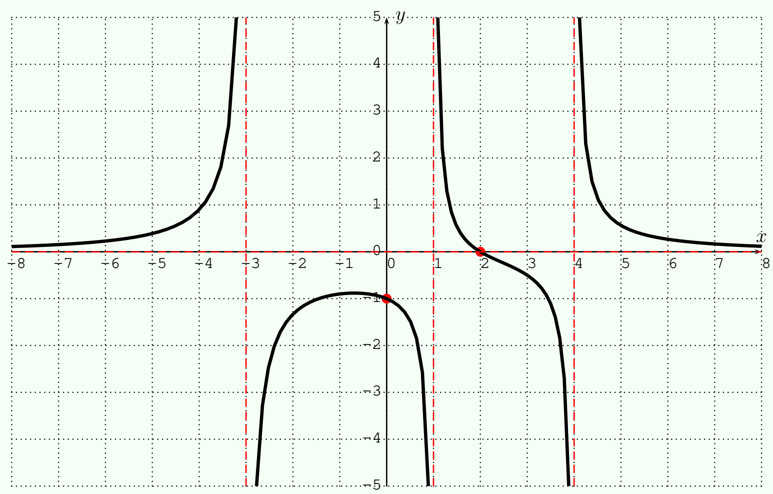
where a is some overall coefficient. The coefficient a can be determined either via the horizontal asymptote or via the y-intercept. In the first case, note that the horozintal asymptote of $f(x) = a\cdot\frac{x^2+5x+4}{x^2+x-2}$ is $y=a$, so that we conclude $a=-1$. For the latter case, note that the y-intercept of $f(x) = a\cdot\frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$ is at $y = f(0) = a\cdot\frac{4\cdot 1}{2\cdot(-1)} = a\cdot(-2)$, which, according to the graph, has to be equal to 2:

$$2 = -2a \implies a = -1$$

Therefore, $f(x) = (-1)\cdot\frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$. □

Example 11.2

The graph of the rational function $f(x) = \frac{p(x)}{q(x)}$ is displayed below, where p is a polynomial of degree 1 and q is a polynomial of degree 3. Assuming that all intercepts and asymptotes are at integer values as indicated (in red), find these intercepts and asymptotes. Use this information to find a formula for $f(x)$.



Solution.

From the graph, we see that the intercepts and asymptotes are:

x-intercept :	$(x, y) = (2, 0)$
y-intercept :	$(x, y) = (0, -1)$
vertical asymptotes :	$x = -3$ and $x = 1$ and $x = 4$
horizontal asymptote :	$y = 0$

Using the x-intercept and the vertical asymptotes, we see that f is of the form

$$f(x) = a \cdot \frac{(x-2)}{(x+3)\cdot(x-1)\cdot(x-4)}$$

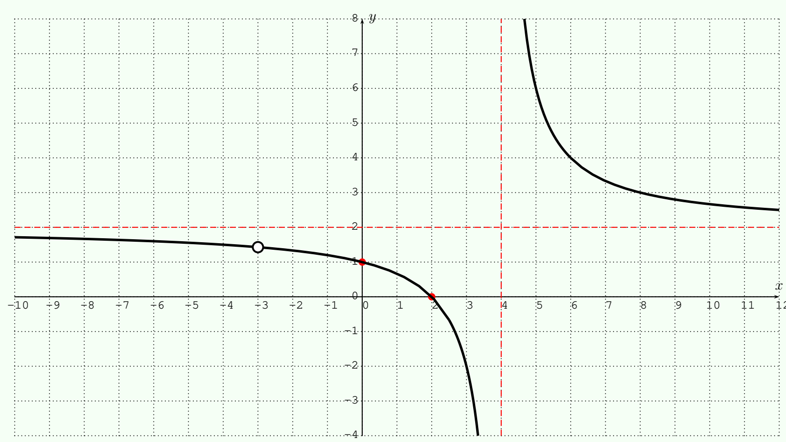
for some coefficient a. Note that the horizontal asymptote is automatically $y = 0$, since the denominator has a higher degree than the numerator. To find the coeffient a, we use the y-intercept, which in this formula is given by $y = f(0) = a \cdot \frac{-2}{3\cdot(-1)\cdot(-4)} = a \cdot \frac{-2}{12} = a \cdot \frac{-1}{6}$. Since the graph shows a y-intercept at $y = -1$, it follows that

$$-1 = a \cdot \frac{-1}{6} \implies a = 6$$

Therefore, $f(x) = 6 \cdot \frac{(x-2)}{(x+3)\cdot(x-1)\cdot(x-4)}$. □

Example 11.3

The graph of the function $y = f(x)$ is displayed below.



Assume that $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where p and q are polynomials of degree 2; that all intercepts and asymptotes are at integer values (indicated in red); and that f has a removable discontinuity at $x = -3$.

a) Use this information to find a formula for $f(x)$.

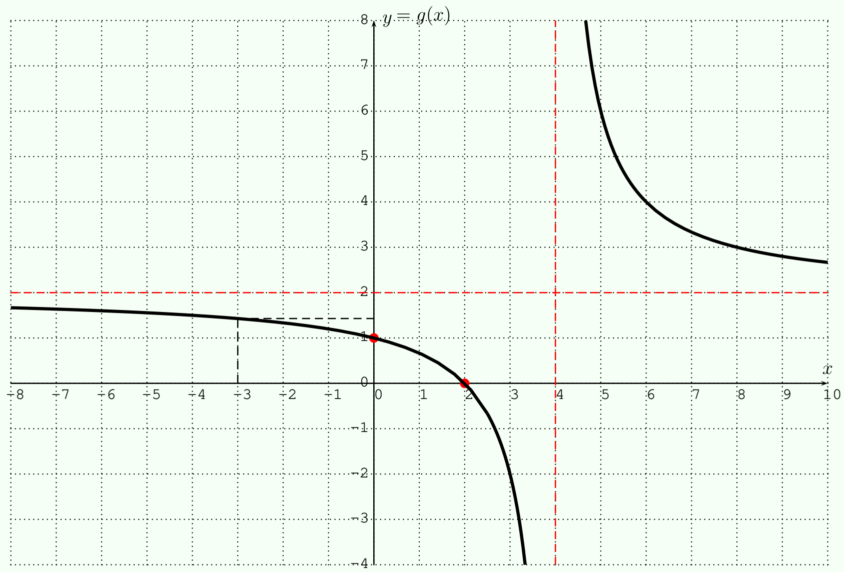
b) Find the coordinates of the removable discontinuity.

Solution.

a) The x-intercept is at $(2, 0)$, the y-intercept is at $(0, 1)$, the vertical asymptote is at $x = 4$, and the horizontal asymptote is at $y = 2$. The discontinuity at $x = -3$ requires a factor of $(x + 3)$ in the numerator and the denominator, so that $f(x) = a \cdot \frac{(x-2)\cdot(x+3)}{(x-4)\cdot(x+3)}$. From the horizontal asymptote $y = 2$ we see that $a = 2$, so that

$$f(x) = 2 \cdot \frac{(x-2)\cdot(x+3)}{(x-4)\cdot(x+3)}$$

b) To find the y-coordinate of the discontinuity, we may try to plug $x = -3$ into the function f. Unfortunately, this does not lead to an answer, since $f(-3)$ is undefined. However, we can see that for all $x \neq -3$, the function f coincides with $g(x) = 2 \cdot \frac{x-2}{x-4}$ after canceling the factor $(x + 3)$. The graph of g is displayed below.



Evaluating g at $x = -3$ gives

$$g(-3) = 2 \cdot \frac{-3-2}{-3-4} = 2 \cdot \frac{-5}{-7} = \frac{10}{7}$$

This shows that the coordinates of the discontinuity of f are $(x, y) = (-3, \frac{10}{7})$. We say that, "as x approaches -3, $f(x)$ approaches $\frac{10}{7}$", which is written as:

$$\text{as } x \to -3, \quad f(x) \to \frac{10}{7}$$

This is also called a limit, written as: $\lim\limits_{x \to -3} f(x) = \frac{10}{7}$.

□

We also want to study a rational function as x becomes arbitrarily large (having large positive or large negative values), or, saying it differently, when x approaches ±infinity ($x \to \infty$ or $x \to -\infty$). We already saw the behavior of some rational functions for this case when we considered horizontal asymptotes. More generally, we can describe the behavior of a rational function when $x \to \pm\infty$ as follows.

Observation 11.4: Asymptotic behavior, slant asymptote

The asymptotic behavior of a rational function $f(x) = \frac{p(x)}{q(x)}$, in the case where the degree of p is greater than the degree of q, can be calculated by performing a long division. If the long division has a quotient $g(x)$ and a remainder $r(x)$, then

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}.$$

Now, since deg(r) <deg(q), the fraction $\frac{r(x)}{q(x)}$ approaches zero as x approaches $\pm\infty$, so that $f(x) \approx g(x)$ for large $|x|$. Thus, $f(x)$ is approximately $g(x)$ for large $|x|$.
If $g(x)$ is a linear function (that is, a polynomial of degree 1), then g is called the **slant asymptote** of f.

Example 11.5

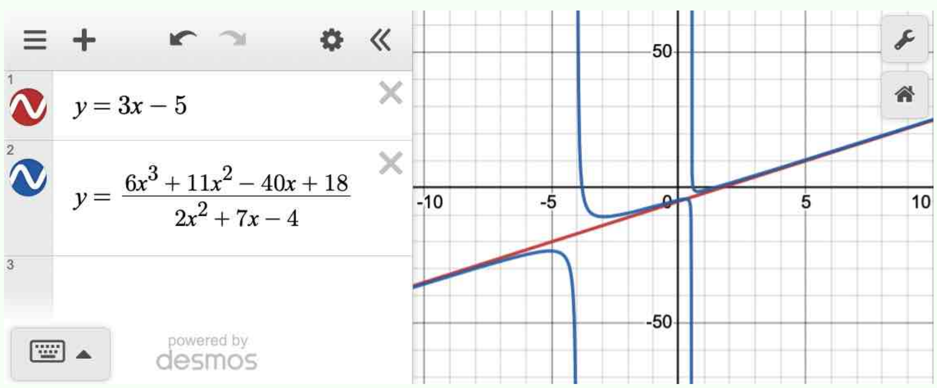
Find the slant asymptote of $f(x) = \frac{6x^3+11x^2-40x+18}{2x^2+7x-4}$.

Solution.

We divide the polynomials via a long division:

$$\begin{array}{r|rrrr}
 & & & 3x & -5 \\ \hline
2x^2+7x-4 & 6x^3 & +11x^2 & -40x & +18 \\
 & -(6x^3 & +21x^2 & -12x) & \\ \hline
 & & -10x^2 & -28x & +18 \\
 & & -(-10x^2 & -35x & +20) \\ \hline
 & & & 7x & -2
\end{array}$$

Therefore, $f(x) = 3x - 5 + \frac{7x-2}{2x^2+7x-4}$, so that for large $|x|$, we have $f(x) \approx 3x - 5$. Thus, the slant asymptote of $f(x)$ is $y = 3x - 5$.



Note that the slant asymptote has positive slope, so that it goes to $+\infty$ when x becomes large (that is $x \to \infty$), while it goes to $-\infty$ when $x \to -\infty$.

We also write this as $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$. □

11.2 *Optional section:* Limits

In Example 11.3 we implicitly calculated the *limit* as the y-coordinate of the removable discontinuity. A full treatment of limits is the subject of a course in calculus, which will provide many more tools to evaluate limits. For now, we will only explore some intuition regarding limits mainly stemming from studying graphs of functions.

Definition 11.6: Limit

Let $y = f(x)$ be a function, which is defined near a number a. We write

$$\boxed{\text{as } x \to a, \quad f(x) \to L}$$

if $f(x)$ approaches L as x approaches a. Alternatively, we also write

$$\boxed{\lim_{x\to a} f(x) = L}$$

and we call L the **limit** of f as x approaches a.

This definition contains several concepts that were not made precise, such as what it means to be "near a", what it means to "approach a number a", or what it means to "approach L as x approaches a".

A precise version of a limit will formally specify that $f(x)$ will be within an arbitrarily small distance from L for all x close enough to a. This is what is done, for example, in the ϵ-δ definition of a limit. The details are topics of a course in calculus, and are beyond the scope of this text.

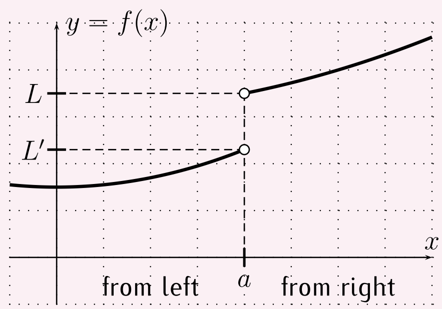
We also consider the case when x approaches a number from one side only, that is, from the right or from the left:

- if $f(x)$ approaches L as x approaches a from the *right*, then we write

$$\boxed{\text{as } x \to a^+, \quad f(x) \to L} \qquad \text{or} \qquad \boxed{\lim_{x\to a^+} f(x) = L}$$

- if $f(x)$ approaches L' as x approaches a from the *left*, then we write

$$\boxed{\text{as } x \to a^-, \quad f(x) \to L'} \qquad \text{or} \qquad \boxed{\lim_{x\to a^-} f(x) = L'}$$



Note 11.7

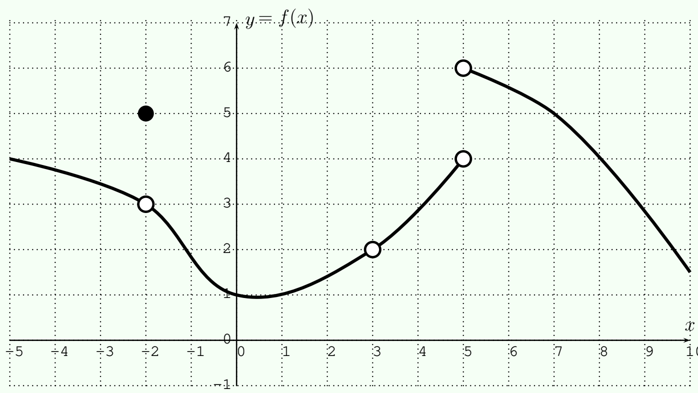
We note that $f(x)$ approaches L when x approaches a, precisely when $f(x)$ approaches L when x approaches a from the right and from the left.

$$\left(\lim_{x\to a} f(x) = L\right) \quad \Leftrightarrow \quad \left(\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = L\right)$$

We explore these concepts in the next examples.

Example 11.8

The graph of $y = f(x)$ is shown below.



Find the limit from the right, the limit from the left, and the (two-sided) limit as x approaches the following numbers.

a) as $x \to 3$ b) as $x \to 5$ c) as $x \to 7$ d) as $x \to -2$

Solution.

a) The limits from the right and from the left approaching 3 are $\lim\limits_{x\to 3^+} f(x) = 2$ and $\lim\limits_{x\to 3^-} f(x) = 2$. Therefore, we also have $\lim\limits_{x\to 3} f(x) = 2$.

b) The limits from the right and left approaching 5 are $\lim\limits_{x\to 5^+} f(x) = 6$ and $\lim\limits_{x\to 5^-} f(x) = 4$. Since these limits differ, the two-sided limit $\lim\limits_{x\to 5} f(x)$ does not exist.

c) The limits approaching 7 are $\lim_{x\to 7^+} f(x) = 5$ and $\lim_{x\to 7^-} f(x) = 5$, and therefore also $\lim_{x\to 7} f(x) = 5$. Note that in this case $f(7)$ is also defined and $f(7) = 5$ coincides with the limit. (We say that f is *continuous at* 7.)

d) The limits approaching -2 are equal, $\lim_{x\to -2^+} f(x) = 3$ and $\lim_{x\to -2^+} f(x) = 3$, and therefore $\lim_{x\to -2} f(x) = 3$. In this case $f(-2)$ is also defined but does not coincide with the limit.

□

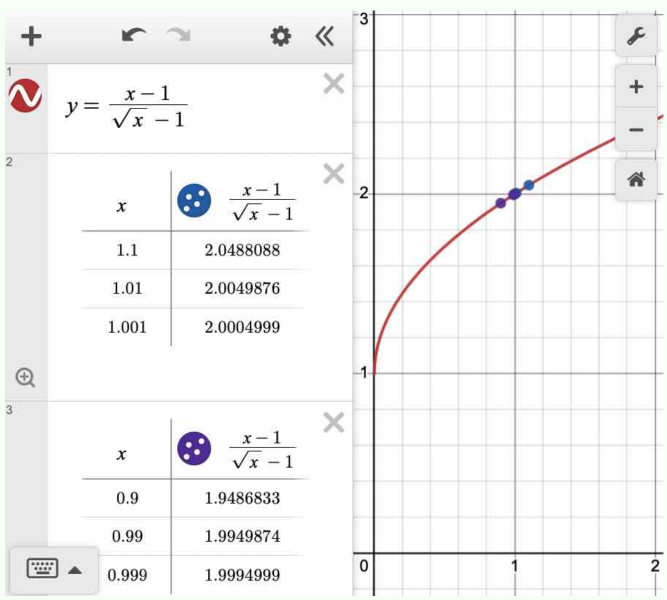
Example 11.9

Use the graphing calculator to identify the stated limits.

a) $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$ b) $\lim_{x\to 2} \frac{x\cdot|x-2|}{x-2}$ c) $\lim_{x\to -3} \frac{x+5}{x+3}$ d) $\lim_{x\to 0^+} x^2 \cdot \ln(x)$

Solution.

a) We plug numbers into the graphing calculator that approach 1 from the right and from the left. Suitable numbers from the right are: 1.1, 1.01, 1.001, etc. Suitable numbers from the left are: 0.9, 0.99, 0.999, etc.



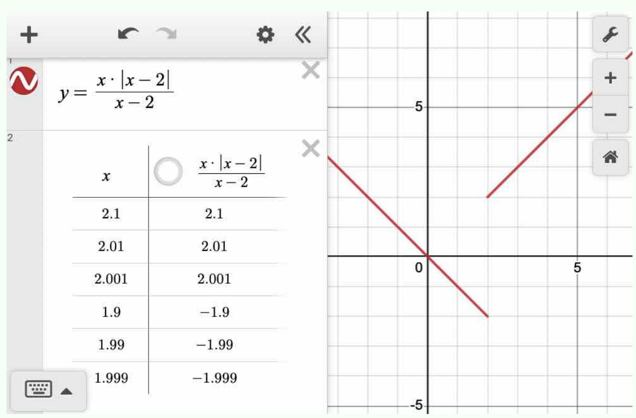
These evaluations appear to indicate that $f(x)$ approaches 2 as x approaches 1.

Note that we only used the graphing calculator to get an idea what the limit might be. A precise evaluation of the limit will require a more thorough analysis of the function at $x = 1$. Such an analysis is beyond the scope of this exercise. Nevertheless, we will provide this analysis for one case (that is for part (a) of this example) in the hope that it might help some readers, but will not do so for parts (b) and (c). Note that for $x \neq 1$:

$$f(x) = \frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1$$

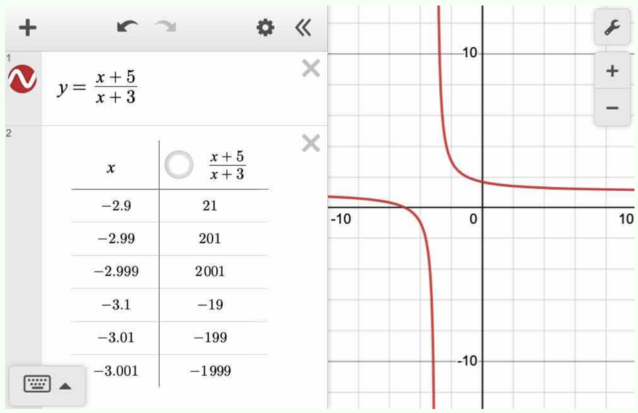
Therefore, the function $y = \sqrt{x}+1$ coincides with $y = f(x)$ for $x \neq 1$, but does not have a discontinuity at $x = 1$. Evaluating $y = \sqrt{x}+1$ at $x = 1$ gives $y = \sqrt{1}+1 = 1+1 = 2$, which shows that $\lim\limits_{x\to 1} f(x) = 2$.

b) Evaluating $\frac{x\cdot|x-2|}{x-2}$ to the right and left of 2 gives:



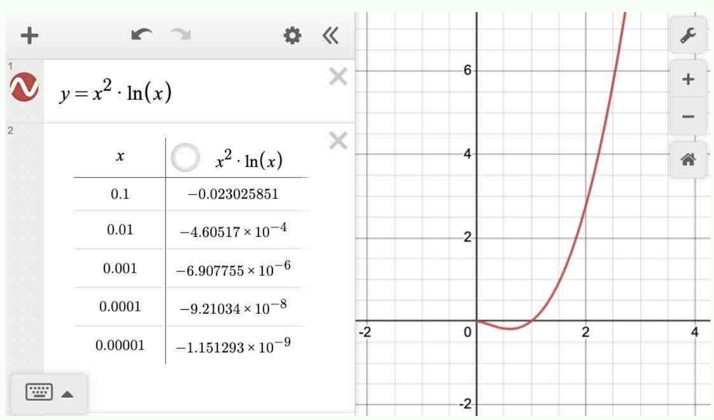
Therefore, $\lim\limits_{x\to 2^+} \frac{x\cdot|x-2|}{x-2} = 2$ and $\lim\limits_{x\to 2^-} \frac{x\cdot|x-2|}{x-2} = -2$. As these limits differ, $\lim\limits_{x\to 2} \frac{x\cdot|x-2|}{x-2}$ does not exist.

c) We evaluate $\frac{x+5}{x+3}$ close to -3.



From this, it is reasonable to conclude that $\lim\limits_{x\to -3^+} \frac{x+5}{x+3} = +\infty$ and $\lim\limits_{x\to -3^-} \frac{x+5}{x+3} = -\infty$, so that $\lim\limits_{x\to -3} \frac{x+5}{x+3}$ does not exist. Note that this also aligns with our knowledge about the vertical asymptote at $x = -3$.

d) Note that $y = x^2 \cdot \ln(x)$ is only defined for $x > 0$, since this is where $\ln(x)$ is defined.



Moreover, when $x \to 0^+$, we know that x^2 approaches 0, but $\ln(x)$ approaches $-\infty$. Interestingly, from the values shown the calculators, it appears that this product approaches $\lim\limits_{x\to 0^+} x^2 \cdot \ln(x) = 0$.

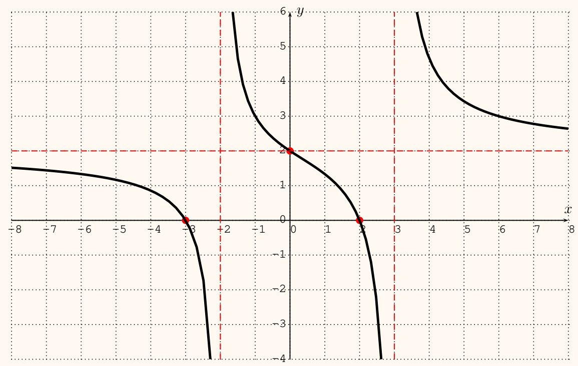
□

11.3 Exercises

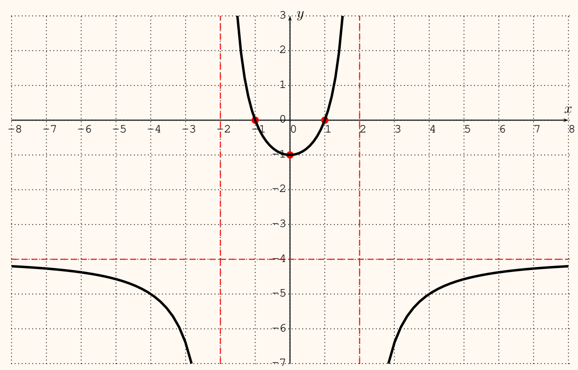
Exercise 11.1

Below are the graphs of rational functions whose numerators and denominators are polynomials of degree 2. All intercepts and asymptotes are at integer values, indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

a)



b)

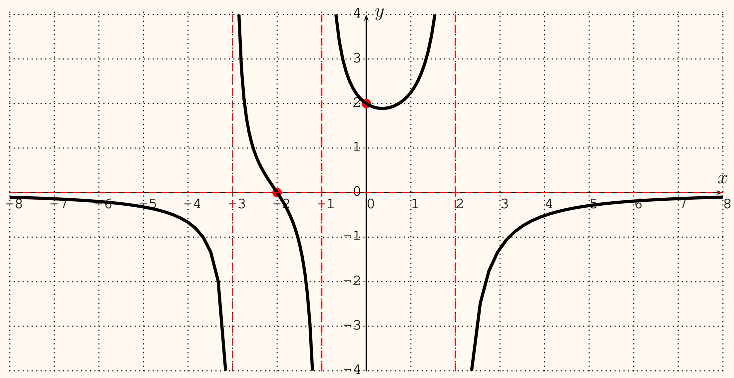


c)

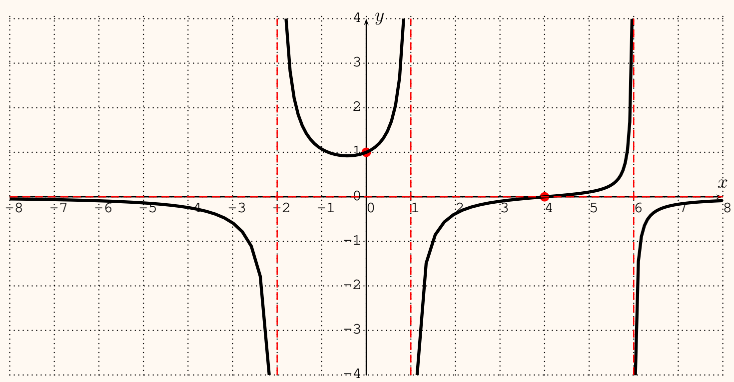
Exercise 11.2

Below are the graphs of rational functions whose numerators are polynomials of degree 1 and whose denominators are polynomials of degree 3. All intercepts and asymptotes are at integer values indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

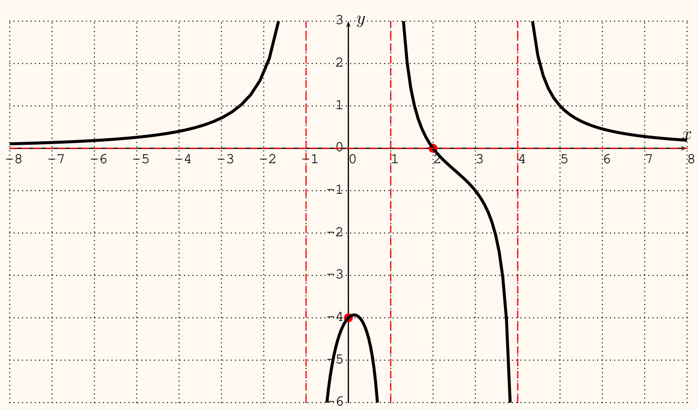
a)



b)



c)



Exercise 11.3

Find the domain of each rational function below. Identify the removable discontinuities and find their x- and y-coordinates.

a) $f(x) = \frac{(x-3)(x-4)}{(x+5)(x-4)}$

b) $f(x) = \frac{3(x+2)(x-5)}{(x+3)(x-5)}$

c) $f(x) = \frac{7(x-2)}{(x+3)(x-2)(x-6)}$

d) $f(x) = \frac{x^2+6x+8}{x^2+x-12}$

e) $f(x) = \frac{x^2-9}{x^2-x-6}$

f) $f(x) = \frac{x^2-4x+3}{x^3+x^2-2x}$

Exercise 11.4

Find the slant asymptote of the rational function.

a) $f(x) = \frac{2x^3+9x^2-20x-21}{2x^2-3x-4}$

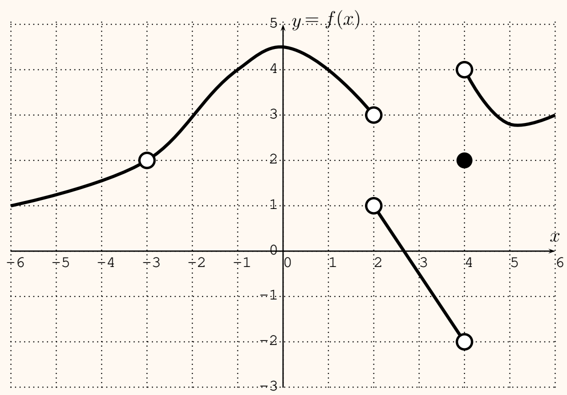
b) $f(x) = \frac{2x^3-13x^2+35x-26}{x^2-4x+6}$

c) $f(x) = \frac{12x^3+10x^2-4x-9}{3x^2+x-2}$

d) $f(x) = \frac{-3x^3-4x^2+20x-16}{x^2+2x-5}$

Exercise 11.5

The graph of the function $y = f(x)$ is shown below.



Find the limits of $f(x)$ as x approaches the values indicated below.

a) $x \to 2^+$

b) $x \to 2^-$

c) $x \to 2$

d) $x \to -3^+$

e) $x \to -3^-$

f) $x \to -3$

g) $x \to -1^+$

h) $x \to -1^-$

i) $x \to -1$

j) $x \to 4^+$

k) $x \to 4^-$

l) $x \to 4$

Exercise 11.6

Choose inputs that approach the given value from the indicated side (right or left). (Note that there is not just one unique answer for this part of the problem!)
Then, use the graphing calculator to compute the corresponding output values and guess what the limit might be.

a) $\lim\limits_{x\to 3^-} \frac{x-3}{|x-3|}$ b) $\lim\limits_{x\to 1^+} \frac{x^3-1}{\sqrt{x}-1}$ c) $\lim\limits_{x\to 2^-} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

d) $\lim\limits_{x\to -5^+} \frac{x^3+5x^2}{|x+5|}$ e) $\lim\limits_{x\to -5^-} \frac{x^3+5x^2}{|x+5|}$ f) $\lim\limits_{x\to 4^-} \frac{x-1}{x-4}$

Chapter 12

Solving inequalities

In this chapter we use our knowledge of functions to solve inequalities. In Section 12.1, we study polynomial inequalities and absolute value inequalities, while in Section 12.2 we solve rational inequalities.

12.1 Polynomial and absolute value inequalities

We will develop a general strategy for solving inequalities involving non-linear functions. Linear inequalities, however, can be solved quite easily by separating the variable x, while keeping in mind that multiplying or dividing a negative number reverses the sign of the inequality.

$$\begin{array}{rccc} & -2x \leq -6 & \Longrightarrow & x \geq 3 \\ \text{but} & 2x \leq 6 & \Longrightarrow & x \leq 3 \end{array}$$

Example 12.1

Solve for x.

a) $-3x + 7 > 19$ b) $2x + 5 \geq 4x - 11$
c) $3 < -6x - 4 \leq 13$ d) $-2x - 1 \leq 3x + 4 < 4x - 20$

Solution.
The first three calculations are as follows:

a) $-3x + 7 > 19 \quad \overset{(-7)}{\Longrightarrow} \quad -3x > 12 \quad \overset{(\div(-3))}{\Longrightarrow} \quad x < -4$

b) $2x + 5 \geq 4x - 11 \quad \overset{(-4x-5)}{\Longrightarrow} \quad -2x \geq -16 \quad \overset{(\div(-2))}{\Longrightarrow} \quad x \leq 8$

c) $3 < -6x - 4 \leq 13 \overset{(+4)}{\Longrightarrow} 7 < -6x \leq 17$
$\overset{(\div(-6))}{\Longrightarrow} \frac{7}{-6} > x \geq \frac{17}{-6} \Longrightarrow -\frac{17}{6} \leq x < -\frac{7}{6}$

Here, the last implication was obtained by switching the right and left terms of the inequality. The solution set is the interval $[-\frac{17}{6}, -\frac{7}{6})$.
For part (d), it is best to consider both inequalities separately.

$$-2x - 1 \leq 3x + 4 \overset{(-3x+1)}{\Longrightarrow} -5x \leq 5 \overset{(\div(-5))}{\Longrightarrow} x \geq -1,$$
$$3x + 4 < 4x - 20 \overset{(-4x-4)}{\Longrightarrow} -x < -24 \overset{(\cdot(-1))}{\Longrightarrow} x > 24.$$

The solution has to satisfy *both* inequalities $x \geq -1$ *and* $x > 24$. Both inequalities are true for $x > 24$ (since then also $x \geq -1$), so that this is in fact the solution: $x > 24$. □

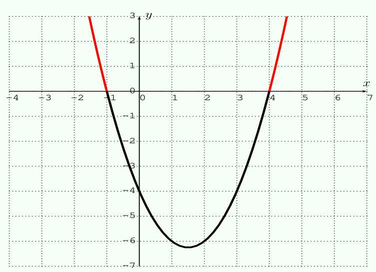
We now consider inequalities with polynomials of higher degree.

Example 12.2

Solve for x: $x^2 - 3x - 4 \geq 0$

Solution.

To get an idea of where $x^2 - 3x - 4 \geq 0$, we graph the left-hand side function $f(x) = x^2 - 3x - 4$.



Note that the output $x^2 - 3x - 4$ is greater or equal to zero when the graph of $f(x)$ is above or on the x-axis, which is marked in red. Since the graph is a parabola, the graph can only switch from above to below the x-axis (and the same from below to above the x-axis) when it intersects the x-axis. These are the roots of the function.
So, we first find the roots of the polynomial, which, in this case, can be

done by factoring.

$$x^2 - 3x - 4 = 0 \implies (x-4)(x+1) = 0 \implies x = 4 \text{ or } x = -1$$

From the graph we see that $f(x) \geq 0$ when $x \leq -1$ or when $x \geq 4$ (the parts of the graph above the x-axis). To show this without using the calculator, we can check one point in each of the intervals $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$:

Check $x = -2$**:**	**Check** $x = 0$**:**	**Check** $x = 5$**:**
$f(-2) = (-2)^2 - 3 \cdot (-2) - 4$	$f(0) = 0^2 - 3 \cdot 0 - 4$	$f(5) = 5^2 - 3 \cdot 5 - 4$
$= 4 + 6 - 4$	$= 0 - 0 - 4$	$= 25 - 15 - 4$
$= 6 \geq 0$	$= -4 \ngeq 0$	$= 6 \geq 0$
TRUE	FALSE	TRUE

The solution set S is therefore

$$S = \{x | x \leq -1 \text{ or } x \geq 4\} = (-\infty, -1] \cup [4, \infty).$$

The numbers -1 and 4 are included in the solution set since this is where we have equality $x^2 - 3x - 4 = 0$, and the original inequality $x^2 - 3x - 4 \geq 0$ includes the equality. □

Note 12.3: Solving inequalities

Analyzing the previous example, we use a three-step approach when dealing with inequalities.

- In *step one* we find the x where the left-hand side and the right-hand side of the inequality change from "$>$" to "$<$" and vice versa. In particular, we check where the two sides are equal.

- In *step two* we check one x in each of the subintervals from step one to decide whether they satisfy the original inequality or not.

For steps one and two we may also use the graphing calculator to gain further insights.

- In *step three* we check which of the endpoints of the intervals are included in the solution set.

Example 12.4

Solve for x.

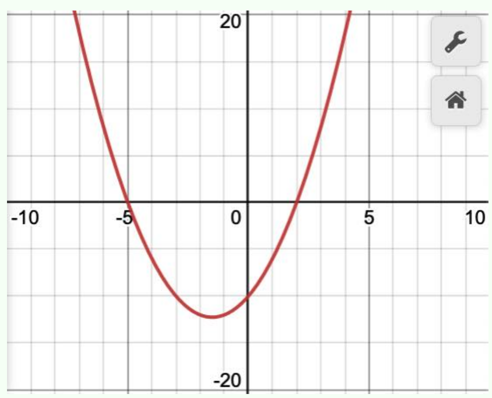
a) $x^2 + 3x - 10 < 0$ b) $x^3 - 9x^2 + 23x - 15 \leq 0$
c) $x^3 + 15x > 7x^2 + 9$ d) $x^4 - x^2 \geq 5(x^3 - x)$

Solution.

a) We can find the roots of the polynomial on the left by factoring.

$$x^2+3x-10=0 \implies (x+5)(x-2)=0 \implies x=-5 \text{ or } x=2$$

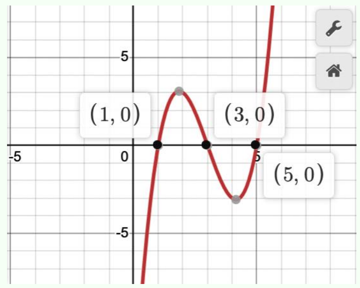
To see where $f(x) = x^2 + 3x - 10$ is < 0, we graph it with the calculator and check numbers in each interval where $f(x) \neq 0$.



-5		2
Check $x = -6$: $f(-6)$ $= (-6)^2 + 3 \cdot (-6) - 10$ $= 36 - 18 - 10$ $= 8 \not< 0$ FALSE	**Check $x = 0$:** $f(0)$ $= 0^2 + 3 \cdot 0 - 10$ $= 0 + 0 - 10$ $= -10 < 0$ TRUE	**Check $x = 3$:** $f(3)$ $= 3^2 + 3 \cdot 3 - 10$ $= 9 + 9 - 10$ $= 8 \not< 0$ FALSE

We see that $f(x) < 0$ when $-5 < x < 2$. The numbers -5 and 2 are not included because the inequality "$<$" does not include equality. The solution set is therefore $S = \{x| -5 < x < 2\} = (-5, 2)$.

b) Here is the graph of the function $f(x) = x^3 - 9x^2 + 23x - 15$ from the graphing calculator.



This graph shows that there are two intervals where $f(x) \leq 0$ (the parts of the graph below the x-axis). To determine the exact intervals, we calculate where $f(x) = x^3 - 9x^2 + 23x - 15 = 0$. The graph suggests that the roots of $f(x)$ are at $x = 1$, $x = 3$, and $x = 5$. This can be confirmed by a calculation:

$$\begin{aligned} f(1) &= 1^3 - 9 \cdot 1^2 + 23 \cdot 1 - 15 = 1 - 9 + 23 - 15 = 0, \\ f(3) &= 3^3 - 9 \cdot 3^2 + 23 \cdot 3 - 15 = 27 - 81 + 69 - 15 = 0, \\ f(5) &= 5^3 - 9 \cdot 5^2 + 23 \cdot 5 - 15 = 125 - 225 + 115 - 15 = 0. \end{aligned}$$

Since f is a polynomial of degree 3, the roots $x = 1, 3, 5$ are all of the roots of f. (Alternatively, we could have divided $f(x)$, for example, by $x - 1$ and used this to completely factor f and with this obtain all the roots of f.) We next check each interval.

1	3	5	
Check $x = 0$: $f(0) = -15 \leq 0$ TRUE	**Check** $x = 2$: $f(2) = 3 \not\leq 0$ FALSE	**Check** $x = 4$: $f(4) = -3 \leq 0$ TRUE	**Check** $x = 6$: $f(6) = 16 \not\leq 0$ FALSE

With this, we can determine the solution set to be the set:

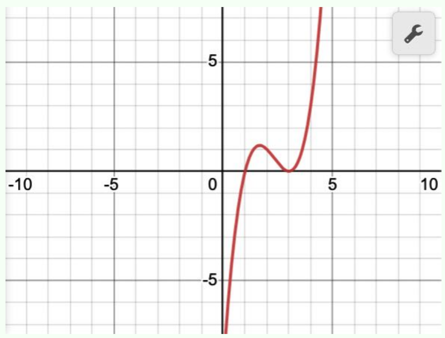
$$\begin{aligned} \text{solution set } S &= \{x \in \mathbb{R} | x \leq 1, \text{ or } 3 \leq x \leq 5\} \\ &= (-\infty, 1] \cup [3, 5]. \end{aligned}$$

Note that we include the roots 1, 3, and 5 in the solution set since the original inequality was "$\leq$" (and not "$<$"), which includes the solutions of the corresponding equality.

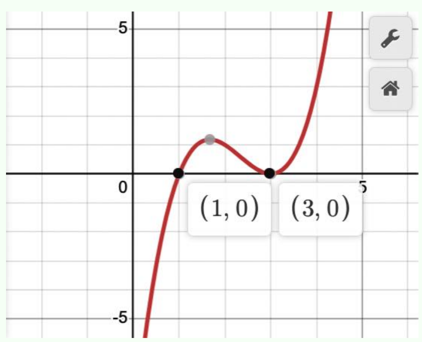
c) We rewrite the inequality in a way that has zero on one side so that we can get a better view of where the corresponding equality holds.

$$x^3 + 15x > 7x^2 + 9 \implies x^3 - 7x^2 + 15x - 9 > 0$$

(Here it does not matter whether we bring the terms to the right or the left side of the inequality sign! The resulting inequality is different, but the solution to the problem is the same.) With this, we now use the graphing calculator to find the graph of the function $f(x) = x^3 - 7x^2 + 15x - 9$.



The graph suggests at least one root (the left-most intersection point), but possibly one or two more roots. To gain a better understanding of whether the graph intersects the x-axis on the right, we rescale the window size of the previous graph.



This viewing window suggests that there are two roots $x = 1$ and $x = 3$. We confirm that these are the only roots with an algebraic computation. First, we check that $x = 1$ and $x = 3$ are indeed roots:

$$f(1) \quad = \quad 1^3 - 7 \cdot 1^2 + 15 \cdot 1 - 9 = 1 - 7 + 15 - 9 = 0,$$

$$f(3) = 3^3 - 7 \cdot 3^2 + 15 \cdot 3 - 9 = 27 - 63 + 45 - 9 = 0.$$

To confirm that these are the *only* roots (and we have not just missed one of the roots that might possibly become visible after sufficiently zooming into the graph), we factor $f(x)$ completely. We divide $f(x)$ by $x - 1$:

$$\begin{array}{r|rrrr}
 & x^2 & -6x & +9 & \\
\hline
x-1 & x^3 & -7x^2 & +15x & -9 \\
 & -(x^3 & -x^2) & & \\
\hline
 & & -6x^2 & +15x & -9 \\
 & & -(-6x^2 & +6x) & \\
\hline
 & & & 9x & -9 \\
 & & & -(9x & -9) \\
\hline
 & & & & 0
\end{array}$$

and use this to factor f:

$$\begin{aligned} f(x) &= x^3 - 7x^2 + 15x - 9 = (x-1)(x^2 - 6x + 9) \\ &= (x-1)(x-3)(x-3) \end{aligned}$$

This shows that 3 is a root of multiplicity 2, and so f has no other roots than $x = 1$ and $x = 3$. The solution set consists of those numbers x for which $f(x) > 0$. We check points in each interval.

$x < 1$	$1 < x < 3$	$x > 3$
Check $x = 0$**:**	**Check** $x = 2$**:**	**Check** $x = 4$**:**
$f(0) = -9 \not> 0$	$f(2) = 1 > 0$	$f(4) = 3 > 0$
FALSE	TRUE	TRUE

From this calculation, as well as from the graph, we see that $f(x) > 0$ when $1 < x < 3$ and when $x > 3$ (the roots $x = 1$ and $x = 3$ are not included as solutions). We can write the solution set in several different ways:

$$\text{solution set } S = \{x | 1 < x < 3 \text{ or } x > 3\} = \{x | 1 < x\} - \{3\},$$

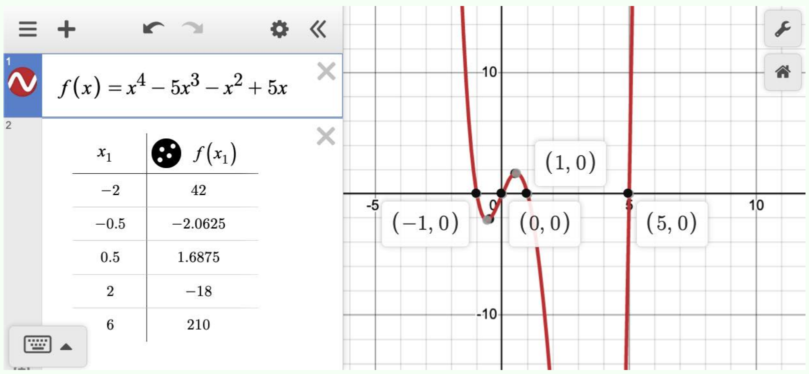
or in interval notation:

$$\text{solution set } S = (1, 3) \cup (3, \infty) = (1, \infty) - \{3\}.$$

d) Again, we move all terms to one side:

$$x^4 - x^2 \geq 5(x^3 - x) \quad \text{(distribute 5)} \quad \Longrightarrow \quad x^4 - x^2 \geq 5x^3 - 5x$$
$$\text{(subtract } 5x^3\text{, add } 5x\text{)} \quad \Longrightarrow \quad x^4 - 5x^3 - x^2 + 5x \geq 0.$$

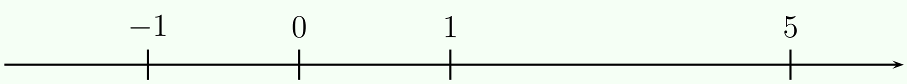
We graph $f(x) = x^4 - 5x^3 - x^2 + 5x$ with the graphing calculator.



The graph suggests the roots $x = -1$, 0, 1, and 5. This can be confirmed by a straightforward calculation.

$$\begin{aligned}
f(-1) &= (-1)^4 - 5\cdot(-1)^3 - (-1)^2 + 5\cdot(-1) = 1 + 5 - 1 - 5 = 0,\\
f(0) &= 0^4 - 5\cdot 0^3 - 0^2 - 5\cdot 0 = 0,\\
f(1) &= 1^4 - 5\cdot 1^3 - 1^2 + 5\cdot 1 = 1 - 5 - 1 + 5 = 0,\\
f(5) &= 5^4 - 5\cdot 5^3 - 5^2 + 5\cdot 5 = 125 - 125 - 25 + 25 = 0.
\end{aligned}$$

The roots $x = -1$, 0, 1, and 5 are the only roots, since f is of degree 4. We check points in each interval.



Check $x = -2$: $f(-2) = (-2)^4 - 5\cdot(-2)^3 - (-2)^2 + 5\cdot(-2)$
$= 42 \geq 0$ TRUE

Check $x = -0.5$: $f(-0.5) = (-0.5)^4 - 5\cdot(-0.5)^3 - (-0.5)^2 - 5\cdot(-0.5)$
$\approx -2.1 \not\geq 0$ FALSE

Check $x = 0.5$: $f(0.5) = 0.5^4 - 5\cdot 0.5^3 - 0.5^2 + 5\cdot 1$
$\approx 1.7 \geq 0$ TRUE

Check $x = 2$:	$f(2) = 2^4 - 5 \cdot 2^3 - 2^2 + 5 \cdot 5 = -18 \not\geq 0$	FALSE
Check $x = 6$:	$f(6) = 6^4 - 5 \cdot 6^3 - 6^2 + 5 \cdot 6 = 210 \geq 0$	TRUE

Since the inequality we want to solve is $f(x) \geq 0$, which includes equality, the zeros of f are included in the solution, and so the solution set is:

$$S = (-\infty, -1] \cup [0, 1] \cup [5, \infty)$$

□

Polynomial inequalities come up, for example, when finding the domain of functions involving a square root, as we will show in the next example.

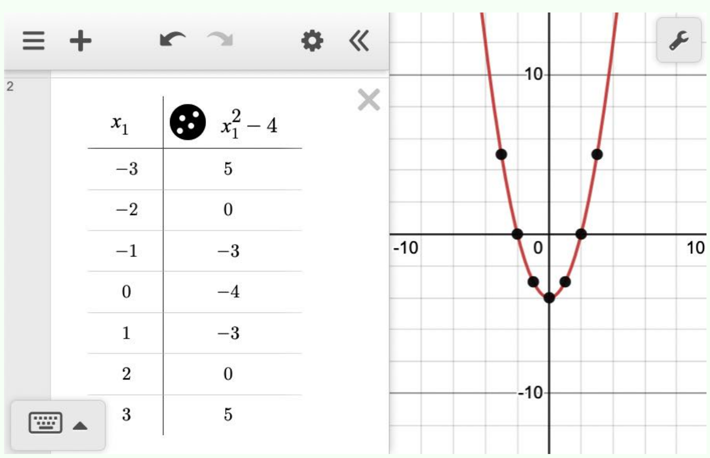
Example 12.5

Find the domain of the given functions.

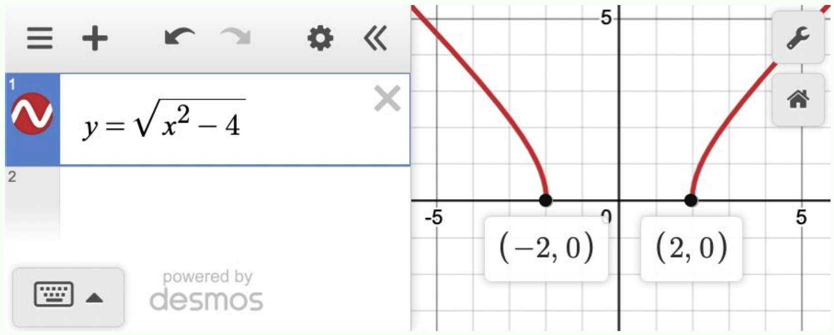
a) $f(x) = \sqrt{x^2 - 4}$ b) $g(x) = \sqrt{x^3 - 5x^2 + 6x}$

Solution.

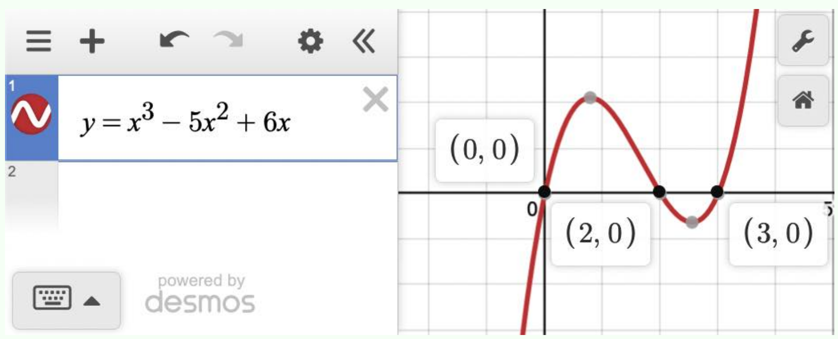
a) The domain of $f(x) = \sqrt{x^2 - 4}$ is given by all x for which the square root is non-negative. In other words, the domain is given by numbers x with $x^2 - 4 \geq 0$. Graphing the function $y = x^2 - 4 = (x+2)(x-2)$, we see that this is precisely the case when $x \leq -2$ or $x \geq 2$.



Therefore, the domain is $D_f = (-\infty, -2] \cup [2, \infty)$. This is also confirmed by the graph of f, which is shown below.



b) For the domain of $g(x) = \sqrt{x^3 - 5x^2 + 6x}$, we need find those x with $x^3 - 5x^2 + 6x \geq 0$. To this end, we graph $y = x^3 - 5x^2 + 6x$ and check for its roots.



From the graph above, we calculate the roots of $y = x^3 - 5x^2 + 6x$ at $x = 0$, $x = 2$, and $x = 3$. Furthermore, the graph shows that $x^3 - 5x^2 + 6x \geq 0$ precisely when $0 \leq x \leq 2$ or $3 \leq x$. The domain is therefore $D_g = [0, 2] \cup [3, \infty)$.

□

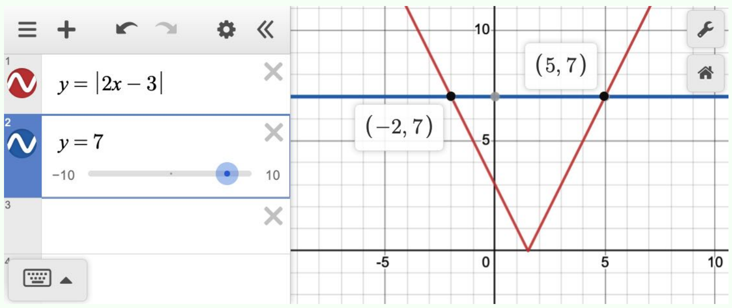
A similar computation to that for polynomial inequalities also applies to absolute value inequalities, which we show in the next example.

Example 12.6

Solve for x: $|2x-3| \geq 7$

Solution.

To analyze $|2x-3| \geq 7$, we graph the function $f(x) = |2x-3|$, as well as the function $g(x) = 7$.



To see where $|2x-3| \geq 7$, we first find the values where $|2x-3| = 7$. Note that $2x-3$ has an absolute value of 7 exactly when $2x-3$ is either 7 or -7.

$$|2x-3| = 7 \quad \Longrightarrow \quad 2x-3 = \pm 7$$

	$\Longrightarrow$	$2x-3=7$
(add 3)	$\Longrightarrow$	$2x=10$
(divide by 2)	$\Longrightarrow$	$x=5$

	$\Longrightarrow$	$2x-3=-7$
(add 3)	$\Longrightarrow$	$2x=-4$
(divide by 2)	$\Longrightarrow$	$x=-2$

We next check in each interval whether $|2x-3| \geq 7$:

$x < -2$	$-2 < x < 5$	$x > 5$
Check $x=-3$:	**Check $x=0$:**	**Check $x=6$:**
$\lvert 2\cdot(-3)-3\rvert \overset{?}{\geq} 7$	$\lvert 2\cdot 0-3\rvert \overset{?}{\geq} 7$	$\lvert 2\cdot 6-3\rvert \overset{?}{\geq} 7$
$\lvert -9\rvert \overset{?}{\geq} 7$	$\lvert -3\rvert \overset{?}{\geq} 7$	$\lvert 9\rvert \overset{?}{\geq} 7$
$9 \overset{?}{\geq} 7$	$3 \overset{?}{\geq} 7$	$9 \overset{?}{\geq} 7$
TRUE	FALSE	TRUE

Since the values at $x=-2$ and $x=5$ give equality, the solution set for $|2x-3| \geq 7$ is given by $S = (-\infty, -2] \cup [5, \infty)$. □

12.2 Rational inequalities

Rational inequalities are solved with a similar three-step process that was used to solve the polynomial and absolute value inequalities before (see Note 12.3 page 212). That is, in step 1, we find possible inputs where the inequality may change its sign (for example at the x-intercepts). In step 2, we check in which of the intervals the given inequality is true, and which are thus part of the solution set. Finally, in step 3, we determine which of the endpoints of the intervals should be included in the solution set.

Example 12.7

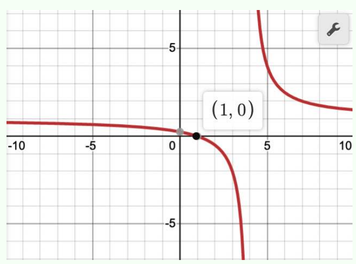
Solve for x.

a) $\frac{x-1}{x-4} \leq 0$ b) $\frac{7x-3}{6x+5} > 0$ c) $\frac{x^2-5x+6}{x^2-5x} \geq 0$

d) $\frac{5}{x-2} \leq 3$ e) $\frac{4}{x+5} < \frac{3}{x-3}$

Solution.

a) We first graph the function $f(x) = \frac{x-1}{x-4}$.



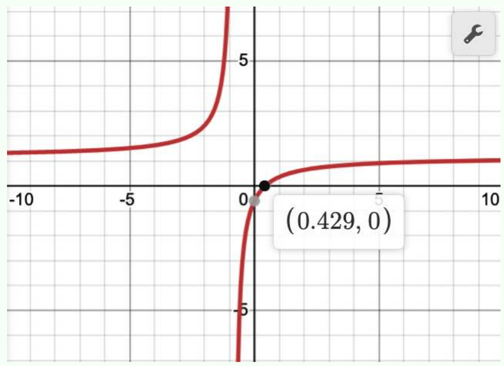
As shown above, the graph changes from above to below the x-axis at the x-intercept $x = 1$, and then changes from below to above the x-axis at the vertical asymptote at $x = 4$. Using both $x = 1$ and $x = 4$, we get the three intervals $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, which we will check as to whether $f(x) \leq 0$ or not.

$1 \qquad 4$

Check $x = 0$**:**	**Check** $x = 2$**:**	**Check** $x = 5$**:**
$\frac{0-1}{0-4} \stackrel{?}{\leq} 0$	$\frac{2-1}{2-4} \stackrel{?}{\leq} 0$	$\frac{5-1}{5-4} \stackrel{?}{\leq} 0$
$\frac{-1}{-4} = \frac{1}{4} \stackrel{?}{\leq} 0$	$\frac{1}{-2} \stackrel{?}{\leq} 0$	$\frac{4}{1} \stackrel{?}{\leq} 0$
FALSE	TRUE	FALSE

Since the inequality is $\frac{x-1}{x-4} \leq 0$, we include the root at $x = 1$ in the solution set. However, we do not include $x = 4$, since this is a vertical asymptote of f and would not give a solution of the inequality, but would rather give an undefined expression on the left-hand side of the inequality. The solution set is therefore $S = [1, 4)$.

b) To solve $\frac{7x-3}{6x+5} > 0$, we graph the function $f(x) = \frac{7x-3}{6x+5}$.



$$x\text{-intercept: } 7x - 3 = 0 \implies 7x = 3 \implies x = \frac{3}{7}$$

$$\text{vertical asymptote: } 6x + 5 = 0 \implies 6x = -5 \implies x = -\frac{5}{6}$$

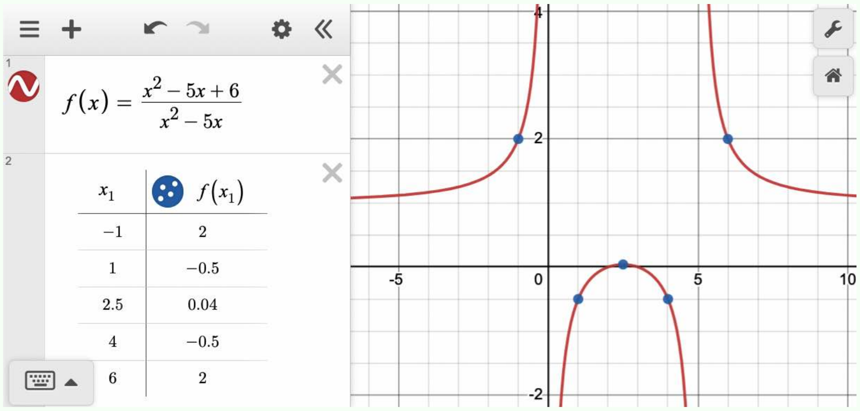
Note that $\frac{3}{7} \approx 0.429$ is indicated in the graph. The vertical asymptote is approximately at $-\frac{5}{6} \approx -0.833$. We can therefore use -1, 0, and 1 to check the inequality $f(x) = \frac{7x-3}{6x+5} > 0$ on the corresponding intervals $(-\infty, -\frac{5}{6})$, $(-\frac{5}{6}, \frac{3}{7})$ and $(\frac{3}{7}, \infty)$.

$(-\infty, -\frac{5}{6})$	$(-\frac{5}{6}, \frac{3}{7})$	$(\frac{3}{7}, \infty)$
Check $x = -1$:	**Check $x = 0$:**	**Check $x = 1$:**
$\frac{7\cdot(-1)-3}{6\cdot(-1)+5} \overset{?}{>} 0$	$\frac{7\cdot 0-3}{6\cdot 0+5} \overset{?}{>} 0$	$\frac{7\cdot 1-3}{6\cdot 1+5} \overset{?}{>} 0$
$\frac{-10}{-1} = 10 \overset{?}{>} 0$	$\frac{-3}{5} \overset{?}{>} 0$	$\frac{4}{11} \overset{?}{>} 0$
TRUE	FALSE	TRUE

For the solution set, we do not include the root of f since the inequality is strict $f(x) > 0$, and we never include the vertical asymptote of f. The solution set is therefore

$$S = \big(-\infty, -\tfrac{5}{6}\big) \cup \big(\tfrac{3}{7}, \infty\big)$$

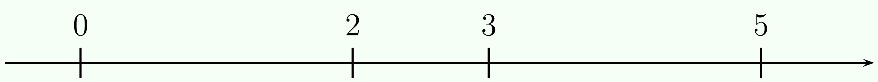
c) Here is the graph of $\frac{x^2-5x+6}{x^2-5x}$ in the standard window.



Factoring numerator and denominator, we can determine vertical asymptotes, holes, and x-intercepts.

$$\frac{x^2-5x+6}{x^2-5x} = \frac{(x-2)(x-3)}{x(x-5)}$$

The vertical asymptotes are at $x = 0$ and $x = 5$, the x-intercepts are at $x = 2$ and $x = 3$. To see where $\frac{x^2-5x+6}{x^2-5x} \geq 0$, we check numbers in each of the corresponding intervals.



Check $x = -1$: $f(-1) = \frac{(-1)^2-5\cdot(-1)+6}{(-1)^2-5\cdot(-1)} = \frac{12}{6} \geq 0$ TRUE

Check $x = 1$: $f(1) = \frac{1^2-5\cdot 1+6}{1^2-5\cdot 1} = \frac{2}{-4} \ngeq 0$ FALSE

Check $x = 2.5$: $f(2.5) = \frac{2.5^2-5\cdot 2.5+6}{2.5^2-5\cdot 2.5} = \frac{-0.25}{-6.25} \geq 0$ TRUE

Check $x = 4$: $f(4) = \frac{4^2-5\cdot 4+6}{4^2-5\cdot 4} = \frac{2}{-4} \ngeq 0$ FALSE

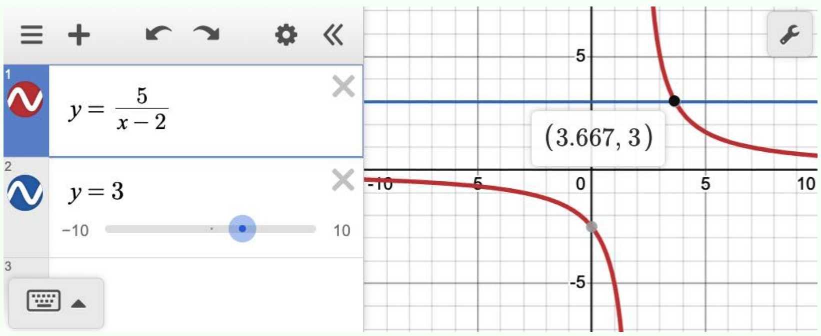
Check $x = 6$: $f(6) = \frac{6^2-5\cdot 6+6}{6^2-5\cdot 6} = \frac{12}{6} \geq 0$ TRUE

Combining all of the above information, we obtain the solution set:

$$\text{solution set } S = (-\infty, 0) \cup [2, 3] \cup (5, \infty)$$

Notice that the $x-$intercepts $x = 2$ and $x = 3$ are included in the solution set, whereas the values $x = 0$ and $x = 5$ associated with the vertical asymptotes are not included, since the fraction is not defined for $x = 0$ and $x = 5$.

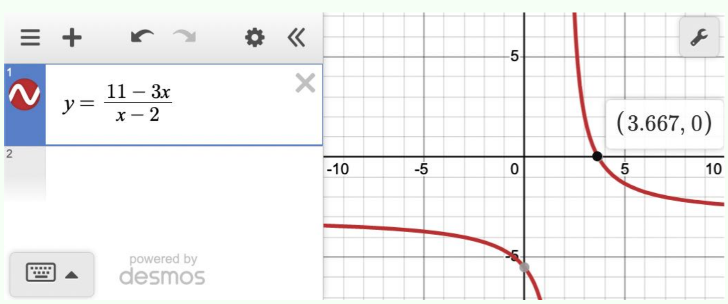
d) To find the numbers x where $\frac{5}{x-2} \leq 3$, we can graph the two functions on the left- and right-hand side of the inequality.



However, this can sometimes be confusing, and we recommend rewriting the inequality so that one side becomes zero. Then, we graph the function on the other side of the new inequality.

$$\frac{5}{x-2} \leq 3 \quad \Longleftrightarrow \quad \frac{5}{x-2} - 3 \leq 0 \quad \Longleftrightarrow \quad \frac{5-3(x-2)}{x-2} \leq 0$$
$$\Longleftrightarrow \quad \frac{5-3x+6}{x-2} \leq 0 \quad \Longleftrightarrow \quad \frac{11-3x}{x-2} \leq 0$$

Therefore, we graph the function $f(x) = \frac{11-3x}{x-2}$.



The vertical asymptote is $x = 2$, and the x-intercept found is thus

$$11 - 3x = 0 \implies \quad 11 = 3x \implies \quad x = \frac{11}{3} \approx 3.667.$$

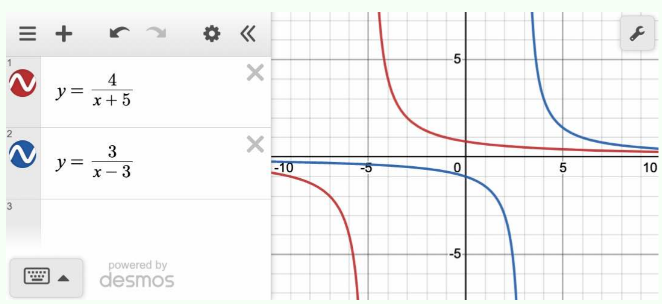
We check the inequality $\frac{11-3x}{x-2} \leq 0$ at 0, 3, and 4.

$x < 2$	$2 < x < \frac{11}{3}$	$x > \frac{11}{3}$
Check $x=0$:	**Check $x=3$:**	**Check $x=4$:**
$\frac{11-3\cdot 0}{0-2} \overset{?}{\leq} 0$	$\frac{11-3\cdot 3}{3-2} \overset{?}{\leq} 0$	$\frac{11-3\cdot 4}{4-2} \overset{?}{\leq} 0$
$\frac{11}{-2} \overset{?}{\leq} 0$	$\frac{2}{1} \overset{?}{\leq} 0$	$\frac{-1}{2} \overset{?}{\leq} 0$
TRUE	FALSE	TRUE

This, together with the fact that f is undefined at 2 and $f(\frac{11}{3}) = 0$, gives the following solution set:

$$S = \Big(-\infty, 2\Big) \cup \Big[\frac{11}{3}, \infty\Big)$$

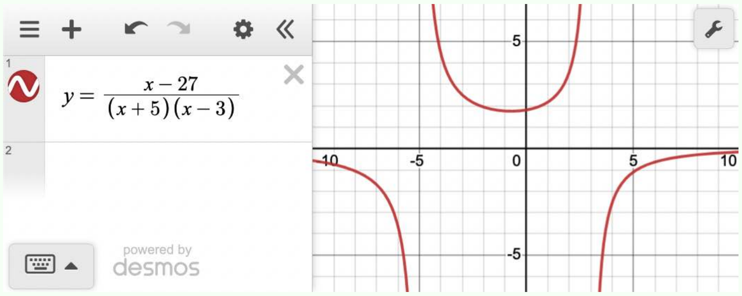
e) We want to find those numbers x for which $\frac{4}{x+5} < \frac{3}{x-3}$. One way to do this is given by graphing both functions $f_1(x) = \frac{4}{x+5}$ and $f_2(x) = \frac{3}{x-3}$, and by trying to determine where $f_1(x) < f_2(x)$. The graphs of f_1 and f_2 are displayed below. Note that it may sometimes not be completely obvious to determine in which intervals f_1 is greater than f_2.



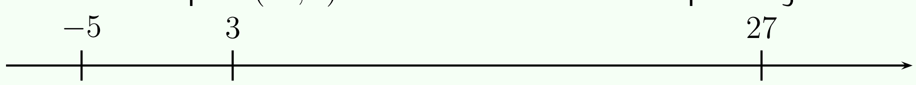
As before, we recommend rewriting the inequality so that one side of the inequality becomes zero:

$$\begin{aligned}
\frac{4}{x+5} < \frac{3}{x-3} \quad &\iff \quad \frac{4}{x+5} - \frac{3}{x-3} < 0 \\
&\iff \quad \frac{4(x-3) - 3(x+5)}{(x+5)(x-3)} < 0 \\
&\iff \quad \frac{4x - 12 - 3x - 15}{(x+5)(x-3)} < 0
\end{aligned}$$

Simplifying this, we get the inequality: $\frac{x-27}{(x+5)(x-3)} < 0$. We therefore graph the function $f(x) = \frac{x-27}{(x+5)(x-3)}$.

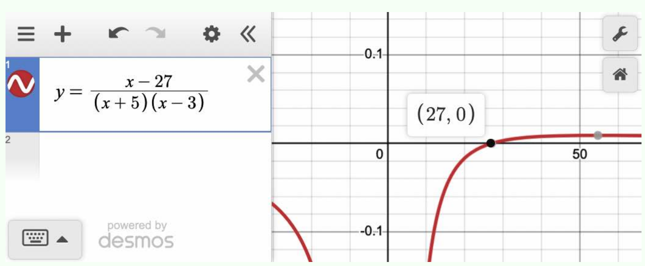


The vertical asymptotes of $f(x) = \frac{x-27}{(x+5)(x-3)}$ are $x = -5$ and $x = 3$. The x-intercept is $(27, 0)$. We next check the corresponding intervals.



Check $x = -6$:	$f(-6) = \frac{(-6)-27}{((-6)+5)\cdot((-6)-3)} = \frac{-33}{9} < 0$	TRUE
Check $x = 0$:	$f(0) = \frac{0-27}{(0+5)\cdot(0-3)} = \frac{-27}{-15} = \frac{27}{15} \not< 0$	FALSE
Check $x = 4$:	$f(4) = \frac{4-27}{(4+5)\cdot(4-3)} = \frac{-23}{9} < 0$	TRUE
Check $x = 28$:	$f(28) = \frac{28-27}{(28+5)\cdot(28-3)} = \frac{1}{825} \not< 0$	FALSE

Note that the graph of f is indeed above the x-axis for $x > 27$.



Therefore, the solution set is

$$\text{solution set } S = \{x | x < -5, \text{ or } 3 < x < 27\} = (-\infty, -5) \cup (3, 27).$$

Here, the x-intercept $x = 27$ is not included in the solution set since the inequality had a "$<$" and not "$\leq$" sign.

□

12.3 Exercises

Exercise 12.1

Solve for x.

a) $5x + 6 \leq 21$
b) $3 + 4x > 10x$
c) $2x + 8 \geq 6x + 24$
d) $9 - 3x < 2x - 13$
e) $-5 \leq 2x + 5 \leq 19$
f) $15 > 7 - 2x \geq 1$
g) $3x + 4 \leq 6x - 2 \leq 8x + 5$
h) $5x + 2 < 4x - 18 \leq 7x + 11$

Exercise 12.2

Solve for x.

a) $x^2 - 5x - 14 > 0$
b) $x^2 - 2x \geq 35$
c) $x^2 - 4 \leq 0$
d) $x^2 + 3x - 3 < 0$
e) $2x^2 + 2x \leq 12$
f) $3x^2 < 2x + 1$
g) $x^2 - 4x + 4 > 0$
h) $x^3 - 2x^2 - 5x + 6 \geq 0$
i) $x^3 + 4x^2 + 3x + 12 < 0$
j) $-x^3 - 4x < -4x^2$
k) $x^4 - 10x^2 + 9 \leq 0$
l) $x^4 - 5x^3 + 5x^2 + 5x < 6$
m) $x^4 - 5x^3 + 6x^2 > 0$
n) $x^5 - 6x^4 + x^3 + 24x^2 - 20x \leq 0$
o) $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 \geq 0$,
p) $x^{11} - x^{10} + x - 1 \leq 0$

Exercise 12.3

Find the domain of the functions below.

a) $f(x) = \sqrt{x^2 - 8x + 15}$
b) $f(x) = \sqrt{9x - x^3}$
c) $f(x) = \sqrt{(x-1)(4-x)}$
d) $f(x) = \sqrt{(x-2)(x-5)(x-6)}$
e) $f(x) = \frac{5}{\sqrt{6-2x}}$
f) $f(x) = \frac{1}{\sqrt{x^2-6x-7}}$

Exercise 12.4

Solve for x.

a) $|2x + 7| > 9$

b) $|6x + 2| < 3$

c) $|5 - 3x| \geq 4$

d) $|-x - 7| \leq 5$

e) $|1 - 8x| \geq 3$

f) $1 > \left|2 + \frac{x}{5}\right|$

Exercise 12.5

Solve for x.

a) $\frac{x+2}{x+4} \geq 0$

b) $\frac{x-5}{2-x} > 0$

c) $\frac{9x-11}{7x+15} \leq 0$

d) $\frac{13x+4}{6x-1} \geq 0$

e) $\frac{7x-2}{3x+8} < 0$

f) $\frac{4x-4}{x^2-4} \geq 0$

g) $\frac{x-2}{x^2-4x-5} < 0$

h) $\frac{x^2-9}{x^2-4} \geq 0$

i) $\frac{x-3}{x+3} \leq 4$

j) $\frac{1}{x+10} > 5$

k) $\frac{2}{x-2} \leq \frac{5}{x+1}$

l) $\frac{x^2}{x+4} \leq x$

Review of polynomials and rational functions

Exercise II.1

Divide the polynomials: $\frac{2x^3+x^2-9x-8}{2x+3}$

Exercise II.2

Find the remainder when dividing $x^3 + 3x^2 - 5x + 7$ by $x + 2$.

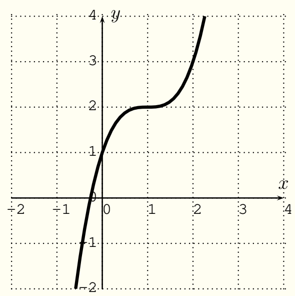
Exercise II.3

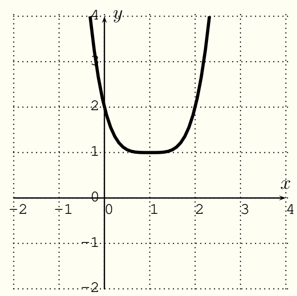
Which of the following is a factor of $x^{400} - 2x^{99} + 1$:

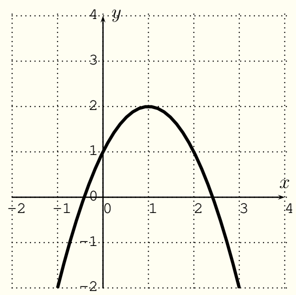
$$x - 1, \quad x + 1, \quad x - 0$$

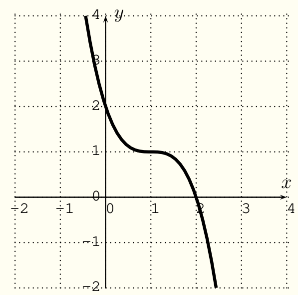
Exercise II.4

Identify the polynomial with its graph.

a) 

b) 

c) 

d) 

i) $f(x) = -x^2 + 2x + 1$, graph: ______

ii) $f(x) = -x^3 + 3x^2 - 3x + 2$, graph: ______

iii) $f(x) = x^3 - 3x^2 + 3x + 1$, graph: ______

iv) $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 2$, graph: ______

Exercise II.5

Sketch a complete graph of the function:

$$f(x) = -2x^3 + 11x^2 - 16x + 3$$

- Include the exact x-intercepts and the y-intercept.
- Approximate the maxima and minima with the graphing calculator.

Exercise II.6

Find all roots of $f(x) = x^3 + 6x^2 + 5x - 12$.
Use this information to factor $f(x)$ completely.

Exercise II.7

Find a polynomial of degree 3 whose roots are 0, 1, and 3, and so that $f(2) = 10$.

Exercise II.8

Find a polynomial of degree 4 with real coefficients, whose roots include -2, 5, and $3 - 2i$.

Exercise II.9

Let $f(x) = \frac{3x^2-12}{x^2-2x-3}$. Sketch the graph of f. Include all vertical and horizontal asymptotes, all holes, and all x- and y-intercepts.

Exercise II.10

Solve for x:

a) $x^2 + 4x > 5$ b) $|6x + 7| \geq 2$ c) $\frac{5x-2}{3x+8} \leq 0$

Part III

Exponential and Logarithmic Functions

Chapter 13

Exponential and logarithmic functions

We now consider functions that differ greatly from polynomials and rational functions in their complexity. More precisely, we will explore exponential and logarithmic functions from a function theoretic point of view.

13.1 Exponential functions and their graphs

We start by recalling the definition of an exponential function and by studying its graph.

Definition 13.1: Exponential function

A function f is called an **exponential function** if it is of the form

$$f(x) = c \cdot b^x$$

for some real number c and positive real number $b > 0$. The constant b is called the **base**.
Since $f(x) = c \cdot b^x$ is defined for all real numbers, the domain of f is $D = \mathbb{R}$.

Example 13.2

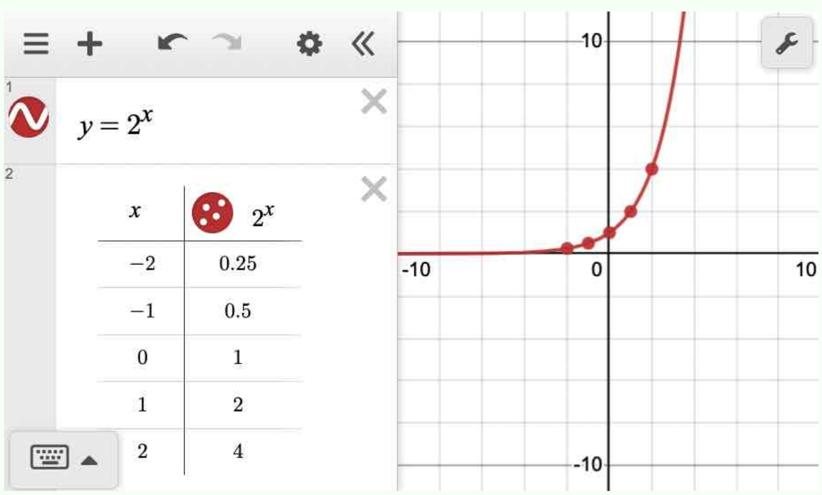
Graph the functions.

$$f(x) = 2^x, \quad g(x) = 3^x, \quad h(x) = 10^x, \quad k(x) = \left(\frac{1}{2}\right)^x, \quad l(x) = \left(\frac{1}{10}\right)^x$$

First, we will graph the function $f(x) = 2^x$ by calculating the function values in a table and then plotting the points in the x-y plane. We can calculate the values by hand, or simply use the table function of the calculator to find the function values.

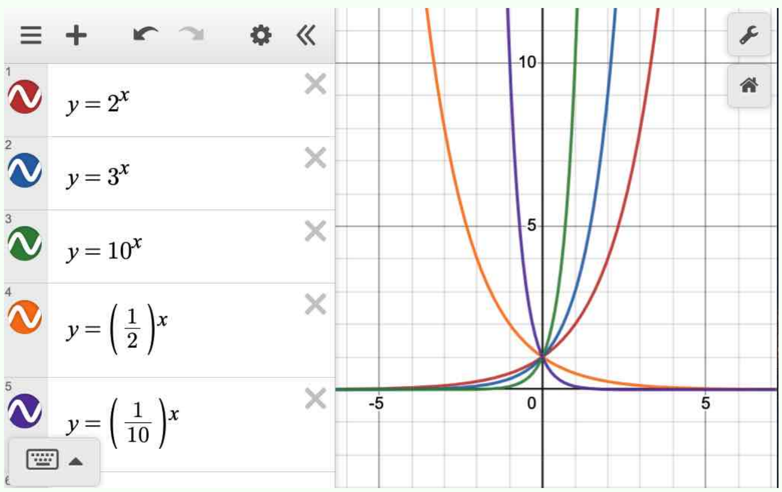
$$\begin{aligned} f(0) &= 2^0 = 1 \\ f(1) &= 2^1 = 2 \\ f(2) &= 2^2 = 4 \\ f(3) &= 2^3 = 8 \\ f(-1) &= 2^{-1} = 0.5 \\ f(-2) &= 2^{-2} = 0.25 \end{aligned}$$

We obtain the following graph.



Similarly, we can compute the table for the other functions g, h, k, and l, and plot them with the graphing calculator.

x	2^x	3^x	10^x	$\left(\frac{1}{2}\right)^x$	$\left(\frac{1}{10}\right)^x$
−2	0.25	0.11111111	0.01	4	100
−1	0.5	0.33333333	0.1	2	10
0	1	1	1	1	1
1	2	3	10	0.5	0.1
2	4	9	100	0.25	0.01
3	8	27	1000	0.125	0.001
4	16	81	10000	0.0625	1×10^{-4}



Note that the function k can also be written as

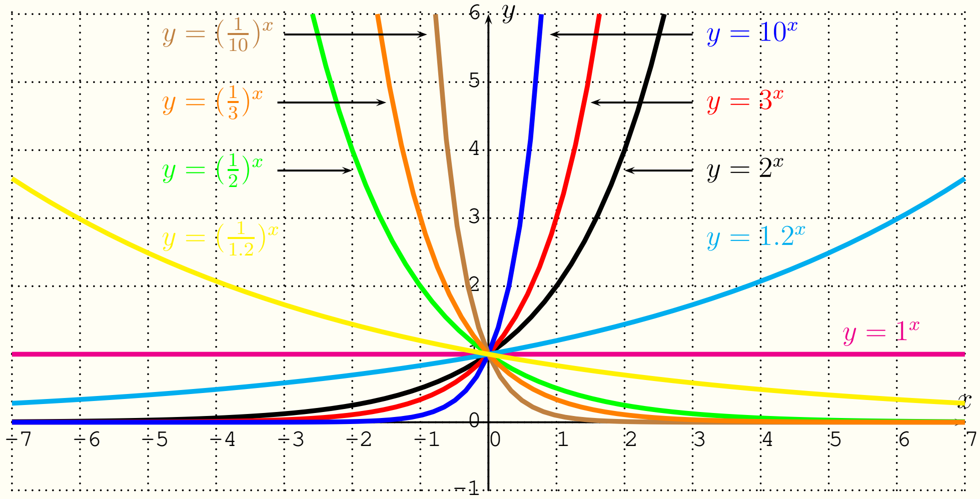
$$k(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x},$$

and similarly, $l(x) = \left(\frac{1}{10}\right)^x = 10^{-x}$.

This example shows that the exponential function has the following properties.

Observation 13.3: Graph of an exponential function

The graph of the exponential function $f(x) = b^x$ with $b > 0$ and $b \neq 1$ has a horizontal asymptote at $y = 0$.



- If $b > 1$, then $f(x)$ approaches $+\infty$ when x approaches $+\infty$, and $f(x)$ approaches 0 when x approaches $-\infty$.
- If $0 < b < 1$, then $f(x)$ approaches 0 when x approaches $+\infty$, and $f(x)$ approaches $+\infty$ when x approaches $-\infty$.

Note that all of these graphs have the horizontal asymptote $y = 0$.

An important base that we will frequently need to consider is the base of e, where e is Euler's number.

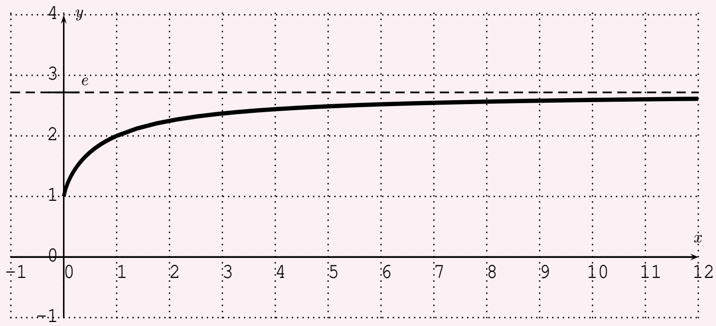
Definition 13.4: Euler's number

Euler's number e is an irrational number that is approximately

$$\boxed{e = 2.718281828459045235\ldots}$$

To be precise, we can define e as the number which is the horizontal asymptote of the function

$f(x) = \left(1 + \frac{1}{x}\right)^x$ when x approaches $+\infty$.



One can show that f has, indeed, a horizontal asymptote, and this limit is defined as e.

$$e := \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Furthermore, one can show that the exponential function with base e has a similar limit expression.

$$e^r = \lim_{x \to \infty} \left(1 + \frac{r}{x}\right)^x \tag{13.1}$$

Alternatively, Euler's number and the exponential function with base e may also be defined using an infinite series, namely, $e^r = 1 + r + \frac{r^2}{1 \cdot 2} + \frac{r^3}{1 \cdot 2 \cdot 3} + \frac{r^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$. These ideas will be explored further in a course in calculus.

We next graph a few functions that use Euler's number.

Example 13.5

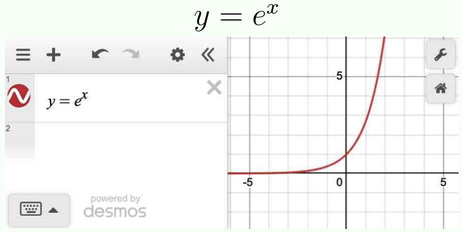
Graph the functions.

a) $y = e^x$ b) $y = e^{-x}$ c) $y = e^{-x^2}$ d) $y = \frac{e^x + e^{-x}}{2}$

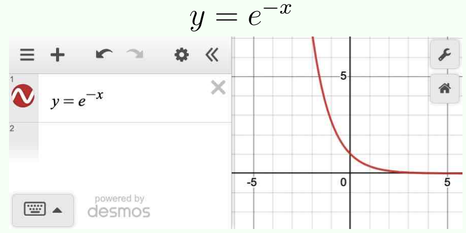
Solution.

Using the calculator, we obtain the desired graphs.

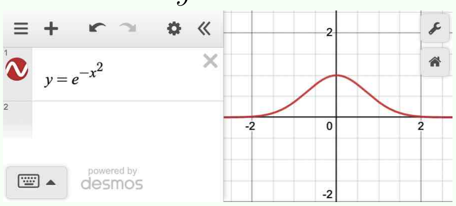
$y = e^x$



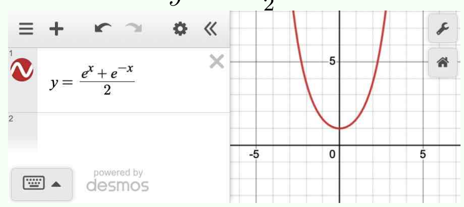
$y = e^{-x}$



$y = e^{-x^2}$



$y = \frac{e^x+e^{-x}}{2}$



The last function $y = \frac{e^x+e^{-x}}{2}$ is called the *hyperbolic cosine*, and is denoted by $\cosh(x) = \frac{e^x+e^{-x}}{2}$. (The *hyperbolic sine*, $\sinh(x) = \frac{e^x-e^{-x}}{2}$, and the *hyperbolic tangent*, $\tanh(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ will be graphed in Exercise 13.1.) □

We now study how different multiplicative factors c affect the shape of an exponential function.

Example 13.6

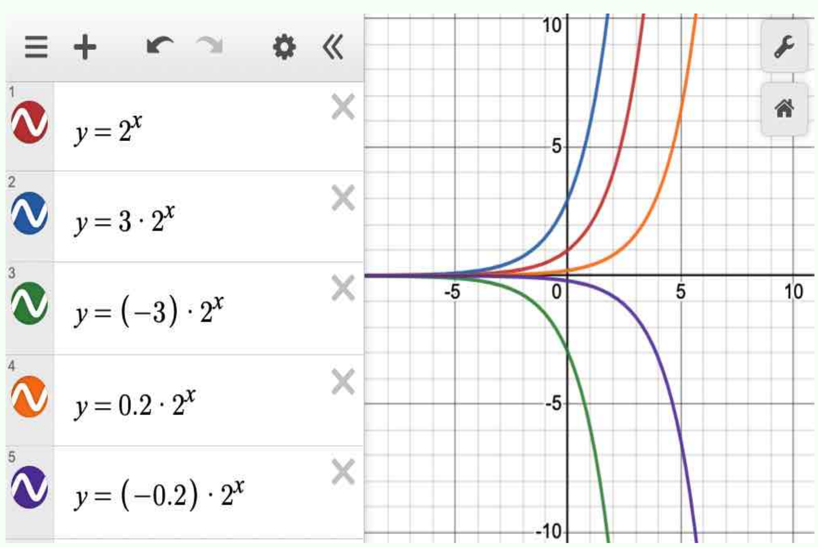
Graph the functions.

a) $y = 2^x$ b) $y = 3 \cdot 2^x$ c) $y = (-3) \cdot 2^x$

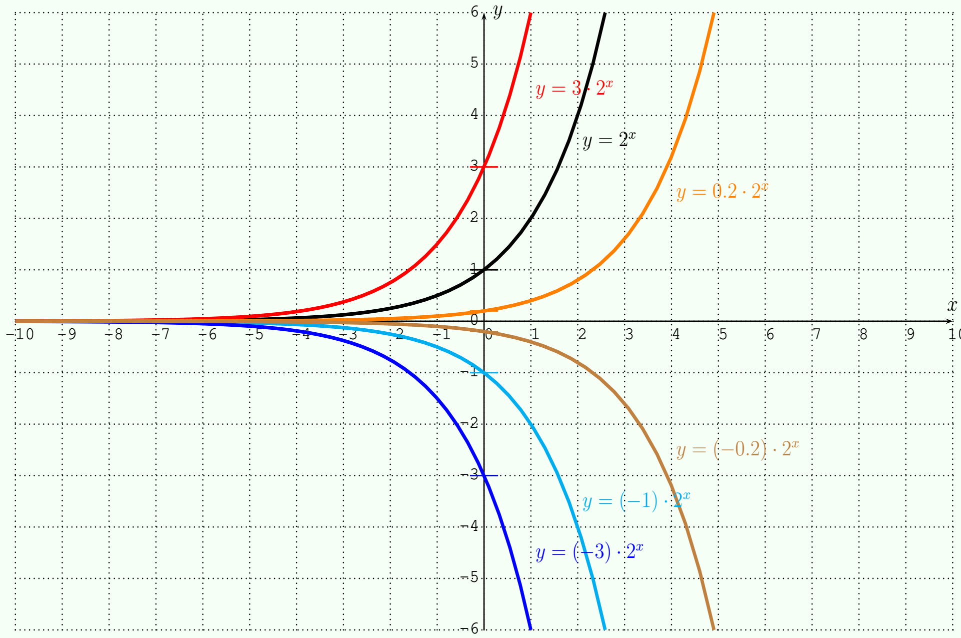
d) $y = 0.2 \cdot 2^x$ e) $y = (-0.2) \cdot 2^x$

Solution.

We graph the functions in the same viewing window.



Here are the graphs of functions $f(x) = c \cdot 2^x$ for various choices of c.



Note that for $f(x) = c \cdot 2^x$, the y-intercept is given at $f(0) = c$. □

Finally, we can combine our knowledge of graph transformations to study exponential functions that are shifted and stretched.

Example 13.7

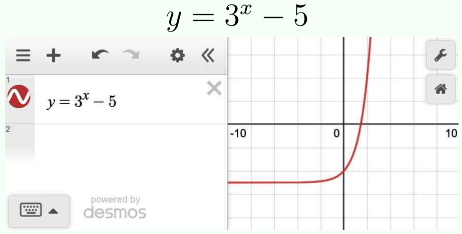
Graph the functions.

a) $y = 3^x - 5$ b) $y = e^{x+4}$ c) $y = \frac{1}{4} \cdot e^{x-3} + 2$

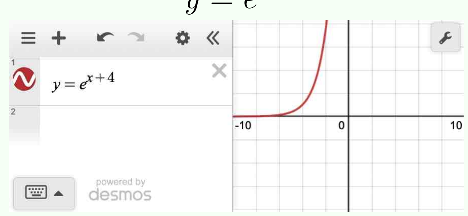
Solution.

The first two graphs are displayed below.

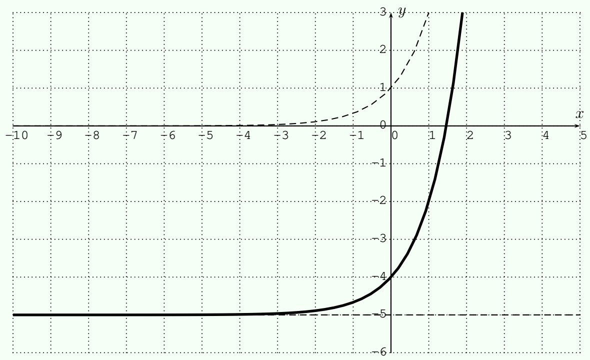
$y = 3^x - 5$



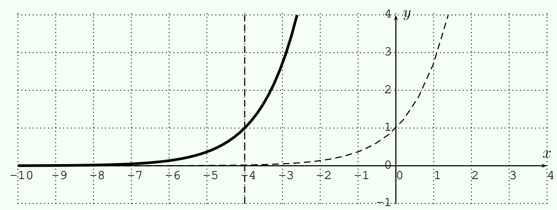
$y = e^{x+4}$



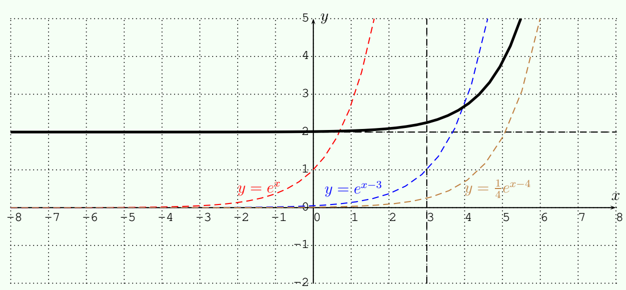
The first graph $y = 3^x - 5$ is the graph of $y = 3^x$ shifted down by 5.

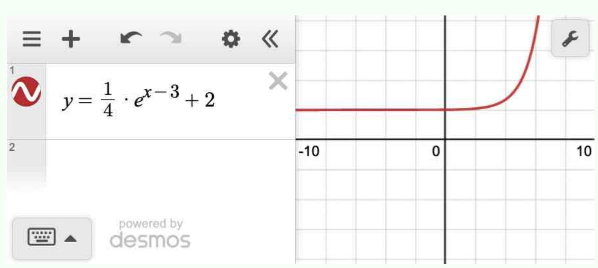


The graph of $y = e^{x+4}$ is the graph of $y = e^x$ shifted to the left by 4.



Finally, $y = \frac{1}{4}e^{x-3} + 2$ is the graph of $y = e^x$ shifted to the right by 3 (see the graph of $y = e^{x-3}$), then compressed by a factor 4 toward the x-axis (see the graph of $y = \frac{1}{4}e^{x-3}$), and then shifted up by 2.





□

13.2 Logarithmic functions and their graphs

The logarithm is defined as the inverse function of an exponential function. From the above, we can see that $y = b^x$ is one-to-one (for $0 < b$, $b \neq 1$), so that it makes sense to define the inverse of $y = b^x$. Specifically, we call the inverse function of $y = b^x$ the logarithm with base b.

Definition 13.8: Logarithmic function

Let $0 < b \neq 1$ be a positive real number that is not equal to 1. For $x > 0$, the **logarithm of x with base b** is defined by the equivalence

$$\boxed{x = b^y \qquad \Longleftrightarrow \qquad y = \log_b(x)} \tag{13.2}$$

Note that this computes the inverse of the exponential function $y = b^x$ with base b (that is, we exchange x and y to get $x = b^y$ and solve for y).
For the particular base $b = 10$ we use the short form

$$\boxed{\log(x) := \log_{10}(x)}$$

For the particular base $b = e$, where $e \approx 2.71828$ is Euler's number, we call the logarithm with base e the **natural logarithm**, and write

$$\boxed{\ln(x) := \log_e(x)}$$

The **logarithmic function** is the function $y = \log_b(x)$ with domain $D = \{x \in \mathbb{R} | x > 0\}$ of all positive real numbers, and range $R = \mathbb{R}$ of all real numbers.

Example 13.9

Rewrite the equation as a logarithmic equation.

a) $3^4 = 81$ b) $10^3 = 1000$ c) $e^x = 17$ d) $2^{7 \cdot a} = 53$

Solution.
We can immediately apply Equation (13.2). For part (a), we have $b = 3$, $y = 4$, and $x = 81$. Therefore we have:

$$3^4 = 81 \quad \Leftrightarrow \quad \log_3(81) = 4$$

Similarly, we obtain the solutions for (b), (c), and (d).

b) $10^3 = 1000 \quad \Leftrightarrow \quad \log(1000) = 3$

c) $e^x = 17 \quad \Leftrightarrow \quad \ln(17) = x$

d) $2^{7a} = 53 \quad \Leftrightarrow \quad \log_2(53) = 7a$ □

Example 13.10

Evaluate the expression by rewriting it as an exponential expression.

a) $\log_2(16)$ b) $\log_5(125)$ c) $\log_{13}(1)$ d) $\log_4(4)$
e) $\log(100,000)$ f) $\log(0.001)$ g) $\ln(e^7)$ h) $\log_b(b^x)$

Solution.

a) If we set $y = \log_2(16)$, then this is equivalent to $2^y = 16$. Since, clearly, $2^4 = 16$, we see that $y = 4$. Therefore, we have $\log_2(16) = 4$.

b) $\log_5(125) = y \quad \Leftrightarrow \quad 5^y = 125 \quad \overset{(\text{since } 5^3=125)}{\Longrightarrow} \quad 3 = y = \log_5(125)$

c) $\log_{13}(1) = y \quad \Leftrightarrow \quad 13^y = 1 \quad \overset{(\text{since } 13^0=1)}{\Longrightarrow} \quad 0 = y = \log_{13}(1)$

d) $\log_4(4) = y \quad \Leftrightarrow \quad 4^y = 4 \quad \overset{(\text{since } 4^1=4)}{\Longrightarrow} \quad 1 = y = \log_4(4)$

e) $\log(100,000) = y \quad \Leftrightarrow \quad 10^y = 100,000 \quad \overset{(\text{since } 10^5=100{,}000)}{\Longrightarrow} \quad 5 = y = \log(100,000)$

f) $\log(0.001) = y \quad \Leftrightarrow \quad 10^y = 0.001 \quad \overset{(\text{since } 10^{-3}=0.001)}{\Longrightarrow} \quad -3 = y = \log(0.001)$

g) $\ln(e^7) = y \quad \Leftrightarrow \quad e^y = e^7 \quad \Longrightarrow \quad 7 = y = \ln(e^7)$

h) $\log_b(b^x) = y \quad \Leftrightarrow \quad b^y = b^x \quad \Longrightarrow \quad x = y = \log_b(b^x)$

Note that the last example, in which we obtained $\log_b(b^x) = x$, combines all of the previous examples. □

In the previous example (in parts (c), (d), and (h)), we were able to find certain elementary logarithms. We record these in the next observation.

Observation 13.11: Basic logarithmic evaluations

We have the elementary logarithms:

$$\boxed{\log_b(b^x) = x} \qquad \boxed{\log_b(b) = 1} \qquad \boxed{\log_b(1) = 0} \tag{13.3}$$

In general, when the argument is not a power of the base, we can use the calculator to approximate the values of a logarithm via the formulas:

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)}} \quad \text{or} \quad \boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}} \tag{13.4}$$

The last two formulas will be proved in Proposition 14.2 below. For now, we want to show how they can be used to calculate any logarithmic expression with any calculator that has either the $\ln$ or the $\log$ function.

Example 13.12

Evaluate: a) $\log_3(13)$ b) $\log_{2.34}(98.765)$

Solution.

a) We calculate $\log_3(13)$ by using the first formula in (13.4).

$$\log_3(13) = \frac{\log(13)}{\log(3)} \approx 2.335$$

Alternatively, we can also calculate this with the second formula in (13.4).

$$\log_3(13) = \frac{\ln(13)}{\ln(3)} \approx 2.335$$

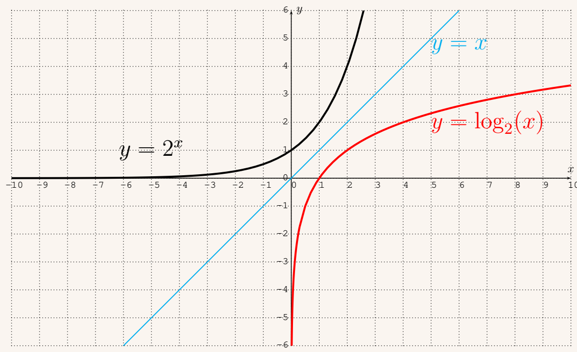
b) We compute $\log_{2.34}(98.765) = \frac{\log(98.765)}{\log(2.34)} \approx 5.402$.

□

We also study the graph of logarithmic functions.

Note 13.13

Consider the graph of $y = 2^x$ from the previous section. Recall that the graph of the inverse of a function is the reflection of the graph of the function about the diagonal line $y = x$. So in this case we have:



Note that the horizontal asymptote $y = 0$ for $y = 2^x$ becomes the vertical asymptote $x = 0$ for $y = \log_2(x)$. The x-intercept of $y = \log_2(x)$ is at $y = 0$, that is $0 = \log_2(x)$, which gives $x = 2^0 = 1$ as the x-intercept.

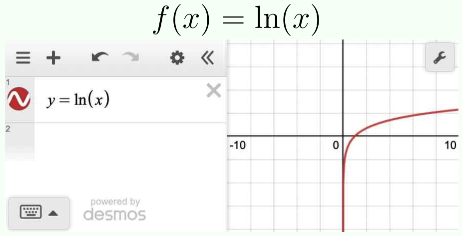
Example 13.14

Graph the functions $f(x) = \ln(x)$, $g(x) = \log(x)$, $h(x) = \log_2(x)$, and $k(x) = \log_{0.5}(x)$. What are the domains of f, g, h, and k? How do these functions differ?

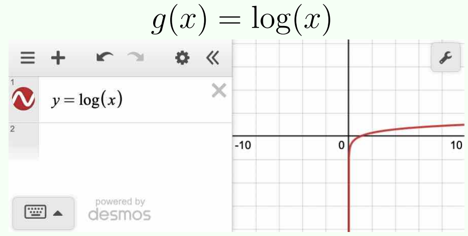
Solution.

We know from the definition that the domain of f, g, and h is all real positive numbers, $D_f = D_g = D_h = D_k = \{x|x > 0\}$. The functions f and g can immediately be entered into the calculator. The standard window gives the following graphs.

$f(x) = \ln(x)$



$g(x) = \log(x)$

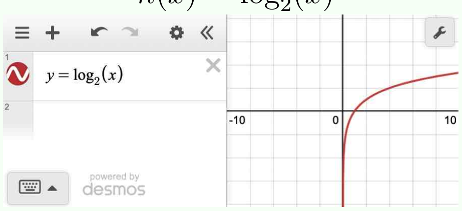


Note that we can rewrite $g(x)$, $h(x)$, and $k(x)$ as a constant times $f(x)$:

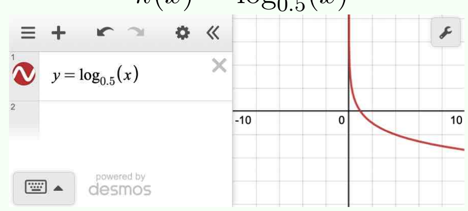
$$\begin{aligned} g(x) &= \log(x) = \log_{10}(x) = \frac{\ln(x)}{\ln(10)} = \frac{1}{\ln(10)} \cdot f(x) \\ h(x) &= \log_2(x) = \frac{\ln(x)}{\ln(2)} = \frac{1}{\ln(2)} \cdot f(x) \\ k(x) &= \log_{0.5}(x) = \frac{\ln(x)}{\ln(0.5)} = \frac{1}{\ln(0.5)} \cdot f(x) \end{aligned}$$

Since $\frac{1}{\ln(10)} \approx 0.434 < 1$, we see that the graph of g is that of f compressed toward the x-axis by a factor $\frac{1}{\ln(10)}$. Similarly, $\frac{1}{\ln(2)} \approx 1.443 > 1$, so that the graph of h is that of f stretched away from the x-axis by a factor $\frac{1}{\ln(2)}$. Finally, $\frac{1}{\ln(0.5)} \approx -1.443$, or more precisely, $\frac{1}{\ln(0.5)} = \frac{1}{\ln(2^{-1})} = -\frac{1}{\ln(2)}$, so that the graph of k is that of h reflected about the x-axis.

$h(x) = \log_2(x)$



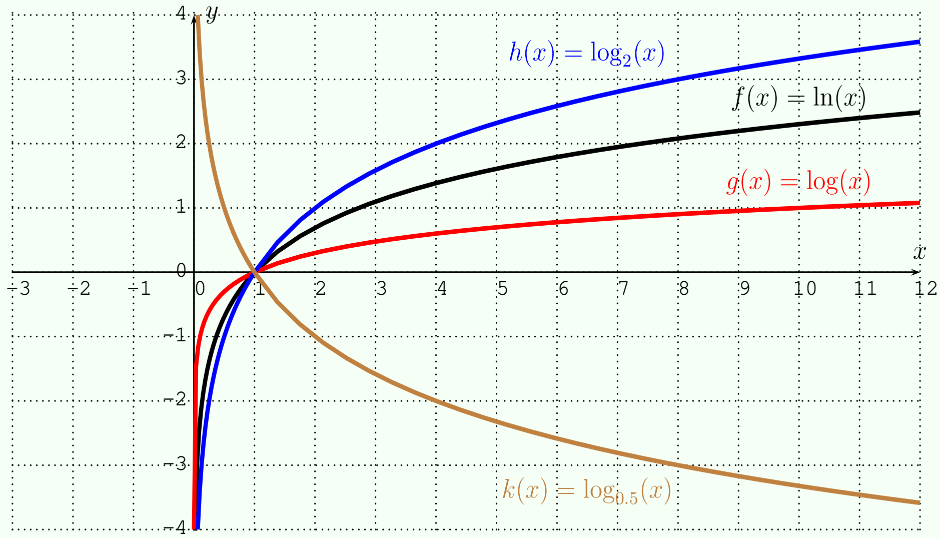
$k(x) = \log_{0.5}(x)$



Note that all these graphs have a vertical asymptote at $x = 0$. Moreover, all of the functions have an x-intercept at $x = 1$:

$$f(1) = g(1) = h(1) = k(1) = 0$$

To visualize the differences between the graphs, we graph them together in one coordinate system.



□

Example 13.15

Graph the given function. State the domain, find the vertical asymptote, and find the x-intercept of the function.

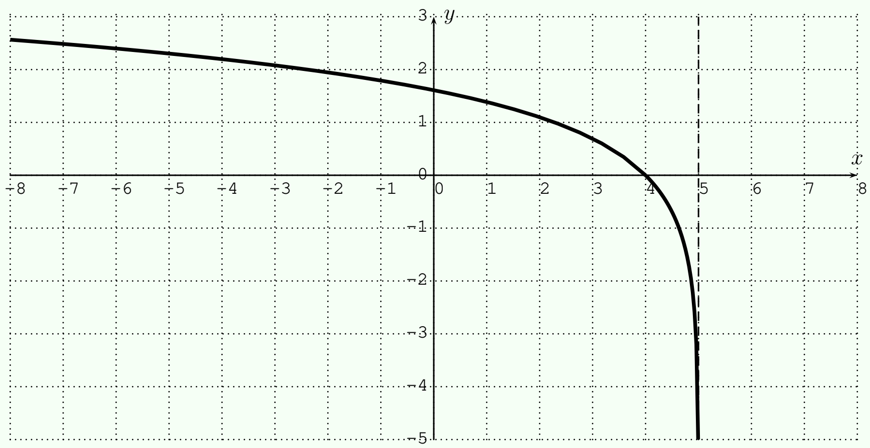
a) $f(x) = \ln(5 - x)$ b) $g(x) = \log_7(2x + 8)$ c) $h(x) = -3 \cdot \ln(x) + 4$

Solution.

a) To determine the domain of $f(x) = \ln(5 - x)$, we have to see for which x the logarithm has a positive argument. More precisely, we need $5-x > 0$, that is, $5 > x$, so that the domain is $D_f = \{x|x < 5\}$.

The calculator displays the following graph:

Note that the graph, as displayed by the calculator, appears to end at a point that is approximately at $(5, -35)$. However, the actual graph of the logarithm *does not* stop at any point, since it has a vertical asymptote at $x = 5$, that is, the graph approaches $-\infty$ as x approaches 5. The calculator only displays an approximation, which may be misleading, since this approximation is determined by the window size and the size of each pixel. We therefore graph the function $f(x) = \ln(5 - x)$ as follows:



The x-intercept is given where $y = 0$, that is

$$0 = \ln(5 - x) \implies 5 - x = e^0 \implies 5 - x = 1 \implies x = 4$$

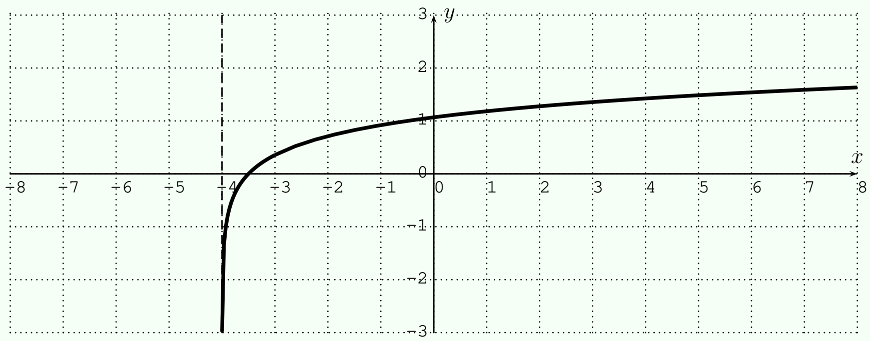
Therefore, the x-intercept is at $(4, 0)$.

b) The domain of $g(x) = \log_7(2x + 8)$ consists of those numbers x for which the argument of the logarithm is positive.

$$2x + 8 > 0 \overset{\text{(subtract 8)}}{\Longrightarrow} 2x > -8 \overset{\text{(divide by 2)}}{\Longrightarrow} x > -4$$

Therefore, the domain is $D_g = \{x | x > -4\}$. The graph of $g(x) =$

$\log_7(2x+8)$ is displayed below.



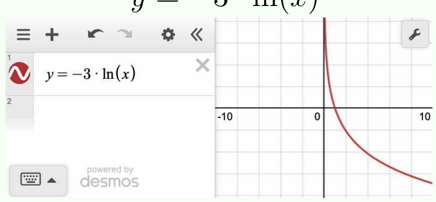
There is a vertical asymptote at $x = -4$. The x-intercept is given at $y = 0$:

$$0 = \log_7(2x+8) \implies 2x+8 = 7^0 \implies 2x+8 = 1$$
$$\implies 2x = -7 \implies x = -\frac{7}{2}$$

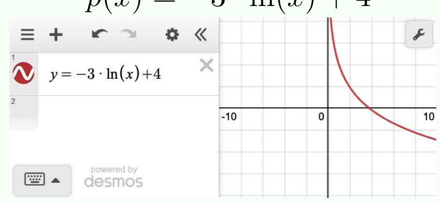
Therefore, the x-intercept is at $(-\frac{7}{2}, 0)$.

c) Using our knowledge of transformations of graphs, we expect that $h(x) = -3 \cdot \ln(x) + 4$ is that of $y = \ln(x)$ reflected and stretched away from the x-axis (by a factor 3), and then shifted up by 4. The stretched and reflected graph is on the left below, whereas the graph of the shifted function h is on the right.

$y = -3 \cdot \ln(x)$



$p(x) = -3 \cdot \ln(x) + 4$



The domain consist of numbers x for which the $\ln(x)$ is defined, that is, $D_p = \{x | x > 0\}$. The vertical asymptote is therefore also at $x = 0$. The x-intercept is computed as follows:

$$y = 0 \implies 0 = -3 \cdot \ln(x) + 4 \implies -4 = -3 \cdot \ln(x)$$

$$\implies \ln(x) = \frac{4}{3} \implies x = e^{\frac{4}{3}}$$

Therefore, the x-intercept is at $(e^{\frac{4}{3}}, 0)$.

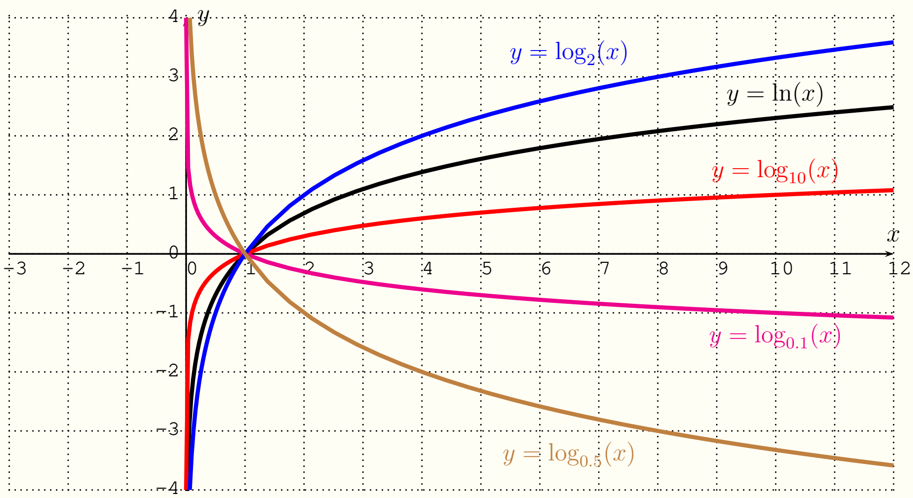
□

In the previous examples we analyzed the graphs of various logarithmic functions. The following is a summary of our findings.

Observation 13.16: Graph of a logarithmic function

The graph of a logarithmic function $y = \log_b(x)$ with base b is that of the natural logarithm $y = \ln(x)$ stretched away from the x-axis, or compressed toward the x-axis when $b > 1$. When $0 < b < 1$, the graph is furthermore reflected about the x-axis.

- The function $y = \log_b(x)$ has domain $D = \{x | x > 0\}$.
- The graph of $y = \log_b(x)$ has a vertical asymptote at $x = 0$.
- The graph of $y = \log_b(x)$ has *no* horizontal asymptote, as $f(x)$ approaches $+\infty$ when x approaches $+\infty$ for $b > 1$, and $f(x)$ approaches $-\infty$ when x approaches $+\infty$ for $0 < b < 1$.
- The x-intercept is given for $y = 0$, which for $y = \log_b(x)$ is at $x = 1$.



13.3 Exercises

Exercise 13.1

Graph the following functions with the calculator.

a) $y = 5^x$	b) $y = 1.01^x$	c) $y = (\frac{1}{3})^x$	d) $y = 0.97^x$
e) $y = 3^{-x}$	f) $y = (\frac{1}{3})^{-x}$	g) $y = e^{x^2}$	h) $y = 0.01^x$
i) $y = 1^x$	j) $y = e^x + 1$	k) $y = \frac{e^x - e^{-x}}{2}$	l) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

The last two functions are known as the *hyperbolic sine*, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and the *hyperbolic tangent*, $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Recall that the *hyperbolic cosine* $\cosh(x) = \frac{e^x + e^{-x}}{2}$ was already graphed in Example 13.5.

Exercise 13.2

Graph the given function. Describe how the graph is obtained by a transformation from the graph of an exponential function $y = b^x$ (for appropriate base b).

a) $y = 0.1 \cdot 4^x$	b) $y = 3 \cdot 2^x$	c) $y = (-1) \cdot 2^x$
d) $y = 0.006 \cdot 2^x$	e) $y = e^{-x}$	f) $y = e^{-x} + 1$
g) $y = (\frac{1}{2})^x + 3$	h) $y = 2^{x-4}$	i) $y = 2^{x+1} - 6$

Exercise 13.3

Use the definition of the logarithm to write the given equation as an equivalent logarithmic equation.

a) $4^2 = 16$	b) $2^8 = 256$	c) $e^x = 7$	d) $10^{-1} = 0.1$
e) $3^x = 12$	f) $5^{7 \cdot x} = 12$	g) $3^{2a+1} = 44$	h) $\left(\frac{1}{2}\right)^{\frac{x}{h}} = 30$

Exercise 13.4

Evaluate the following expressions *without* using a calculator.

a) $\log_7(49)$	b) $\log_3(81)$	c) $\log_2(64)$	d) $\log_{50}(2500)$
e) $\log_2(0.25)$	f) $\log(1000)$	g) $\ln(e^4)$	h) $\log_{13}(13)$
i) $\log(0.1)$	j) $\log_6(\frac{1}{36})$	k) $\ln(1)$	l) $\log_{\frac{1}{2}}(8)$

Exercise 13.5

Using a calculator, approximate the following expressions to the nearest thousandth.

a) $\log_3(50)$ b) $\log_3(12)$ c) $\log_{17}(0.44)$ d) $\log_{0.34}(200)$

Exercise 13.6

State the domain of the function f and find any vertical asymptote(s) and x-intercept(s). Use the results to sketch the graph.

a) $f(x) = \log(x)$
b) $f(x) = \log(x + 7)$
c) $f(x) = \ln(x + 5) - 1$
d) $f(x) = \ln(3x - 6)$
e) $f(x) = 2 \cdot \log(x + 4)$
f) $f(x) = -4 \cdot \log(x + 2)$
g) $f(x) = \log_3(7x + 5)$
h) $f(x) = \ln(-6x + 14)$
i) $f(x) = \log_{0.4}(x)$
j) $f(x) = \log_3(-5x) - 2$
k) $f(x) = \log |x|$
l) $f(x) = \log |x + 2|$

Chapter 14

Properties of logarithms and logarithmic equations

We now study more algebraic properties of the logarithm. We then use this to solve logarithmic equations.

14.1 Algebraic properties of the logarithms

Recall the well-known identities for exponential expressions.

Review 14.1: Exponential identities

We have the following identities:

$$\boxed{\begin{aligned} b^{x+y} &= b^x \cdot b^y \\ b^{x-y} &= \tfrac{b^x}{b^y} \\ (b^x)^n &= b^{nx} \end{aligned}} \tag{14.1}$$

Writing the above identities in terms of $f(x) = b^x$, these can also be expressed as $f(x+y) = f(x)f(y)$, $f(x-y) = f(x)/f(y)$, and $f(nx) = f(x)^n$.

Since the logarithm is the inverse function of the exponential, there are some logarithmic identities that correspond to (14.1).

Proposition 14.2: Logarithmic identities

The logarithm behaves well with respect to products, quotients, and exponentiation. Indeed, for all positive real numbers $0 < b \neq 1$, $x > 0$, $y > 0$, and real numbers n, we have:

$$\boxed{\begin{gathered} \log_b(x \cdot y) = \log_b(x) + \log_b(y) \\ \log_b(\tfrac{x}{y}) = \log_b(x) - \log_b(y) \\ \log_b(x^n) = n \cdot \log_b(x) \end{gathered}} \tag{14.2}$$

In terms of the logarithmic function $g(x) = \log_b(x)$, the properties in the table above can be written: $g(xy) = g(x) + g(y)$, $g(x/y) = g(x) - g(y)$, and $g(x^n) = n \cdot g(x)$.
Furthermore, for another positive real number $0 < a \neq 1$, we have the *change of base formula*:

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}} \tag{14.3}$$

In particular, we have the formulas from Equation (13.4) on page 243 when taking the base $a = 10$ and $a = e$:

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)}} \quad \text{and} \quad \boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Proof. We start with the first formula $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$. If we call $u = \log_b(x)$ and $v = \log_b(y)$, then the equivalent exponential formulas are $b^u = x$ and $b^v = y$. With this, we have

$$x \cdot y = b^u \cdot b^v = b^{u+v}.$$

Rewriting this in logarithmic form, we obtain

$$\log_b(x \cdot y) = u + v = \log_b(x) + \log_b(y).$$

This is what we needed to show.

Next, we prove the formula $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$. We abbreviate $u = \log_b(x)$ and $v = \log_b(y)$ as before, and their exponential forms are $b^u = x$ and $b^v = y$. Therefore, we have

$$\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}.$$

Rewriting this again in logarithmic form, we obtain the desired result.

$$\log_b\left(\frac{x}{y}\right) = u - v = \log_b(x) - \log_b(y)$$

For the third formula, $\log_b(x^n) = n \cdot \log_b(x)$, we write $u = \log_b(x)$, that is in exponential form $b^u = x$. Then:

$$x^n = (b^u)^n = b^{n \cdot u} \quad \Longrightarrow \quad \log_b(x^n) = n \cdot u = n \cdot \log_b(x)$$

For the last formula (14.3), we write $u = \log_b(x)$, that is, $b^u = x$. Applying the logarithm with base a to $b^u = x$ gives $\log_a(b^u) = \log_a(x)$. As we have just shown before, $\log_a(b^u) = u \cdot \log_a(b)$. Combining these identities with the initial definition $u = \log_b(x)$, we obtain

$$\log_a(x) = \log_a(b^u) = u \cdot \log_a(b) = \log_b(x) \cdot \log_a(b)$$

Dividing both sides by $\log_a(b)$ gives the result $\frac{\log_a(x)}{\log_a(b)} = \log_b(x)$. □

Example 14.3

Combine the terms using the properties of logarithms so as to write as one logarithm.

a) $\frac{1}{2}\ln(x) + \ln(y)$ b) $\frac{2}{3}(\log(x^2y) - \log(xy^2))$

c) $2\ln(x) - \frac{1}{3}\ln(y) - \frac{7}{5}\ln(z)$ d) $5 + \log_2(a^2 - b^2) - \log_2(a + b)$

Solution.

Recall that a fractional exponent can also be rewritten with an nth root.

$$\boxed{x^{\frac{1}{2}} = \sqrt{x}} \quad \text{and} \quad \boxed{x^{\frac{1}{n}} = \sqrt[n]{x}} \quad \Longrightarrow \quad x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

We apply the rules from Proposition 14.2.

a) $\frac{1}{2}\ln(x) + \ln(y) = \ln(x^{\frac{1}{2}}) + \ln(y) = \ln(x^{\frac{1}{2}}y) = \ln(\sqrt{x} \cdot y)$

b) $\frac{2}{3}(\log(x^2y) - \log(xy^2)) = \frac{2}{3}\left(\log\left(\frac{x^2y}{xy^2}\right)\right) = \frac{2}{3}\left(\log\left(\frac{x}{y}\right)\right)$

$$= \log\left(\left(\frac{x}{y}\right)^{\frac{2}{3}}\right) = \log\left(\sqrt[3]{\frac{x^2}{y^2}}\right)$$

c) $2\ln(x) - \frac{1}{3}\ln(y) - \frac{7}{5}\ln(z) = \ln(x^2) - \ln(\sqrt[3]{y}) - \ln(\sqrt[5]{z^7}) = \ln\left(\frac{x^2}{\sqrt[3]{y} \cdot \sqrt[5]{z^7}}\right)$

d) $5 + \log_2(a^2 - b^2) - \log_2(a + b) = \log_2(2^5) + \log_2(a^2 - b^2) - \log_2(a + b)$

$$= \log_2\left(\frac{2^5 \cdot (a^2 - b^2)}{a + b}\right) = \log_2\left(\frac{32 \cdot (a + b)(a - b)}{a + b}\right) = \log_2(32 \cdot (a - b))$$

□

Example 14.4

Write the expressions in terms of elementary logarithms $u = \log_b(x)$, $v = \log_b(y)$, and, in part (c), also $w = \log_b(z)$. Assume that $x, y, z > 0$.

a) $\ln(\sqrt{x^5} \cdot y^2)$ b) $\log\left(\sqrt{\sqrt{x} \cdot y^3}\right)$ c) $\log_2\left(\sqrt[3]{\frac{x^2}{y\sqrt{z}}}\right)$

Solution.

In a first step, we rewrite the expression with fractional exponents, and then apply the rules from Proposition 14.2.

a)

$$\begin{aligned}\ln(\sqrt{x^5} \cdot y^2) &= \ln(x^{\frac{5}{2}} \cdot y^2) = \ln(x^{\frac{5}{2}}) + \ln(y^2)\\ &= \frac{5}{2}\ln(x) + 2\ln(y) = \frac{5}{2}u + 2v\end{aligned}$$

b)

$$\begin{aligned}\log\left(\sqrt{\sqrt{x} \cdot y^3}\right) &= \log\left(\left(x^{\frac{1}{2}}y^3\right)^{\frac{1}{2}}\right) = \frac{1}{2}\log\left(x^{\frac{1}{2}}y^3\right)\\ &= \frac{1}{2}\left(\log(x^{\frac{1}{2}}) + \log(y^3)\right) = \frac{1}{2}\left(\frac{1}{2}\log(x) + 3\log(y)\right)\\ &= \frac{1}{4}\log(x) + \frac{3}{2}\log(y) = \frac{1}{4}u + \frac{3}{2}v\end{aligned}$$

c)

$$\begin{aligned}\log_2\left(\sqrt[3]{\frac{x^2}{y\sqrt{z}}}\right) &= \log_2\left(\left(\frac{x^2}{y \cdot z^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\log_2\left(\frac{x^2}{y \cdot z^{\frac{1}{2}}}\right)\\ &= \frac{1}{3}\left(\log_2(x^2) - \log_2(y) - \log_2(z^{\frac{1}{2}})\right)\\ &= \frac{1}{3}\left(2\log_2(x) - \log_2(y) - \frac{1}{2}\log_2(z)\right)\\ &= \frac{2}{3}\log_2(x) - \frac{1}{3}\log_2(y) - \frac{1}{6}\log_2(z)\\ &= \frac{2}{3}u - \frac{1}{3}v - \frac{1}{6}w\end{aligned}$$

□

14.2 Solving logarithmic equations

We can solve exponential and logarithmic equations by applying logarithms and exponentials. Since the exponential and logarithmic functions are invertible (they are inverses of each other), these functions necessarily have to be one-to-one functions. As an algebraic expression, this means that:

Observation 14.5: $y = b^x$ and $y = \log_b(x)$ are one-to-one

The exponential and the logarithmic functions are one-to-one:

$$\boxed{b^x = b^y \quad \Leftrightarrow \quad x = y} \tag{14.4}$$

$$\boxed{\log_b(x) = \log_b(y) \quad \Leftrightarrow \quad x = y} \tag{14.5}$$

In the following examples, we use the above to solve equations that involve logarithms.

Example 14.6

Solve for x.

a) $\log_6(3x-5) = \log_6(x-1)$ b) $\log_2(x+5) = \log_2(x+3) + 4$
c) $\log(x) + \log(x+4) = \log(5)$ d) $\log_3(x-2) + \log_3(x+6) = 2$
e) $\ln(x+2) + \ln(x-3) = \ln(7)$

Solution.

a) We can use Equation (14.5) as follows.

$$\begin{aligned} \log_6(3x-5) = \log_6(x-1) \quad &\Longrightarrow \quad 3x - 5 = x - 1 \overset{(-x+5)}{\Longrightarrow} \quad 2x = 4 \\ &\Longrightarrow \quad x = 2 \end{aligned}$$

An immediate check shows $x = 2$ is indeed a solution, since $\log_6(3 \cdot 2 - 5) = \log_6(1)$ and $\log_6(2-1) = \log_6(1)$.

b) We have to solve $\log_2(x+5) = \log_2(x+3) + 4$. To combine the right-hand side, recall that 4 can be written as a logarithm, $4 = \log_2(2^4) = \log_2 16$. With this remark we can now solve the equation for x.

$$\log_2(x+5) = \log_2(x+3)+4 \implies \log_2(x+5) = \log_2(x+3)+\log_2(16)$$
$$\implies \log_2(x+5) = \log_2(16\cdot(x+3)) \implies x+5 = 16(x+3)$$
$$\implies x+5 = 16x+48 \overset{(-16x-5)}{\implies} -15x = 43 \implies x = -\frac{43}{15}$$

c) We start by combining the logarithms.

$$\begin{aligned}
\log(x)+\log(x+4) = \log(5) &\implies \log(x\cdot(x+4)) = \log(5) \\
&\overset{\text{remove log}}{\implies} x(x+4) = 5 \\
&\implies x^2+4x-5 = 0 \\
&\implies (x+5)(x-1) = 0 \\
&\implies x = -5 \text{ or } x = 1
\end{aligned}$$

Since the equation became a quadratic equation, we ended up with two possible solutions $x = -5$ and $x = 1$. However, since $x = -5$ would give a negative value inside a logarithm in our original equation $\log(x)+\log(x+4) = \log(5)$, we need to exclude this solution. The only solution is $x = 1$.

We note that the incorrect solution $x = -5$ is introduced in the very first implication, since -5 in fact *is* a perfectly well-defined solution of the equation $\log(x\cdot(x+4)) = \log(5)$,

$$\log((-5)\cdot(-5+4)) = \log((-5)\cdot(-1)) = \log(5),$$

whereas -5 is not a solution of $\log(x)+\log(x+4) = \log(5)$, since $\log(-5)+\log(-5+4)$ is undefined.

d) Using that $2 = \log_3(3^2)$:

$$\begin{aligned}
\log_3(x-2)+\log_3(x+6) = 2 &\implies \log_3((x-2)(x+6)) = \log_3(3^2) \\
&\implies (x-2)(x+6) = 3^2 \\
&\implies x^2+4x-12 = 9 \\
&\implies x^2+4x-21 = 0 \\
&\implies (x+7)(x-3) = 0 \\
&\implies x = -7 \text{ or } x = 3
\end{aligned}$$

We exclude $x = -7$, since we would obtain a negative value inside a logarithm, so that the solution is $x = 3$.

e) We combine the left-hand side of $\ln(x+2)+\ln(x-3)=\ln(7)$ to get

$$\begin{aligned} \ln((x+2)\cdot(x-3)) = \log(7) \quad &\Longrightarrow \quad (x+2)\cdot(x-3)=7 \\ &\Longrightarrow \quad x^2-3x+2x-6=7 \\ &\Longrightarrow \quad x^2-x-13=0 \end{aligned}$$

To solve this, we need to use the quadratic formula (8.1).

$$\begin{aligned} x^2-x-13=0 \implies x &= \frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot(-13)}}{2\cdot 1} \\ &= \frac{1\pm\sqrt{1+52}}{2} = \frac{1\pm\sqrt{53}}{2} \end{aligned}$$

To see which of these are actual solutions of $\ln(x+2)+\ln(x-3)=\ln(7)$, note that we have to plug $x=\frac{1\pm\sqrt{53}}{2}$ into $x+2$ and $x-3$ and make sure these are positive:

$$\begin{aligned} \frac{1+\sqrt{53}}{2}+2 \approx 6.14 > 0 \text{ and } \frac{1+\sqrt{53}}{2}-3 \approx 1.14 > 0 \\ \frac{1-\sqrt{53}}{2}+2 \approx -1.14 < 0 \text{ and } \frac{1-\sqrt{53}}{2}-3 \approx -6.14 < 0 \end{aligned}$$

Thus, $\frac{1-\sqrt{53}}{2}$ is not a solution (since, for example, $\ln(\frac{1-\sqrt{53}}{2}+2)$ is undefined), and the only solution is $x=\frac{1+\sqrt{53}}{2}$.

□

In Examples 14.6 (c)–(e) our calculations showed that the given equalities had two *possible* solutions. After checking these with the original equation, we saw that one was an actual solution (making the equation true), while the other was not (and therefore was rejected). In general, it may turn out that all of the possible solutions are actual solutions, or none of the possible solutions are actual solutions. This is demonstrated in the next example.

Example 14.7

Solve for x.

a) $\log_3(x+1)+\log_3(7-x)=\log_3(12)$
b) $\log_5(x-7)+\log_5(2-x)=\log_5(4)$

Solution.

a) Combining the logarithms gives $\log_3((x+1)(7-x))=\log_3(12)$, which implies

$$\begin{aligned}(x+1)(7-x)=12 &\implies 7x-x^2+7-x=12\\ &\implies 0=x^2-6x+5\\ &\implies 0=(x-1)(x-5)\\ &\implies x=1, x=5\end{aligned}$$

Since both give positive arguments in the logarithms, we have, indeed, two solutions $x=1$ and $x=5$.

b) We get $\log_5((x-7)(2-x))=\log_5(4)$, and thus $(x-7)(2-x)=4$, which can be rewritten as $2x-x^2-14+7x=4$, and thus as $0=x^2-9x+18$. Factoring yields $0=(x-3)(x-6)$, which has the two possible solutions $x=3$ and $x=6$. However, 3 is not a solution, since $2-3=-1<0$; and 6 is not a solution since $2-6=-4<0$. We conclude that there is no solution.

□

14.3 Exercises

Exercise 14.1

Combine the terms and write your answer as one logarithm.

a) $3\ln(x) + \ln(y)$
b) $\log(x) - \frac{2}{3}\log(y)$
c) $\frac{1}{3}\log(x) - \log(y) + 4\log(z)$
d) $\log(xy^2z^3) - \log(x^4y^3z^2)$
e) $\frac{1}{4}\ln(x) - \frac{1}{2}\ln(y) + \frac{2}{3}\ln(z)$
f) $-\ln(x^2-1) + \ln(x-1)$
g) $5\ln(x) + 2\ln(x^4) - 3\ln(x)$
h) $\log_5(a^2+10a+9) - \log_5(a+9) + 2$

Exercise 14.2

Write the expressions in terms of elementary logarithms $u = \log_b(x)$, $v = \log_b(y)$, and $w = \log_b(z)$ (whichever are applicable). Assume that $x, y, z > 0$.

a) $\log(x^3 \cdot y)$
b) $\log(\sqrt[3]{x^2} \cdot \sqrt[4]{y^7})$
c) $\log\left(\sqrt{x \cdot \sqrt[3]{y}}\right)$
d) $\ln\left(\frac{x^3}{y^4}\right)$
e) $\ln\left(\frac{x^2}{\sqrt{y}\cdot z^2}\right)$
f) $\log_3\left(\sqrt{\frac{x\cdot y^3}{\sqrt{z}}}\right)$
g) $\log_2\left(\frac{\sqrt[4]{x^3\cdot z}}{y^3}\right)$
h) $\log\left(\frac{100\sqrt[5]{z}}{y^2}\right)$
i) $\ln\left(\sqrt[3]{\frac{\sqrt{y}\cdot z^4}{e^2}}\right)$

Exercise 14.3

Solve for x without using a calculator.

a) $\ln(2x+4) = \ln(5x-5)$
b) $\ln(x+6) = \ln(x-2) + \ln(3)$
c) $\log_2(x+5) = \log_2(x) + 5$
d) $\log(x) + 1 = \log(5x+380)$
e) $\log(x+5) + \log(x) = \log(6)$
f) $\log_2(x) + \log_2(x-6) = 4$
g) $\log_6(x) + \log_6(x-16) = 2$
h) $\log_5(x-24) + \log_5(x) = 2$
i) $\log_4(x) + \log_4(x+6) = 2$
j) $\log_2(x+3) + \log_2(x+5) = 3$

Chapter 15

Exponential equations and applications

We now turn to exponential equations, and discuss the application of population growth in Section 15.2. In the next chapter, we will study two more common applications of exponential functions.

15.1 Exponential equations

Recall from Observation 14.5 that both the exponential and the logarithmic functions are one-to-one:

$$b^x = b^y \quad \Leftrightarrow \quad x = y$$
$$\log_b(x) = \log_b(y) \quad \Leftrightarrow \quad x = y$$

In Section 14.2 we used the second equivalence to solve logarithmic equations. Now we use the first equivalence to solve exponential equations. Note that we can immediately apply this to exponential equations with a common base.

Example 15.1

Solve for x.

a)	$2^{x+7} = 32$	b)	$10^{2x-8} = 0.01$
c)	$7^{2x-3} = 7^{5x+4}$	d)	$5^{3x+1} = 25^{4x-7}$

Solution.

In these examples, we can always write both sides of the equation as an exponential expression with the same base.

a) $2^{x+7} = 32 \implies 2^{x+7} = 2^5 \implies x + 7 = 5 \implies x = -2$

b) $10^{2x-8} = 0.01 \implies 10^{2x-8} = 10^{-2} \implies 2x - 8 = -2$
$\implies 2x = 6 \implies x = 3$

Here it is useful to recall the powers of 10, which were also used to solve the equation above.

$$\begin{array}{rcl} 10^4 &=& 10,000 \\ 10^3 &=& 1000 \\ 10^2 &=& 100 \\ 10^1 &=& 10 \\ 10^0 &=& 1 \\ 10^{-1} &=& 0.1 \\ 10^{-2} &=& 0.01 \\ 10^{-3} &=& 0.001 \\ 10^{-4} &=& 0.0001 \end{array} \qquad \text{In general } (n \geq 1): \begin{cases} 10^n = 1\underbrace{00\cdots00}_{n \text{ zeros}} \\ 10^{-n} = \underbrace{0.0\cdots00}_{n \text{ zeros}}1 \end{cases}$$

c) $7^{2x-3} = 7^{5x+4} \implies 2x - 3 = 5x + 4 \overset{(-5x+3)}{\implies} -3x = 7$
$\implies x = -\frac{7}{3}$

d) $5^{3x+1} = 25^{4x-7} \implies 5^{3x+1} = 5^{2\cdot(4x-7)}$
$\implies 3x + 1 = 2 \cdot (4x - 7)$
$\implies 3x + 1 = 8x - 14$
$\overset{(-8x-1)}{\implies} -5x = -15$
$\implies x = 3$

By a similar reasoning, we can solve equations involving logarithms whenever the bases coincide. □

To solve exponential equations that do not have a common base on both sides, we need to apply the logarithm, as stated in the following note.

Note 15.2

An equation between two exponential expressions with the same base can be simplified using the fact that the exponential is one-to-one.

$$b^{f(x)} = b^{g(x)} \quad \Longrightarrow \quad f(x) = g(x)$$

To solve an equation between two exponential expressions with different bases, we first apply a logarithm and then solve for x. Indeed, using the identity $\log_b(x^n) = n \cdot \log_b(x)$ from (14.2), we can rewrite an exponent as a coefficient and solve from there:

$$\begin{aligned} a^{f(x)} = b^{g(x)} \quad &\Longrightarrow \quad \log(a^{f(x)}) = \log(b^{f(x)}) \\ &\Longrightarrow \quad f(x) \cdot \log(a) = g(x) \cdot \log(b) \end{aligned}$$

Example 15.3

Solve for x.

a)	$3^{x+5} = 8$	b)	$13^{2x-4} = 6$	c)	$5^{x-7} = 2^x$
d)	$5.1^x = 2.7^{2x+6}$	e)	$17^{x-2} = 3^{x+4}$	f)	$7^{2x+3} = 11^{3x-6}$

Solution.

We solve these equations by applying a logarithm (both $\log$ or $\ln$ will work for solving the equation), and then we use the identity $\log_b(x^n) = n \cdot \log_b(x)$ from (14.2).

a) $3^{x+5} = 8 \implies \ln 3^{x+5} = \ln 8 \implies (x+5) \cdot \ln 3 = \ln 8$

$$\implies x + 5 = \frac{\ln 8}{\ln 3} \implies x = \frac{\ln 8}{\ln 3} - 5 \approx -3.11$$

b) $13^{2x-4} = 6 \implies \ln 13^{2x-4} = \ln 6 \implies (2x-4) \cdot \ln 13 = \ln 6$

$$\begin{aligned} &\implies 2x - 4 = \frac{\ln 6}{\ln 13} \implies 2x = \frac{\ln 6}{\ln 13} + 4 \\ &\implies x = \frac{\frac{\ln 6}{\ln 13} + 4}{2} = \frac{\ln 6}{2 \cdot \ln 13} + 2 \approx 2.35 \end{aligned}$$

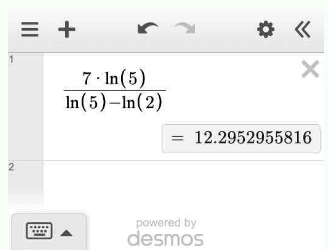
c) $5^{x-7} = 2^x \implies \ln 5^{x-7} = \ln 2^x \implies (x-7) \cdot \ln 5 = x \cdot \ln 2$

At this point, the calculation will proceed differently than the calculations in parts (a) and (b). Since x appears on both sides of $(x-7)\cdot\ln 5 = x\cdot\ln 2$, we need to separate terms involving x from terms without x. That is, we need to distribute $\ln 5$ on the left:

$$(x-7)\cdot\ln 5 = x\cdot\ln 2 \implies x\cdot\ln 5 - 7\cdot\ln 5 = x\cdot\ln 2$$

Next, we separate the terms with x from those without x by adding $7\cdot\ln 5$ and subtracting $x\cdot\ln 2$ to both sides:

$$\begin{aligned} &\implies && x\cdot\ln 5 - x\cdot\ln 2 = 7\cdot\ln 5 \\ &\implies && x\cdot(\ln 5 - \ln 2) = 7\cdot\ln 5 \\ &\implies && x = \frac{7\cdot\ln 5}{\ln 5 - \ln 2} \approx 12.30 \end{aligned}$$



We apply the same solution strategy that we used in (c) for the remaining parts (d)-(f).

d) $5.1^x = 2.7^{2x+6} \implies \ln 5.1^x = \ln 2.7^{2x+6}$

$$\begin{aligned} &\implies && x\cdot\ln 5.1 = (2x+6)\cdot\ln 2.7 \\ &\implies && x\cdot\ln 5.1 = 2x\cdot\ln 2.7 + 6\cdot\ln 2.7 \\ &\implies && x\cdot\ln 5.1 - 2x\cdot\ln 2.7 = 6\cdot\ln 2.7 \\ &\implies && x\cdot(\ln 5.1 - 2\cdot\ln 2.7) = 6\cdot\ln 2.7 \\ &\implies && x = \frac{6\cdot\ln 2.7}{\ln 5.1 - 2\cdot\ln 2.7} \approx -16.68 \end{aligned}$$

e) $17^{x-2} = 3^{x+4} \implies \ln 17^{x-2} = \ln 3^{x+4}$

$$\implies \quad (x-2)\cdot\ln 17 = (x+4)\cdot\ln 3$$

$$\begin{aligned}
&\Longrightarrow \quad x \cdot \ln 17 - 2 \cdot \ln 17 = x \cdot \ln 3 + 4 \cdot \ln 3 \\
&\Longrightarrow \quad x \cdot \ln 17 - x \cdot \ln 3 = 2 \cdot \ln 17 + 4 \cdot \ln 3 \\
&\Longrightarrow \quad x \cdot (\ln 17 - \ln 3) = 2 \cdot \ln 17 + 4 \cdot \ln 3 \\
&\Longrightarrow \quad x = \frac{2 \cdot \ln 17 + 4 \cdot \ln 3}{\ln 17 - \ln 3} \approx 5.80
\end{aligned}$$

f) $7^{2x+3} = 11^{3x-6} \implies \ln 7^{2x+3} = \ln 11^{3x-6}$

$$\begin{aligned}
&\Longrightarrow \quad (2x+3) \cdot \ln 7 = (3x-6) \cdot \ln 11 \\
&\Longrightarrow \quad 2x \cdot \ln 7 + 3 \cdot \ln 7 = 3x \cdot \ln 11 - 6 \cdot \ln 11 \\
&\Longrightarrow \quad 2x \cdot \ln 7 - 3x \cdot \ln 11 = -3 \cdot \ln 7 - 6 \cdot \ln 11 \\
&\Longrightarrow \quad x \cdot (2 \cdot \ln 7 - 3 \cdot \ln 11) = -3 \cdot \ln 7 - 6 \cdot \ln 11 \\
&\Longrightarrow \quad x = \frac{-3 \cdot \ln 7 - 6 \cdot \ln 11}{2 \cdot \ln 7 - 3 \cdot \ln 11} \approx 6.13
\end{aligned}$$

□

Before we get to specific applications of exponential functions, we pause to explain how we can identify the base b and the coefficient c of an exponential function $f(x) = c \cdot b^x$.

Note 15.4

Let $f(x) = c \cdot b^x$ be an exponential function. Then, the parameters c and b in the function f are uniquely determined by knowing the function values $f(x_1)$ and $f(x_2)$ for any two distinct inputs x_1 and x_2.

Example 15.5

Let $f(x) = c \cdot b^x$. Determine the constant c and base b under the given conditions.

a) $f(0) = 5$, $f(1) = 20$ b) $f(0) = 3$, $f(4) = 48$
c) $f(2) = 160$, $f(7) = 5$ d) $f(-2) = 55$, $f(1) = 7$

Solution.

a) Applying $f(0) = 5$ to $f(x) = c \cdot b^x$, we get

$$5 = f(0) = c \cdot b^0 = c \cdot 1 = c$$

Indeed, in general, we always have $f(0) = c$ for any exponential function. The base b is then determined by substituting the second equation $f(1) = 20$.

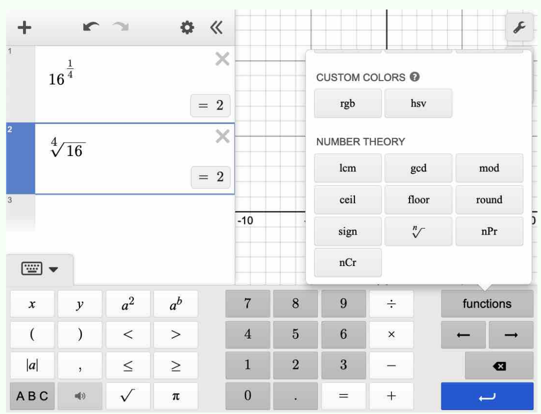
$$20 = f(1) = c \cdot b^1 = 5 \cdot b \quad \overset{(\div 5)}{\Longrightarrow} \quad b = 4$$

Therefore, $f(x) = 5 \cdot 4^x$. Note that in the last implication, we used that the base must be positive.

b) As before, we get $3 = f(0) = c \cdot b^0 = c$, and

$$48 = f(4) = c \cdot b^4 = 3 \cdot b^4 \quad \overset{(\div 3)}{\Longrightarrow} \quad 16 = b^4$$
$$\overset{(\text{exponentiate by } \frac{1}{4})}{\Longrightarrow} \quad b = 16^{\frac{1}{4}} = 2$$

Recall that $\sqrt[4]{a} = a^{\frac{1}{4}}$, and so the 4th root can be calculated with the graphing calculator either via the exponent $\frac{1}{4}$ or via the 4th root.



Therefore, $f(x) = 3 \cdot 2^x$.

c) When $f(0)$ is not given, it is easiest to solve for b first. We can see this as follows. Since $160 = f(2) = c \cdot b^2$ and $5 = f(7) = c \cdot b^7$, the quotient of these equations eliminates c.

$$\frac{160}{5} = \frac{c \cdot b^2}{c \cdot b^7} = \frac{1}{b^5} \quad \Longrightarrow \quad 32 = b^{-5}$$
$$\overset{(\text{exponentiate by } (-\frac{1}{5}))}{\Longrightarrow} \quad b = 32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}} = \frac{1}{2}$$

Then c is determined by any of the original equations.

$$160 = f(2) = c \cdot b^2 = c \cdot \left(\frac{1}{2}\right)^2 = c \cdot \frac{1}{4} \quad \Longrightarrow \quad c = 4 \cdot 160 = 640$$

Therefore, $f(x) = 640 \cdot \left(\frac{1}{2}\right)^x$.

d) This solution is similar to part (c).

$$\frac{55}{7} = \frac{f(-2)}{f(1)} = \frac{c \cdot b^{-2}}{c \cdot b^1} = \frac{1}{b^3} \quad \Longrightarrow \quad b^3 = \frac{7}{55}$$
$$\Longrightarrow \quad b = \left(\frac{7}{55}\right)^{\frac{1}{3}} \approx 0.503$$

$$55 = f(-2) = c \cdot b^{-2} = c \cdot \left(\left(\frac{7}{55}\right)^{\frac{1}{3}}\right)^{-2} = c \cdot \left(\frac{7}{55}\right)^{\frac{-2}{3}}$$
$$\Longrightarrow \quad c = \frac{55}{\left(\frac{7}{55}\right)^{\frac{-2}{3}}} = 55 \cdot \left(\frac{7^{\frac{2}{3}}}{55^{\frac{2}{3}}}\right)$$
$$= 55^{\frac{1}{3}} \cdot 7^{\frac{2}{3}} = \sqrt[3]{55 \cdot 7^2} = \sqrt[3]{2695} \approx 13.916$$

Therefore, $f(x) = \sqrt[3]{2695} \cdot \left(\sqrt[3]{\frac{7}{55}}\right)^x$.

□

15.2 Applications of exponential functions

Exponential functions express situations where the growths of a quantity is proportional to the amount of the quantity at a given time. This makes exponential functions an important toy model for many applications. In this text we will use exponential functions to model the following:

- population growths or decline
- compound interest on an investment
- radioactive decay

In this section we will focus on population growth and decline, and we will study compound interest and radioacitve decay in the next chapter.

Example 15.6

The mass of a bacteria sample is $2 \cdot 1.02^t$ grams after t hours.

a) What is the mass of the bacteria sample after 4 hours?
b) When will the mass reach 10 grams?

Solution.

a) The formula for the mass y in grams after t hours is $y(t) = 2 \cdot 1.02^t$. Therefore, after 4 hours, the mass in grams is:

$$y(4) = 2 \cdot 1.02^4 \approx 2.16$$

b) We are seeking the number of hours t for which $y = 10$ grams. Therefore, we have to solve:

$$10 = 2 \cdot 1.02^t \quad \overset{(\div 2)}{\Longrightarrow} \quad 5 = 1.02^t$$

We need to solve for the variable in the exponent. In general, to solve for a variable in the exponent requires an application of a logarithm on both sides of the equation.

$$5 = 1.02^t \quad \overset{\text{(apply log)}}{\Longrightarrow} \quad \log(5) = \log(1.02^t)$$

Recall an important property that we can use to solve for t:

$$\log(x^t) = t \cdot \log(x) \tag{15.1}$$

Using (15.1), we can now solve for t as follows:

$$\begin{aligned} \log(5) = \log(1.02^t) \quad &\Longrightarrow \quad \log(5) = t \cdot \log(1.02) \\ &\overset{\text{(divide by }\log(1.02))}{\Longrightarrow} \quad t = \frac{\log(5)}{\log(1.02)} \approx 81.3 \end{aligned}$$

After approximately 81.3 hours, the mass will be 10 grams.

□

Example 15.7

The population size of a country was 12.7 million in the year 2010, and 14.3 million in the year 2020.

a) Assuming an exponential growth for the population size, find the formula for the population depending on the year t.

b) What will the population size be in the year 2025, assuming the formula holds until then?

c) When will the population reach 18 million?

Solution.

a) The growth is assumed to be exponential, so that $y(t) = c \cdot b^t$ describes the population size depending on the year t, where we set $t = 0$ corresponding to the year 2010. Then the example describes $y(0) = c$ as $c = 12.7$, which we assume in units of millions of people. To find the base b, we substitute the data of $t = 10$ and $y(t) = 14.3$ into $y(t) = c \cdot b^t$.

$$14.3 = 12.7 \cdot b^{10} \quad \Longrightarrow \quad \frac{14.3}{12.7} = b^{10} \quad \Longrightarrow \quad \left(\frac{14.3}{12.7}\right)^{\frac{1}{10}} = (b^{10})^{\frac{1}{10}} = b$$
$$\Longrightarrow \quad b = \left(\frac{14.3}{12.7}\right)^{\frac{1}{10}} \approx 1.012$$

The formula for the population size is $y(t) \approx 12.7 \cdot 1.012^t$.

b) We calculate the population size in the year 2025 by setting $t = 2025 - 2010 = 15$:

$$y(15) = 12.7 \cdot 1.012^{15} \approx 15.2$$

c) We seek t so that $y(t) = 18$. We solve for t using the logarithm.

$$18 = 12.7 \cdot 1.012^t \quad \Longrightarrow \quad \frac{18}{12.7} = 1.012^t$$
$$\Longrightarrow \quad \log\left(\frac{18}{12.7}\right) = \log(1.012^t)$$

$$\implies \quad \log\left(\frac{18}{12.7}\right) = t \cdot \log(1.012)$$

$$\implies \quad t = \frac{\log\left(\frac{18}{12.7}\right)}{\log(1.012)} \approx 29.2$$

Adding 29.2 years to the year 2010, we see that the population will reach 18 million in the year 2039.

□

In many instances the exponential function $f(x) = c \cdot b^x$ is given via a rate of growth r.

Definition 15.8: Rate of growth

An exponential function with a **rate of growth** r is a function $f(x) = c \cdot b^x$ with base

$$\boxed{b = e^r}$$

Note 15.9

Some textbooks use a different convention than the one given in Definition 15.8 for the rate of growth. Indeed, sometimes a function with rate of growth r is defined as an exponential function with base $b = 1 + r$, whereas we use a base $b = e^r$. Since e^r can be expanded as $e^r = 1 + r + \frac{r^2}{2} + \dots$, this shows that the two versions only vary by a difference of order 2 (that is they differ by $\frac{r^2}{2}$ plus higher powers of r), and so, for small r, the base $1 + r$ and the base e^r are approximately equal.

Example 15.10

The number of PCs that are sold in the US in the year 2021 is approximately 350 million. Assuming that the number grows exponentially at a constant rate of 3.6% per year, how many PCs will be sold in the year 2027?

Solution.

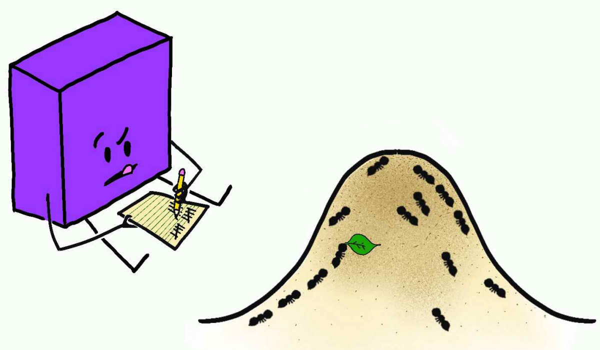
Since the rate of growth is $r = 3.6\% = 0.036$, we obtain a base of $b = e^r = e^{0.036}$. Therefore, we will model the number of PCs sold (in millions of PCs) by the function $y(t) = c \cdot (e^{0.036})^t = c \cdot e^{0.036 \cdot t}$. If we set $t = 0$ for the year 2021, we find that $c = 350$, so the number of sales is given by $y(t) = 350 \cdot e^{0.036 \cdot t}$. Since the year 2027 corresponds to $t = 2027 - 2021 = 6$, we can calculate the number of sales in the year 2027 as

$$y(4) = 350 \cdot e^{0.036} \approx 434.$$

Approximately 434 million PCs will be sold in the year 2027. □

Example 15.11

The size of an ant colony is decreasing at a rate of 1% per month. How long will it take until the colony has reached 80% of its original size?



Solution.

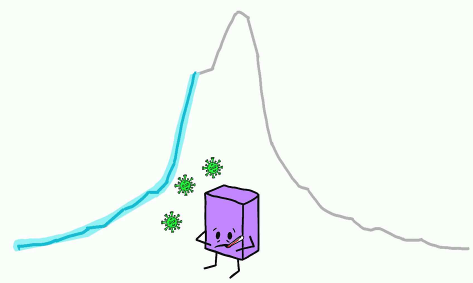
Since the population size is decreasing, the rate is negative, that is $r = -1\% = -0.01$. We therefore obtain the base $b = e^r = e^{-0.01}$. We have a colony size of $y(t) = c \cdot e^{-0.01 \cdot t}$ after t months, where c is the original size. We need to find t so that the size is at 80% of its original size c, that is, $y(t) = 80\% \cdot c = 0.8 \cdot c$.

$$\begin{aligned} 0.8 \cdot c = c \cdot e^{-0.01 \cdot t} \quad &\overset{(\div c)}{\Longrightarrow} \quad 0.8 = e^{-0.01 \cdot t} \\ &\Longrightarrow \quad \ln(0.8) = \ln(e^{-0.01 \cdot t}) \\ &\Longrightarrow \quad \ln(0.8) = -0.01 \cdot t \cdot \underbrace{\ln(e)}_{=1} \\ &\Longrightarrow \quad t = \frac{\ln(0.8)}{-0.01} \approx 22.3 \end{aligned}$$

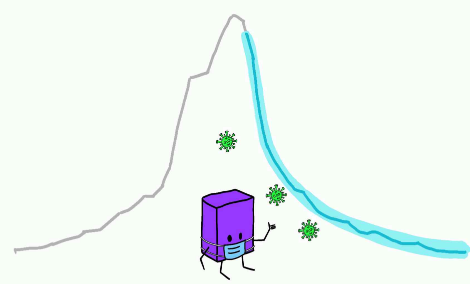
After approximately 22.3 months, the ant colony has decreased to 80% of its original size. □

Example 15.12

a) The number of flu cases in the fall was increasing at a rate of 9.8% per week. How long did it take for the number of flu cases to double?



b) The number of flu cases in the spring was decreasing at a rate of 15% per week. How long did it take for the number of flu cases to decrease to a quarter of its size?



Solution.

a) The rate of change is $r = 9.8\% = 0.098$ per week, so that the number of flu cases is an exponential function with base $b = e^{0.098}$. Therefore, $f(x) = c \cdot e^{0.098 \cdot x}$ denotes the number of flu cases, with c being the initial number of cases at the time corresponding to $x = 0$. In order for the number of flu cases to double, $f(x)$ has to reach twice its initial size, that is:

$$\begin{aligned} f(x) = 2c \quad &\implies \quad 2c = c \cdot e^{0.098 \cdot x} \\ &\overset{(\div c)}{\implies} \quad 2 = e^{0.098 \cdot x} \end{aligned}$$

$$\Longrightarrow \quad \ln(2) = \ln(e^{0.098\cdot x})$$
$$\Longrightarrow \quad \ln(2) = 0.098 \cdot x \ln(e)$$
$$\Longrightarrow \quad x = \frac{\ln(2)}{0.098} \approx 7.07$$

Therefore, it took about 7.07 weeks until the number of flu cases doubled.

b) Since the number of flu cases was decreasing, the rate of growth is negative, $r = -15\% = -0.15$ per week, so that we have an exponential function with base $b = e^r = e^{-0.15}$. To reach a quarter of its initial number of flu cases, we set $f(x) = c \cdot e^{-0.15\cdot x}$ equal to $\frac{1}{4}c$.

$$\frac{1}{4}c = c \cdot e^{-0.15\cdot x} \quad \overset{(\div c)}{\Longrightarrow} \quad \frac{1}{4} = e^{-0.15\cdot x}$$
$$\Longrightarrow \quad \ln(\frac{1}{4}) = -0.15 \cdot x \cdot \ln(e)$$
$$\Longrightarrow \quad x = \frac{\ln(\frac{1}{4})}{-0.15} \approx 9.24$$

It therefore took about 9.24 weeks until the number of flu cases decreased to a quarter.

□

15.3 Exercises

Exercise 15.1

Solve for x without using a calculator.

a)	$6^{x-2} = 36$	b)	$2^{3x-8} = 16$
c)	$10^{5-x} = 0.0001$	d)	$5^{5x+7} = \frac{1}{125}$
e)	$2^x = 64^{x+1}$	f)	$4^{x+3} = 32^x$
g)	$13^{4+2x} = 1$	h)	$3^{x+2} = 27^{x-3}$
i)	$25^{7x-4} = 5^{2-3x}$	j)	$9^{5+3x} = 27^{8-2x}$

Exercise 15.2

Solve for x. First find the exact answer as an expression involving logarithms. Then approximate the answer to the nearest hundredth using a calculator.

a) $4^x = 57$	b) $9^{x-2} = 7$	c) $2^{x+1} = 31$
d) $3.8^{2x+7} = 63$	e) $5^{x+5} = 8^x$	f) $3^{x+2} = 0.4^x$
g) $1.02^{2x-9} = 4.35^x$	h) $4^{x+1} = 5^{x+2}$	i) $9^{3-x} = 4^{x-6}$
j) $2.4^{7-2x} = 3.8^{3x+4}$	k) $4^{9x-2} = 9^{2x-4}$	l) $1.95^{-3x-4} = 1.2^{4-7x}$

Exercise 15.3

Assuming that $f(x) = c \cdot b^x$ is an exponential function, find the constants c and b from the given conditions.

a) $f(0) = 4$, $f(1) = 12$	b) $f(0) = 5$, $f(3) = 40$
c) $f(0) = 3200$, $f(6) = 0.0032$	d) $f(3) = 12$, $f(5) = 48$
e) $f(-1) = 4$, $f(2) = 500$	f) $f(2) = 3$, $f(4) = 15$

Exercise 15.4

The number of downloads of a certain software application was 8.4 million in the year 2017 and 13.6 million in the year 2022.

a) Assuming an exponential growth for the number of downloads, find the formula for the downloads depending on the year t.

b) Assuming the number of downloads will follow the formula from part (a), what will the number of downloads be in the year 2026?

c) In what year will the number of downloaded applications reach the 25 million barrier?

Exercise 15.5

The population size of a city was $79,000$ in the year 1998 and $136,000$ in the year 2013. Assume that the population size follows an exponential function.

a) Find the formula for the population size.

b) What is the population size in the year 2030?

c) What is the population size in the year 2035?

d) When will the city reach its expected maximum capacity of one million residents?

Exercise 15.6

The population of a city decreases at a rate of 2.3% per year. After how many years will the population be at 90% of its current size? Round your answer to the nearest tenth.

Exercise 15.7

A big company plans to expand its franchise and, with this, its number of employees. For tax reasons, it is most beneficial to expand the number of employees at a rate of 5% per year. If the company currently has 4730 employees, how many years will it take until the company has 6000 employees? Round your answer to the nearest hundredth.

Exercise 15.8

An ant colony has a population size of 4000 ants and is increasing at a rate of 3% per week. How long will it take until the ant population has doubled? Round your answer to the nearest tenth.

Exercise 15.9

The size of a beehive is decreasing at a rate of 15% per month. How long will it take for the beehive to be at half of its current size? Round your answer to the nearest hundredth.

Exercise 15.10

If the population size of the world is increasing at a rate of 0.5% per year, how long does it take until the world population doubles in size? Round your answer to the nearest tenth.

Chapter 16

More applications: Compound interest and half-life

We have already encountered some applications of exponential functions in Section 15.2. In this chapter we give two more applications that come from finance (computing compound interest) and from physics (radioactive decay).

16.1 Compound interest

An important application of the exponential function is given by calculating the interest and the current value of an investment. We start with a motivating example in the following note.

Note 16.1

- We invest an initial amount of $P = \$500$ for 1 year at a rate of $r = 6\%$. The initial amount P is also called the **principal**.

  After 1 year, we receive the principal P together with the interest $r \cdot P$ generated from the principal. The final amount A after 1 year is therefore

  $$A = \$500 + 6\% \cdot \$500 = \$500 \cdot (1 + 0.06) = \$530.$$

- We change the setup of the previous example by taking a quarterly compounding. This means that instead of receiving interest on the principal once at the end of the year, we receive the interest 4 times within the year (after each quarter). However, we now receive only $\frac{1}{4}$ of the interest rate of 6%. We break down the amount received after each quarter.

$$\begin{aligned}
\text{after first quarter: } & 500 \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015 \\
\text{after second quarter: } & (500 \cdot 1.015) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^2 \\
\text{after third quarter: } & (500 \cdot 1.015^2) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^3 \\
\text{after fourth quarter: } & (500 \cdot 1.015^3) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^4 \\
& \implies A \approx 530.68
\end{aligned}$$

Note that in the second quarter, we receive interest on the amount we had after the first quarter, and so on. So, in fact, we keep receiving interest on the interest of the interest, etc. For this reason, the final amount received after 1 year $A = \$530.68$ is slightly higher when compounded quarterly than when compounded annually (where $A = \$530.00$).

- We make yet another variation to the above setup. Instead of investing money for 1 year, we invest the principal for 10 years at a quarterly compounding. We then receive interest every quarter for a total of $4 \cdot 10 = 40$ quarters.

$$\begin{aligned}
\text{after first quarter: } & 500 \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015 \\
\text{after second quarter: } & (500 \cdot 1.015) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^2 \\
\text{after third quarter: } & (500 \cdot 1.015^2) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^3 \\
& \vdots \\
\text{after fortieth quarter: } & (500 \cdot 1.015^{39}) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^{40} \\
& \implies A \approx 907.01
\end{aligned}$$

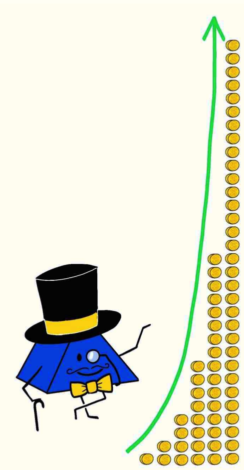
We state our observations from the previous example in the following general observation.

Observation 16.2: Value of an investment compounded n times

A principal (=initial amount) P is invested for t years at a rate r and compounded n times per year. The final amount A is given by

$$\boxed{A = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}} \quad \text{where}$$

$$\begin{cases} P = \text{principal (=initial) amount} \\ A = \text{final amount} \\ r = \text{annual interest rate} \\ n = \text{number of} \\ \qquad \text{compounding periods per year} \\ t = \text{number of years} \end{cases}$$



We can consider performing the compounding in smaller and smaller time intervals. Instead of quarterly compounding, we may take monthly compounding, or daily, hourly, secondly compounding or compounding in even smaller time intervals. Note that, in this case, the number of compounding periods n in the above formula tends to infinity. In the limit when the time intervals go to zero, we obtain what is called *continuous compounding*.

Observation 16.3: Value of an investment compounded continuously

A principal amount P is invested for t years at a rate r and with **continuous compounding**. The final amount A is given by

$$\boxed{A = P \cdot e^{r \cdot t}} \quad \text{where} \quad \begin{cases} P = \text{principal amount} \\ A = \text{final amount} \\ r = \text{annual interest rate} \\ t = \text{number of years} \end{cases}$$

Note 16.4

The reason the exponential function appears in the above formula is that the exponential is the limit of the previous formula in Observation 16.2, when n approaches infinity; compare this with Equation (13.1) on page 237.

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

A more detailed discussion of limits will be provided in a calculus course.

Example 16.5

Determine the final amount received on an investment under the given conditions.

a)	$\$700$,	compounded monthly,	at 4%,	for 3 years
b)	$\$2500$,	compounded semi-annually,	at 5.5%,	for 6 years
c)	$\$1200$,	compounded continuously,	at 3%,	for 2 years

Solution.

a) We can immediately apply the formula in which we substitute the given values of $P = 700$, $n = 12$ (because "monthly" means compounded 12 times per year), $r = 4\% = 0.04$, and $t = 3$. Therefore, we calculate

$$A = 700 \cdot \left(1 + \frac{0.04}{12}\right)^{12\cdot 3} = 700 \cdot \left(1 + \frac{0.04}{12}\right)^{36} \approx 789.09$$

b) We have $P = 2500$, $n = 2$, $r = 5.5\% = 0.055$, and $t = 6$.

$$A = 2500 \cdot \left(1 + \frac{0.055}{2}\right)^{2\cdot 6} \approx 3461.96$$

c) We have $P = 1200$, $r = 3\% = 0.03$, $t = 2$, and we use the formula for continuous compounding.

$$A = 1200 \cdot e^{0.03\cdot 2} = 1200 \cdot e^{0.06} \approx 1274.20$$

□

Instead of asking to find the final amount, we may also ask about any of the other variables in the above formulas for investments.

Example 16.6

a) Find the amount P that needs to be invested at 4.275% compounded annually for 5 years to give a final amount of $\$3000$. (This amount P is also called the **present value** of the future amount of $\$3000$ in 5 years.)

b) At what rate do we have to invest $\$800$ for 6 years compounded quarterly to obtain a final amount of $\$1200$?

c) For how long do we have to invest $\$1000$ at a rate of 2.5% compounded continuously to obtain a final amount of $\$1100$?

d) For how long do we have to invest at a rate of 3.2% compounded monthly until the investment doubles its value?

Solution.

a) We have the following data: $r = 4.275\% = 0.04275$, $n = 1$, $t = 5$, and $A = 3000$. We want to find the present value P. Substituting the given numbers into the appropriate formula, we can solve this for P.

$$3000 = P \cdot \left(1 + \frac{0.04275}{1}\right)^{1\cdot 5} \implies 3000 = P \cdot (1.04275)^5$$

$$\overset{\text{(divide by } 1.04275^5)}{\implies} P = \frac{3000}{1.04275^5} \approx 2433.44$$

Therefore, if we invest $\$2433.44$ today under the given conditions, then this will be worth $\$3000$ in 5 years.

b) Substituting the given numbers ($P = 800$, $t = 6$, $n = 4$, $A = 1200$) into the formula gives:

$$1200 = 800 \cdot \left(1 + \frac{r}{4}\right)^{4\cdot 6} \overset{\text{(divide by 800)}}{\implies} \frac{1200}{800} = \left(1 + \frac{r}{4}\right)^{24}$$

$$\implies \left(1 + \frac{r}{4}\right)^{24} = \frac{3}{2}$$

Next, we have to get the exponent 24 to the right side. This is done by taking a power of $\frac{1}{24}$, or in other words, by taking the 24th root, $\sqrt[24]{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{1}{24}}$.

$$\left(\left(1+\frac{r}{4}\right)^{24}\right)^{\frac{1}{24}} = \left(\frac{3}{2}\right)^{\frac{1}{24}} \quad\Longrightarrow\quad \left(1+\frac{r}{4}\right)^{24\cdot\frac{1}{24}} = \left(\frac{3}{2}\right)^{\frac{1}{24}}$$
$$\Longrightarrow\quad 1+\frac{r}{4} = \left(\frac{3}{2}\right)^{\frac{1}{24}} \quad\Longrightarrow\quad \frac{r}{4} = \left(\frac{3}{2}\right)^{\frac{1}{24}} - 1$$
$$\Longrightarrow\quad r = 4\cdot\left(\left(\frac{3}{2}\right)^{\frac{1}{24}} - 1\right)$$

Plugging this into the calculator gives $r \approx 0.06815 = 6.815\%$. Therefore, the rate should be about 6.815%.

c) Again, we substitute the given values, $P = 1000$, $r = 2.5\% = 0.025$, $A = 1100$, but now we use the formula for continuous compounding.

$$1100 = 1000\cdot e^{0.025\cdot t} \quad\Longrightarrow\quad \frac{1100}{1000} = e^{0.025\cdot t} \quad\Longrightarrow\quad e^{0.025\cdot t} = 1.1$$

To solve for the variable t in the exponent, we need to apply the logarithm. Here, it is most convenient to apply the natural logarithm, because $\ln(x)$ is the inverse of the exponential e^x with base e. Thus, by applying $\ln$ to both sides, we see that

$$\ln(e^{0.025\cdot t}) = \ln(1.1) \quad\Longrightarrow\quad 0.025\cdot t\cdot\ln(e) = \ln(1.1)$$

Note that we have used that $\log_b(x^n) = n\cdot\log_b(x)$ for any number n as we have seen in Proposition 14.2. Using that $\ln(e) = 1$ (which is the special case of the second equation in (13.3) on page 243 for the base $b = e$), the above becomes

$$0.025\cdot t = \ln(1.1) \quad\Longrightarrow\quad t = \frac{\ln(1.1)}{0.025} \approx 3.81$$

Therefore, we have to wait 4 years until the investment is worth (more than) $\$1100$.

d) We are given that $r = 3.2\% = 0.032$ and $n = 12$, but no initial amount P is provided. We are seeking to find the time t when the investment doubles. This means that the final amount A is twice the initial amount P, or as a formula: $A = 2\cdot P$. Substituting this into the investment formula and solving gives the wanted answer.

$$2P = P \cdot \left(1 + \frac{0.032}{12}\right)^{12\cdot t} \quad \overset{\text{(divide by } P)}{\Longrightarrow} \quad 2 = \left(1 + \frac{0.032}{12}\right)^{12\cdot t}$$

$$\overset{\text{(apply ln)}}{\Longrightarrow} \quad \ln(2) = \ln\left(\left(1 + \frac{0.032}{12}\right)^{12\cdot t}\right)$$

$$\Longrightarrow \quad \ln(2) = 12 \cdot t \cdot \ln\left(1 + \frac{0.032}{12}\right)$$

$$\overset{\text{(divide by } 12\cdot\ln\left(1+\frac{0.032}{12}\right))}{\Longrightarrow} \quad t = \frac{\ln(2)}{12 \cdot \ln\left(1 + \frac{0.032}{12}\right)} \approx 21.69$$

So, after approximately 21.69 years, the investment will have doubled in value.

□

16.2 Half-life

Recall from Definition 15.8 on page 270 that a function with rate of growth r is an exponential function $f(x) = c \cdot b^x$ with base $b = e^r$. Instead of using the rate of growth, there are other ways to specify the base of an exponential function. One way to specify the base is given by the notion of *half-life*. We give a motivating example in the following note.

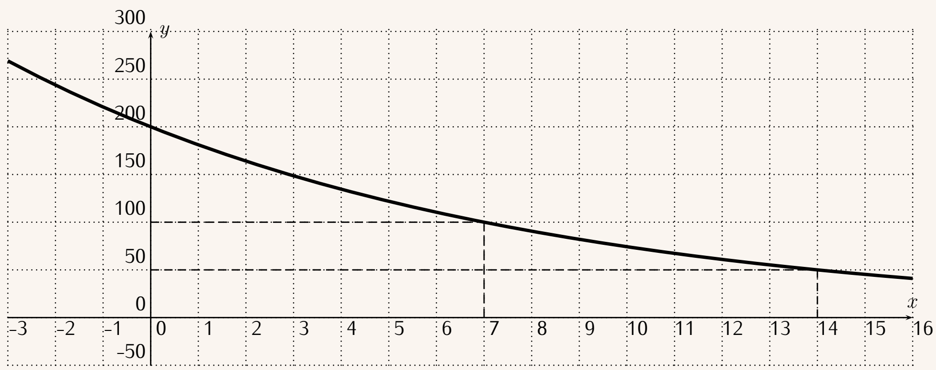
Note 16.7

Consider the function $f(x) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{x}{7}}$. We calculate the function values $f(x)$, for $x = 0$, 7, 14, 21, and 28.

$$\begin{aligned}
f(0) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{0}{7}} = 200 \cdot 1 = 200 \\
f(7) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{7}{7}} = 200 \cdot \frac{1}{2} = 100 \\
f(14) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{14}{7}} = 200 \cdot \frac{1}{4} = 50
\end{aligned}$$

$$f(21) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{21}{7}} = 200 \cdot \frac{1}{8} = 25$$
$$f(28) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{28}{7}} = 200 \cdot \frac{1}{16} = 12.5$$

From this calculation, we can see how the function values of f behave: starting from $f(0) = 200$, the function takes half of its value whenever x is increased by 7. For this reason, we say that f has a *half-life* of 7. (The general definition will be given below.) The graph of the function is displayed below.



We collect the ideas that are displayed in the above example in the definition and observation below.

Definition 16.8: Half-life

Let f be an exponential function $f(x) = c \cdot b^x$ with a domain of all real numbers, $D = \mathbb{R}$. Then we say that f has a **half-life** of h if the base is given by

$$\boxed{b = \left(\frac{1}{2}\right)^{\frac{1}{h}}} \tag{16.1}$$

Note that we can also write h in terms of b. Converting (16.1) into a logarithmic equation gives $\frac{1}{h} = \log_{\frac{1}{2}}(b) = \frac{\log b}{\log \frac{1}{2}}$, so that $h = \frac{\log \frac{1}{2}}{\log b} = \log_b\left(\frac{1}{2}\right)$.

Observation 16.9: Graphical interpretation of half-life

Let f be the exponential function given for some real constants $c > 0$ and half-life $h > 0$, that is

$$f(x) = c \cdot \left(\left(\frac{1}{2}\right)^{\frac{1}{h}}\right)^{x} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}}.$$

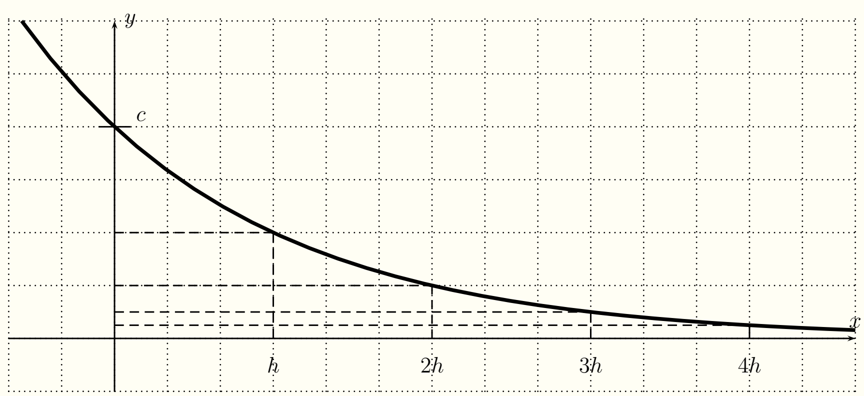
Then we can calculate $f(x+h)$ as follows:

$$\begin{aligned} f(x+h) &= c \cdot \left(\frac{1}{2}\right)^{\frac{x+h}{h}} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}+\frac{h}{h}} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}+1} \\ &= c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}} \cdot \left(\frac{1}{2}\right)^{1} = \frac{1}{2} \cdot f(x) \end{aligned}$$

To summarize, f has the following property:

$$\boxed{f(x+h) = \frac{1}{2} f(x)} \qquad \text{for all } x \in \mathbb{R}. \tag{16.2}$$

The above equation shows that whenever we add an amount of h to an input x, the effect on f is that the function value decreases by half its previous value. This is also displayed in the graph below.



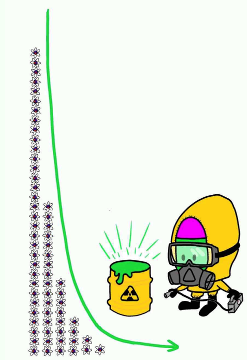
We will sometimes use a different letter for the input variable. In particular, the function $f(x) = c \cdot (\frac{1}{2})^{\frac{x}{h}}$ is the same as the function $f(t) = c \cdot (\frac{1}{2})^{\frac{t}{h}}$.

Many radioactive isotopes decay with well-known half-lives.[1]

Example 16.10

a) Chromium-51 has a half-life of 27.7 days. How much of 3 grams of chromium-51 will remain after 90 days?

b) An isotope decays within 20 hours from 5 grams to 2.17 grams. Find the half-life of the isotope.



Solution.

a) We use the above formula $y = c\cdot\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where $c = 3$ grams is the initial amount of chromium-51, $h = 27.7$ days is the half-life of chromium-51, and $t = 90$ days is time that the isotope decayed. Substituting these numbers into the formula for y, we obtain:

$$y = 3 \cdot \left(\frac{1}{2}\right)^{\frac{90}{27.7}} \approx 0.316$$

Therefore, after 90 days, 0.316 grams of the chromium-51 remains.

b) We have an initial amount of $c = 5$ grams and a remaining amount of $y = 2.17$ grams after $t = 20$ hours. The half-life can be obtained as follows.

$$\begin{aligned} 2.17 = 5 \cdot \left(\frac{1}{2}\right)^{\frac{20}{h}} \quad &\overset{(\div 5)}{\Longrightarrow} \quad 0.434 = \left(\frac{1}{2}\right)^{\frac{20}{h}} \\ &\overset{(\text{apply } \ln)}{\Longrightarrow} \quad \ln(0.434) = \ln\left(0.5^{\frac{20}{h}}\right) \\ &\Longrightarrow \quad \ln(0.434) = \frac{20}{h} \cdot \ln(0.5) \\ &\overset{(\times \frac{h}{\ln(0.434)})}{\Longrightarrow} \quad h = \frac{20 \cdot \ln(0.5)}{\ln(0.434)} \approx 16.6 \end{aligned}$$

Therefore, the half-life of the isotope is approximately 16.6 hours.

□

[1] Half-lives are taken from http://en.wikipedia.org/wiki/List_of_radioactive_nuclides_by_half-life

Note 16.11: Half-life of carbon-14

An important isotope is the radioisotope carbon-14. It decays with a half-life of 5730 years with an accuracy of ± 40 years. For definiteness we will take 5730 years as the half-life of carbon-14.

The half-life of carbon-14 is 5730 years.

One can use the knowledge of the half-life of carbon-14 in dating organic materials via the so called **carbon dating method**. Carbon-14 is produced by a plant during the process of photosynthesis at a fixed level until the plant dies. Therefore, by measuring the remaining amount of carbon-14 in a dead plant, one can determine the date when the plant died. Furthermore, since humans and animals consume plants, the same argument can be applied to determine their (approximate) dates of death.

Example 16.12

a) A dead tree trunk has 86% of its original carbon-14. (Approximately) how many years ago did the tree die?

b) A dead animal at an archeological site has lost 41.3% of its carbon-14. When did the animal die?

Solution.

a) Using the function $y = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$, where c is the amount of carbon-14 that was produced by the tree until it died, y is the remaining amount to date, t is the time that has passed since the tree has died, and h is the half-life of carbon-14. Since 86% of the carbon-14 is left, we have $y = 86\% \cdot c$. Substituting the half-life $h = 5730$ of carbon-14, we can solve for t.

$$0.86 \cdot c = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} \quad \overset{(\div c)}{\Longrightarrow} \quad 0.86 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\overset{\text{(apply ln)}}{\Longrightarrow} \quad \ln(0.86) = \ln\left(0.5^{\frac{t}{5730}}\right)$$

$$\Longrightarrow \quad \ln(0.86) = \frac{t}{5730} \cdot \ln(0.5)$$

$$\overset{(\times \frac{5730}{\ln(0.5)})}{\Longrightarrow} \quad \frac{5730}{\ln(0.5)} \cdot \ln(0.86) = t$$
$$\Longrightarrow \quad t \approx 1247$$

Therefore, the tree died approximately 1247 years ago.

b) Since 41.3% of the carbon-14 is gone, $100\% - 41.3\% = 58.7\%$ remains. Using $y = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$ with $y = 58.7\% \cdot c$ and $h = 5730$, we obtain

$$0.587 \cdot c = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} \quad \overset{(\div c)}{\Longrightarrow} \quad 0.587 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$\overset{(\text{apply } \ln)}{\Longrightarrow} \quad \ln(0.587) = \ln\left(0.5^{\frac{t}{5730}}\right)$$
$$\Longrightarrow \quad \ln(0.587) = \frac{t}{5730} \cdot \ln(0.5)$$
$$\overset{(\times \frac{5730}{\ln(0.5)})}{\Longrightarrow} \quad \frac{5730}{\ln(0.5)} \cdot \ln(0.587) = t$$
$$\Longrightarrow \quad t \approx 4404$$

The animal died 4404 years ago.

□

16.3 Exercises

Exercise 16.1

An investment of $\$5000$ was locked in for 30 years. According to the agreed-upon conditions, the investment will be worth $\$5000 \cdot 1.08^t$ after t years.

a) How much is the investment worth after 5 years?
b) After how many years will the investment be worth $\$20,000$?

Exercise 16.2

Determine the final amount in a savings account under the given conditions.

a)	$\$700$,	compounded quarterly,	at 3%,	for 7 years
b)	$\$1400$,	compounded annually,	at 2.25%,	for 5 years
c)	$\$1400$,	compounded continuously,	at 2.25%,	for 5 years
d)	$\$500$,	compounded monthly,	at 3.99%,	for 2 years
e)	$\$5000$,	compounded continuously,	at 7.4%,	for 3 years
f)	$\$1600$,	compounded daily,	at 3.333%,	for 1 year
g)	$\$750$,	compounded semi-annually,	at 4.9%,	for 4 years

Exercise 16.3

a) Find the amount P that needs to be invested at a rate of 5% compounded quarterly for 6 years to give a final amount of $\$2000$.

b) Find the present value P of a future amount of $A = \$3500$ invested at 6% compounded annually for 3 years.

c) Find the present value P of a future amount of $\$1000$ invested at a rate of 4.9% compounded continuously for 7 years.

d) At what rate do we have to invest $\$1900$ for 4 years compounded monthly to obtain a final amount of $\$2250$?

e) At what rate do we have to invest $\$1300$ for 10 years compounded continuously to obtain a final amount of $\$2000$?

f) For how long do we have to invest $\$3400$ at a rate of 5.125% compounded annually to obtain a final amount of $\$3700$?

g) For how long do we have to invest $\$1000$ at a rate of 2.5% compounded continuously to obtain a final amount of $\$1100$?

h) How long do you have to invest a principal at a rate of 6.75% compounded daily until the investment doubles its value?

i) A certain amount of money has tripled its value while being in a savings account that has an interest rate of 8% compounded continuously. For how long was the money in the savings account?

Exercise 16.4

An unstable element decays at a rate of 5.9% per minute. If 40mg of this element has been produced, how long will it take until 2mg of the element are left? Round your answer to the nearest thousandth.

Exercise 16.5

A substance decays radioactively with a half-life of 232.5 days. How much of 6.8 grams of this substance is left after 1 year?

Exercise 16.6

Fermium-252 decays in 10 minutes to 76.1% of its original mass. Find the half-life of fermium-252.

Exercise 16.7

How long do you have to wait until 15mg of beryllium-7 have decayed to 4mg if the half-life of beryllium-7 is 53.12 days?

Exercise 16.8

If Pharaoh Ramses II died in the year 1213 BC, then what percent of the carbon-14 was left in the mummy of Ramses II in the year 2000?

Exercise 16.9

In order to determine the age of a piece of wood, the amount of carbon-14 was measured. It was determined that the wood had lost 33.1% of its carbon-14. How old is this piece of wood?

Exercise 16.10

Archaeologists uncovered a bone at an ancient resting ground. It was determined that 62% of the carbon-14 was left in the bone. How old is the bone?

Review of exponential and logarithmic functions

Exercise III.1

Let $f(x) = \ln(3x+7)$. Find the domain of f, the asymptote(s) of f, and the x-intercept(s). Use this information to sketch a graph of f.

Exercise III.2

Combine to an expression with only one logarithm.

a) $\dfrac{2}{3}\ln(x) + 4\ln(y)$ b) $\dfrac{1}{2}\log_2(x) - \dfrac{3}{4}\log_2(y) + 3\log_2(z)$

Exercise III.3

Assuming that $x, y > 0$, write the following expressions in terms of $u = \log(x)$ and $v = \log(y)$:

a) $\log\left(\dfrac{\sqrt[3]{x^4}}{y^2}\right)$ b) $\log\left(x\sqrt{y^5}\right)$ c) $\log\left(\sqrt[5]{xy^4}\right)$

Exercise III.4

Solve without using the calculator: $\log_3(x) + \log_3(x-8) = 2$

Exercise III.5

a) Find the exact solution of the equation: $6^{x+2} = 7^x$
b) Use the calculator to approximate your solution from part (a).

Exercise III.6

The population of a country grows exponentially at a rate of 2.6% per year. If the population was 35.7 million in the year 2020, then what is the population size of this country in the year 2027?

Exercise III.7

The number of bees in a beehive decreases exponentially at a rate of 1.4% per month. How long will it take until half of the bees are left?

Exercise III.8

How much do you have to invest today at 3% compounded quarterly to obtain $\$2000$ in return in 3 years?

Exercise III.9

45mg of fluorine-18 decay in 3 hours to 14.4mg. Find the half-life of fluorine-18.

Exercise III.10

A bone has lost 35% of its carbon-14. How old is the bone?

Part IV

Trigonometric Functions

Chapter 17

Trigonometric functions reviewed

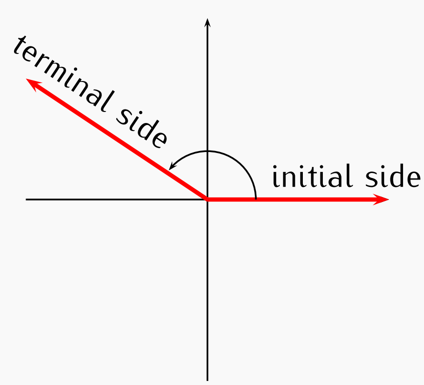
In the next chapters, we will study trigonometric functions, such as $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$ in terms of their function theoretic aspects.

17.1 Review of unit circle trigonometry

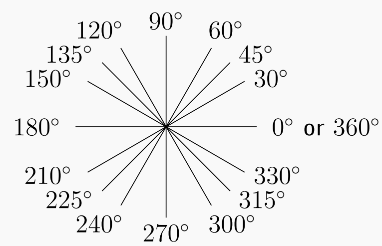
We start by reviewing some basic definitions and facts about trigonometry on the unit circle.

Review 17.1: Angle in standard position

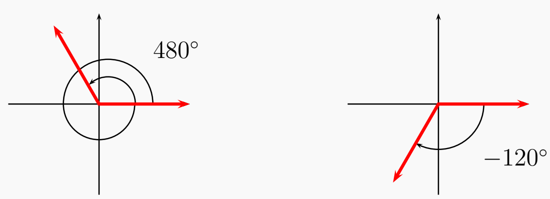
An angle in the plane is in **standard position** if its vertex is at the origin and the initial side is at the positive x-axis.



The angle is measured in the counterclockwise direction, where a full rotation measures as $360°$.



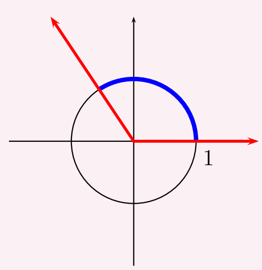
Angles greater than $360°$ will span more than a full rotation, angles less than $0°$ will rotate in the clockwise direction. Adding or subtracting $360°$ will give the same terminal side.



Besides degree measure, we will also need to use radians for the measure of an angle.

Definition 17.2: Radian

The **radian** measure of an angle is the length of the arc (shown below in blue) on the unit circle from the initial side to the terminal side.



Note that a full rotation has a radian measure of 2π, and that degrees and radians are linearly related via the conversion formula:

$$\boxed{\pi = 180°} \tag{17.1}$$

Example 17.3

Convert from radian to degree measure and vice versa.

a) $\frac{5\pi}{4}$ b) $\frac{11\pi}{6}$ c) $150°$ d) $240°$ e) $315°$

Solution.
Replace π with $180°$ and simplify as needed.

a) $\frac{5\pi}{4} = \frac{5\cdot 180°}{4} = 225°$

b) $\frac{11\pi}{6} = \frac{11\cdot 180°}{6} = 330°$

Conversely, using $180° = \pi$, we get that $1° = \frac{\pi}{180}$. With this, we have:

c) $150° = 150 \cdot \frac{\pi}{180} = \frac{150\pi}{180} = \frac{5\pi}{6}$ (cancel with 30)

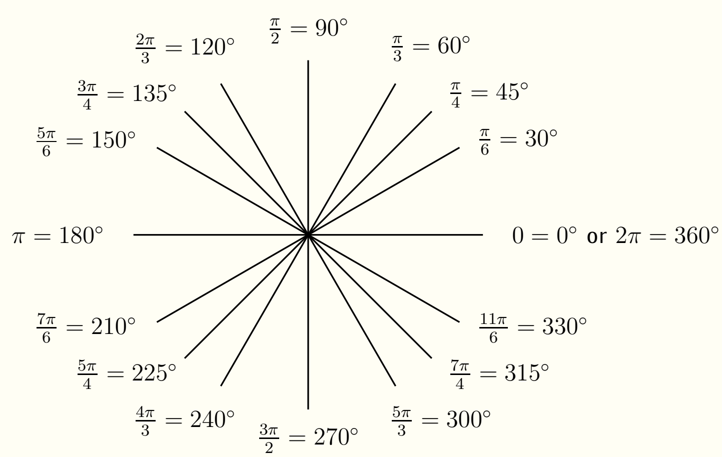
d) $240° = 240 \cdot \frac{\pi}{180} = \frac{4\pi}{3}$ (cancel with 60)

e) $315° = 315 \cdot \frac{\pi}{180} = \frac{7\pi}{4}$ (cancel with 45)

□

Observation 17.4: Radian and degree for multiples of $30°$ and $45°$

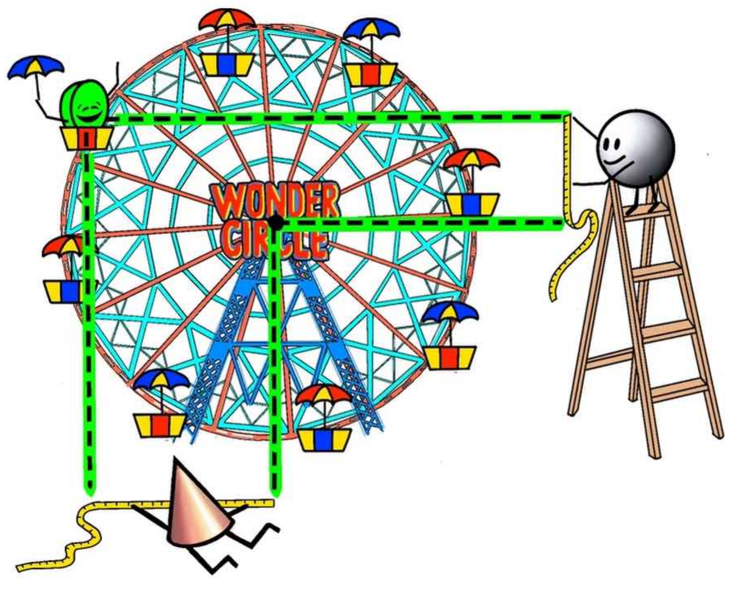
Below are all angles that are multiples of $30°$ or $45°$ between $0°$ and $360°$ in both degree and radian measure.



We next define the "sine", "cosine", and "tangent" of an angle.

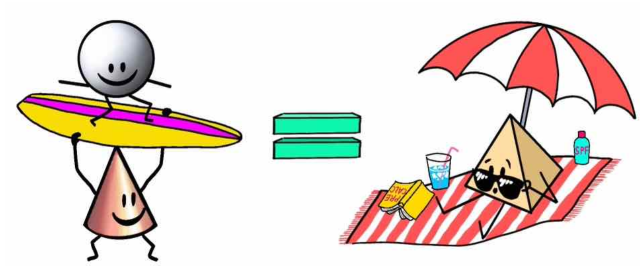
Motivation: Sine and cosine of an angle

Imagine riding in a Ferris wheel.



We may be interested in calculating the distance from the ground to a passenger car when its position around the wheel is rotated at an angle x. Alternatively, we can measure the vertical displacement of the passenger car from the center of the Ferris wheel. Note that, in the above picture, this distance is measured by the "silver sphere" figure atop the ladder. (We think of placing a coordinate system so that the center of the Ferris wheel is the origin $(0, 0)$ of the coordinate system.) If the Ferris wheel has a radius of 1, then the number $\sin(x)$ for a given angle x is precisely what measures this vertical displacement of the passenger car from the center of the wheel.
Similarly, we may ask how far the passenger car is displaced horizontally from the center of the Ferris wheel. Note that this is what the "copper cone" figure at the bottom left is measuring in the above picture. This distance is calculated by $\cos(x)$ for the angle x in the case in which the Ferris wheel has a radius of 1.

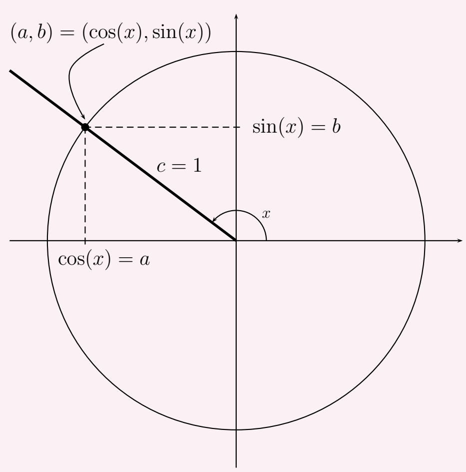
The sine over the cosine is the tangent: $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$.



Using the idea of vertical and horizontal displacement from the center of the Ferris wheel, we now define the trigonometric functions of an angle x.

Definition 17.5: Trigonometric functions

For an angle x, let $P(a, b)$ be the intersection point of the terminal side of x with the unit circle.
Then, we define the **cosine** of x to be the horizontal coordinate a of P, that is, $\cos(x) = a$. We define the **sine** of x to be the vertical coordinate b of P, that is, $\sin(x) = b$. Moreover, we define the **tangent** of x to be the quotient $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{b}{a}$.



$$\boxed{\begin{aligned} \cos(x) &= a \\ \sin(x) &= b \\ \tan(x) &= \tfrac{b}{a} \end{aligned}} \tag{17.2}$$

Moreover, we define three more trigonometric functions, which are called the **secant**, the **cosecant**, and the **cotangent**, respectively.

$$\boxed{\begin{aligned} \sec(x) &= \frac{1}{\cos(x)} = \frac{1}{a} \\ \csc(x) &= \frac{1}{\sin(x)} = \frac{1}{b} \\ \cot(x) &= \frac{\cos(x)}{\sin(x)} = \frac{a}{b} \end{aligned}} \tag{17.3}$$

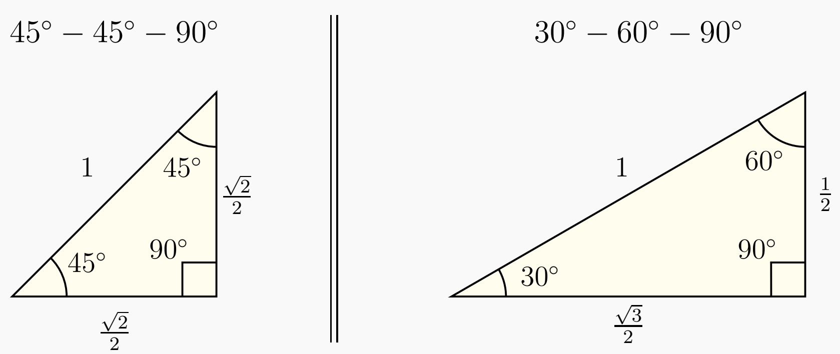
Many elementary facts and identities immediately follow from the above definition, and we will come back to these in Chapter 21. In the following section we will instead show how to compute the trigonometric function values for suitable angles (angles that are multiples of $30°$ or $45°$).

17.2 Computing trigonometric function values

First, we show how to compute $\sin(x)$, $\cos(x)$, and $\tan(x)$ for certain angles x by hand. One way to do so is by using the special $45° - 45° - 90°$ and $30° - 60° - 90°$ triangles. We will review these triangles now.

Review 17.6: Special right triangles

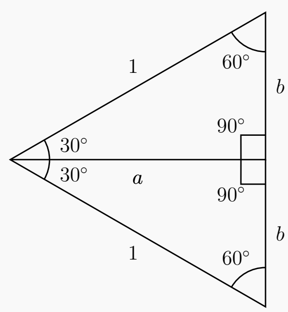
Consider right triangles with angles either $45° - 45° - 90°$ or $30° - 60° - 90°$. If the hypotenuse of the triangles are taken to be 1, then the other side lengths are given as follows:



Proof. In the $45° - 45° - 90°$ triangle, denote the side lengths by a, b, and c. By assumption, the hypotenuse is $c = 1$, and by symmetry, the two side lengths $a = b$ are equal. Using the Pythagorean theorem $a^2 + b^2 = c^2$, we get $a^2 + a^2 = 1^2$, which gives

$$2a^2 = 1 \implies a^2 = \frac{1}{2} \implies a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

For the $30° - 60° - 90°$ triangle, we again denote the side lengths by a, b, and c, with the hypotenuse $c = 1$ by assumption. Now reflect the triangle on its edge opposite to the $60°$ angle.



The outer triangle is an equilateral triangle having all side lengths equal to 1, so that $2b = 1$, or $b = \frac{1}{2}$. Finally, we find a from the Pythagorean theorem $a^2 + b^2 = c^2$, that is, $a^2 + (\frac{1}{2})^2 = 1^2$, so that:

$$a^2 + \frac{1}{4} = 1 \implies a^2 = 1 - \frac{1}{4} = \frac{3}{4} \implies a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

□

We use the above Review 17.6 to compute some trigonometric function values by hand.

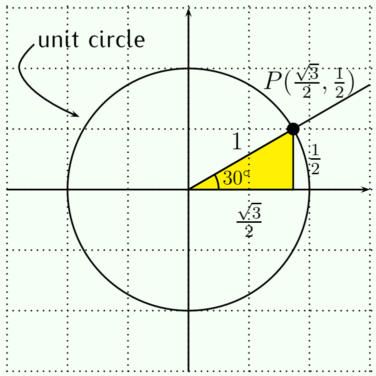
Example 17.7

Find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for the angles

a) $x = 30°$ b) $x = 45°$ c) $x = 60°$ d) $x = 90°$
e) $x = 150°$ f) $x = 225°$ g) $x = 300°$

Solution.

a) We place a $30° - 60° - 90°$ right triangle (drawn below in yellow) in the coordinate plane as shown below.



$$\cos(30°) = a = \frac{\sqrt{3}}{2}$$

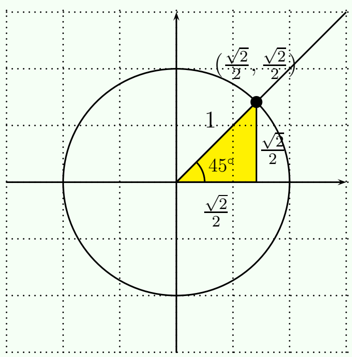
$$\sin(30°) = b = \frac{1}{2}$$

$$\tan(30°) = \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

The coordinates of the intersection point P of the terminal side of $30°$ with the unit circle are $(a, b) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, so these give the $\cos(30°) = a = \frac{\sqrt{3}}{2}$ and $\sin(30°) = b = \frac{1}{2}$.

b) For the $45°$ angle, we place a $45° - 45° - 90°$ right triangle (drawn in yellow) in the coordinate plane as shown below.

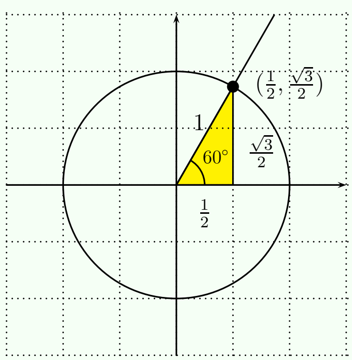


$$\cos(45°) = a = \frac{\sqrt{2}}{2}$$

$$\sin(45°) = b = \frac{\sqrt{2}}{2}$$

$$\tan(45°) = \frac{b}{a} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

c) We place a $30° - 60° - 90°$ triangle with the $60°$ angle at the origin.

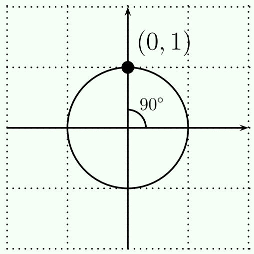


$$\cos(60°) = a = \frac{1}{2}$$

$$\sin(60°) = b = \frac{\sqrt{3}}{2}$$

$$\tan(60°) = \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

d) The terminal side of a $90°$ angle is the positive y-axis, intersecting with the unit circle at $(0, 1)$.

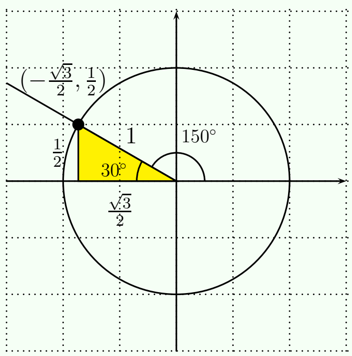


$$\cos(90°) = a = 0$$

$$\sin(90°) = b = 1$$

$\tan(90°) = \frac{b}{a}$ is undefined (since $a = 0$)

e) To obtain the intersection point for terminal side of $150°$ with the unit circle, we place a $30° - 60° - 90°$ triangle in quadrant II. Thus, the coordinates of the intersection point $P(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ are in quadrant II and therefore have a negative x-coordinate and a positive y-coordinate.

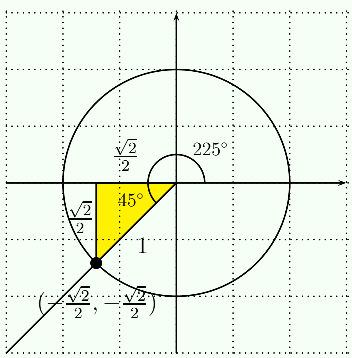


$$\cos(150°) = a = -\frac{\sqrt{3}}{2}$$

$$\sin(150°) = b = \frac{1}{2}$$

$$\tan(150°) = \frac{b}{a} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$
$$= -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

f)



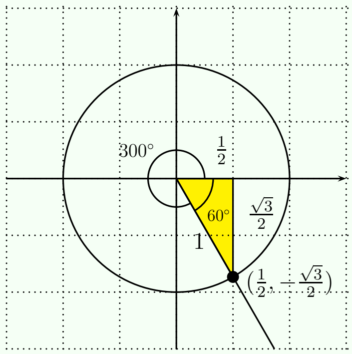
$$\cos(225°) = a = \frac{-\sqrt{2}}{2}$$

$$\sin(225°) = b = \frac{-\sqrt{2}}{2}$$

$$\tan(225°) = \frac{b}{a} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

Note that for an angle in quadrant III, both coordinates are negative, and thus both sine and cosine are negative.

g)



$$\cos(300°) = a = \frac{1}{2}$$

$$\sin(300°) = b = -\frac{\sqrt{3}}{2}$$

$$\tan(300°) = \frac{b}{a} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1}$$

$$= -\sqrt{3}$$

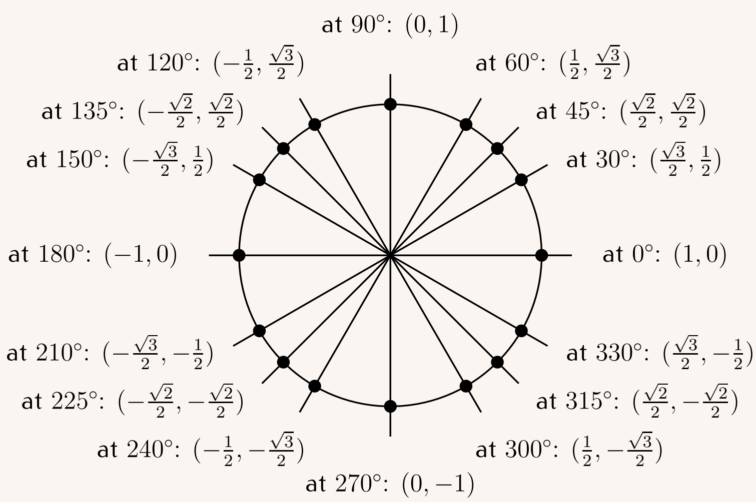
For an angle in quadrant IV, the cosine is positive and the sine is negative.

□

Note 17.8

The method from the previous example can be used to obtain the sine, cosine, and tangent of any angle x that is a multiple of $30°$ or $45°$.

- The sine, cosine, and tangent are given by the coordinates of the intersection of the terminal side of x with the unit circle. These intersection points for angles x from $0°$ to $360°$ (where x are multiples of $30°$ or $45°$) are:



- This gives the trigonometric function values for all angles that are multiples of $30°$ or $45°$. In particular:

x	$0° = 0$	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

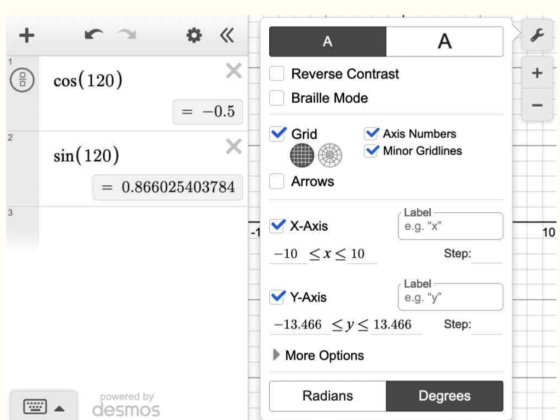
- Angles that differ by $360°$ have the same terminal side and, therefore, the same trigonometric function values.

$$\boxed{\sin(x \pm 360°) = \sin(x)} \quad \boxed{\cos(x \pm 360°) = \cos(x)} \tag{17.4}$$

When using a scientific calculator, the computations of the trigonometric function values becomes significantly easier. However, some of the answers from the calculator may require an appropriate interpretation. We demonstrate this in the following observation.

Observation 17.9: Trigonometric function values with the calculator

We want to use a calculator to find $\cos(120°)$ and $\sin(120°)$. We enter "cos(120)" and "sin(120)" into the calculator and, since our angles are in degree, we check in our settings that we have the degree mode selected. (The mode can be changed with the wrench symbol .)



Note that the calculator correctly shows the cosine $\cos(120°) = -0.5 = -\frac{1}{2}$. However, for sine, we only get an approximation $\sin(120°) \approx 0.866$, which *we have to interpret* as $\sin(120°) = \frac{\sqrt{3}}{2}$. Fortunately, there are only a few possible numbers that appear as the coordinates of the points on the unit circle in Note 17.8. Up to sign, these are the following:

$$\boxed{0 \qquad \frac{1}{2} \qquad \frac{\sqrt{2}}{2} \approx 0.707 \qquad \frac{\sqrt{3}}{2} \approx 0.866 \qquad 1} \tag{17.5}$$

Thus, these are the possible sine and cosine values (up to ±-sign) for the angles shown in Note 17.8.
For the tangent $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we need to look at quotients of the above numbers. These are (compare Example 17.7):

$$\boxed{0 \qquad \frac{\sqrt{3}}{3} \approx 0.577 \qquad 1 \qquad \sqrt{3} \approx 1.732} \tag{17.6}$$

We demonstrate the above observation on the method for finding trigonometric function values with the calculator in the following example.

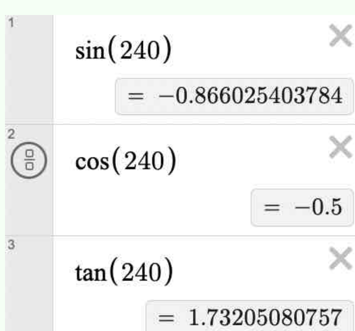
Example 17.10

Use the calculator to find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for:

a) $x = 240°$ b) $x = 495°$ c) $x = \frac{11\pi}{6}$ d) $x = -\frac{9\pi}{4}$

Solution.
We use the calculator and interpret approximations as shown in (17.5) and (17.6).

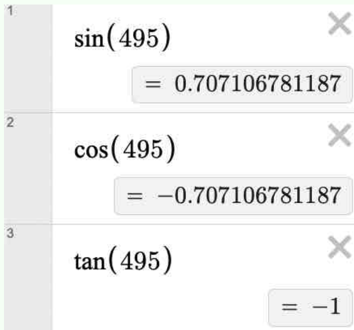
a)



$\sin(240°) = -\frac{\sqrt{3}}{2}$

$\cos(240°) = -\frac{1}{2}$

$\tan(240°) = \sqrt{3}$

b) 

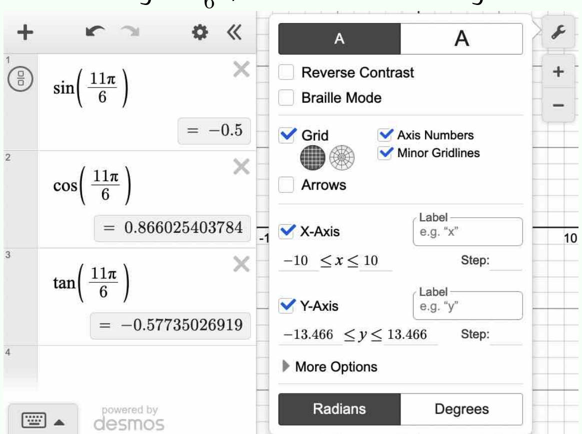
$\sin(495°) = \frac{\sqrt{2}}{2}$

$\cos(495°) = -\frac{\sqrt{2}}{2}$

$\tan(495°) = -1$

Note that $495° - 360° = 135°$, so that $495°$ and $135°$ have the same trigonometric function values.

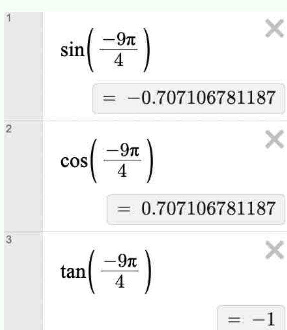
c) For the angle $\frac{11\pi}{6}$, we need to change the calculator to radian mode.



$\sin(\frac{11\pi}{6}) = -\frac{1}{2}$

$\cos(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2}$

$\tan(\frac{11\pi}{6}) = -\frac{\sqrt{3}}{3}$

d) 

$\sin(-\frac{9\pi}{4}) = -\frac{\sqrt{2}}{2}$

$\cos(-\frac{9\pi}{4}) = \frac{\sqrt{2}}{2}$

$\tan(-\frac{9\pi}{4}) = -1$

□

We can also compute exact trigonometric function values for at least some angles other than those in Note 17.8. One possible way to do so is to use the addition and subtraction of angles formulas. We will state these formulas in this section and show they apply in our setting. A proof for these formulas will be given in Section 21.2.

Proposition 17.11: Addition and subtraction of angles formulas

For any angles α and β, we have the following **addition and subtraction of angles formulas**:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

Example 17.12

Find the exact values of the trigonometric functions.

a) $\cos(75^\circ)$ b) $\sin\left(\frac{11\pi}{12}\right)$ c) $\tan(15^\circ)$

Solution.

a) Note that 75° is not one of the angles we computed in Note 17.8, but it is the sum of two such angles, since $75^\circ = 30^\circ + 45^\circ$. To compute $\cos(75^\circ)$, we therefore use the formula for $\cos(\alpha+\beta)$ with $\alpha = 30^\circ$ and $\beta = 45^\circ$.

$$\begin{aligned}
\cos(75^\circ) &= \cos(30^\circ+45^\circ) = \cos(30^\circ)\cos(45^\circ) - \sin(30^\circ)\sin(45^\circ) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

b) Again we want to write the angle $\frac{11\pi}{12}$ as a sum or difference of angles from Note 17.8. It might be a bit easier to first convert the angle from radian into degree:

$$\frac{11\pi}{12} = \frac{11\cdot 180}{12} = 165^\circ$$

Now, there are several ways in which we can write 165° as a sum or difference of known angles: $165^\circ = 45^\circ + 120^\circ$ or $165^\circ = 210^\circ - 45^\circ$, etc. We will use the difference $165^\circ = 210^\circ - 45^\circ$ for our computation, but we note that other choices would work just as well. We thus get:

$$\begin{aligned}\sin(165^\circ) &= \sin(210^\circ - 45^\circ) = \sin 210^\circ \cos 45^\circ - \cos 210^\circ \sin 45^\circ \\ &= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \Big(-\frac{\sqrt{3}}{2}\Big)\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

c) For $\tan(15^\circ)$ we note that $15^\circ = 60^\circ - 45^\circ$. We get that:

$$\begin{aligned}\tan(15^\circ) = \tan(60^\circ - 45^\circ) &= \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} \\ &= \frac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1} = \frac{\sqrt{3}-1}{1+\sqrt{3}}\end{aligned}$$

To fully simplify this expression, we rationalize the denominator by multiplying $1-\sqrt{3}$ to both numerator and denominator:

$$\begin{aligned}\tan(15^\circ) &= \frac{\sqrt{3}-1}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{\sqrt{3}-\sqrt{3}^2-1+\sqrt{3}}{1^2-\sqrt{3}^2} \\ &= \frac{2\sqrt{3}-3-1}{1-3} = \frac{2\sqrt{3}-4}{-2} = -\sqrt{3}+2 = 2-\sqrt{3}\end{aligned}$$

□

Using Proposition 17.11 we can also obtain certain identities among the trigonometric functions.

Example 17.13

Rewrite $\cos(x + \frac{\pi}{2})$ by using the addition formula.

Solution.

$$\cos\Big(x+\frac{\pi}{2}\Big) = \cos x\cdot\cos\frac{\pi}{2} - \sin x\cdot\sin\frac{\pi}{2} = \cos x\cdot 0 + \sin x\cdot 1 = \sin(x)$$

□

Another set of useful formulas concerns the trigonometric function values of half- and double-angles.

Proposition 17.14: Half- and double-angle formulas

Let α be an angle. Then we have the **half-angle formulas**:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

Here, the signs "$\pm$" are determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies. (For more on the signs, see also page 310.)
Furthermore, we have the **double-angle formulas**:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Here is an example involving the half-angle identities.

Example 17.15

Find the trigonometric functions using the half-angle formulas.

a) $\sin(22.5°)$ b) $\cos\left(\frac{7\pi}{8}\right)$ c) $\tan\left(\frac{\pi}{8}\right)$

Solution.

a) Since $22.5° = \frac{45°}{2}$, we use the half-angle formula with $\alpha = 45°$.

$$\begin{aligned}
\sin(22.5°) &= \sin\left(\frac{45°}{2}\right) = \pm\sqrt{\frac{1-\cos 22.5°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\
&= \pm\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}
\end{aligned}$$

Since $22.5°$ is in the first quadrant, the sine is positive, so that $\sin(22.5°) = \frac{\sqrt{2-\sqrt{2}}}{2}$.

b) Note that $\frac{7\pi}{8} = \frac{7\cdot 180°}{2} = 157.5°$ and $157.5° = \frac{315°}{2}$, so that we use the half-angle formulas for $\alpha = 315°$.

$$\cos(157.5°) = \cos\left(\frac{315°}{2}\right) = \pm\sqrt{\frac{1+\cos 315°}{2}}$$

Now, $157.5°$ is in the second quadrant, so that the cosine is negative. Using that $\cos(315°) = \frac{\sqrt{2}}{2}$, we thus get

$$\cos(157.5°) = -\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

c) Note that $\frac{\pi}{8} = \frac{180°}{8} = 22.5°$ and that $22.5° = \frac{45°}{2}$. We therefore get (using the first formula for $\tan\frac{\alpha}{2}$ from Proposition 17.14):

$$\tan(22.5°) = \tan\left(\frac{45°}{2}\right) = \frac{1-\cos 45°}{\sin 45°} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{2}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} - 1 = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

□

We end this section by noting where the trigonometric functions are positive or negative.

Note 17.16: Signs by quadrant

Following the notation from Defintion 17.5, let x be an angle, and let $P(a,b)$ be the intersection point of the terminal side of x with the unit circle. Then, the horizontal coordinate $a = \cos(x)$ of P is positive when P is in quadrant I and IV, whereas the vertical coordinate $b = \sin(x)$ of P is positive when P is in quadrant I and II. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ the sign of $\tan(x)$ is determined by the signs of $\sin(x)$ and $\cos(x)$. Thus the trigonometric functions are positive/negative according to the chart:

Quadrant II	Quadrant I
$\sin(x)$ is positive	$\sin(x)$ is positive
$\cos(x)$ is negative	$\cos(x)$ is positive
$\tan(x)$ is negative	$\tan(x)$ is positive
Quadrant III	Quadrant IV
$\sin(x)$ is negative	$\sin(x)$ is negative
$\cos(x)$ is negative	$\cos(x)$ is positive
$\tan(x)$ is positive	$\tan(x)$ is negative

17.3 Exercises

Exercise 17.1

Convert from radian to degree.

a) $\frac{\pi}{4}$ b) $\frac{2\pi}{3}$ c) $\frac{5\pi}{6}$ d) $\frac{7\pi}{4}$ e) $\frac{3\pi}{2}$ f) $\frac{5\pi}{4}$ g) $\frac{13\pi}{6}$ h) $-\frac{5\pi}{3}$

Exercise 17.2

Convert from degree to radian.

a) $120°$ b) $60°$ c) $300°$ d) $135°$

e) $90°$ f) $225°$ g) $480°$ h) $-150°$

Exercise 17.3

Find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for the following angles.

a) $x = 150°$	b) $x = 45°$	c) $x = 210°$	d) $x = 60°$
e) $x = 30°$	f) $x = 300°$	g) $x = 90°$	h) $x = 315°$
i) $x = 225°$	j) $x = 180°$	k) $x = 120°$	l) $x = 270°$
m) $x = 405°$	n) $x = -135°$	o) $x = -240°$	p) $x = 690°$
q) $x = \frac{5\pi}{3}$	r) $x = \frac{\pi}{6}$	s) $x = \frac{4\pi}{3}$	t) $x = \frac{5\pi}{6}$
u) $x = \frac{7\pi}{3}$	v) $x = \frac{7\pi}{4}$	w) $x = -\frac{\pi}{2}$	x) $x = \frac{13\pi}{3}$

Exercise 17.4

Find the trigonometric function values by using the addition and subtraction formulas.

a) $\sin(75°)$	b) $\cos(15°)$	c) $\tan(105°)$	d) $\sin(195°)$
e) $\cos(345°)$	f) $\sin(15°)$	g) $\cos(285°)$	h) $\tan(165°)$
i) $\cos\left(\frac{11\pi}{12}\right)$	j) $\sin\left(\frac{\pi}{12}\right)$	k) $\tan\left(\frac{13\pi}{12}\right)$	l) $\sin\left(\frac{23\pi}{12}\right)$

Exercise 17.5

Find the exact trigonometric function values by using the half-angle formulas.

a) $\cos(22.5°)$	b) $\sin(15°)$	c) $\cos(15°)$	d) $\tan(15°)$
e) $\sin(7.5°)$	f) $\tan(105°)$	g) $\sin\left(\frac{3\pi}{8}\right)$	h) $\cos\left(\frac{11\pi}{12}\right)$

Exercise 17.6

Simplify the function f using the addition and subtraction formulas.

a) $f(x) = \sin\left(x + \frac{\pi}{2}\right)$	b) $f(x) = \cos\left(x - \frac{\pi}{4}\right)$	c) $f(x) = \tan(\pi - x)$
d) $f(x) = \sin\left(\frac{\pi}{6} - x\right)$	e) $f(x) = \cos\left(\frac{2\pi}{3} - x\right)$	f) $f(x) = \cos\left(x + \frac{11\pi}{12}\right)$

Chapter 18

Graphing trigonometric functions

We now turn to function theoretic aspects of the trigonometric functions defined in the last chapter. In particular, we will study the graphs of trigonometric functions.

18.1 Graphs of $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$

To graph the functions $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$, we review a few trigonometric function values in the table below. Here, the angles x, which are the inputs of the trigonometric functions, are most conveniently taken in *radian* measure.

Note 18.1

We showed in the previous chapter how these function values were defined and how they can be computed, in particular, with the help of a calculator.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\dots$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\dots$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\dots$
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0	$\dots$

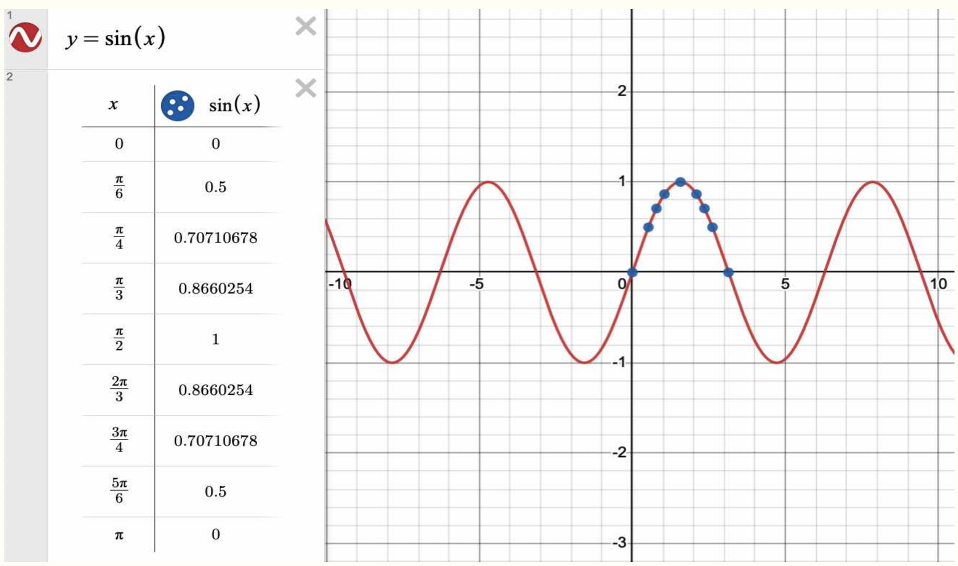
We start with the graph of the sine function.

Observation 18.2: Graph of $y = \sin(x)$

Graph the function $y = \sin(x)$.

Solution.

We graph the function $y = \sin(x)$ by plotting the corresponding (x, y) values in the plane. This can be done by hand, for example, by using the values from the above table and connecting the dots. While this is a great exercise, we will instead use the graphing calculator to do the main work for us.



From this, we make the following observations. The domain of $y = \sin(x)$ is all real numbers, $D = \mathbb{R}$, since $\sin(x)$ is defined for any angle x (see Definition 17.5).
Next, the graph is bounded (in the y-direction) between -1 and $+1$,

$$-1 \leq \sin(x) \leq 1 \quad \text{for all } x.$$

This follows from Definition 17.5, where we defined $\sin(x) = \frac{b}{r}$ with $-r \leq b \leq r$. Therefore, the range of $y = \sin(x)$ is $R = [-1, 1]$.
Moreover, the graph shows that $y = \sin(x)$ is a *periodic* function, that is, a function that repeats its output values after adding a fixed constant

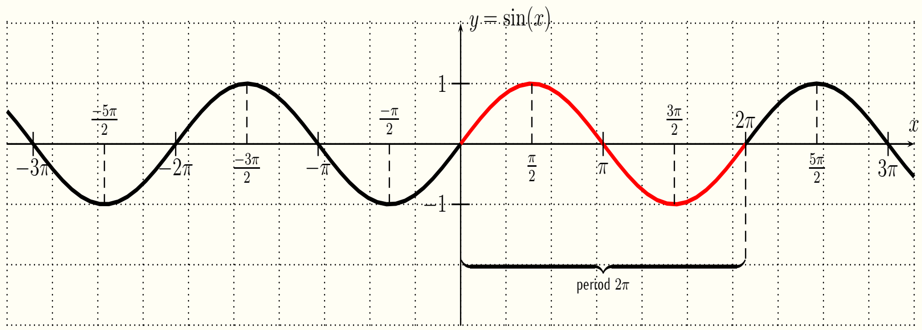
to the input. More precisely, $y = f(x)$ is *periodic* if there is a number $P \neq 0$ called a *period* so that

$$f(x + P) = f(x) \quad \text{for all } x.$$

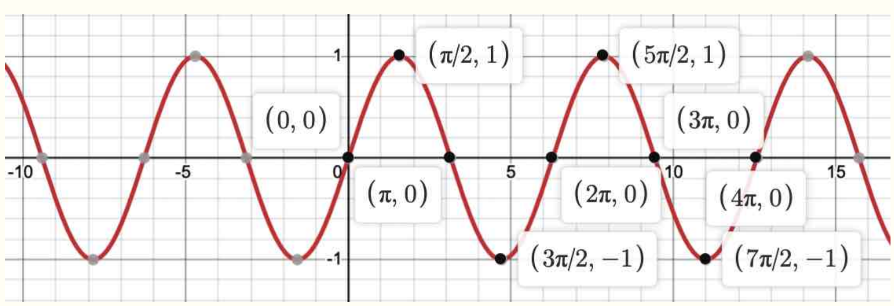
Note that $y = \sin(x)$ has a period of $P = 2\pi$, since the function does not change its value when adding $360° = 2\pi$ to its argument (and this is, in fact, the smallest positive number with that property):

$$\sin(x + 2\pi) = \sin(x)$$

The graph of $y = \sin(x)$ has the following specific values:



This can also be seen with the graphing calculator by clicking on the points of interest.



We say that $y = \sin(x)$ has a period of 2π, and an amplitude of 1. ☐

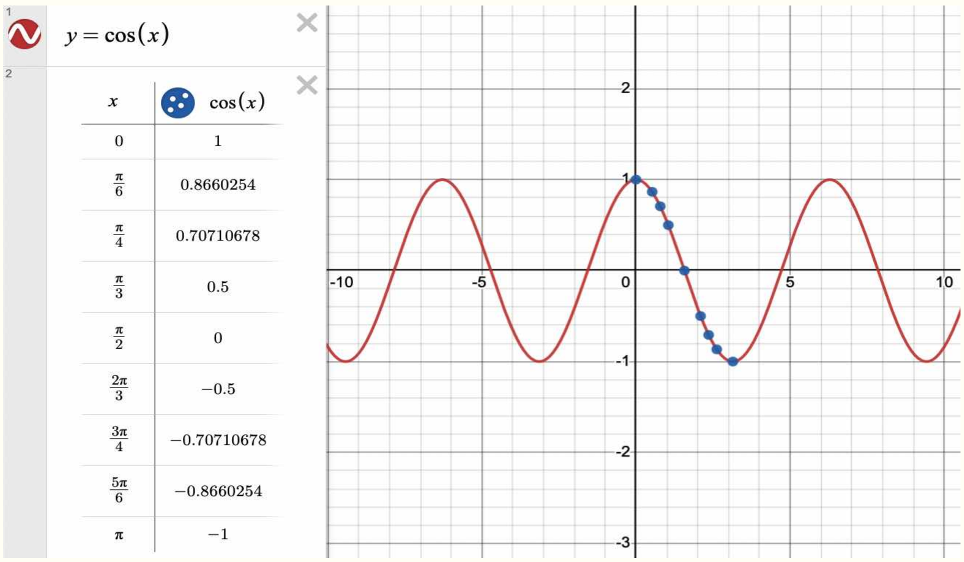
Next, we graph the cosine.

Observation 18.3: Graph of $y = \cos(x)$

Graph the function $y = \cos(x)$.

Solution.

Using the graphing calculator, we obtain the graph below.



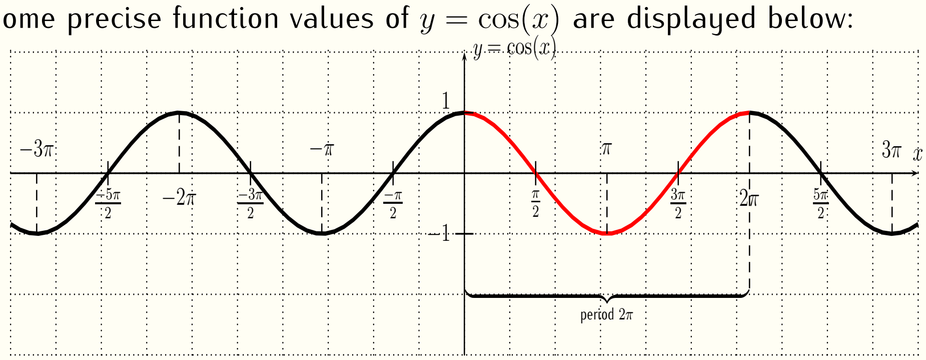
Just as we argued for the $\sin(x)$, we have the following similar observations for $y = \cos(x)$.

The domain of $y = \cos(x)$ is all real numbers, $D = \mathbb{R}$. The range is $R = [-1, 1]$, that is, $\cos(x)$ is bounded between -1 and 1.

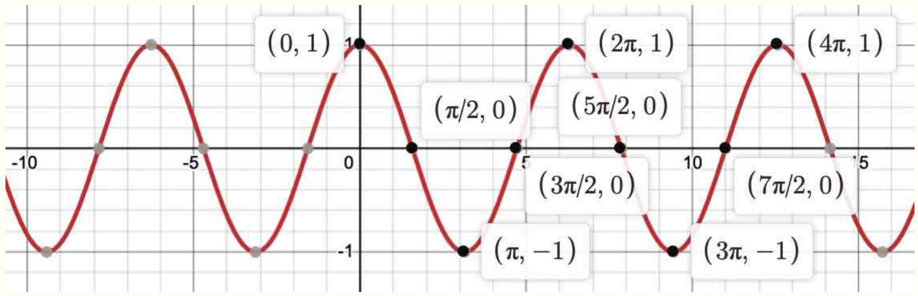
The function $y = \cos(x)$ is a periodic function with period $P = 2\pi$, that is,

$$\cos(x + 2\pi) = \cos(x) \quad \text{for all } x.$$

Some precise function values of $y = \cos(x)$ are displayed below:



These values can also be seen with the graphing calculator:



We say that $y = \cos(x)$ has a period of 2π, and an amplitude of 1. □

Note 18.4

Many properties of $\sin$ and $\cos$ can be observed from the above graphs (as well as from the unit circle definition). For example, the graph of $y = \cos(x)$ appears to be that of $y = \sin(x)$ shifted to the left by $\frac{\pi}{2}$. Algebraically, this can be expressed with the following identity:

$$\boxed{\cos(x) = \sin\left(x + \frac{\pi}{2}\right)} \tag{18.1}$$

Moreover, the graph of $y = \sin(x)$ appears to be symmetric with respect to the origin, the graph of $y = \cos(x)$ appears to be symmetric with respect to the y-axis. Algebraically, this means that the sine and cosine functions satisfy the following relations:

$$\boxed{\sin(-x) = -\sin(x)} \quad \text{and} \quad \boxed{\cos(-x) = \cos(x)} \tag{18.2}$$

We will show in Observations 21.8 and 21.9, that these identities are indeed true.
Note, in particular, that this means that $y = \sin(x)$ is an odd function, while $y = \cos(x)$ is an even function (see Observation 4.24).

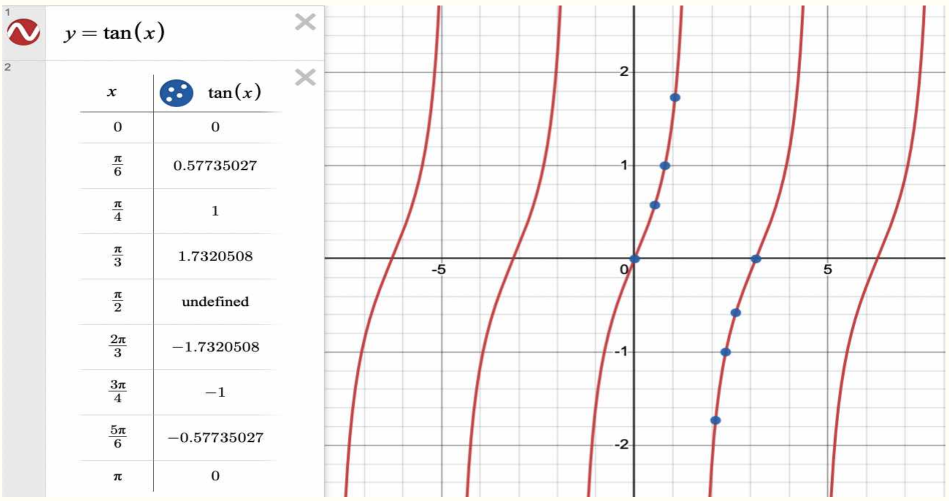
Finally, we also graph the tangent.

Observation 18.5: Graph of $y = \tan(x)$

Graph of $y = \tan(x)$.

Solution.

We graph the function $y = \tan(x)$ with the graphing calculator.



First, we observe that the tangent is periodic with a period of π:

$$\tan(x + \pi) = \tan(x) \tag{18.3}$$

Zooming into this graph, we see that $y = \tan(x)$ has vertical asymptotes $x = \frac{\pi}{2} \approx 1.6$ and $x = \frac{-\pi}{2} \approx -1.6$. This is also supported by the fact that $\tan(\frac{\pi}{2})$ and $\tan(-\frac{\pi}{2})$ are undefined. Since $y = \tan(x)$ is periodic, there are, in fact, infinitely many vertical asymptotes: $x = \frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2}, \frac{-5\pi}{2}, \dots$, or, in short

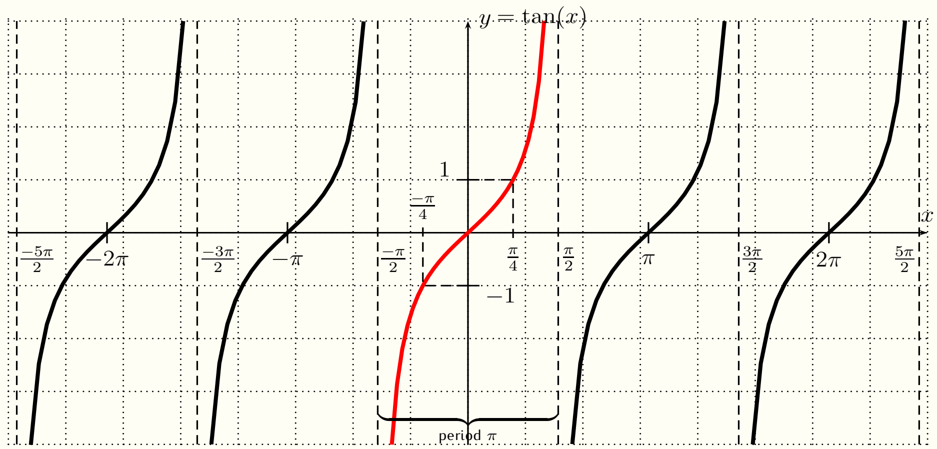
$$\text{asymptotes of } y = \tan(x): \quad x = n \cdot \frac{\pi}{2}, \text{ where } n = \pm 1, \pm 3, \pm 5, \dots$$

In particular, the domain of $y = \tan(x)$ is

$$D = \mathbb{R} - \left\{ x \ : \ x = n \cdot \frac{\pi}{2}, \text{ where } n = \pm 1, \pm 3, \pm 5, \dots \right\}$$

The range of $y = \tan(x)$ is all real numbers $R = \mathbb{R}$.

The graph of $y = \tan(x)$ with some more specific function values is shown below.



Furthermore, the tangent is an odd function, since it is symmetric with respect to the origin (see Observation 4.24):

$$\boxed{\tan(-x) = -\tan(x)} \tag{18.4}$$

□

18.2 Amplitude, period, and phase shift

Recall from Section 4.3 how adding or multiplying constants affects the graph of the function, such as:

- graph of $f(x) + c$ is the graph of $f(x)$ shifted up by c (or down when $c < 0$)
- graph of $f(x + c)$ is the graph of $f(x)$ shifted to the left by c (or to the right when $c < 0$)
- the graph of $c \cdot f(x)$ (for $c > 0$) is the graph of $f(x)$ stretched away from the x-axis by a factor c (or compressed when $0 < c < 1$)
- the graph of $f(c \cdot x)$ (for $c > 0$) is the graph of $f(x)$ compressed toward the y-axis by a factor c (or stretched away the y-axis when $0 < c < 1$)

With this, we can graph some variations of the basic trigonometric functions.

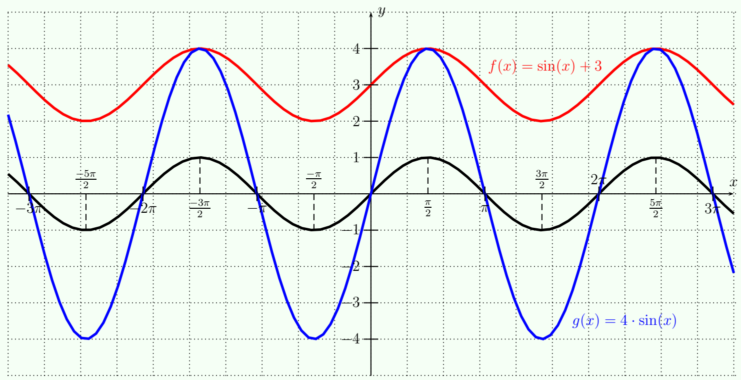
Example 18.6

Graph the functions:

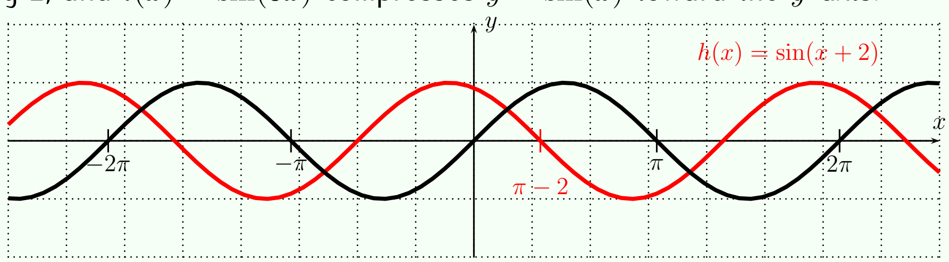
$$f(x) = \sin(x) + 3,\ \ g(x) = 4 \cdot \sin(x),\ \ h(x) = \sin(x+2),\ \ i(x) = \sin(3x)$$
$$j(x) = 2 \cdot \cos(x) + 3,\ \ k(x) = \cos(2x - \pi),\ \ l(x) = \tan(x+2) + 3$$

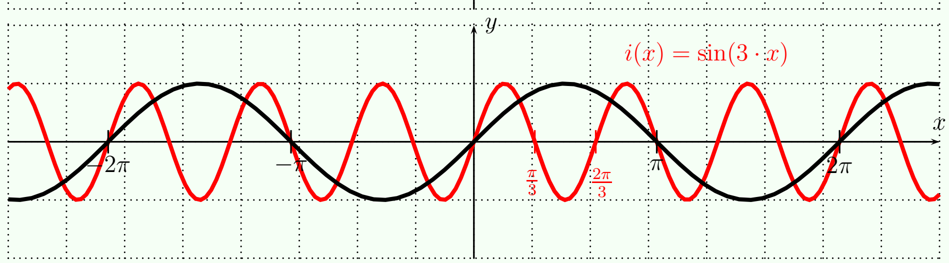
Solution.

The functions f, g, h, and i have graphs that are variations of the basic $y = \sin(x)$ graph. The graph of $f(x) = \sin(x) + 3$ shifts the graph of $y = \sin(x)$ up by 3, whereas the graph of $g(x) = 4 \cdot \sin(x)$ stretches $y = \sin(x)$ away from the x-axis.

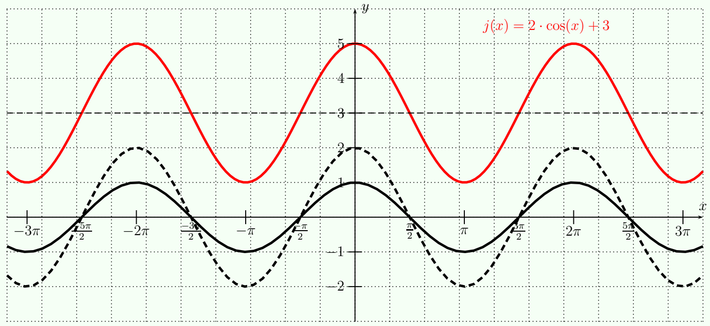


The graph of $h(x) = \sin(x+2)$ shifts the graph of $y = \sin(x)$ to the left by 2, and $i(x) = \sin(3x)$ compresses $y = \sin(x)$ toward the y-axis.

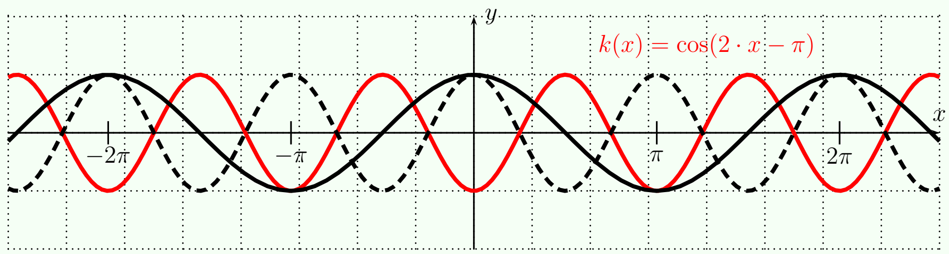




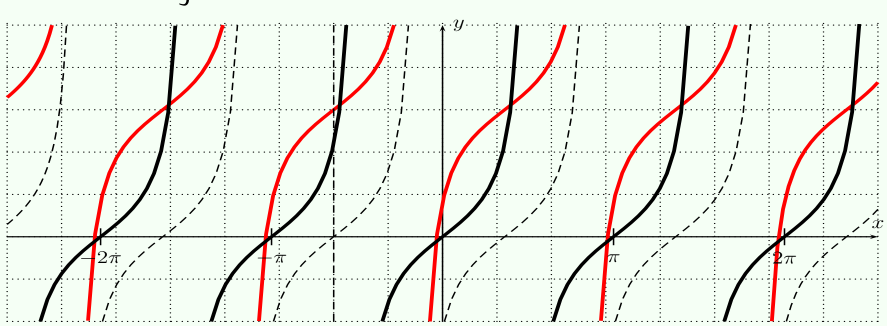
Next, $j(x) = 2 \cdot \cos(x) + 3$ has a graph of $y = \cos(x)$ stretched by a factor 2 away from the x-axis, and shifted up by 3.



For the graph of $k(x) = \cos(2x - \pi) = \cos(2 \cdot (x - \frac{\pi}{2}))$, we need to compress the graph of $y = \cos(x)$ by a factor 2 (we obtain the graph of the function $y = \cos(2x)$), and then shift this by $\frac{\pi}{2}$ to the right.



We will explore this case below in more generality. In fact, whenever $y = \cos(b \cdot x + c) = \cos(b \cdot (x - \frac{-c}{b}))$, the graph of $y = \cos(x)$ is shifted to the right by $\frac{-c}{b}$, and compressed by a factor b, so that it has a period of $\frac{2\pi}{b}$.
Finally, $l(x) = \tan(x + 2) + 3$ shifts the graph of $y = \tan(x)$ up by 3 and to the left by 2.



□

We collect some of the above observations in the following definition.

Definition 18.7: Amplitude, period, phase shift

Let f be one of the functions:

$$f(x) = a \cdot \sin(b \cdot x + c) \quad \text{or} \quad f(x) = a \cdot \cos(b \cdot x + c)$$

We define the **amplitude** A, the **period** P, and the **phase shift** S to be

$$\boxed{A = |a|} \quad \text{amplitude} \tag{18.5}$$

$$\boxed{P = \left|\frac{2\pi}{b}\right|} \quad \text{period} \tag{18.6}$$

$$\boxed{S = \frac{-c}{b}} \quad \text{phase shift} \tag{18.7}$$

In physical applications, the period is sometimes denoted by $T = P = \left|\frac{2\pi}{b}\right|$ and the **frequency** f is defined as the reciprocal, $f = \frac{1}{P}$.

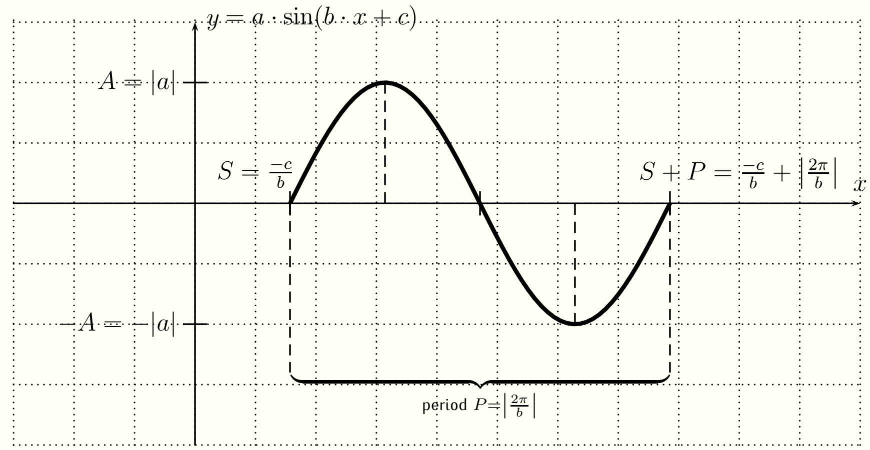
Using the amplitude, period, and phase shift, we know precisely how the shape of a sine or cosine function has been shifted or stretched.

Observation 18.8: Graphing $\sin$ or $\cos$ over one full period

With the above definition, we analyze the graph of, for example, $f(x) = a \cdot \sin(b \cdot x + c)$ with positive $a > 0$ and $b > 0$ as follows.

- First, consider $g(x) = a \cdot \sin(b \cdot x)$, that is, the function where we put $c = 0$. The graph of g is that of $y = \sin(x)$ stretched by a factor A away from the x-axis, and stretched (away or toward the y-axis) in such a way that it has a period of $P = \frac{2\pi}{b}$.

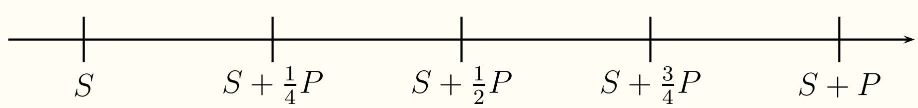
- Then, the graph of $f(x) = a \cdot \sin(b \cdot x + c) = a \cdot \sin\left(b \cdot \left(x + \frac{c}{b}\right)\right)$ is that of $g(x) = a \cdot \sin(b \cdot x)$ shifted by the phase shift $S = \frac{-c}{b}$. In other words, one full period of the graph starts at $(S, 0)$ and ends at $(S + P, 0)$.



We thus have the following strategy for graphing $f(x) = a \cdot \sin(b \cdot x + c)$ over one full period, starting at the phase shift S:

1. Mark the starting point of the period at $(S, 0) = (\frac{-c}{b}, 0)$.
2. Mark the endpoint of the period at $(S + P, 0) = (\frac{-c}{b} + \left|\frac{2\pi}{b}\right|, 0)$.
3. Draw the graph of $f(x) = a \cdot \sin(b \cdot x + c)$ from S to $S + P$ as the graph of $y = \sin(x)$ from 0 to 2π stretched and shifted so that it starts at S and ends at $S + P$, and so that it has an amplitude of A.

Note that the root(s), the maxima, and the minima of the graph within the drawn period are in equal distance from each other. The values of these can be found, for example, by calculating the midpoint between $(S, 0)$ and $(S + P, 0)$ and midpoints between these and the resulting midpoint:



A similar graph can be obtained for $f(x) = a \cdot \cos(b \cdot x + c)$ by replacing the graph of $y = \sin(x)$ with the graph of $y = \cos(x)$ over the period from 0 to 2π.
Moreover, note that when a and b are not positive, appropriate reflections may have to be applied as well.

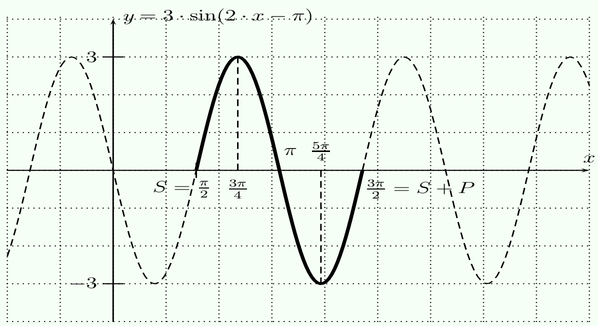
Example 18.9

Find the amplitude, period, and phase shift, and sketch the graph over one full period starting at the phase shift. Label all roots, maxima, and minima.

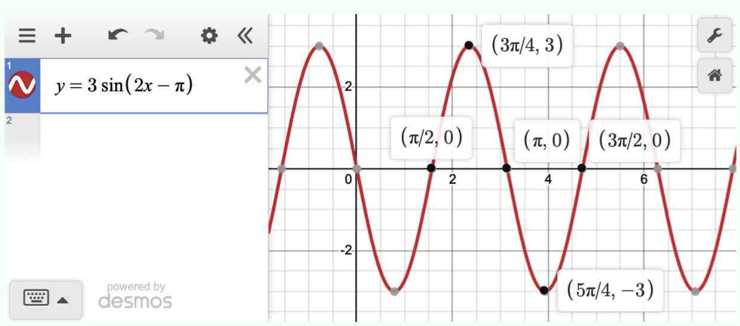
a) $f(x) = 3 \cdot \sin(2x - \pi)$ b) $f(x) = 4 \cdot \cos(5x - \frac{\pi}{2})$
c) $f(x) = 6 \cdot \sin(3x + \pi)$ d) $f(x) = -2 \cdot \cos(\frac{x}{2} - \frac{\pi}{4})$
e) $f(x) = 5 \cdot \sin\left(-2x + \frac{\pi}{3}\right)$

Solution.

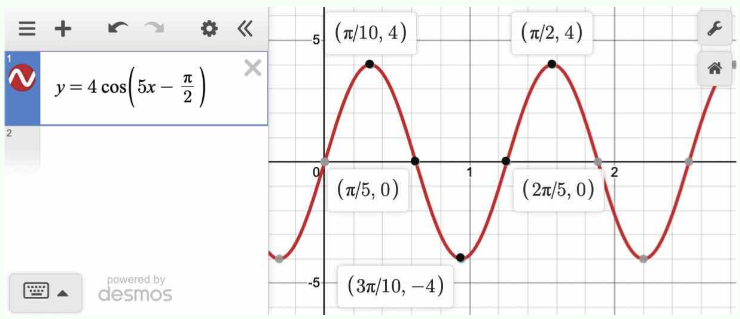
a) The amplitude is $A = |a| = 3$, the period is $P = \left|\frac{2\pi}{b}\right| = \left|\frac{2\pi}{2}\right| = \pi$, and the phase shift is $S = \frac{-c}{b} = \frac{-(-\pi)}{2} = \frac{\pi}{2}$. We graph one full period from $S = \frac{\pi}{2}$ to $S + P = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$.



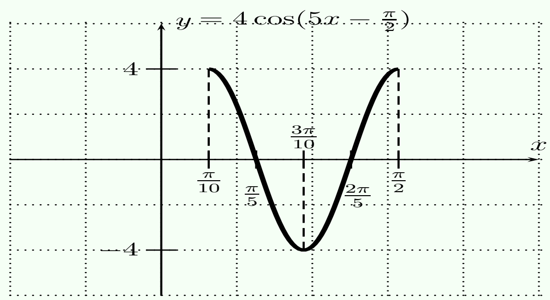
Note that the zero at the center of this period is given at $(\frac{\pi}{2} + \frac{3\pi}{2}) \div 2 = \frac{4\pi}{2} \cdot \frac{1}{2} = \pi$. The maximum is at $(\frac{\pi}{2} + \pi) \div 2 = \frac{3\pi}{2} \cdot \frac{1}{2} = \frac{3\pi}{4}$. The minimum is at $(\pi + \frac{3\pi}{2}) \div 2 = \frac{5\pi}{2} \cdot \frac{1}{2} = \frac{5\pi}{4}$. This is also confirmed with the graphing calculator.



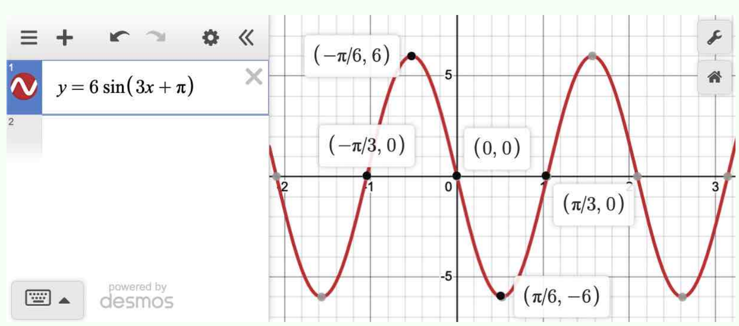
b) For $f(x) = 4 \cdot \cos(5x - \frac{\pi}{2})$, we have the amplitude $A = 4$, period $P = \frac{2\pi}{5}$, and phase shift $S = \frac{-(-\pi/2)}{5} = \frac{\pi}{2} \cdot \frac{1}{5} = \frac{\pi}{10}$. Therefore, the endpoint of the period is at $S + P = \frac{\pi}{10} + \frac{2\pi}{5} = \frac{5\pi}{10} = \frac{\pi}{2}$.



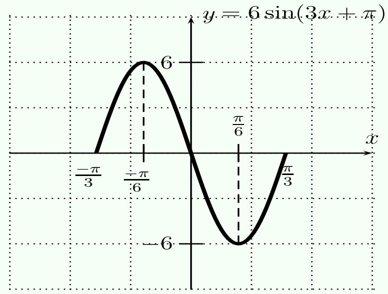
The minimum is at $\left(\frac{\pi}{10} + \frac{\pi}{2}\right) \div 2 = \frac{6\pi}{10} \cdot \frac{1}{2} = \frac{3\pi}{10}$, the roots are at $\left(\frac{\pi}{10} + \frac{3\pi}{10}\right) \div 2 = \frac{4\pi}{10} \cdot \frac{1}{2} = \frac{\pi}{5}$ and $\left(\frac{3\pi}{10} + \frac{\pi}{2}\right) \div 2 = \frac{8\pi}{10} \cdot \frac{1}{2} = \frac{2\pi}{5}$.



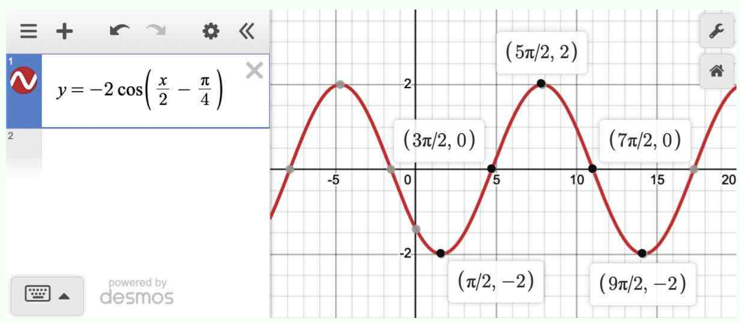
c) For $f(x) = 6 \cdot \sin(3x + \pi)$, we have the amplitude $A = 6$, period $P = \frac{2\pi}{3}$, and phase shift $S = \frac{-\pi}{3}$. Therefore, the endpoint of the period is at $S + P = \frac{-\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$.



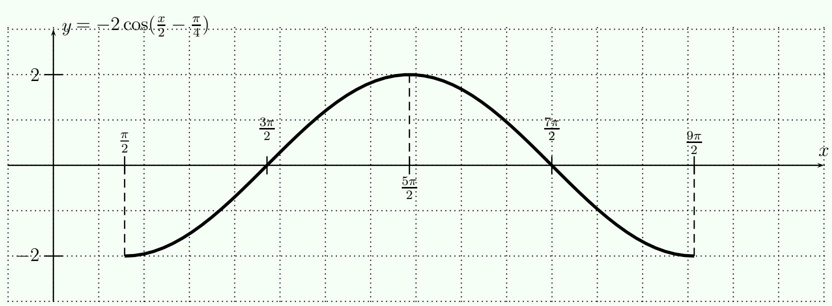
The root is at $\left(\frac{-\pi}{3}+\frac{\pi}{3}\right)\div 2=0$, the maximum is at $\left(\frac{-\pi}{3}+0\right)\div 2=\frac{-\pi}{3}\cdot\frac{1}{2}=\frac{-\pi}{6}$, the minimum is at $\left(0+\frac{\pi}{3}\right)\div 2=\frac{\pi}{3}\cdot\frac{1}{2}=\frac{\pi}{6}$.



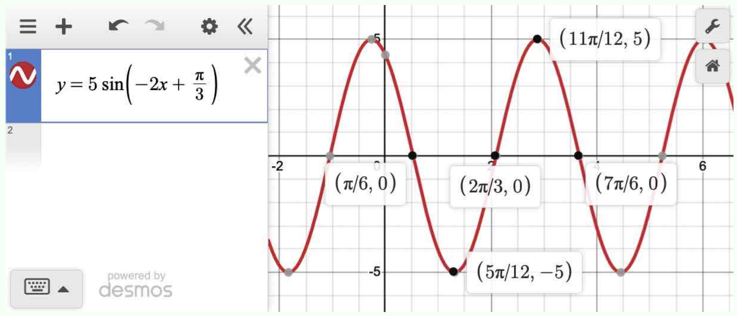
d) For $f(x)=-2\cdot\cos(\frac{x}{2}-\frac{\pi}{4})$, we have the amplitude $A=|-2|=2$, period $P=\frac{2\pi}{\frac{1}{2}}=2\pi\cdot\frac{2}{1}=4\pi$, and phase shift $S=\frac{-(-\pi/4)}{\frac{1}{2}}=\frac{\pi}{4}\cdot\frac{2}{1}=\frac{\pi}{2}$. Therefore, the endpoint of the period is at $S+P=\frac{\pi}{2}+4\pi=\frac{9\pi}{2}$.



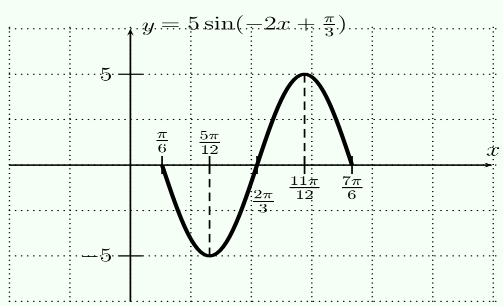
Since the coefficient $a=-2$ is negative, we need to graph the cosine reflected over the x-axis. We have the maximum at $\left(\frac{\pi}{2}+\frac{9\pi}{2}\right)\div 2=\frac{10\pi}{2}\cdot\frac{1}{2}=\frac{5\pi}{2}$, the roots are at $\left(\frac{\pi}{2}+\frac{5\pi}{2}\right)\div 2=\frac{6\pi}{2}\cdot\frac{1}{2}=\frac{3\pi}{2}$ and $\left(\frac{5\pi}{2}+\frac{9\pi}{2}\right)\div 2=\frac{14\pi}{2}\cdot\frac{1}{2}=\frac{7\pi}{2}$.



e) For $f(x) = 5 \cdot \sin\left(-2x + \frac{\pi}{3}\right)$, we have the amplitude $A = 5$, period $P = \left|\frac{2\pi}{-2}\right| = \pi$, and phase shift $S = \frac{-(\pi/3)}{-2} = \frac{\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{6}$. Therefore, the endpoint of the period is at $S + P = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$.

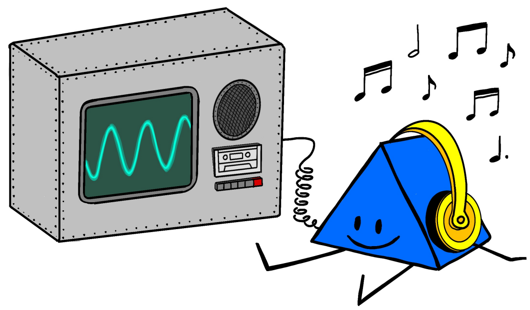


To graph $f(x)$, we recall from (18.2) that $\sin(-x) = -\sin(x)$, so that $f(x)$ can be rewritten as $f(x) = 5 \cdot \sin\left(-(2x - \frac{\pi}{3})\right) = -5 \cdot \sin\left(2x - \frac{\pi}{3}\right)$. Writing $f(x)$ in this way gives the same period and phase shift as before, since now $P = \left|\frac{2\pi}{2}\right| = \pi$ and $S = \frac{-(-\pi/3)}{2} = \frac{\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{6}$. Since $f(x) = -5 \cdot \sin\left(2x - \frac{\pi}{3}\right)$ has a negative leading coefficient, we need to reflect the sine graph over the x-axis. Thus, we get the root at $\left(\frac{\pi}{6} + \frac{7\pi}{6}\right) \div 2 = \frac{8\pi}{6} \cdot \frac{1}{2} = \frac{8\pi}{12} = \frac{2\pi}{3}$; the minimum is at $\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) \div 2 = \frac{5\pi}{6} \cdot \frac{1}{2} = \frac{5\pi}{12}$, the maximum is at $\left(\frac{2\pi}{3} + \frac{7\pi}{6}\right) \div 2 = \frac{11\pi}{6} \cdot \frac{1}{2} = \frac{11\pi}{12}$.

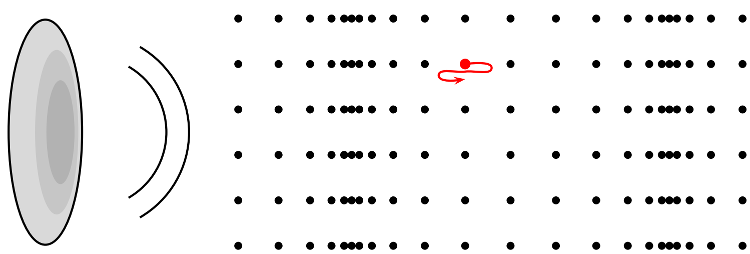


□

Application: Sound waves



Sound is created from a source, for example, from a loudspeaker. A vibrating membrane within the loudspeaker creates vibrations of the molecules in the surrounding air, creating regions of higher and lower pressures.[1] These differences in pressures transmit via sound waves and are perceived by the brain as sound.



The above picture shows a vibrating molecule (in red) in regions of variable pressure. Note that it is not any overall motion of the molecules that create the sound wave, but molecules vibrating at their equilibrium positions. The displacement $s(t)$ of a molecule from its equilibrium position is given by a $\sin$ or $\cos$ function in time.

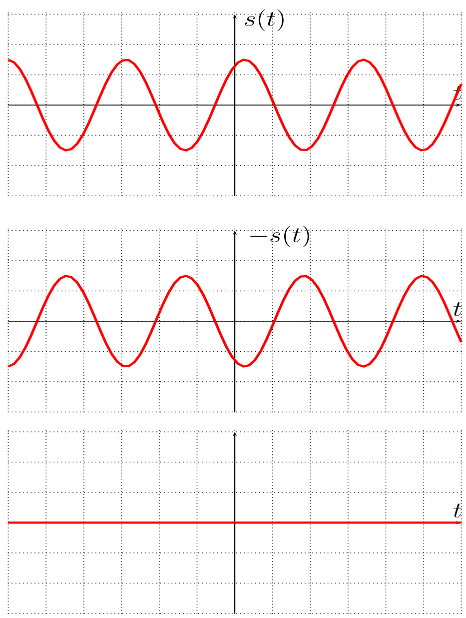
$$s(t) = a \cdot \cos(\omega t + c) \tag{18.8}$$

In the formula above, ω is called the angular frequency, and so the period is given by $P = \frac{2\pi}{\omega}$. If we recall that the frequency $f = \frac{1}{P}$ is the reciprocal of the period, we see that $\omega = \frac{2\pi}{P} = 2\pi f$.

[1] For more information, see https://openstax.org/books/university-physics-volume-1/pages/17-1-sound-waves

We can interpret the amplitute, period, and phase shift from (18.8) in terms of the perceived sound as follows:

- Sound waves can be perceived by humans for frequencies ranging from 20 Hz to $20,000$ Hz. The higher the frequency (or the smaller the period), the higher the perceived pitch of the sound.
- The amplitude is perceived as the volume of the sound.
- While the phase shift cannot be perceived directly from the sound, we can see the effect of the phase shift from applications such as noise canceling headphones. The idea for this is that for a given displacement $s(t)$ of a molecule, if we can create a displacement in the opposite direction, then the overall effect is no displacement at all, which yields a cancellation of the sound.



18.3 Exercises

Exercise 18.1

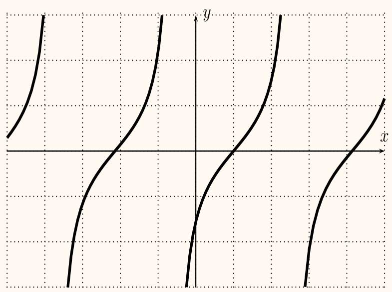
Graph the function and describe how the graph can be obtained from one of the basic graphs $y = \sin(x)$, $y = \cos(x)$, or $y = \tan(x)$.

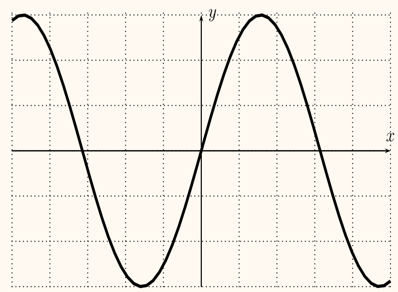
a) $f(x) = \sin(x) + 2$ b) $f(x) = \cos(x - \pi)$ c) $f(x) = \tan(x) - 4$
d) $f(x) = 5 \cdot \sin(x)$ e) $f(x) = \cos(2 \cdot x)$ f) $f(x) = \sin(x - 2) - 5$

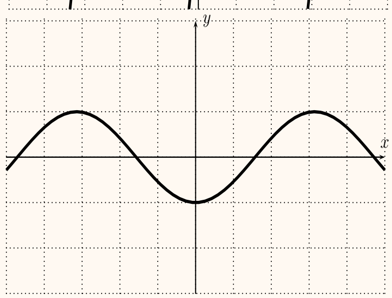
Exercise 18.2

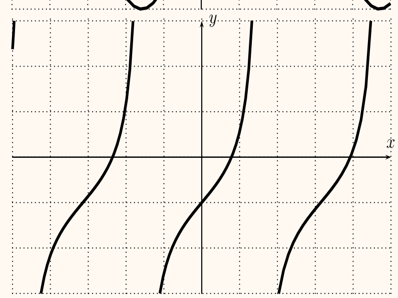
Identify the formulas with the graphs.

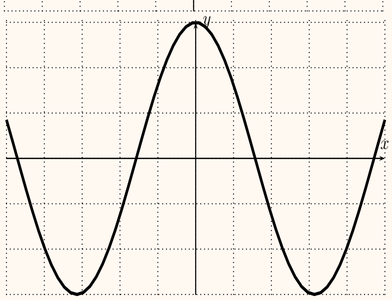
$f(x) = \sin(x) + 2$, $g(x) = \tan(x - 1)$, $h(x) = 3\sin(x)$
$i(x) = 3\cos(x)$, $j(x) = \cos(x - \pi)$, $k(x) = \tan(x) - 1$

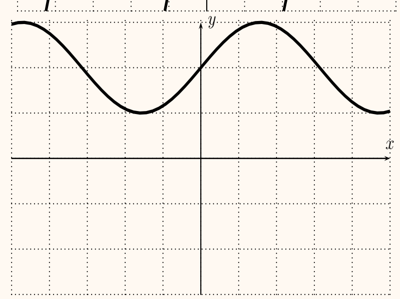
a) 

b) 

c) 

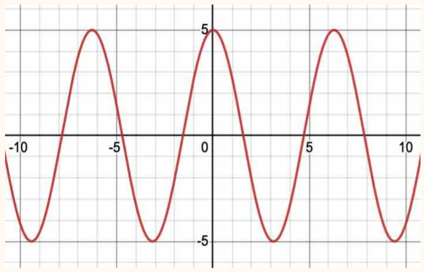
d) 

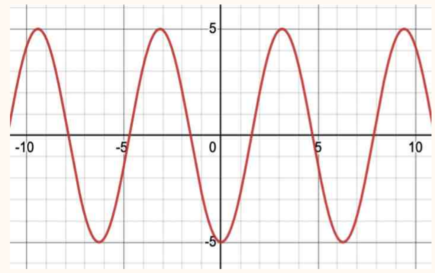
e) 

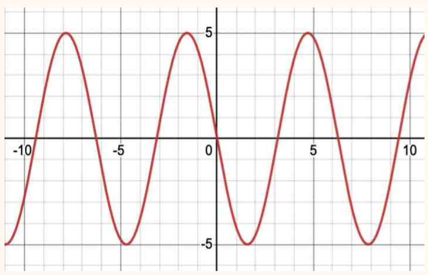
f) 

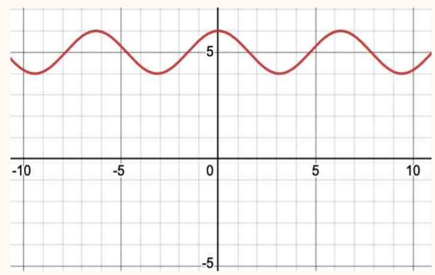
Exercise 18.3

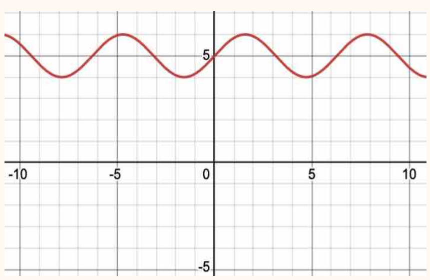
Find the formula of a function whose graph is the one displayed below.

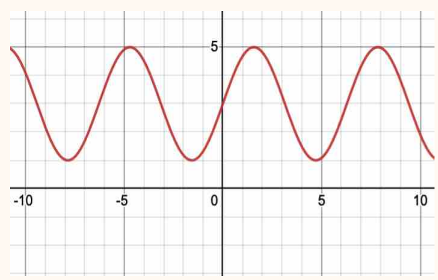
a) 

b) 

c) 

d) 

e) 

f) 

Exercise 18.4

Find the amplitude, period, and phase shift of the function.

a) $f(x) = 5\sin(2x + \pi)$

b) $f(x) = 3\sin(4x - \frac{\pi}{2})$

c) $f(x) = 4\sin(6x)$

d) $f(x) = 2\cos(7x + \frac{\pi}{4})$

e) $f(x) = 8\cos(2x - 3\pi)$

f) $f(x) = 3\sin(\frac{x}{4})$

g) $f(x) = -4\cos(5x + \frac{\pi}{3})$

h) $f(x) = 7\sin(\frac{1}{2}x - \frac{6\pi}{5})$

i) $f(x) = \cos(-2x)$

j) $f(x) = 6\cos(\pi x - \pi)$

Exercise 18.5

Find the amplitude, period, and phase shift of the function. Use this information to graph the function over a full period. Label all roots, maxima, and minima of the function.

a) $y = 5\cos(2x)$
b) $y = -4\sin(\pi x)$
c) $y = 4\sin(5x - \pi)$
d) $y = 6\cos(2x - \pi)$
e) $y = 5\sin(2x - \frac{\pi}{2})$
f) $y = 7\cos(3x - \frac{\pi}{2})$
g) $y = 5\sin(3x - \frac{\pi}{4})$
h) $y = 3\sin(4x + \pi)$
i) $y = 2\cos(5x + \pi)$
j) $y = 4\sin(2x + \frac{\pi}{2})$
k) $y = 3\cos(6x + \frac{\pi}{2})$
l) $y = 3\cos\left(2x + \frac{\pi}{4}\right)$
m) $y = 7\sin\left(\frac{1}{4}x + \frac{\pi}{4}\right)$
n) $y = -2\sin\left(\frac{1}{5}x - \frac{\pi}{10}\right)$
o) $y = \frac{1}{3}\cos\left(\frac{14}{5}x - \frac{6\pi}{5}\right)$

Chapter 19

Inverse trigonometric functions

The inverse trigonometric functions are the inverse functions of the $y = \sin x$, $y = \cos x$, and $y = \tan x$ functions restricted to appropriate domains. In this chapter we give a precise definition of these functions.

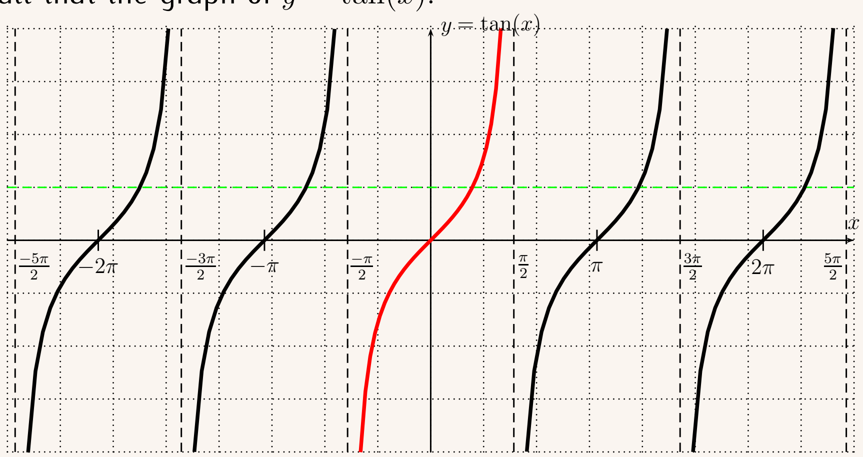
19.1 The functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$

We start with the inverse to the tangent function $y = \tan(x)$.

The function $y = \tan^{-1}(x)$

Note 19.1

Recall that the graph of $y = \tan(x)$:



The graph has vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$. Note that $y = \tan(x)$ is not a one-to-one function in the sense of defintion 6.1 on page 92. (For example, the horizontal line $y = 1$ intersects the graph at $x = \frac{\pi}{4}$, $x = \frac{\pi}{4} \pm \pi$, $x = \frac{\pi}{4} \pm 2\pi$, etc.) However, when we restrict the function to the domain $D = (\frac{-\pi}{2}, \frac{\pi}{2})$, that is, the red part of the above graph, then the restricted function *is* one-to-one, and for this restricted function, we may take its inverse function.

Definition 19.2: Inverse tangent function

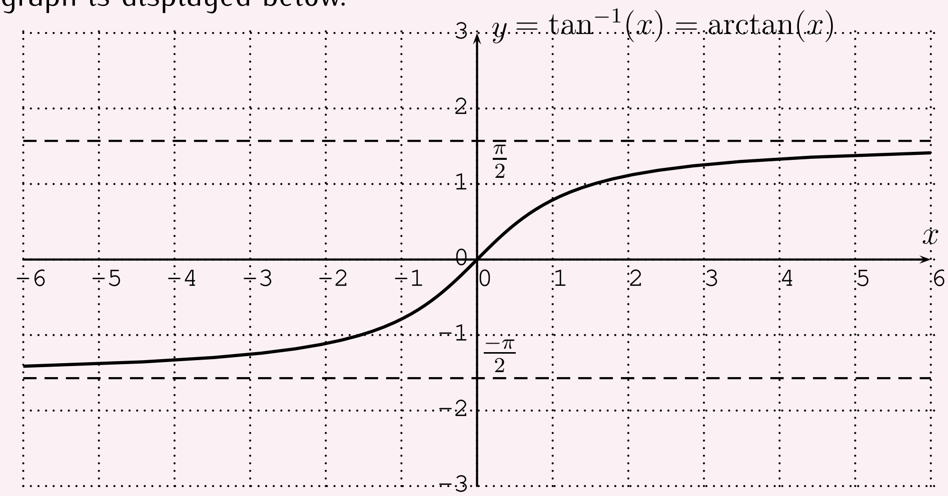
The inverse of the function $y = \tan(x)$ with restricted domain $D = (\frac{-\pi}{2}, \frac{\pi}{2})$ and range $R = \mathbb{R}$ is called the **inverse tangent** function. It is defined by

$$\boxed{x = \tan(y) \iff y = \tan^{-1}(x)} \quad \text{, for } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Alternatively, the inverse tangent function is also written as the **arc-tangent** function:

$$y = \tan^{-1}(x) = \arctan(x)$$

The arctangent reverses the input and output of the tangent function, so that the arctangent has domain $D = \mathbb{R}$ and range $R = (\frac{-\pi}{2}, \frac{\pi}{2})$. The graph is displayed below.



The inverse tangent function has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

Note 19.3: $\tan^{-1}$ is odd

We note that the inverse tangent function is an odd function:

$$\boxed{\tan^{-1}(-x) = -\tan^{-1}(x)} \tag{19.1}$$

This can be seen by observing that the tangent $y = \tan(x)$ is an odd function (that is $\tan(-x) = -\tan(x)$), and this also is confirmed by the the symmetry of the graph of $y = \tan^{-1}(x)$ with respect to the origin $(0, 0)$.

Note 19.4

The exponent notation of $\tan^{-1}(x)$ is unfortunately somewhat inconsistent, since the exponent can refer to two different concepts. Indeed, writing $\tan^{-1}(x) = \arctan(x)$ means that we consider the *inverse function* of the $\tan(x)$ function. However, when we write $\tan^2(x)$, we mean

$$\tan^2(x) = (\tan(x))^2 = \tan(x) \cdot \tan(x)$$

Therefore, $\tan^{-1}(x)$ is the inverse function of $\tan(x)$ with respect to the *composition* operation, whereas $\tan^2(x)$ is the square with respect to the *product* in $\mathbb{R}$. Note also that the inverse function of the tangent with respect to the product in $\mathbb{R}$ is $y = \frac{1}{\tan(x)} = \cot(x)$, which is the cotangent.

The next example calculates some inverse tangent function values.

Example 19.5

Compute the inverse tangent function values.

a) $\tan^{-1}(\sqrt{3})$ b) $\tan^{-1}(-1)$ c) $\tan^{-1}(4.3)$

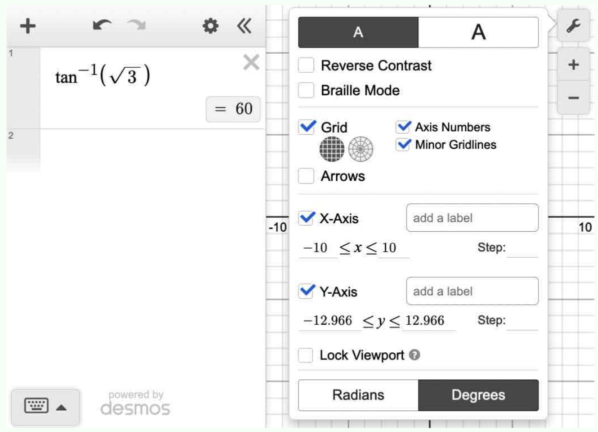
Solution.

a) We will first show how to compute $\tan^{-1}(\sqrt{3})$ without the use of a calculator, and then more easily with the use of a calculator. Recall the exact values of the tangent function from Section 17.1:

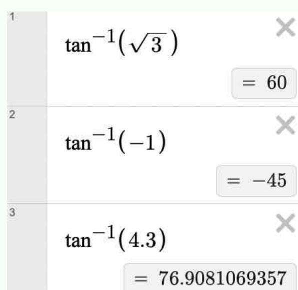
x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

Since, by Definition 19.2, $y = \tan^{-1}(x)$ is given by $x = \tan(y)$ (for $-\frac{\pi}{2} < y < \frac{\pi}{2}$), we can rewrite $y = \tan^{-1}(\sqrt{3})$ as $\sqrt{3} = \tan(y)$. The above table shows that $\tan(\frac{\pi}{3}) = \sqrt{3}$ and so, $\tan^{-1}(\sqrt{3}) = y = \frac{\pi}{3} = 60°$.

Alternatively, we can use a calculator. Many calculators will not display the exact radian measure for the angle, but only an approximation. Nevertheless, we can use degree measure and then convert to radian if needed.



b) Similarly, we can compute the other values with the calculator.



We see that $\tan^{-1}(-1) = -45° = -\frac{\pi}{4}$.

c) For $\tan^{-1}(4.3)$, we do not have an exact value that appears in our table above. However, we can still find an approximate answer using the calculator, $\tan^{-1}(4.3) \approx 76.91$.

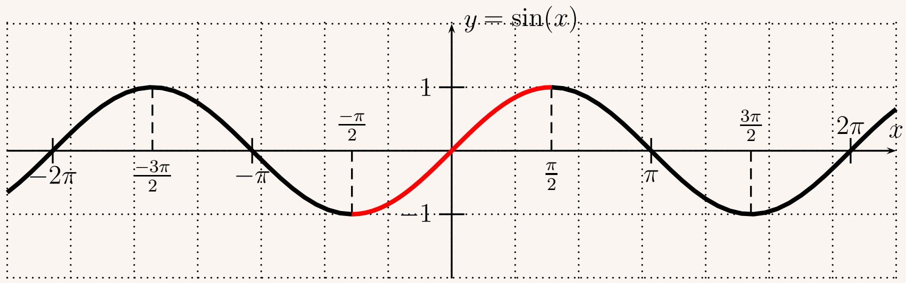
□

The function $y = \sin^{-1}(x)$

Next, we define the inverse sine function.

Note 19.6

We recall the graph of the $y = \sin(x)$ function, and note that it is *not* one-to-one.



However, when restricting the sine to the domain $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (drawn in the red part in the above graph), the restricted function *is* one-to-one. Note furthermore, that when restricting the domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, the range is $[-1, 1]$, and therefore we cannot extend this to a larger domain in a way such that the function remains a one-to-one function. We use the domain $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ to define the inverse sine function.

Definition 19.7: Inverse sine function

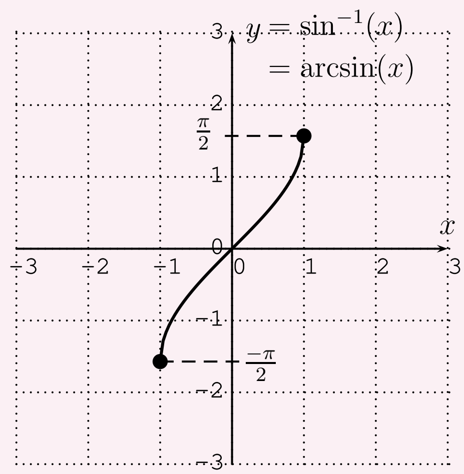
The inverse of the function $y = \sin(x)$ with restricted domain $D = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and range $R = [-1, 1]$ is called the **inverse sine** function. It is defined by

$$\boxed{x = \sin(y) \iff y = \sin^{-1}(x)} \text{ , for } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Alternatively, the inverse sine function is also written as the **arcsine** function:

$$y = \sin^{-1}(x) = \arcsin(x)$$

The arcsine reverses the input and output of the sine function, so that the arcsine has domain $D = [-1, 1]$ and range $R = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. The graph of the arcsine is drawn below.



Note 19.8: $\sin^{-1}$ is odd

The inverse sine function is odd:

$$\boxed{\sin^{-1}(-x) = -\sin^{-1}(x)} \tag{19.2}$$

This can again be seen by observing that the sine $y = \sin(x)$ is an odd function (that is, $\sin(-x) = -\sin(x)$), and is also confirmed by the symmetry of the graph with respect to the origin $(0, 0)$.

We calculate some function values of the inverse sine.

Example 19.9

Compute the inverse sine function values.

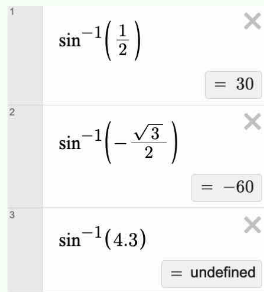
a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ c) $\sin^{-1}(4.3)$

Solution.

a) We may either use the definition or a calculator to evaluate the expressions. Since $y = \sin^{-1}(\frac{1}{2})$ is equivalent to $\frac{1}{2} = \sin(y)$, we need to find such a y with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. For this, recall the known values of the sine.

x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

We see that $y = \frac{\pi}{6}$, so that $\sin^{-1}(\frac{1}{2}) = y = \frac{\pi}{6} = 30°$. Alternatively, we can obtain the answers using the calculator in degree mode.



b) From the calculator we get that $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -60° = -\frac{\pi}{3}$.

c) As for $\sin^{-1}(4.3)$, we note that this is undefined, since the $\sin^{-1}$ has a domain of $[-1, 1]$ and so is only defined for x with $-1 \le x \le 1$.

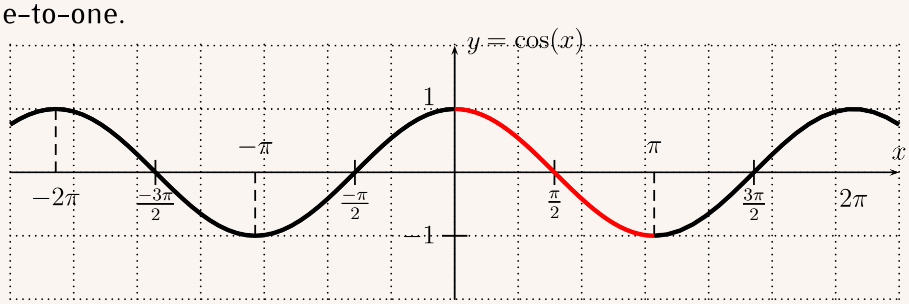
□

The function $y = \cos^{-1}(x)$

Lastly, we define the inverse cosine.

Note 19.10

Recall the graph of $y = \cos(x)$, and note again that the function is *not* one-to-one.



Again, we need to restrict the cosine to a smaller domain so that the restricted function becomes one-to-one. By convention, the cosine is restricted to the domain $[0, \pi]$ (see the red part above). This provides a function that is one-to-one, which is used to define the inverse cosine.

Definition 19.11: Inverse cosine function

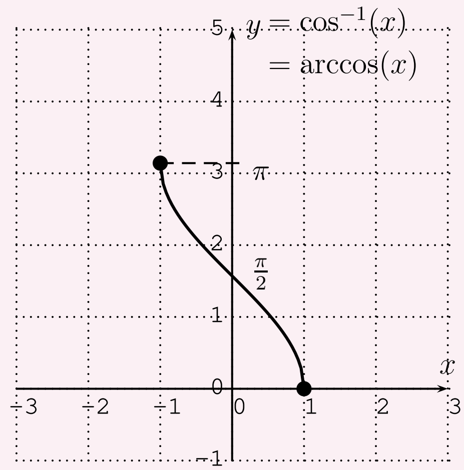
The inverse of the function $y = \cos(x)$ with restricted domain $D = [0, \pi]$ and range $R = [-1, 1]$ is called the **inverse cosine** function. It is defined by

$$\boxed{x = \cos(y) \quad \iff \quad y = \cos^{-1}(x)} \quad \text{, for } y \in [0, \pi]$$

Alternatively, the inverse cosine function is also written as the **arccosine** function:

$$y = \cos^{-1}(x) = \arccos(x)$$

The arccosine reverses the input and output of the cosine function, so that the arccosine has domain $D = [-1, 1]$ and range $R = [0, \pi]$. The graph of the arccosine is drawn below.



Note 19.12: $\cos^{-1}$ is neither even nor odd

The inverse cosine function is *neither even nor odd*. That is, the function $\cos^{-1}(-x)$ *cannot* be computed by simply taking $\pm\cos^{-1}(x)$. But it does have some symmetry given algebraically by the more complicated relation

$$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}(x)} \tag{19.3}$$

Proof of Equation (19.3). We can see that if we shift the graph down by $\frac{\pi}{2}$ the resulting function is odd. That is to say the function with the rule $\cos^{-1}(x) - \frac{\pi}{2}$ is odd:

$$\cos^{-1}(-x) - \frac{\pi}{2} = -(\cos^{-1}(x) - \frac{\pi}{2}),$$

which yields 19.3 upon distributing and adding $\frac{\pi}{2}$.

Another, more formal approach, is as follows. The right relation of (21.12) on page 367 states that we have the relation $\cos(\pi - y) = -\cos(y)$ for all y. Let $-1 \leq x \leq 1$, and denote by $y = \cos^{-1}(x)$. This means that y is the number $0 \leq y \leq \pi$ with $\cos(y) = x$. Then we have

$$-x = -\cos(y) = \cos(\pi - y) \qquad \text{(by Equation (21.12))}$$

Applying $\cos^{-1}$ to both sides gives:

$$\cos^{-1}(-x) = \cos^{-1}(\cos(\pi - y)) = \pi - y$$

The last equality follows, since cos and $\cos^{-1}$ are inverse to each other, and $0 \leq y \leq \pi$, so that $0 \leq \pi - y \leq \pi$ are also in the range of the $\cos^{-1}$. Rewriting $y = \cos^{-1}(x)$ gives the wanted result:

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

□

Example 19.13

Compute the inverse cosine function values.

a) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ b) $\cos^{-1}\left(-\frac{1}{2}\right)$ c) $\cos^{-1}(4.3)$

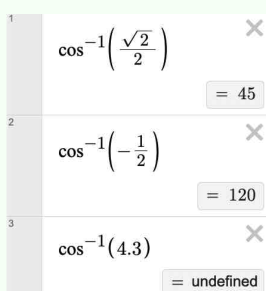
Solution.

a) Evaluating these expressions by hand requires the use of specific values of the cosine function. We recall the known values of the cosine.

x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Since $y = \cos^{-1}(x)$ is given by $x = \cos(y)$ for $0 \leq y \leq \pi$, we see that for $y = \cos^{-1}(\frac{\sqrt{2}}{2})$, we need a y with $\frac{\sqrt{2}}{2} = \cos(y)$. According to the above table, we get $y = \frac{\pi}{4}$, so that $\cos^{-1}(x) = y = \frac{\pi}{4} = 45°$.

Alternatively, we can obtain the answer with the calculator.



b) Using the calculator, we obtain $\cos^{-1}(-\frac{1}{2}) = 120° = \frac{2\pi}{3}$. Note that this is *not* the same as the negative of $\cos^{-1}(\frac{1}{2}) = 60°$, but the identity (19.3) holds: $\cos^{-1}(-\frac{1}{2}) = 180° - \cos^{-1}(\frac{1}{2})$, that is, $120° = 180° - 60°$.

c) $\cos^{-1}(4.3)$ is undefined, since the domain of $y = \cos^{-1}(x)$ is $D = [-1, 1]$.

□

19.2 Exercises

Exercise 19.1

Graph the function with the calculator. Use both radian and degree mode to display your graph. Zoom to an appropriate window for each mode to display a graph which includes the main features of the graph.

a) $y = \sin^{-1}(x)$ b) $y = \cos^{-1}(x)$ c) $y = \tan^{-1}(x)$

Exercise 19.2

Find the exact value of the inverse trigonometric function.

a) $\tan^{-1}(\sqrt{3})$ b) $\sin^{-1}(\frac{1}{2})$ c) $\cos^{-1}(\frac{1}{2})$ d) $\tan^{-1}(0)$
e) $\cos^{-1}(\frac{\sqrt{2}}{2})$ f) $\cos^{-1}(-\frac{\sqrt{2}}{2})$ g) $\sin^{-1}(-1)$ h) $\tan^{-1}(-\sqrt{3})$
i) $\cos^{-1}(-\frac{\sqrt{3}}{2})$ j) $\sin^{-1}(-\frac{\sqrt{2}}{2})$ k) $\sin^{-1}(-\frac{\sqrt{3}}{2})$ l) $\tan^{-1}(-\frac{1}{\sqrt{3}})$

Exercise 19.3

Find the inverse trigonometric function value using the calculator. Approximate your answer to the nearest hundredth.

- For parts (a)-(f), write your answer in radian mode.

a) $\cos^{-1}(0.2)$ b) $\sin^{-1}(-0.75)$ c) $\cos^{-1}(\frac{1}{3})$
d) $\tan^{-1}(100,000)$ e) $\tan^{-1}(-2)$ f) $\cos^{-1}(-2)$

- For parts (g)-(l), write your answer in degree mode.

g) $\cos^{-1}(0.68)$	h) $\tan^{-1}(-1)$	i) $\sin^{-1}(\frac{\sqrt{2}+\sqrt{6}}{4})$
j) $\tan^{-1}(100,000)$	k) $\cos^{-1}(\frac{\sqrt{2-\sqrt{2}}}{2})$	l) $\tan^{-1}(2+\sqrt{3}-\sqrt{6}-\sqrt{2})$

Chapter 20

Solving trigonometric equations

Using the inverse trigonometric functions, we now solve equations that contain $\sin$, $\cos$, or $\tan$.

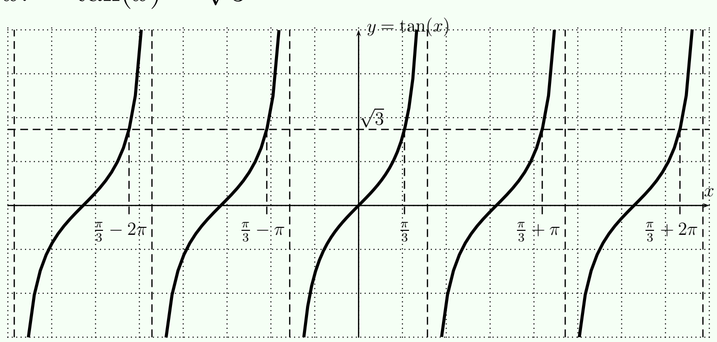
20.1 Basic trigonometric equations

In this section, we solve equations such as $\tan(x) = \sqrt{3}$. We can easily check that $x = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ solves this equation. However, there are other solutions, such as $x = \frac{4\pi}{3}$ or $x = \frac{7\pi}{3}$. Below, we will find all solutions of equations of the form $\sin(x) = c$, $\cos(x) = c$, and $\tan(x) = c$. We start with equations involving the tangent.

The equation $\tan(x) = c$

Example 20.1

Solve for x: $\quad \tan(x) = \sqrt{3}$



Solution.
There is an obvious solution given by $x = \tan^{-1}(\sqrt{3}) = 60^\circ = \frac{\pi}{3}$, as we studied in the last section. However, there are more solutions, as the graph of $\tan(x) = \sqrt{3}$ above shows. To find the other solutions, we look for all points where the graph of the $y = \tan(x)$ intersects with the horizontal line $y = \sqrt{3}$. Since the function $y = \tan(x)$ is periodic with period π, we see that the other solutions of $\tan(x) = \sqrt{3}$ besides $x = \frac{\pi}{3}$ are

$$\frac{\pi}{3}+\pi, \quad \frac{\pi}{3}+2\pi, \quad \frac{\pi}{3}+3\pi, \dots, \quad \text{and} \quad \frac{\pi}{3}-\pi, \quad \frac{\pi}{3}-2\pi, \quad \frac{\pi}{3}-3\pi, \dots$$

In general, we write the solution as

$$x = \frac{\pi}{3} + n \cdot \pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

The graph also shows that these are *all* solutions of $\tan(x) = \sqrt{3}$. □

In a similar fashion, we can solve the equation $\tan(x) = c$ by replacing $\sqrt{3}$ in the above example with c. We get the following general solution of $\tan(x) = c$.

Observation 20.2: Solving $\tan(x) = c$

To solve $\tan(x) = c$, we first determine one solution $x = \tan^{-1}(c)$. Then the general solution is given by

$$\boxed{x = \tan^{-1}(c) + n \cdot \pi} \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots \qquad (20.1)$$

Example 20.3

Solve for x:

a) $\tan(x) = 1$ b) $\tan(x) = -1$ c) $\tan(x) = 5.1$ d) $\tan(x) = -3.7$

Solution.

a) First, we find $\tan^{-1}(1) = 45^\circ = \frac{\pi}{4}$. The general solution is thus:

$$x = \frac{\pi}{4} + n \cdot \pi \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

b) We first compute one solution, $\tan^{-1}(-1) = -45° = -\frac{\pi}{4}$. The general solution of $\tan(x) = -1$ is therefore,

$$x = -\frac{\pi}{4} + n \cdot \pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

For parts (c) and (d), we do not have an exact solution, so that the solution can only be approximated with the calculator.

c)

$$x = \tan^{-1}(5.1) + n\pi \quad \approx 1.377 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2$$

d)

$$x = \tan^{-1}(-3.7) + n\pi \quad \approx -1.307 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2$$

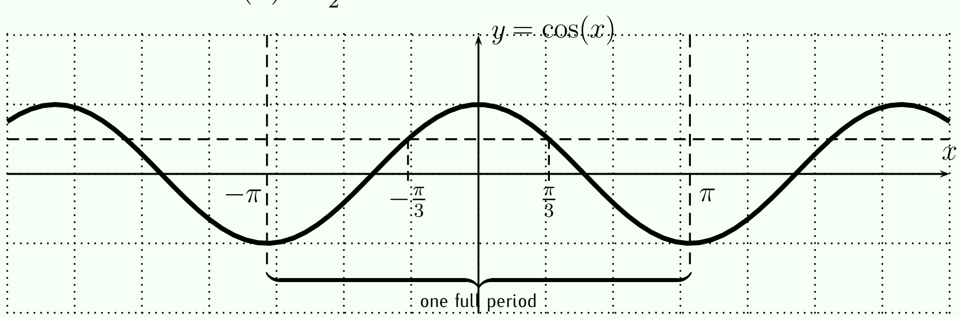
□

The equation $\cos(x) = c$

Next, we consider equations that contain a cosine. We start again by solving a specific example from which we infer the general solution.

Example 20.4

Solve for x: $\quad \cos(x) = \frac{1}{2}$



Solution.

We have the obvious solution to the equation $x = \cos^{-1}(\frac{1}{2}) = 60° = \frac{\pi}{3}$. However, since $\cos(-x) = \cos(x)$, there is another solution given by taking $x = -\frac{\pi}{3}$:

$$\cos\Big(-\frac{\pi}{3}\Big) = \cos\Big(\frac{\pi}{3}\Big) = \frac{1}{2}$$

Moreover, the $y = \cos(x)$ function is periodic with period 2π, that is, we have $\cos(x + 2\pi) = \cos(x)$. Thus, all of the following numbers are solutions of the equation $\cos(x) = \frac{1}{2}$:

$$\begin{array}{lccccccc} & \dots, & \frac{\pi}{3} - 4\pi, & \frac{\pi}{3} - 2\pi, & \frac{\pi}{3}, & \frac{\pi}{3} + 2\pi, & \frac{\pi}{3} + 4\pi, & \dots, \\ \text{and:} & \dots, & -\frac{\pi}{3} - 4\pi, & -\frac{\pi}{3} - 2\pi, & -\frac{\pi}{3}, & -\frac{\pi}{3} + 2\pi, & -\frac{\pi}{3} + 4\pi, & \dots. \end{array}$$

From the graph we see that there are only two solutions of $\cos(x) = \frac{1}{2}$ within one period. Thus, the above list constitutes *all* solutions of the equation. With this observation, we may write the general solution as:

$$\begin{array}{rl} & x = \frac{\pi}{3} + 2n\cdot\pi \\ \text{or} & x = -\frac{\pi}{3} + 2n\cdot\pi \end{array} \quad \text{where } n = 0, \pm1, \pm2, \pm3, \dots$$

In short, we write this as: $x = \pm\frac{\pi}{3} + 2n\cdot\pi$ with $n = 0, \pm1, \pm2, \pm3, \dots$.

□

We generalize this example as follows.

Observation 20.5: Solving $\cos(x) = c$

To solve $\cos(x) = c$, we first determine one solution $x = \cos^{-1}(c)$. Then the general solution is given by

$$\boxed{\begin{array}{rl} & x = \cos^{-1}(c) + 2n\cdot\pi \\ \text{or} & x = -\cos^{-1}(c) + 2n\cdot\pi \end{array}} \quad \text{where } n = 0, \pm1, \pm2, \pm3, \dots \qquad (20.2)$$

In short, we can also write this as

$$x = \pm\cos^{-1}(c) + 2n\cdot\pi \quad \text{where } n = 0, \pm1, \pm2, \pm3, \dots$$

Example 20.6

Solve for x.

a) $\cos(x) = -\frac{\sqrt{2}}{2}$ b) $\cos(x) = 0.6$ c) $\cos(x) = -3$ d) $\cos(x) = -1$

Solution.

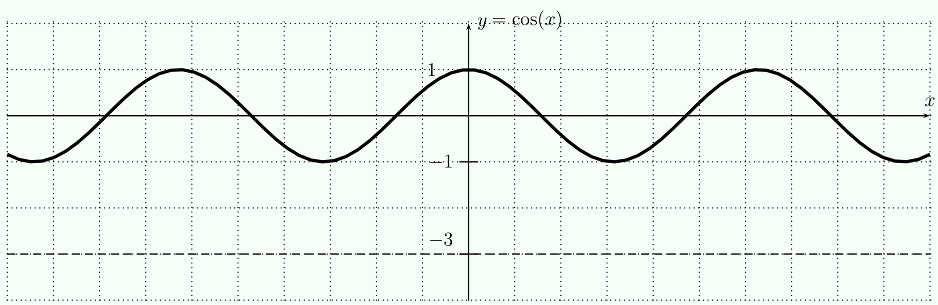
a) First, we need to find $\cos^{-1}(-\frac{\sqrt{2}}{2}) = 135° = \frac{3\pi}{4}$. The general solution is therefore,

$$x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

b) We calculate $\cos^{-1}(0.6) \approx 0.927$ with the calculator. The general solution is therefore,

$$x = \pm\cos^{-1}(0.6) + 2n\pi \quad \approx \pm 0.927 + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

c) Since the cosine is always $-1 \leq \cos(x) \leq 1$, the cosine can never be -3. Therefore, there is *no solution* to the equation $\cos(x) = -3$. This can also be seen from the graph, which does not intersect with the horizontal line $y = -3$.



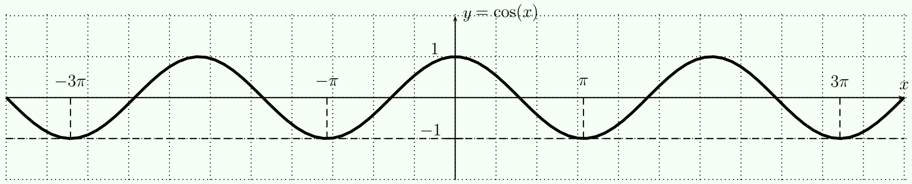
d) A special solution of $\cos(x) = -1$ is $\cos^{-1}(-1) = 180° = \pi$, so that the general solution is

$$x = \pm\pi + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

However, since $-\pi + 2\pi = +\pi$, the solutions $\pi + 2n\pi$ and $-\pi + 2n\pi$ (for $n = 0, \pm 1, \pm 2, \dots$) can be identified with each other, and there is only *one* solution in each period. Thus, the general solution can be written as

$$x = \pi + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

The graphs of $y = \cos(x)$ and $y = -1$ confirm this solution.



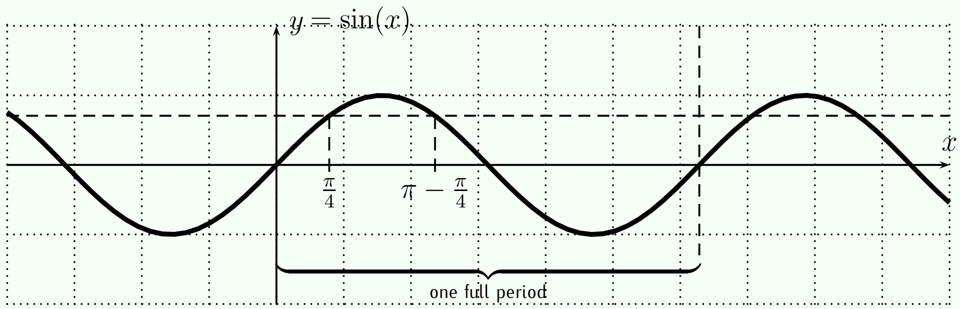
□

The equation $\sin(x) = c$

Finally, we consider equations with a sine.

Example 20.7

Solve for x: $\quad \sin(x) = \frac{\sqrt{2}}{2}$



Solution.

First, we can find one obvious solution $x = \sin^{-1}(\frac{\sqrt{2}}{2}) = 45° = \frac{\pi}{4}$. Furthermore, another solution appears to be given at an input with the same distance $\frac{\pi}{4}$ from π, that is at $\pi - \frac{\pi}{4}$:

$$\sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

In fact, we have the general identity $\sin(\pi - x) = \sin(x)$ for any x, which will be shown in (21.12). These are all solutions within one period, as can be confirmed from the graph above. The function $y = \sin(x)$ is periodic with period 2π, so that adding $2n \cdot \pi$ for any $n = 0, \pm 1, \pm 2, \ldots$

gives all solutions of $\sin(x) = \frac{\sqrt{2}}{2}$. This means that the general solution is given by:

$$\begin{array}{rl} & x = \frac{\pi}{4} + 2n \cdot \pi \\ \text{or} & x = (\pi - \frac{\pi}{4}) + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We can evaluate the second solution a bit further, since $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$, so that the final answer is:

$$\begin{array}{rl} & x = \frac{\pi}{4} + 2n \cdot \pi \\ \text{or} & x = \frac{3\pi}{4} + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

□

We have the following general statement.

Observation 20.8: Solving $\sin(x) = c$

To solve $\sin(x) = c$, we first determine one solution $x = \sin^{-1}(c)$. Then the general solution is given by

$$\boxed{\begin{array}{rl} & x = \sin^{-1}(c) + 2n \cdot \pi \\ \text{or} & x = (\pi - \sin^{-1}(c)) + 2n \cdot \pi \end{array}} \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{20.3}$$

Example 20.9

Solve for x.

a) $\sin(x) = \frac{1}{2}$ b) $\sin(x) = -\frac{1}{2}$ c) $\sin(x) = -\frac{5}{7}$ d) $\sin(x) = -1$

Solution.

a) First, we calculate $\sin^{-1}(\frac{1}{2}) = 30° = \frac{\pi}{6}$. A second solution is then given by $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The general solution is therefore,

$$\begin{array}{rl} & x = \frac{\pi}{6} + 2n \cdot \pi \\ \text{or} & x = \frac{5\pi}{6} + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) First, we calculated $\sin^{-1}(-\frac{1}{2}) = -30° = -\frac{\pi}{6}$. We find a second solution by taking $\pi - (-\frac{\pi}{6}) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$. We thus state the general solution as

$$\begin{aligned} & x = -\tfrac{\pi}{6} + 2n\cdot\pi \\ \text{or} \quad & x = \tfrac{7\pi}{6} + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

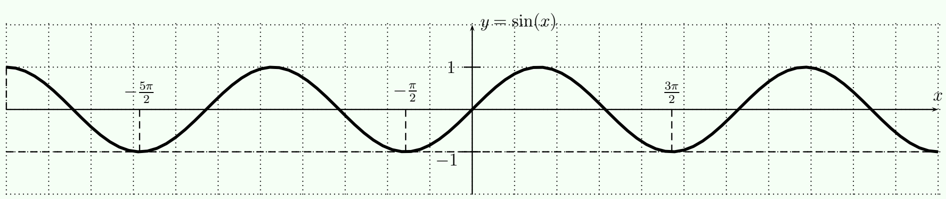
c) We do not have an exact value of $\sin^{-1}(-\frac{5}{7})$, so that we either need to leave it as is, or approximate it with the calculator to be $\sin^{-1}(-\frac{5}{7}) \approx -0.796$. A second solution is given by $\pi - \sin^{-1}(-\frac{5}{7}) \approx 3.937$. We get the solution:

$$\begin{aligned} & x \approx -0.796 + 2n\cdot\pi \\ \text{or} \quad & x \approx 3.937 + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

d) We calculate $\sin^{-1}(-1) = -90° = -\frac{\pi}{2}$, and $\pi - (-\frac{\pi}{2}) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$. The solution is therefore,

$$\begin{aligned} & x = -\tfrac{\pi}{2} + 2n\cdot\pi \\ \text{or} \quad & x = \tfrac{3\pi}{2} + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Note however, that $\frac{3\pi}{2} = -\frac{\pi}{2} + 2\pi$, so that the various solutions can be identified with each other. This can also be seen on the graph of $y = \sin(x)$:



A complete solution is therefore given by

$$x = -\frac{\pi}{2} + 2n\cdot\pi, \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Writing the solution in this way has the advantage that it does not repeat any of the solutions, and is therefore preferred.

□

Summary

We summarize the different formulas we used to solve the basic trigonometric equations in the following table.

Solve: $\sin(x) = c$	Solve: $\cos(x) = c$	Solve: $\tan(x) = c$
First, find one solution: $\sin^{-1}(c)$	First, find one solution: $\cos^{-1}(c)$	First, find one solution: $\tan^{-1}(c)$
The general solution is: $\boxed{\begin{array}{l} x = \sin^{-1}(c) + 2n\pi \\ x = (\pi - \sin^{-1}(c)) + 2n\pi \end{array}}$ where $n = 0, \pm 1, \pm 2, \ldots$	The general solution is: $\boxed{\begin{array}{l} x = \cos^{-1}(c) + 2n\pi \\ x = -\cos^{-1}(c) + 2n\pi \end{array}}$ where $n = 0, \pm 1, \pm 2, \ldots$	The general solution is: $\boxed{x = \tan^{-1}(c) + n\pi}$ where $n = 0, \pm 1, \pm 2, \ldots$

Example 20.10

Find the general solution of the equation, and state at least 6 distinct solutions.

a) $\sin(x) = -\frac{1}{2}$ b) $\cos(x) = -\frac{\sqrt{3}}{2}$

Solution.

a) We already calculated the general solution in Example 20.9(b). The solution is

$$\begin{array}{ll} & x = -\frac{\pi}{6} + n \cdot 2\pi \\ \text{or} & x = \frac{7\pi}{6} + n \cdot 2\pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We simplify the solutions for $n = 0, 1, -1$:

$$\begin{aligned} n = 0: \quad x &= -\frac{\pi}{6} + 0 \cdot 2\pi = -\frac{\pi}{6} \\ n = 1: \quad x &= -\frac{\pi}{6} + 1 \cdot 2\pi = -\frac{\pi}{6} + \frac{12\pi}{6} = \frac{11\pi}{6} \\ n = -1: \quad x &= -\frac{\pi}{6} + (-1) \cdot 2\pi = -\frac{\pi}{6} - \frac{12\pi}{6} = -\frac{13\pi}{6} \end{aligned}$$

and

$$n = 0: \quad x = \frac{7\pi}{6} + 0 \cdot 2\pi = \frac{7\pi}{6}$$

$$n = 1: \quad x = \frac{7\pi}{6} + 1 \cdot 2\pi = \frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6}$$
$$n = -1: \quad x = \frac{7\pi}{6} + (-1) \cdot 2\pi = \frac{7\pi}{6} - \frac{12\pi}{6} = -- \frac{5\pi}{6}$$

b) Since $\cos^{-1}(-\frac{\sqrt{3}}{2}) = 150^\circ = \frac{5\pi}{6}$, the solutions of $\cos(x) = -\frac{\sqrt{3}}{2}$ are:

$$x = \pm\frac{5\pi}{6} + n \cdot 2\pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We write the 6 solutions with $n = 0, +1, -1$, and for each use the two distinct first terms $+\frac{5\pi}{6}$ and $-\frac{5\pi}{6}$.

$$n = 0: \quad x = +\frac{5\pi}{6} + 0 \cdot 2\pi = \frac{5\pi}{6}$$
$$n = 1: \quad x = +\frac{5\pi}{6} + 1 \cdot 2\pi = \frac{5\pi}{6} + 2\pi = \frac{5\pi + 12\pi}{6} = \frac{17\pi}{6}$$
$$n = -1: \quad x = +\frac{5\pi}{6} + (-1) \cdot 2\pi = \frac{5\pi}{6} - 2\pi = \frac{5\pi - 12\pi}{6} = \frac{-7\pi}{6}$$

and

$$n = 0: \quad x = -\frac{5\pi}{6} + 0 \cdot 2\pi = -\frac{5\pi}{6}$$
$$n = 1: \quad x = -\frac{5\pi}{6} + 1 \cdot 2\pi = -\frac{5\pi}{6} + 2\pi = \frac{-5\pi + 12\pi}{6} = \frac{7\pi}{6}$$
$$n = -1: \quad x = -\frac{5\pi}{6} + (-1) \cdot 2\pi = -\frac{5\pi}{6} - 2\pi = \frac{-5\pi - 12\pi}{6} = \frac{-17\pi}{6}$$

Further solutions can be found by taking values $n = +2, -2, +3, -3, \ldots$. □

20.2 Equations involving trigonometric functions

The previous section explained how to solve the basic trigonometric equations

$$\sin(x) = c, \quad \cos(x) = c, \quad \text{and} \quad \tan(x) = c.$$

The next examples can be reduced to these basic equations.

Example 20.11

Solve for x.

a) $2\sin(x) - \sqrt{3} = 0$ b) $\sec(x) = -\sqrt{2}$ c) $7\cot(x) + 3 = 0$

Solution.

a) Solving for $\sin(x)$, we get

$$2\sin(x) - \sqrt{3} = 0 \overset{(+\sqrt{3})}{\Longrightarrow} 2\sin(x) = \sqrt{3} \overset{(\div 2)}{\Longrightarrow} \sin(x) = \frac{\sqrt{3}}{2}$$

One solution of $\sin(x) = \frac{\sqrt{3}}{2}$ is $\sin^{-1}(\frac{\sqrt{3}}{2}) = 60° = \frac{\pi}{3}$. Another solution is given by $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. The general solution is

$$\begin{array}{ll} & x = \frac{\pi}{3} + 2n\pi \\ \text{or} & x = \frac{2\pi}{3} + 2n\pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) Recall that $\sec(x) = \frac{1}{\cos(x)}$. Therefore,

$$\sec(x) = -\sqrt{2} \implies \frac{1}{\cos(x)} = -\sqrt{2} \overset{\text{(reciprocal)}}{\Longrightarrow} \cos(x) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

A special solution of $\cos(x) = -\frac{\sqrt{2}}{2}$ is $\cos^{-1}(-\frac{\sqrt{2}}{2}) = 135° = \frac{3\pi}{4}$. The general solution is

$$x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

c) Recall that $\cot(x) = \frac{1}{\tan(x)}$. So

$$\begin{aligned} 7\cot(x) + 3 = 0 &\overset{(-3)}{\Longrightarrow} 7\cot(x) = -3 \overset{(\div 7)}{\Longrightarrow} \cot(x) = -\frac{3}{7} \\ &\Longrightarrow \frac{1}{\tan(x)} = -\frac{3}{7} \overset{\text{(reciprocal)}}{\Longrightarrow} \tan(x) = -\frac{7}{3} \end{aligned}$$

The solution is

$$x = \tan^{-1}\left(-\frac{7}{3}\right) + n\pi \approx -1.166 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

□

For the next problems we combine quadratic functions with trigonometric functions. It is customary to use the following notation.

Convention 20.12: Square of a trigonometric function

We denote the square of a trigonometric function as follows:

$$\sin^2\alpha := (\sin\alpha)^2 \qquad \cos^2\alpha := (\cos\alpha)^2 \qquad \tan^2\alpha := (\tan\alpha)^2$$

In order to solve quadratic trigonometric equations, it can be helpful to substitute u for a trigonometric expression first, then solve for u, and finally apply the rules from the previous section to solve for the wanted variable. This method is shown in the next example.

Example 20.13

Solve for x.

a) $2\sin^2(x)+\sqrt{3}\sin(x)=0$ b) $2\cos^2(x)-1=0$
c) $\tan^2(x)+2\tan(x)+1=0$

Solution.

a) We first need to solve $2\sin^2(x)+\sqrt{3}\sin(x)=0$ for $\sin(x)$. In this case, this can be done either by factoring $\sin(x)$ directly, that is, by writing $\sin(x)\cdot(2\sin(x)+\sqrt{3})=0$, or, more thoroughly, by substituting $u=\sin(x)$, and then solving for u, for which we get:

$$2u^2+\sqrt{3}u=0 \overset{\text{(factor } u)}{\Longrightarrow} u\cdot(2u+\sqrt{3})=0 \implies u=0 \text{ or } 2u+\sqrt{3}=0$$

We get two trigonometric equations that we need to solve:

$\sin(x)=0$	$2\sin(x)+\sqrt{3}=0$ $\implies 2\sin(x)=-\sqrt{3}$ $\implies \sin(x)=-\frac{\sqrt{3}}{2}$
then: $\sin^{-1}(0)=0^\circ=0$	then: $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-60^\circ=-\frac{\pi}{3}$
and: $\pi-0=\pi$	and: $\pi-(-\frac{\pi}{3})=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$
$\implies x=0+2n\pi$ or $x=\pi+2n\pi$ where $n=0,\pm1,\pm2,\dots$	$\implies x=-\frac{\pi}{3}+2n\pi$ or $x=\frac{4\pi}{3}+2n\pi$ where $n=0,\pm1,\pm2,\dots$

The general solution is therefore,

$$\begin{aligned} x &= 0 + 2n\pi, \quad \text{or} \quad x = -\frac{\pi}{3} + 2n\pi, \\ \text{or } x &= \pi + 2n\pi, \quad \text{or} \quad x = \frac{4\pi}{3} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

b) Substituting $u = \cos(x)$, we get

$$\begin{aligned} 2u^2 - 1 = 0 \quad &\overset{(+1)}{\Longrightarrow} \quad 2u^2 = 1 \quad \overset{(\div 2)}{\Longrightarrow} \quad u^2 = \frac{1}{2} \\ &\Longrightarrow \quad u = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \\ &\Longrightarrow \quad u = +\frac{\sqrt{2}}{2} \quad \text{or} \quad u = -\frac{\sqrt{2}}{2} \end{aligned}$$

For each of the two cases, we need to solve the corresponding trigonometric equation after replacing $u = \cos(x)$.

$\cos(x) = \frac{\sqrt{2}}{2}$	$\cos(x) = -\frac{\sqrt{2}}{2}$
then: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ = \frac{\pi}{4}$	then: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135^\circ = \frac{3\pi}{4}$
$\Longrightarrow x = \pm\frac{\pi}{4} + 2n\pi$ where $n = 0, \pm 1, \pm 2, \dots$	$\Longrightarrow x = \pm\frac{3\pi}{4} + 2n\pi$ where $n = 0, \pm 1, \pm 2, \dots$

Thus, the general solution is,

$$x = \pm\frac{\pi}{4} + 2n\pi, \quad \text{or} \quad x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \dots$$

c) Substituting $u = \tan(x)$, we have to solve the equation

$$u^2 + 2u + 1 = 0 \overset{\text{(factor)}}{\Longrightarrow} (u+1)(u+1) = 0 \Longrightarrow u + 1 = 0 \overset{(-1)}{\Longrightarrow} u = -1$$

Resubstituting $u = \tan(x)$, we have to solve $\tan(x) = -1$. Using the fact that $\tan^{-1}(-1) = -45^\circ = -\frac{\pi}{4}$, we have the general solution

$$x = -\frac{\pi}{4} + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \dots$$

□

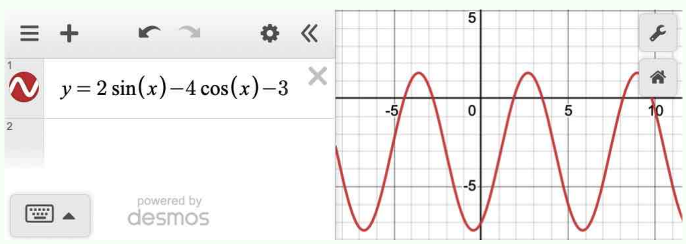
Example 20.14

Solve the equation with the calculator. Approximate the solution to the nearest thousandth.

a) $2\sin(x) = 4\cos(x) + 3$ b) $5\cos(2x) = \tan(x)$

Solution.

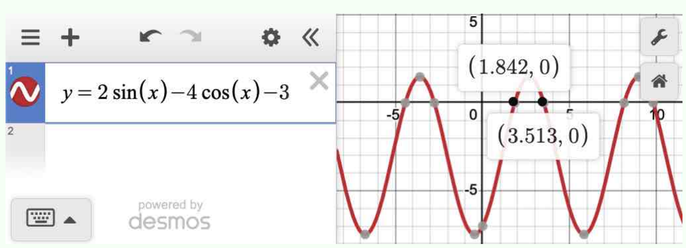
a) We rewrite the equation as $2\sin(x) - 4\cos(x) - 3 = 0$, and use the calculator to find the graph of the function $f(x) = 2\sin(x) - 4\cos(x) - 3$. The zeros of the function f are the solutions of the initial equation. The graph that we obtain is displayed below.



The graph indicates that the function $f(x) = 2\sin(x) - 4\cos(x) - 3$ is periodic. This can be confirmed by observing that both $\sin(x)$ and $\cos(x)$ are periodic with period 2π, and thus also $f(x)$.

$$f(x + 2\pi) = 2\sin(x + 2\pi) - 4\cos(x + 2\pi) - 3$$
$$= 2\sin(x) - 4\cos(x) - 3 = f(x)$$

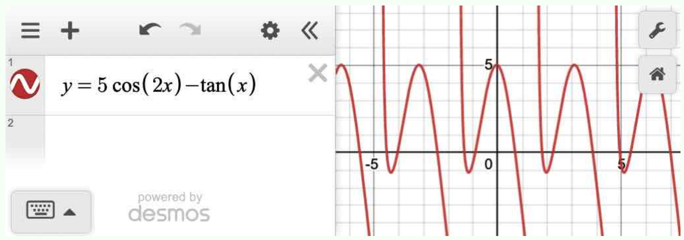
The solution of $f(x) = 0$ can be approximated by clicking on the roots.



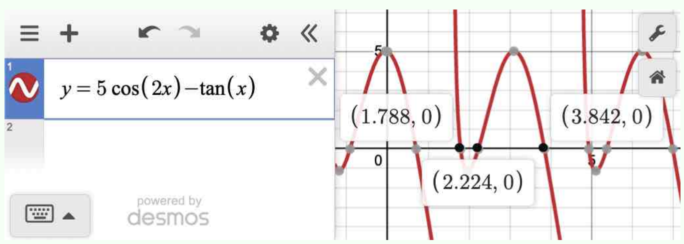
The general solution is thus

$$x \approx 1.842 + 2n\pi \quad \text{or} \quad x \approx 3.513 + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

b) We rewrite the equation as $5\cos(2x) - \tan(x) = 0$ and graph the function $f(x) = 5\cos(2x) - \tan(x)$ in the standard window.



Note again that the function f is periodic. The period of $\cos(2x)$ is $\frac{2\pi}{2} = \pi$ (see Definition 18.7 on page 321), and the period of $\tan(x)$ is also π (see Equation (18.3) on page 317). Thus, f is also periodic with period π. The solutions in one period are approximated by finding the zeros with the calculator.



The general solution is given by any of these numbers, with possibly an additional shift by any multiple of π.

$$x \approx 1.788 + n\pi \quad \text{or} \quad x \approx 2.224 + n\pi \quad \text{or} \quad x \approx 3.842 + n\pi,$$
$$\text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

□

20.3 Exercises

Exercise 20.1

Find all solutions of the equation, and simplify as much as possible. Do not approximate the solution.

a) $\tan(x) = \frac{\sqrt{3}}{3}$ b) $\sin(x) = \frac{\sqrt{3}}{2}$ c) $\sin(x) = -\frac{\sqrt{2}}{2}$ d) $\cos(x) = \frac{\sqrt{3}}{2}$
e) $\cos(x) = 0$ f) $\cos(x) = -0.5$ g) $\cos(x) = 1$ h) $\sin(x) = 5$
i) $\sin(x) = 0$ j) $\sin(x) = -1$ k) $\tan(x) = -\sqrt{3}$ l) $\cos(x) = 0.2$

Exercise 20.2

Find all solutions of the equation. Approximate your solution with the calculator.

a) $\tan(x) = 6.2$ b) $\cos(x) = 0.45$ c) $\sin(x) = 0.91$
d) $\cos(x) = -.772$ e) $\tan(x) = -0.2$ f) $\sin(x) = -0.06$

Exercise 20.3

Find at least 5 distinct solutions of the equation.

a) $\tan(x) = -1$ b) $\cos(x) = \frac{\sqrt{2}}{2}$ c) $\sin(x) = -\frac{\sqrt{3}}{2}$ d) $\tan(x) = 0$
e) $\cos(x) = 0$ f) $\cos(x) = 0.3$ g) $\sin(x) = 0.4$ h) $\sin(x) = -1$

Exercise 20.4

Solve for x. State the general solution without approximation.

a) $\tan(x) - 1 = 0$ b) $2\sin(x) = 1$ c) $2\cos(x) + \sqrt{3} = 0$
d) $\sqrt{2}\cos(x) - 1 = 0$ e) $\sec(x) = -2$ f) $\cot(x) = \sqrt{3}$

Exercise 20.5

Solve for x. State the general solution without approximation.

a) $2\sin^2(x) - \sqrt{2}\sin(x) = 0$
b) $\tan^2(x) + \tan(x) = 0$
c) $2\cos^2(x) + \sqrt{3}\cos(x) = 0$
d) $\sin^2(x) + \sin(x) = 0$
e) $\tan^2(x) - 3 = 0$
f) $4\cos^2(x) - 1 = 0$
g) $4\sin^2(x) - 3 = 0$
h) $\cos(x)\sin(x) + \sin(x) = 0$
i) $\tan(x)\cos(x) + \sqrt{3}\cos(x) = 0$
j) $\cos^2(x) + 7\cos(x) + 6 = 0$
k) $4\cos^2(x) - 4\cos(x) + 1 = 0$
l) $2\sin^2(x) + 11\sin(x) = -5$
m) $2\sin^2(x) + \sin(x) - 1 = 0$
n) $2\cos^2(x) - 3\cos(x) + 1 = 0$
o) $2\cos^2(x) + 9\cos(x) = 5$
p) $\tan^3(x) - \tan(x) = 0$

Exercise 20.6

Use the calculator to find all solutions of the given equation. Approximate the answer to the nearest thousandth.

a) $2\cos(x) = 2\sin(x) + 1$
b) $7\tan(x) \cdot \cos(2x) = 1$
c) $4\cos^2(3x) + \cos(3x) = \sin(3x) + 2$
d) $\sin(x) + \tan(x) = \cos(x)$

Chapter 21

Trigonometric identities

In this section, we state and summarize various important identities of trigonometric functions, some of which we have already used in previous sections. We will look at four kinds of identities:

1. Reciprocal identities and quotient identities
2. Pythagorean identities
3. Identities involving signs
4. Identities from adding $\frac{\pi}{2}$ or π to an angle
5. Addition, subtraction of angles formulas, half- and double-angle formulas

21.1 Reciprocal, Pythagorean, and sign identities

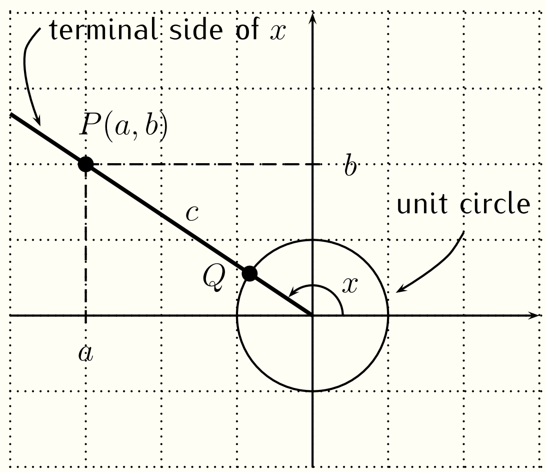
We start by recalling the definition of the trigonometric functions. In fact, going beyond the unit circle, we will restate the definition in a slightly more general setting, that is, stating the trigonometric functions for *any* point on the terminal side of the angle.

Observation 21.1: $\sin$, $\cos$, $\tan$ via point on the terminal side

Let x be an angle. Consider the terminal side of the angle x, and assume that the point $P(a, b)$ is a point on the terminal side of x (not necessarily on the unit circle). Let c be the distance from P to the origin $(0,0)$. Note that the the Pythagorean theorem states that

$$a^2+b^2=c^2 \quad \Longrightarrow \quad \boxed{c=\sqrt{a^2+b^2}} \tag{21.1}$$

Now, dividing the coordinates of P by c gives another point Q also on the terminal side of x with coordinates $(\frac{a}{c},\frac{b}{c})$, and moreover, Q is on the unit circle, since its distance to the origin is $\sqrt{(\frac{a}{c})^2+(\frac{b}{c})^2}=\sqrt{\frac{a^2}{c^2}+\frac{b^2}{c^2}}=\sqrt{\frac{a^2+b^2}{c^2}}=\sqrt{\frac{c^2}{c^2}}=\sqrt{1}=1$.



Therefore, the trigonometric function values of x are given by the coordinates of $Q(\frac{a}{c},\frac{b}{c})$, that is:

$$\boxed{\sin(x)=\frac{b}{c}} \qquad \boxed{\cos(x)=\frac{a}{c}} \qquad \boxed{\tan(x)=\frac{b}{a}} \tag{21.2}$$

where we used that $\tan(x)=\frac{\sin(x)}{\cos(x)}=\frac{\frac{b}{c}}{\frac{a}{c}}=\frac{b}{c}\cdot\frac{c}{a}=\frac{b}{a}$. Moreover, the cosecant, the secant, and the cotangent are given by:

$$\csc(x)=\frac{1}{\sin(x)}=\frac{c}{b} \qquad \sec(x)=\frac{1}{\cos(x)}=\frac{c}{a} \qquad \cot(x)=\frac{\cos(x)}{\sin(x)}=\frac{a}{b}$$

1. Reciprocal identities and quotient identities

From the above observation, we have the following immediate identities between the trigonometric functions:

Observation 21.2: Reciprocal identities and quotient identities

The following reciprocal identities hold true (whenever both sides are defined):

$$\boxed{\sin(x) = \frac{1}{\csc(x)}} \quad \boxed{\cos(x) = \frac{1}{\sec(x)}} \quad \boxed{\tan(x) = \frac{1}{\cot(x)}} \tag{21.3}$$

$$\boxed{\csc(x) = \frac{1}{\sin(x)}} \quad \boxed{\sec(x) = \frac{1}{\cos(x)}} \quad \boxed{\cot(x) = \frac{1}{\tan(x)}} \tag{21.4}$$

The quotient identities hold true:

$$\boxed{\tan(x) = \frac{\sin(x)}{\cos(x)}} \qquad \boxed{\cot(x) = \frac{\cos(x)}{\sin(x)}} \tag{21.5}$$

Example 21.3

Write the expression as one of the six trigonometric functions.

a) $\sin(x) \cdot \cot(x)$ b) $\frac{\cot(x)}{\csc(x)\cos(x)} \cdot \frac{\tan(x)}{\sin(x)}$

Solution.

a) $\sin(x) \cdot \cot(x) = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x)$

b) We rewrite in terms of $\sin(x)$ and $\cos(x)$ and cancel:

$$\frac{\cot(x)}{\csc(x)\cos(x)} \cdot \frac{\tan(x)}{\sin(x)} = \frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}\cos(x)} \cdot \frac{\frac{\sin(x)}{\cos(x)}}{\sin(x)} = \frac{1}{\cos(x)} = \sec(x)$$

□

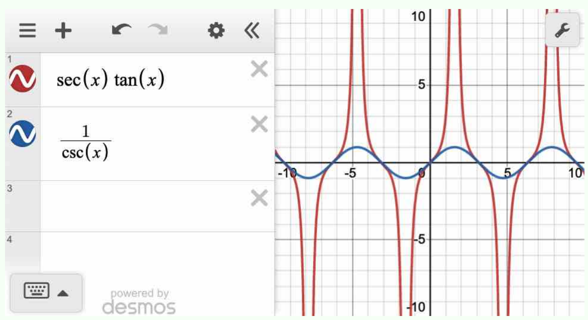
Example 21.4

Determine whether the identity is true or false.

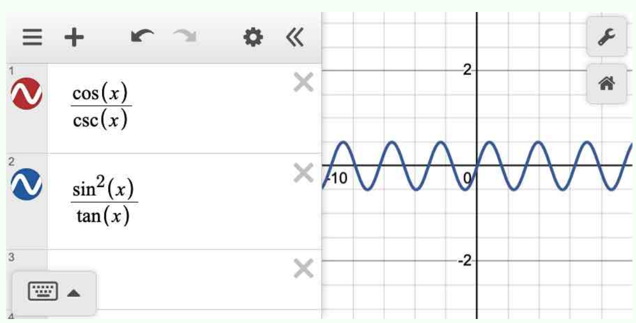
a) $\sec(x) \cdot \tan(x) = \frac{1}{\csc(x)}$ b) $\frac{\cos(x)}{\csc(x)} = \frac{\sin^2(x)}{\tan(x)}$

Solution.

a) Since $\sec(x) \cdot \tan(x) = \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)}$ and $\frac{1}{\csc(x)} = \frac{1}{\frac{1}{\sin(x)}} = \sin(x)$, we see that the expression on the right-hand side is different from the expression on the left-hand side, and so, the identity is false. We can also check this by graphing both sides with the calculator, which, indeed, shows that the two expressions are different.



b) Note that the calculator appears to show the same graph for $\frac{\cos(x)}{\csc(x)}$ and $\frac{\sin^2(x)}{\tan(x)}$.



To confirm this, calculate $\frac{\cos(x)}{\csc(x)} = \frac{\cos(x)}{\frac{1}{\sin(x)}} = \cos(x) \cdot \sin(x)$ and $\frac{\sin^2(x)}{\tan(x)} = \frac{\sin^2(x)}{\frac{\sin(x)}{\cos(x)}} = \sin^2(x)\frac{\cos(x)}{\sin(x)} = \sin(x) \cdot \cos(x)$, which are, indeed, equal. Therefore, the identity is true.

$\square$

2. Pythagorean identities

The next identities come from the Pythagorean theorem.

Observation 21.5: Pythagorean identities

The Pythagorean identities hold true:

$$\boxed{\sin^2(x) + \cos^2(x) = 1} \tag{21.6}$$

$$\boxed{\sec^2(x) = 1 + \tan^2(x)} \tag{21.7}$$

$$\boxed{\csc^2(x) = 1 + \cot^2(x)} \tag{21.8}$$

Proof. In the notation of Observation 21.1, where $P(a, b)$ is a point on the terminal side of x with distance c to the origin, the Pythagorean theorem states $a^2 + b^2 = c^2$ (see (21.1)). Therefore, from (21.2), we have

$$\sin^2(x) + \cos^2(x) = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1$$

Similarly, $1 + \tan^2(x) = 1 + \left(\frac{b}{a}\right)^2 = \frac{a^2+b^2}{a^2} = \frac{c^2}{a^2} = \left(\frac{c}{a}\right)^2 = \sec^2(x)$, and $1 + \cot^2(x) = 1 + \left(\frac{a}{b}\right)^2 = \frac{b^2+a^2}{b^2} = \frac{c^2}{b^2} = \left(\frac{c}{b}\right)^2 = \csc^2(x)$. □

Example 21.6

Simplify the expression as much as possible.

a) $(\cos(x) - 1) \cdot (\cos(x) + 1)$ b) $\frac{1-\sec^2(x)}{\cot(x)}$ c) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}$

Solution.

a) $(\cos(x) - 1) \cdot (\cos(x) + 1) = \cos^2(x) - 1 \overset{(21.6)}{=} -\sin^2(x)$

b) $\frac{1-\sec^2(x)}{\cot(x)} \overset{(21.7)}{=} \frac{-\tan^2(x)}{\frac{1}{\tan(x)}} = -\tan^2(x) \cdot \frac{\tan(x)}{1} = -\tan^3(x)$

c) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} = \frac{\sin^2(x)}{\sin(x)\cos(x)} + \frac{\cos^2(x)}{\sin(x)\cos(x)} = \frac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)} = \frac{1}{\sin(x)\cos(x)}$

□

Example 21.7

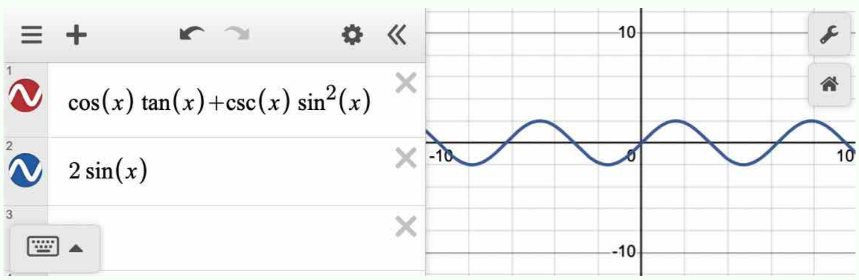
Determine whether the identity is true or false.

a) $\cos(x)\tan(x) + \csc(x)\sin^2(x) = 2\sin(x)$

b) $\frac{\cos^2(x)-1}{1-\sec^2(x)} = \frac{\cos(x)+1}{2}$

Solution.

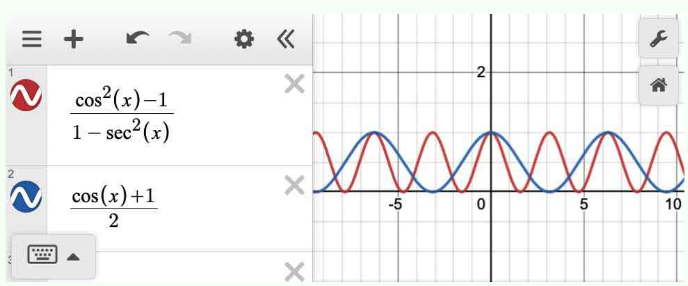
a) We use the calculator to check for differences between the right- and left-hand sides of the equation. The two sides appear to be equal.



To verify the identity, we compute:

$$\cos(x)\tan(x) + \csc(x)\sin^2(x) = \cos(x)\frac{\sin(x)}{\cos(x)} + \frac{1}{\sin(x)}\sin^2(x)$$
$$= \sin(x) + \sin(x) = 2\sin(x)$$

b) The calculator shows that the two sides differ. The identity is false.



□

3. Identities involving signs

When changing an angle to its negative angle, the trigonometric functions also transform in a well-behaved manner. We now state and prove these transformation identities.

Observation 21.8: Identities involving signs

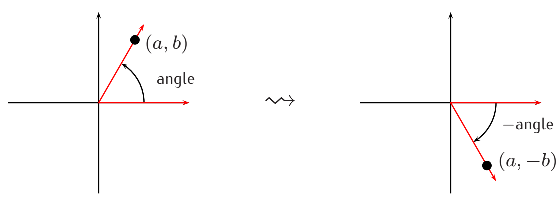
The definition of $\sin(x)$ and $\cos(x)$ gives the following behavior under sign change.

$$\boxed{\sin(-x) = -\sin(x)} \qquad \boxed{\cos(-x) = \cos(x)} \tag{21.9}$$

$$\boxed{\csc(-x) = -\csc(x)} \qquad \boxed{\sec(-x) = \sec(x)} \tag{21.10}$$

$$\boxed{\tan(-x) = -\tan(x)} \qquad \boxed{\cot(-x) = -\cot(x)} \tag{21.11}$$

Proof. The negative of an angle has a terminal side that is reflected about the x-axis, so that the cosine (which is the x-coordinate of a point on the terminal side) stays the same, and the sine (which is the y-coordinate of a point on the terminal side) becomes the negative of the original angle.



While the above picture is for an angle in the first quadrant, the argument holds in general. Convince yourself that the same holds in other quadrants as well!

The other identities (21.10) and (21.11) then follow from the reciprocal and quotient identities (21.4) and (21.5). □

21.2 *Optional section:* Further identities revisited

To give a more complete picture, we now state and provide a proof for some further identities. Several of these identities have already been encountered in previous sections.

4. Identities from adding $\frac{\pi}{2}$ or π to an angle

Observation 21.9: Identities from adding $\frac{\pi}{2}$ or π to an angle

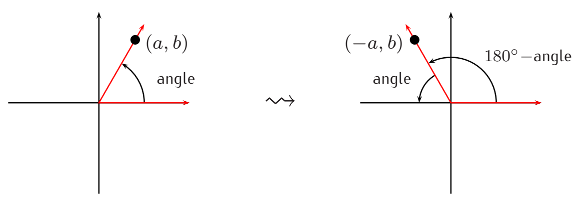
$$\boxed{\sin(\pi - x) = \quad \sin(x)} \qquad \boxed{\cos(\pi - x) = -\cos(x)} \tag{21.12}$$

$$\boxed{\sin(x + \frac{\pi}{2}) = \quad \cos x} \qquad \boxed{\cos(x + \frac{\pi}{2}) = -\sin(x)} \tag{21.13}$$

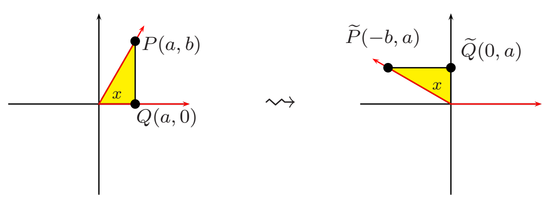
$$\boxed{\sin(x - \frac{\pi}{2}) = -\cos x} \qquad \boxed{\cos(x - \frac{\pi}{2}) = \quad \sin(x)} \tag{21.14}$$

Proof. Just as in the proof to Observation 21.8, we will give the argument in general, but only provide pictures for an angle in the first quadrant. Convince yourself that the same argument holds in other quadrants as well!

Taking $(\pi-\text{angle}) = (180°-\text{angle})$ reflects the terminal side of the angle about the y-axis. Therefore, the cosine becomes negative, and the sine stays the same.



For the identity $\sin(x + \frac{\pi}{2}) = \cos x$, note that adding $\frac{\pi}{2} = 90°$ rotates the terminal side by $90°$. If $P(a, b)$ are coordinates on the terminal side of the angle, then consider the triangle $\triangle OQP$, given by the points $O(0, 0)$ and $Q(a, 0)$. Note that triangle $\triangle OQP$ is congruent to triangle $\triangle O\widetilde{Q}\widetilde{P}$ where $\widetilde{Q}(0, a)$ and $\widetilde{P}(-b, a)$, and they have the same angle x at the origin.



Note that the point $\widetilde{P}(-b, a)$ lies on the terminal side of $x + 90°$. Thus, $\sin(x + 90°) = a$ is the vertical coordinate of $\widetilde{P}(-b, a)$, which equals $\cos(x) = a$, the horizontal coordinate of $P(a, b)$. We have shown that $\sin(x + \frac{\pi}{2}) = \cos x$ for all x.

We check the remaining identities with the identities we have already proved. Applying $\cos(u) = \sin(u + \frac{\pi}{2})$ to $u = x + \frac{\pi}{2}$ gives:

$$\cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + \pi) \overset{(21.12)}{=} \sin(\pi - (x + \pi)) \overset{(21.9)}{=} \sin(-x) = -\sin(x)$$

This proves (21.13). For (21.14), note that:

$$-\cos(x) \overset{(21.12)}{=} \cos(\pi - x) \overset{(21.9)}{=} \cos(x-\pi) \overset{(21.13)}{=} \sin(x-\pi+\frac{\pi}{2}) = \sin(x-\frac{\pi}{2})$$

$$\sin(x) \overset{(21.12)}{=} \sin(\pi - x) \overset{(21.9)}{=} -\sin(x-\pi) \overset{(21.13)}{=} \cos(x-\pi+\frac{\pi}{2}) = \cos(x-\frac{\pi}{2})$$

□

Example 21.10

Simplify the expression as much as possible.

a) $\cos(x+\pi)$ b) $\tan(x+\frac{\pi}{2})$

Solution.

a) $\cos(x+\pi) = \cos(\pi-(-x)) \overset{(21.12)}{=} -\cos(-x) \overset{(21.9)}{=} -\cos(x)$

b) $\tan(x+\frac{\pi}{2}) = \frac{\sin(x+\frac{\pi}{2})}{\cos(x+\frac{\pi}{2})} \overset{(21.13)}{=} \frac{\cos(x)}{-\sin(x)} = -\cot(x)$

□

5. Addition, subtraction of angles formulas, half- and double-angle formulas

We end this section by revisiting the addition and subtraction of angles formulas, and the half- and double-angle formulas. In fact, we will give a proof of these identities. We first recall the identities and give an example.

Proposition 21.11: Addition and subtraction of angles formulas

For any angles α and β, we have the following **addition and subtraction of angles formulas**:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

Proposition 21.12: Half- and double-angle formulas

Let α be an angle. Then we have the **half-angle formulas**:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

Here, the signs "$\pm$" are determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies. (For more on the signs, see also page 310.)
Furthermore, we have the **double-angle formulas**:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha-\sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha-1 \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Example 21.13

Find the trigonometric functions of 2α when α has the properties below.

a) $\sin(\alpha)=\frac{3}{5}$, and α is in quadrant II
b) $\tan(\alpha)=\frac{12}{5}$, and α is in quadrant III

Solution.

a) From $\sin^2(\alpha)+\cos^2(\alpha)=1$, we find that $\cos^2(\alpha)=1-\sin^2(\alpha)$, and since α is in the second quadrant, $\cos(\alpha)$ is negative, so that

$$\begin{aligned}
\cos(\alpha) &= -\sqrt{1-\sin^2(\alpha)} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\sqrt{1-\frac{9}{25}} \\
&= -\sqrt{\frac{25-9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5},
\end{aligned}$$

and

$$\tan(\alpha)=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{3}{5}}{\frac{-4}{5}}=\frac{3}{5}\cdot\frac{5}{-4}=-\frac{3}{4}$$

From this we can calculate the solution by plugging these values into the double-angle formulas.

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{(-4)}{5} = \frac{-24}{25}\\
\cos(2\alpha) &= \cos^2(\alpha)-\sin^2(\alpha) = \left(\frac{-4}{5}\right)^2-\left(\frac{3}{5}\right)^2 = \frac{16}{25}-\frac{9}{25}=\frac{7}{25}\\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\left(\frac{-3}{4}\right)}{1-\left(\frac{-3}{4}\right)^2} = \frac{\frac{-3}{2}}{1-\frac{9}{16}} = \frac{\frac{-3}{2}}{\frac{16-9}{16}}\\
&= \frac{-3}{2}\cdot\frac{16}{7} = \frac{-24}{7}
\end{aligned}$$

b) Similar to the calculation in part (a), we first calculate $\sin(\alpha)$ and $\cos(\alpha)$, which are both negative in the third quadrant. Recall from Equation (21.6) on page 364 that $\sec^2\alpha = 1+\tan^2\alpha$, where $\sec\alpha = \frac{1}{\cos\alpha}$. Therefore,

$$\sec^2\alpha = 1+\left(\frac{12}{5}\right)^2 = 1+\frac{144}{25} = \frac{25+144}{25} = \frac{169}{25} \implies \sec\alpha = \pm\frac{13}{5}$$

Since $\cos(\alpha)$ is negative (in quadrant III), so is $\sec(\alpha)$, so that we get,

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{-\frac{13}{5}} = -\frac{5}{13}$$

Furthermore, $\sin^2\alpha = 1-\cos^2\alpha$, and $\sin\alpha$ is negative (in quadrant III), we have

$$\begin{aligned}
\sin\alpha &= -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\left(-\frac{5}{13}\right)^2} = -\sqrt{1-\frac{25}{169}}\\
&= -\sqrt{\frac{169-25}{169}} = -\sqrt{\frac{144}{169}} = -\frac{12}{13}
\end{aligned}$$

Thus, we obtain the solution as follows:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha = 2\cdot\frac{(-12)}{13}\cdot\frac{(-5)}{13} = \frac{120}{169}\\
\cos(2\alpha) &= \cos^2(\alpha)-\sin^2(\alpha) = \left(\frac{-5}{13}\right)^2-\left(\frac{-12}{13}\right)^2
\end{aligned}$$

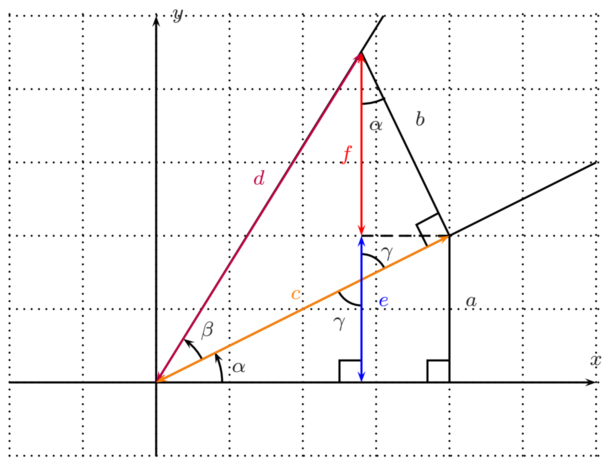
$$
\begin{aligned}
&= \frac{25}{169} - \frac{144}{169} = \frac{-119}{169} \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\frac{12}{5}}{1-\left(\frac{12}{5}\right)^2} = \frac{\frac{24}{5}}{1-\frac{144}{25}} = \frac{\frac{24}{5}}{\frac{25-144}{25}} \\
&= \frac{24}{5}\cdot\frac{25}{-119} = \frac{120}{-119}
\end{aligned}
$$

□

We now give a proof of Proposition 21.11.

Proof of Proposition 21.11. We start with the proof of the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are angles between 0 and $\frac{\pi}{2} = 90°$. We prove the addition formulas (for $\alpha, \beta \in (0, \frac{\pi}{2})$) in a quite elementary way, and then show that the addition formulas also hold for arbitrary angles α and β.

To find $\sin(\alpha+\beta)$, consider the following setup.

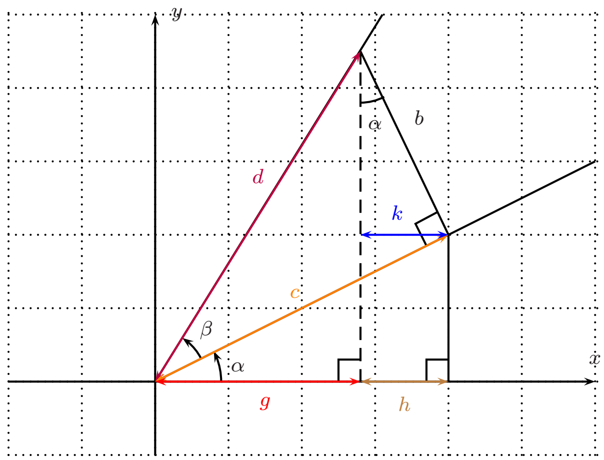


Note that there are vertically opposite angles, labelled by γ, which are therefore equal. These angles are angles in two right triangles, with the third angle being α. We therefore see that the angle α appears again as the angle among the sides b and f. With this, we can now calculate $\sin(\alpha+\beta)$.

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{e+f}{d} = \frac{e}{d} + \frac{f}{d} = \frac{a}{d} + \frac{f}{d} = \frac{a}{c}\cdot\frac{c}{d} + \frac{f}{b}\cdot\frac{b}{d} \\
&= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)
\end{aligned}
$$

The above figure displays the situation when $\alpha+\beta \leq \frac{\pi}{2}$. There is a similar figure for $\frac{\pi}{2} < \alpha+\beta < \pi$. (We recommend as an exercise to draw the corresponding figure for the case of $\frac{\pi}{2} < \alpha+\beta < \pi$.)

Next, we prove the addition formula for $\cos(\alpha+\beta)$. The following figure depicts the relevant objects.



We calculate $\cos(\alpha+\beta)$ as follows.

$$\begin{aligned}\cos(\alpha+\beta) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{g}{d} = \frac{g+h}{d} - \frac{h}{d} = \frac{g+h}{d} - \frac{k}{d} = \frac{g+h}{c}\cdot\frac{c}{d} - \frac{k}{b}\cdot\frac{b}{d} \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{aligned}$$

Again, there is a corresponding figure when the angle $\alpha+\beta$ is greater than $\frac{\pi}{2}$. (We encourage the student to check the addition formula for this situation as well.)

We therefore have proved the addition formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are angles between 0 and $\frac{\pi}{2}$, which we will now extend to all angles α and β. First, note that the addition formulas are trivially true when α or β are 0. (Check this!) Next, by observing that $\sin(x)$ and $\cos(x)$ can be converted to each other via shifts of $\frac{\pi}{2}$, (that is, by using the identities (21.13) and (21.14)), we obtain that

$$\begin{aligned}\sin(x+\frac{\pi}{2}) &= \cos x, & \cos(x+\frac{\pi}{2}) &= -\sin(x), \\ \sin(x-\frac{\pi}{2}) &= -\cos x, & \cos(x-\frac{\pi}{2}) &= \sin(x).\end{aligned}$$

With this, we extend the addition identities for α by $\pm\frac{\pi}{2}$ as follows:

$$\begin{aligned}\sin\left((\alpha+\frac{\pi}{2})+\beta\right) &= \sin(\alpha+\beta+\frac{\pi}{2}) = \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \\ &= \sin(\alpha+\frac{\pi}{2})\cos(\beta)+\cos(\alpha+\frac{\pi}{2})\sin(\beta), \\ \sin\left((\alpha-\frac{\pi}{2})+\beta\right) &= \sin(\alpha+\beta-\frac{\pi}{2}) = -\cos(\alpha+\beta) = -\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) \\ &= \sin(\alpha-\frac{\pi}{2})\cos(\beta)+\cos(\alpha-\frac{\pi}{2})\sin(\beta), \\ \cos\left((\alpha+\frac{\pi}{2})+\beta\right) &= \cos(\alpha+\beta+\frac{\pi}{2}) = -\sin(\alpha+\beta) = -\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \\ &= \cos(\alpha+\frac{\pi}{2})\cos(\beta)-\sin(\alpha+\frac{\pi}{2})\sin(\beta), \\ \cos\left((\alpha-\frac{\pi}{2})+\beta\right) &= \cos(\alpha+\beta-\frac{\pi}{2}) = \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ &= \cos(\alpha-\frac{\pi}{2})\cos(\beta)-\sin(\alpha-\frac{\pi}{2})\sin(\beta).\end{aligned}$$

There are similar proofs to extend the identities for β. An induction argument shows the validity of the addition formulas for arbitrary angles α and β.

The remaining formulas now follow via the use of trigonometric identities.

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}}$$

This shows that $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$. For the relations with $\alpha-\beta$, we use the fact that sin and tan are odd functions, whereas cos is an even function. See identities (18.2) and (18.4).

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))=\sin(\alpha)\cos(-\beta)+\cos(\alpha)\sin(-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,\\
\cos(\alpha-\beta) &= \cos(\alpha+(-\beta))=\cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta,\\
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta))=\frac{\tan(\alpha)+\tan(-\beta)}{1-\tan(\alpha)\tan(-\beta)}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
\end{aligned}$$

This completes the proof of the proposition. □

Finally, using Proposition 21.11, we also prove Proposition 21.12.

Proof of Proposition 21.12. We start with the double angle formulas. Using Proposition 21.11, we have:

$$\begin{aligned}
\sin(2\alpha) &= \sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha=2\sin\alpha\cos\alpha\\
\cos(2\alpha) &= \cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha=\cos^2\alpha-\sin^2\alpha\\
\tan(2\alpha) &= \tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}=\frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Notice that $\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$ can be rewritten using $\sin^2\alpha+\cos^2\alpha=1$ as follows:

$$\begin{aligned}
&\cos^2\alpha-\sin^2\alpha = (1-\sin^2\alpha)-\sin^2\alpha=1-2\sin^2\alpha\\
\text{and}\quad &\cos^2\alpha-\sin^2\alpha = \cos^2\alpha-(1-\cos^2\alpha)=2\cos^2\alpha-1
\end{aligned}$$

This shows the double-angle formulas. These formulas can now be used to prove the half-angle formulas.

$$\begin{aligned}
\cos(2\alpha) &= 1-2\sin^2\alpha \implies 2\sin^2\alpha=1-\cos(2\alpha) \implies \sin^2\alpha=\frac{1-\cos(2\alpha)}{2}\\
&\implies \sin\alpha=\pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \overset{\text{replace }\alpha\text{ by }\frac{\alpha}{2}}{\implies} \sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}\\
\cos(2\alpha) &= 2\cos^2\alpha-1 \implies 2\cos^2\alpha=1+\cos(2\alpha) \implies \cos^2\alpha=\frac{1+\cos(2\alpha)}{2}\\
&\implies \cos\alpha=\pm\sqrt{\frac{1+\cos(2\alpha)}{2}} \overset{\text{replace }\alpha\text{ by }\frac{\alpha}{2}}{\implies} \cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}$$

in particular: $$\tan\frac{\alpha}{2}=\frac{\sin(\frac{\alpha}{2})}{\cos(\frac{\alpha}{2})}=\frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

For the first two formulas for $\tan\frac{\alpha}{2}$ we simplify $\sin(2\alpha)\cdot\tan(\alpha)$ and $(1+\cos(2\alpha))\cdot\tan(\alpha)$ as follows.

$$\begin{aligned}
\sin(2\alpha)\cdot\tan(\alpha) &= 2\sin\alpha\cos\alpha\cdot\frac{\sin\alpha}{\cos\alpha}=2\sin^2\alpha=1-\cos(2\alpha)\\
&\implies \tan(\alpha)=\frac{1-\cos(2\alpha)}{\sin(2\alpha)} \overset{\text{replace }\alpha\text{ by }\frac{\alpha}{2}}{\implies} \tan(\frac{\alpha}{2})=\frac{1-\cos(\alpha)}{\sin(\alpha)}\\
(1+\cos(2\alpha))\cdot\tan(\alpha) &= 2\cos^2\alpha\cdot\frac{\sin\alpha}{\cos\alpha}=2\sin\alpha\cos\alpha=\sin(2\alpha)\\
&\implies \tan(\alpha)=\frac{\sin(2\alpha)}{1+\cos(2\alpha)} \overset{\text{replace }\alpha\text{ by }\frac{\alpha}{2}}{\implies} \tan(\frac{\alpha}{2})=\frac{\sin(\alpha)}{1+\cos(\alpha)}
\end{aligned}$$

This completes the proof of the proposition. □

21.3 Exercises

Exercise 21.1

Write the expression as one of the six trigonometric functions.

a) $\cos(x) \cdot \tan(x)$ b) $\sec(x) \cdot \cot(x)$ c) $\frac{\csc(x)}{\sec(x)}$

d) $\tan(x) \cdot \frac{\cot(x)}{\sin(x)}$ e) $\frac{\cot(x)}{\csc(x)}$ f) $\frac{\sin(x)}{\cot(x)} \cdot \csc^2(x)$

Exercise 21.2

Determine if the identity is true or false. If the identity is true, then give an argument for why it is true.

a) $\cos(x) \cdot \csc(x) = \sin(x) \cdot \sec(x)$

b) $\frac{\sin(x)}{\cot(x)} = \frac{\tan(x)}{\csc(x)}$

c) $\frac{\csc(x)}{\sin(x)} = \frac{\cot(x)}{\tan(x)}$

d) $\sin(x) \cdot \cos(x) \cdot \csc^2(x) = \frac{\csc(x)}{\sec(x)}$

Exercise 21.3

Simplify the expression as much as possible.

a) $\frac{\cos^2(x)-1}{\sin(x)}$ b) $\frac{1-\sin^2(x)}{\cot(x)}$

c) $1 + \frac{\cos^2(x)}{\sin^2(x)}$ d) $\frac{\tan^2(x)}{\sec^2(x)} - 1$

e) $\cos(x) + \frac{\sin^2(x)}{\cos(x)}$ f) $\sec(x) - \frac{\tan^2(x)}{\sec(x)}$

g) $(1 + \sin(x)) \cdot (1 - \sin(x))$ h) $(1 - \sec(x)) \cdot (1 + \sec(x))$

i) $(\csc(x) - 1) \cdot (\csc(x) + 1)$ j) $\frac{\sec(x)}{\tan(x)} - \frac{\tan(x)}{\sec(x)}$

k) $\cos^4(x) - \sin^4(x)$ l) $\tan^4(x) - \sec^4(x)$

Exercise 21.4

Determine whether the identity is true or false. If the identity is true, then give an argument for why it is true.

a) $\sin(x) - \sin(x)\cos^2(x) = \sin^3(x)$
b) $\cot^2(x) - \csc^2(x) = \tan^2(x) - \sec^2(x)$
c) $\tan^2(x) + \sec^2(x) = 1$
d) $\sin^3(x) - \sin(x) = -\sin(x) \cdot \cos^2(x)$
e) $\sin(x) \cdot (\cos(x) - \sin(x)) = \cos^2(x)$
f) $(\sin(x) - \cos(x))^2 = 1 - 2\sin(x)\cos(x)$

Exercise 21.5

Simplify the expression as much as possible.

a) $\sin(x + \pi)$ b) $\tan(\pi - x)$ c) $\cot(x + \frac{\pi}{2})$ d) $\cos(x + \frac{3\pi}{2})$

Exercise 21.6

Find the exact values of the trigonometric functions of $\frac{\alpha}{2}$ and of 2α by using the half-angle and double-angle formulas.

a) $\sin(\alpha) = \frac{4}{5}$, and α in quadrant I
b) $\cos(\alpha) = \frac{7}{13}$, and α in quadrant IV
c) $\sin(\alpha) = \frac{-3}{5}$, and α in quadrant III
d) $\tan(\alpha) = \frac{4}{3}$, and α in quadrant III
e) $\tan(\alpha) = \frac{-5}{12}$, and α in quadrant II
f) $\cos(\alpha) = \frac{-2}{3}$, and α in quadrant II

Review of trigonometric functions

Exercise IV.1

a) Convert from radian to degree: $\frac{4\pi}{3}$
b) Convert from degree to radian: $315°$

Exercise IV.2

Fill in all the trigonometric function values in the table below.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$					
$\cos(x)$					
$\tan(x)$					

Exercise IV.3

Find the exact value of the trigonometric function.

a) $\sin\left(\frac{5\pi}{4}\right)$ b) $\cos\left(\frac{11\pi}{6}\right)$ c) $\cos(300°)$
d) $\tan(\frac{7\pi}{6})$ e) $\tan(120°)$ f) $\tan(\frac{3\pi}{2})$

Exercise IV.4

a) Use the addition and subtraction of angles formulas to find $\cos\left(\frac{\pi}{12}\right)$.
b) Use the half-angles formulas to find $\cos\left(\frac{3\pi}{8}\right)$.

Exercise IV.5

Find the amplitude, period, and the phase shift of the given function. Draw the graph over a one-period interval. Label all maxima, minima, and intercepts.

a) $y = 3\cos\left(4x - \pi\right)$ b) $y = 5\sin\left(x + \frac{\pi}{2}\right)$

Exercise IV.6

Find the exact value and write it in radian.

a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ c) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

Exercise IV.7

Solve for x:

a) $2\sin(x) + \sqrt{3} = 0$ b) $\sqrt{3}\tan(x) - 1 = 0$

Exercise IV.8

Solve for x:

a) $\tan^2(x) - 3 = 0$ b) $4\cos^2(x) - 1 = 0$

Exercise IV.9

Solve for x.

a) $2\sin^2(x) + \sin(x) = 0$ b) $2\cos^2(x) + \sqrt{2}\cos(x) = 0$

Exercise IV.10

Verify the identity: $\tan^2(x)\cos(x) - \sec(x) = -\cos(x)$

Part V

Vectors, Complex Numbers, and Sequences

Chapter 22

Vectors in the plane

So far, we have discussed functions in general, as well as specific examples, such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. In the next chapters we study vectors, complex numbers, sequences, and series. We start in this chapter with vectors in the plane.

22.1 Introduction to vectors

Vectors are used in many applications, as they are useful to describe concepts that have a direction and a magnitude. Examples of these include:

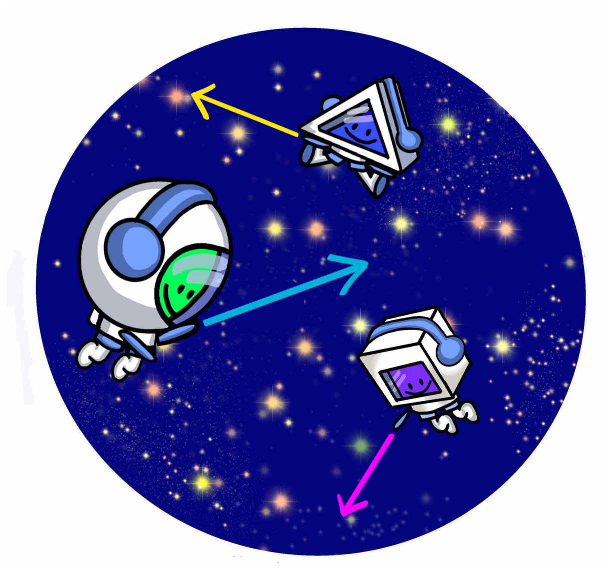
- the velocity of a (moving) object,
- the acceleration of an object,
- the force applied to an object.

Definition 22.1: Geometric vector

A **geometric vector** is a geometric object that is given by a **magnitude** and a **direction**. We denote a vector by $\vec{v}$, that is, by placing an arrow on top of the symbol for the variable.

Note 22.2

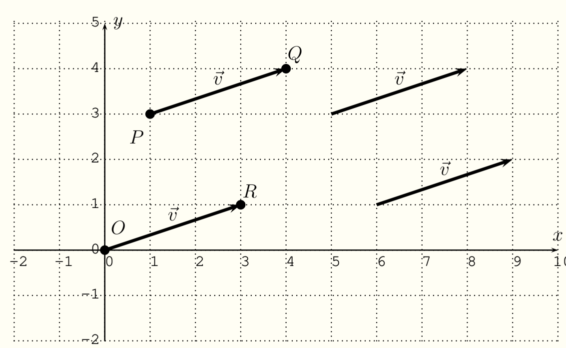
Vectors are often represented by directed line segments $\vec{v} = \overrightarrow{PQ}$. Two directed line segments represent the same vector if one can be moved to the other by parallel translation (without changing its direction or magnitute).



We will now study vectors in the plane $\mathbb{R}^2$ in more detail.

Observation 22.3: Vectors at the origin

A vector $\vec{v} = \overrightarrow{PQ}$ in the plane $\mathbb{R}^2$ can be represented by arranging the starting point of $\vec{v}$ to the origin $O(0,0)$.



If R is given in coordinates by $R(a,b)$, then we also write for $\vec{v} = \overrightarrow{OR}$,

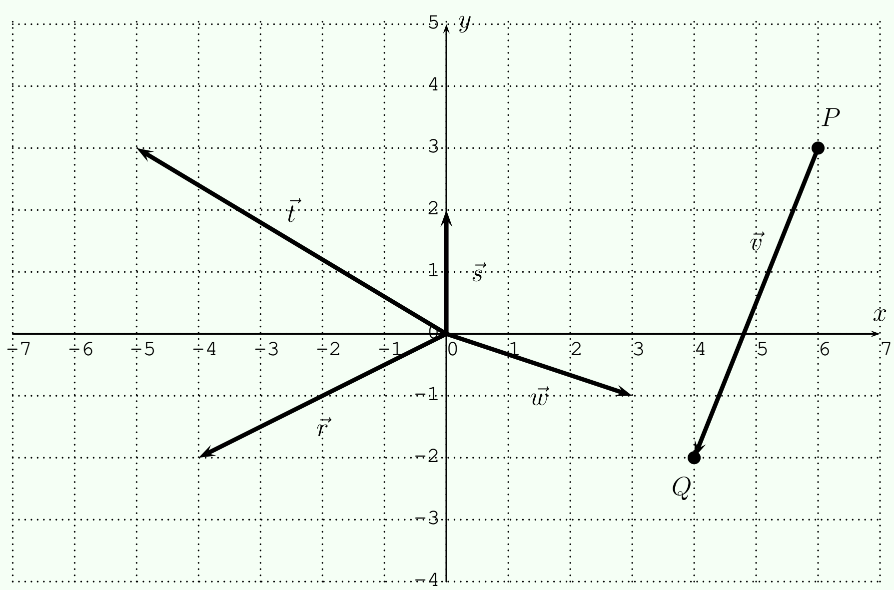
$$\boxed{\vec{v} = \langle a, b \rangle} \quad \text{or, alternatively,} \quad \boxed{\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}}. \tag{22.1}$$

Example 22.4

Graph the vectors $\vec{v}, \vec{w}, \vec{r}, \vec{s}, \vec{t}$ in the plane, where $\vec{v} = \overrightarrow{PQ}$ with $P(6,3)$ and $Q(4,-2)$, and

$$\vec{w} = \langle 3, -1 \rangle, \quad \vec{r} = \langle -4, -2 \rangle, \quad \vec{s} = \langle 0, 2 \rangle, \quad \vec{t} = \langle -5, 3 \rangle.$$

Solution.

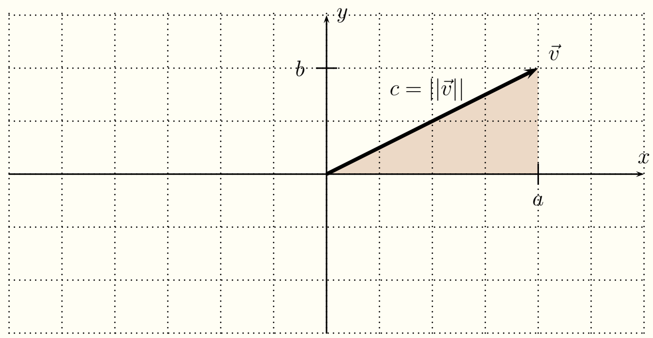


□

We now compute the magnitude and the direction angle of a vector given in coordinates, $\vec{v} = \langle a, b \rangle$.

Observation 22.5: Magnitude

Let $\vec{v} = \langle a, b\rangle = \overrightarrow{OR}$ be a vector in the plane, where $R(a,b)$ is the point with coordinates (a,b).



The **magnitude** of $\vec{v}$ is the length c of the line segment $\overrightarrow{OR}$. The Pythagorean theorem for the shaded right triangle above gives

$$a^2 + b^2 = c^2 \qquad \Longrightarrow \qquad c = \sqrt{a^2+b^2}$$

The magnitude of $\vec{v}$ is usually denoted by $||\vec{v}||$. Therefore, we have:

$$\boxed{||\vec{v}|| = \sqrt{a^2+b^2}} \tag{22.2}$$

Example 22.6

Find the magnitude of the given vectors.

a) $\vec{v} = \langle 8, -6\rangle$ b) $\vec{v} = \langle -5, -5\rangle$ c) $\vec{v} = \langle 4, 4\sqrt{3}\rangle$

Solution.

a) $||\vec{v}|| = ||\langle 8,-6\rangle|| = \sqrt{8^2 + (-6)^2} = \sqrt{64+36} = \sqrt{100} = 10$

b) $||\vec{v}|| = ||\langle -5,-5\rangle|| = \sqrt{(-5)^2 + (-5)^2} = \sqrt{25+25} = \sqrt{50} = \sqrt{25\cdot 2} = 4\sqrt{2}$

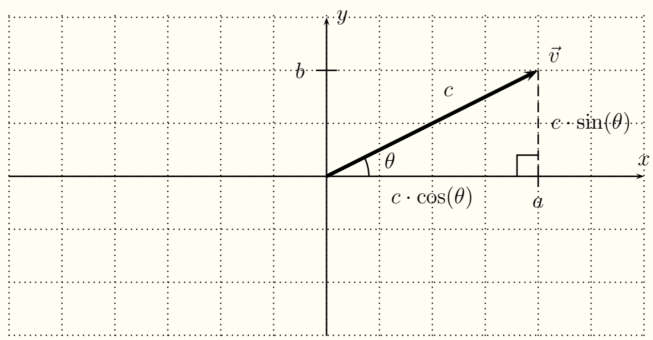
c) $||\vec{v}|| = ||\langle 4, 4\sqrt{3}\rangle|| = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{4^2 + 4^2\sqrt{3}^2} = \sqrt{16+16\cdot 3} = \sqrt{16+48} = \sqrt{64} = 8$

□

Next, we identify the direction angle of a vector $\vec{v} = \langle a, b \rangle$.

Observation 22.7: Direction angle

Let $\vec{v} = \langle a, b \rangle = \overrightarrow{OR}$ be a vector in the plane, where $R(a, b)$ is the point with coordinates (a, b).



The **direction angle** of $\vec{v}$ is the angle θ (read as "theta") determined by the line segment $\overrightarrow{OR}$. Denoting by $c = ||\vec{v}||$, the length of the vector $\vec{c}$, then, by (21.2), we have $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, and so, $\tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \cdot \frac{c}{a} = \frac{b}{a}$:

$$\boxed{\tan\theta = \frac{b}{a}} \tag{22.3}$$

Note that we can recover the vector $\vec{v} = \langle a, b \rangle$ in coordinate form from the magnitude $||\vec{v}|| = c$ and the angle θ, since $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$ gives

$$a = c \cdot \cos(\theta) \qquad \text{and} \qquad b = c \cdot \sin(\theta) \tag{22.4}$$

and therefore:

$$\vec{v} = \langle a, b \rangle = \langle\ ||\vec{v}|| \cdot \cos(\theta)\ ,\ ||\vec{v}|| \cdot \sin(\theta)\ \rangle \tag{22.5}$$

To find the angle θ from identity $\tan\theta = \frac{b}{a}$, we need to be a bit careful, because there is more than one angle θ whose tangent is $\frac{b}{a}$. This is illustrated in the next example.

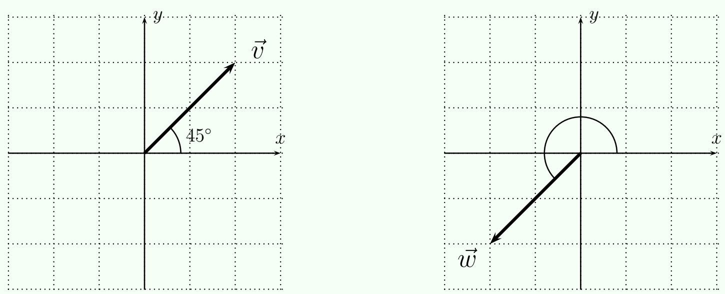
Example 22.8

Find the direction angle for the vectors

$$\vec{v} = \langle 2, 2\rangle \qquad \text{and} \qquad \vec{w} = \langle -2, -2\rangle$$

Solution.

For $\vec{v} = \langle 2, 2\rangle$, the direction angle satisfies $\tan(\theta) = \frac{2}{2} = 1$, and we can compute $\theta = \tan^{-1}(1) = 45^\circ$. This fits well with the depiction of $\vec{v}$ in the plane:



Now, the direction angle for $\vec{w} = \langle -2, -2\rangle$ satisfies $\tan(\theta) = \frac{-2}{-2} = 1$, but $\tan^{-1}(1) = 45^\circ$ is *not* the angle for $\vec{w}$, since $\vec{w}$ is in the third quadrant (while 45° is in the first quadrant). The issue is that the tangent function has the same output values when adding 180°, that is, $\tan(x + 180^\circ) = \tan(x)$ for all angle x. On the other hand, the outputs of the $\tan^{-1}$ function are in the interval $(-\frac{\pi}{2}, \frac{\pi}{2}) = (-90^\circ, 90^\circ)$ (see Definition 19.2) which are in the first and fourth quadrant. We therefore need to add 180° to the $\tan^{-1}(\frac{b}{a})$ whenever the vector lies in the second or third quadrant. We thus get:

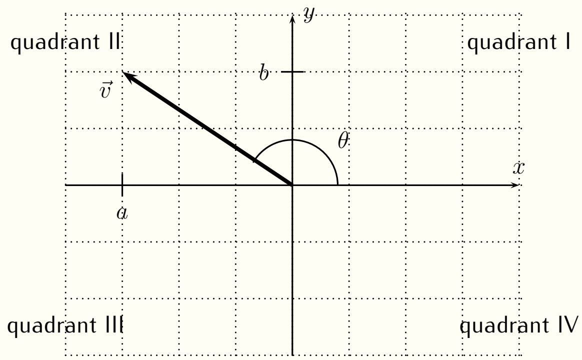
$$\text{(angle of } \vec{v}\,) = 45^\circ$$
$$\text{(angle of } \vec{w}\,) = 45^\circ + 180^\circ = 225^\circ$$

□

We can thus summarize the formulas for the magnitude and direction angle as follows.

Observation 22.9: Magnitude and direction angle

Let $\vec{v} = \langle a, b\rangle = \overrightarrow{OR}$ be a vector in the plane $\mathbb{R}^2$ pointing from the origin to $R(a, b)$.



Then the magnitude and direction angle of $\vec{v}$ are given by:

$$\boxed{||\vec{v}|| = \sqrt{a^2 + b^2}} \quad \text{and} \tag{22.6}$$

$$\boxed{\theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & \text{if } R \text{ is in quadrant I or IV} \\ \tan^{-1}(\frac{b}{a}) + 180^\circ & \text{if } R \text{ is in quadrant II or III} \end{cases}} \tag{22.7}$$

Here R is the endpoint of the vector $\vec{v} = \overrightarrow{OR}$ when placed at the origin $(0, 0)$.

Example 22.10

Find the magnitude and direction angle of the given vectors.

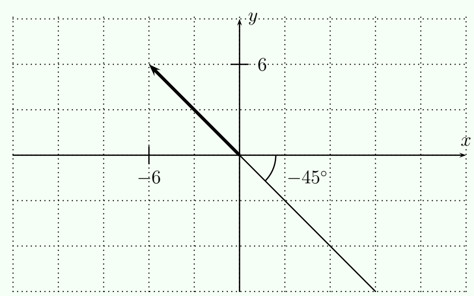
a) $\langle -6, 6\rangle$ b) $\langle 4, -3\rangle$ c) $\langle -2\sqrt{3}, -2\rangle$
d) $\langle 8, 4\sqrt{5}\rangle$ e) $\overrightarrow{PQ}$, where $P(9, 2)$ and $Q(3, 10)$

Solution.

a) We use formulas (22.6) and (22.7). The magnitude of $\vec{v} = \langle -6, 6\rangle$ is

$$||\vec{v}|| = \sqrt{(-6)^2 + 6^2} = \sqrt{36 + 36} = \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

The direction angle θ is given by $\tan(\theta) = \frac{6}{-6} = -1$. Since $\tan^{-1}(-1) = -\tan^{-1}(1) = -45°$ is in the fourth quadrant, but $\vec{v} = \langle -6, 6\rangle$ drawn at the origin $O(0,0)$ has its endpoint in the second quadrant, we see that the angle $\theta = -45° + 180° = 135°$.



b) The magnitude of $\vec{v} = \langle 4, -3\rangle$ is

$$||\vec{v}|| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

The direction angle is given by $\tan(\theta) = \frac{-3}{4}$. Since $\tan^{-1}(\frac{-3}{4}) \approx -36.9°$ is in the fourth quadrant, and $\vec{v} = \langle 4, -3\rangle$ is indeed in the fourth quadrant, we see that

$$\theta = \tan^{-1}\left(\frac{-3}{4}\right) \approx -36.9°.$$

Sometimes it may be preferable to describe the angle between $0°$ and $360°$. To obtain such and angle for $\vec{v}$, we can add $360°$, which gives and angle of $\approx -36.9° + 360° = 323.1°$ for $\vec{v}$.

c) The magnitude of $\vec{v} = \langle -2\sqrt{3}, -2\rangle$ is

$$||\vec{v}|| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{4\cdot 3 + 4} = \sqrt{12+4} = \sqrt{16} = 4.$$

The direction angle is given by $\tan(\theta) = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Note that $\tan^{-1}(\frac{\sqrt{3}}{3}) = 30°$ is in the first quadrant, whereas $\vec{v} = \langle -2\sqrt{3}, -2\rangle$ is in the third quadrant. Therefore, the angle is given by adding an additional $180°$ to the angle:

$$\theta = 30° + 180° = 210°$$

d) The magnitude of $\vec{v} = \langle 8, 4\sqrt{5}\rangle$ is

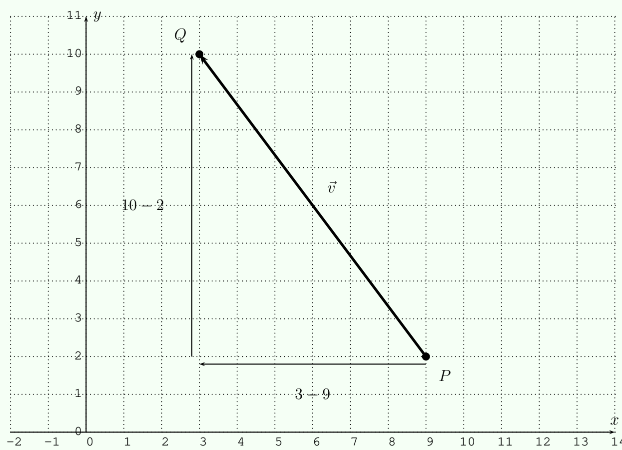
$$||\vec{v}|| = \sqrt{8^2 + (4\sqrt{5})^2} = \sqrt{64 + 16 \cdot 5} = \sqrt{64 + 80} = \sqrt{144} = 12.$$

The direction angle is given by the equation $\tan(\theta) = \frac{4\sqrt{5}}{8} = \frac{\sqrt{5}}{2}$. Since both $\tan^{-1}(\frac{\sqrt{5}}{2}) \approx 48.2°$ and the endpoint of $\vec{v}$ (represented with beginning point at the origin) are in the first quadrant, we have:

$$\theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) \approx 48.2°.$$

e) We first need to find the vector $\vec{v} = \overrightarrow{PQ}$ in the form $\vec{v} = \langle a, b\rangle$. The vector in the plane below shows that $\vec{v}$ is given by

$$\vec{v} = \langle 3 - 9, 10 - 2\rangle = \langle -6, 8\rangle$$



From this we calculate the magnitude to be

$$||\vec{v}|| = \sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

The direction angle is given by $\tan(\theta) = \frac{8}{-6} = -\frac{4}{3}$. Note that $\tan^{-1}(-\frac{4}{3}) \approx -53.1°$ is in quadrant IV, whereas $\vec{v} = \langle -6, 8\rangle$ has its endpoint in quadrant II (when representing $\vec{v}$ with starting point at the origin $O(0, 0)$). Therefore, the direction angle is

$$\theta = \tan^{-1}\left(-\frac{4}{3}\right) + 180° \approx 126.9°$$

□

22.2 Operations on vectors

There are two basic operations on vectors, which are the *scalar multiplication* and the *vector addition*. First, let's look at the scalar multiplication.

Definition 22.11: Scalar multiplication

The **scalar multiplication** of a real number r with a vector $\vec{v} = \langle a, b\rangle$ is defined to be the vector given by multiplying r to each coordinate.

$$\boxed{r \cdot \langle a, b\rangle := \langle r \cdot a, r \cdot b\rangle} \tag{22.8}$$

We study the effect of scalar multiplication with an example.

Example 22.12

Multiply and graph the vectors.

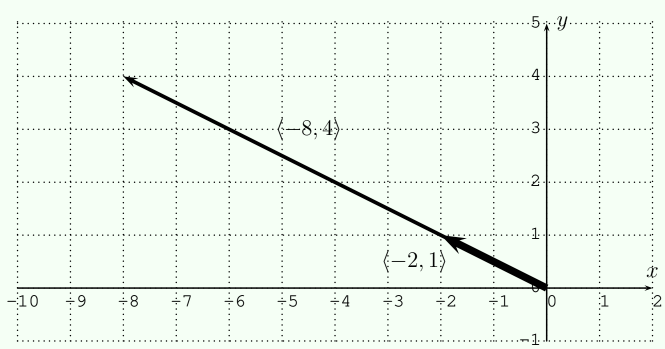
a) $4 \cdot \langle -2, 1\rangle$ b) $(-3) \cdot \langle -6, -2\rangle$

Solution.

a) The calculation is straightforward.

$$4 \cdot \langle -2, 1\rangle = \langle 4 \cdot (-2), 4 \cdot 1\rangle = \langle -8, 4\rangle$$

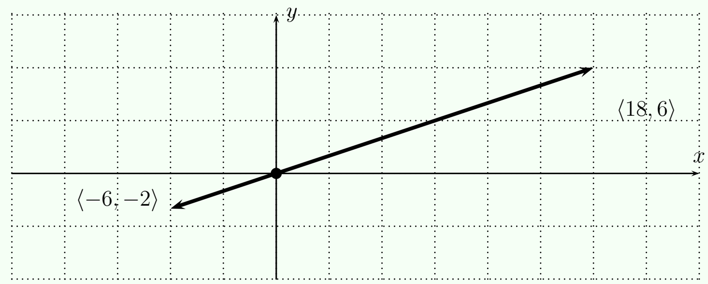
The vectors are displayed below. We see that $\langle -2, 1\rangle$ and $\langle -8, 4\rangle$ both have the same direction angle, and the latter stretches the former by a factor 4.



b) Algebraically, we calculate the scalar multiplication as follows:

$$(-3) \cdot \langle -6, -2\rangle = \langle (-3) \cdot (-6), (-3) \cdot (-2)\rangle = \langle 18, 6\rangle$$

Furthermore, graphing the vectors gives:



We see that the direction angle of the two vectors differs by $180°$. Indeed, $\langle 18, 6\rangle$ is obtained from $\langle -6, -2\rangle$ by reflecting it at the origin $O(0, 0)$ and then stretching the result by a factor 3.

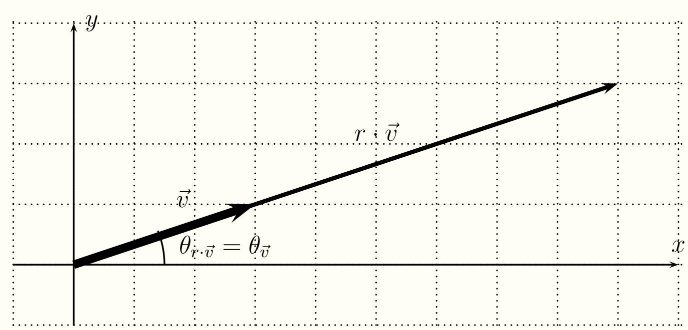
□

We see from the above example, that scalar multiplication by a positive number c does not change the angle of the vector, but it multiplies the magnitude by c.

Observation 22.13: Geometric interpretation of scalar multiplication

Let $\vec{v}$ be a vector with magnitude $||\vec{v}||$ and angle $\theta_{\vec{v}}$. Then, for a positive scalar, $r > 0$, the scalar multiple $r \cdot \vec{v}$ has the same angle as $\vec{v}$, and a magnitude that is r times the magnitude of $\vec{v}$:

$$\text{for } r > 0: \quad ||r \cdot \vec{v}|| = r \cdot ||\vec{v}|| \quad \text{and} \quad \theta_{r\cdot\vec{v}} = \theta_{\vec{v}}$$



Definition 22.14: Unit vector

A vector $\vec{u}$ is called a **unit vector** if it has a magnitude of 1.

$$\vec{u} \text{ is a unit vector} \quad \Longleftrightarrow \quad ||\vec{u}|| = 1$$

There are two special unit vectors, which are the vectors pointing in the x- and the y-direction.

$$\boxed{\vec{i} := \langle 1, 0\rangle} \quad \text{and} \quad \boxed{\vec{j} := \langle 0, 1\rangle} \tag{22.9}$$

Example 22.15

Find a unit vector in the direction of $\vec{v}$.

a) $\langle 8, 6\rangle$ b) $\langle -2, 3\sqrt{7}\rangle$

Solution.

a) Note that the magnitude of $\vec{v} = \langle 8, 6\rangle$ is

$$||\langle 8, 6\rangle|| = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$$

Therefore, if we multiply $\langle 8, 6\rangle$ by $\frac{1}{10}$ we obtain $\frac{1}{10} \cdot \langle 8, 6\rangle = \langle \frac{8}{10}, \frac{6}{10}\rangle = \langle \frac{4}{5}, \frac{3}{5}\rangle$, which (according to Observation 22.13 above) has the same direction angle as $\langle 8, 6\rangle$, and has a magnitude of 1:

$$\left|\left| \frac{1}{10} \cdot \langle 8, 6\rangle \right|\right| = \frac{1}{10} \cdot ||\langle 8, 6\rangle|| = \frac{1}{10} \cdot 10 = 1$$

b) The magnitude of $\langle -2, 3\sqrt{7}\rangle$ is

$$||\langle -2, 3\sqrt{7}\rangle|| = \sqrt{(-2)^2 + (3\sqrt{7})^2} = \sqrt{4 + 9 \cdot 7} = \sqrt{4 + 63} = \sqrt{67}.$$

Therefore, we have the unit vector

$$\begin{aligned} \frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7}\rangle &= \langle \frac{-2}{\sqrt{67}}, \frac{3\sqrt{7}}{\sqrt{67}}\rangle \\ &= \langle \frac{-2\sqrt{67}}{67}, \frac{3\sqrt{7}\sqrt{67}}{67}\rangle = \langle \frac{-2\sqrt{67}}{67}, \frac{3\sqrt{469}}{67}\rangle \end{aligned}$$

which again has the same direction angle as $\langle -2, 3\sqrt{7}\rangle$. We can also check that $\frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7}\rangle$ is a *unit vector*, because $||\frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7}\rangle|| = \frac{1}{\sqrt{67}} \cdot ||\langle -2, 3\sqrt{7}\rangle|| = \frac{1}{\sqrt{67}} \cdot \sqrt{67} = 1$.

□

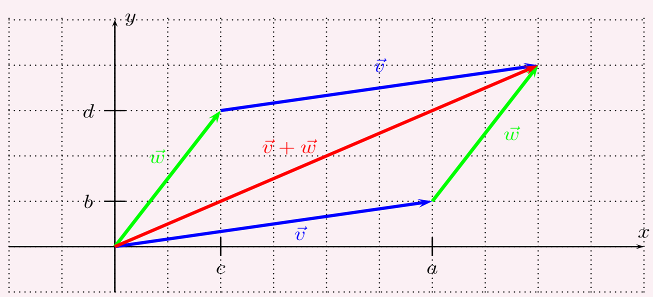
The second operation on vectors is *vector addition*, which we discuss now.

Definition 22.16: Vector addition

Let $\vec{v} = \langle a, b\rangle$ and $\vec{w} = \langle c, d\rangle$ be two vectors. Then the **vector addition** $\vec{v} + \vec{w}$ is defined by component-wise addition:

$$\boxed{\langle a, b\rangle + \langle c, d\rangle := \langle a + c, b + d\rangle} \tag{22.10}$$

In terms of the plane, the vector addition corresponds to placing the vectors $\vec{v}$ and $\vec{w}$ as the edges of a parallelogram, so that $\vec{v}+\vec{w}$ becomes its diagonal. This is displayed below.



Example 22.17

Perform the vector addition and simplify as much as possible.

a) $\langle 3, -5\rangle + \langle 6, 4\rangle$ b) $5 \cdot \langle -6, 2\rangle - 7 \cdot \langle 1, -3\rangle$ c) $4\vec{i} + 9\vec{j}$

d) find $2\vec{v} + 3\vec{w}$ for $\vec{v} = -6\vec{i} - 4\vec{j}$ and $\vec{w} = 10\vec{i} - 7\vec{j}$
e) find $-3\vec{v} + 5\vec{w}$ for $\vec{v} = \langle 8, \sqrt{3}\rangle$ and $\vec{w} = \langle 0, 4\sqrt{3}\rangle$

Solution.

We can find the answer by direct algebraic computation.

a) $\langle 3, -5\rangle + \langle 6, 4\rangle = \langle 3 + 6, -5 + 4\rangle = \langle 9, -1\rangle$

b) $5 \cdot \langle -6, 2\rangle - 7 \cdot \langle 1, -3\rangle = \langle -30, 10\rangle + \langle -7, 21\rangle = \langle -37, 31\rangle$

c) $4\vec{i} + 9\vec{j} = 4 \cdot \langle 1, 0\rangle + 9 \cdot \langle 0, 1\rangle = \langle 4, 0\rangle + \langle 0, 9\rangle = \langle 4, 9\rangle$

From the last calculation, we see that every vector can be written as a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

$$\boxed{\langle a, b\rangle = a \cdot \vec{i} + b \cdot \vec{j}} \tag{22.11}$$

We will use this equation for the next example (d). Here, $\vec{v} = -6\vec{i} - 4\vec{j} = \langle -6, -4\rangle$ and $\vec{w} = 10\vec{i} - 7\vec{j} = \langle 10, -7\rangle$. Therefore, we obtain:

d) $$\begin{aligned} 2\vec{v} + 3\vec{w} &= 2 \cdot \langle -6, -4\rangle + 3 \cdot \langle 10, -7\rangle \\ &= \langle -12, -8\rangle + \langle 30, -21\rangle = \langle 18, -29\rangle \end{aligned}$$

e) $$\begin{aligned} -3\vec{v} + 5\vec{w} &= -3 \cdot \langle 8, \sqrt{3}\rangle + 5 \cdot \langle 0, 4\sqrt{3}\rangle \\ &= \langle -24, -3\sqrt{3}\rangle + \langle 0, 20\sqrt{3}\rangle = \langle -24, 17\sqrt{3}\rangle \end{aligned}$$

Note that the answer could also be written as $-3\vec{v} + 5\vec{w} = -24\vec{i} + 17\sqrt{3}\vec{j}$. □

In many applications in the sciences, vectors play an important role, since many quantities are naturally described by vectors. Examples for this in physics include the velocity $\vec{v}$, acceleration $\vec{a}$, and the force $\vec{F}$ applied to an object.

Example 22.18

The forces $\vec{F_1}$ and $\vec{F_2}$ are applied to an object. Find the resulting total force $\vec{F} = \vec{F_1} + \vec{F_2}$. Determine the magnitude and direction angle of the total force $\vec{F}$. Approximate these values as necessary. Recall that the international system of units for force is the *newton* $[1N = 1\frac{kg \cdot m}{s^2}]$.

a) $\vec{F_1}$ has magnitude 3 newtons, and angle $\theta_1 = 45°$,
$\vec{F_2}$ has magnitude 5 newtons, and angle $\theta_2 = 135°$

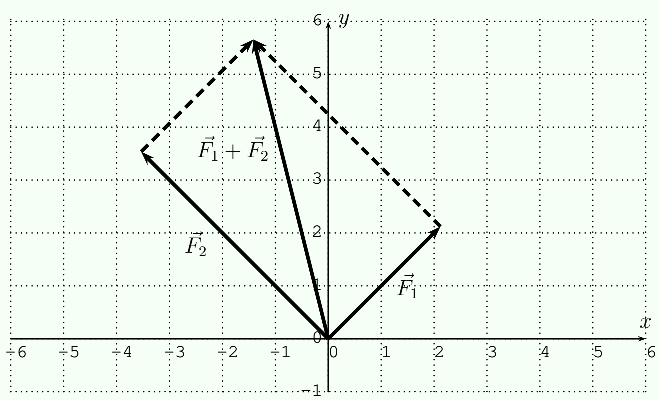
b) $||\vec{F_1}|| = 7$ newtons, and $\theta_1 = \frac{\pi}{6}$, and
$||\vec{F_2}|| = 4$ newtons, and $\theta_2 = \frac{5\pi}{3}$

Solution.

a) The vectors $\vec{F_1}$ and $\vec{F_2}$ are given by their magnitudes and direction angles. However, the addition of vectors (in Definition 22.16) is defined in terms of their components. Therefore, our first task is to find the vectors in component form. As was stated in Equation (22.5) on page 383, the vectors are calculated by $\vec{v} = \langle\ ||\vec{v}|| \cdot \cos(\theta)\ ,\ ||\vec{v}|| \cdot \sin(\theta)\ \rangle$. Therefore,

$$\begin{aligned}
\vec{F_1} &= \langle 3 \cdot \cos(45°), 3 \cdot \sin(45°) \rangle \\
&= \langle 3 \cdot \frac{\sqrt{2}}{2}, 3 \cdot \frac{\sqrt{2}}{2} \rangle = \langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle \\
\vec{F_2} &= \langle 5 \cdot \cos(135°), 5 \cdot \sin(135°) \rangle \\
&= \langle 5 \cdot \frac{-\sqrt{2}}{2}, 5 \cdot \frac{\sqrt{2}}{2} \rangle = \langle \frac{-5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle
\end{aligned}$$

The total force is the sum of the forces.



$$\begin{aligned}
\vec{F} &= \vec{F_1} + \vec{F_2} = \langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle + \langle \frac{-5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle \\
&= \langle \frac{3\sqrt{2}}{2} + \frac{-5\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} \rangle = \langle \frac{3\sqrt{2} - 5\sqrt{2}}{2}, \frac{3\sqrt{2} + 5\sqrt{2}}{2} \rangle \\
&= \langle \frac{-2\sqrt{2}}{2}, \frac{8\sqrt{2}}{2} \rangle = \langle -\sqrt{2}, 4\sqrt{2} \rangle
\end{aligned}$$

The total force applied in components is $\vec{F} = \langle -\sqrt{2}, 4\sqrt{2} \rangle$. It has a magnitude of $||\vec{F}|| = \sqrt{(-\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{2 + 16 \cdot 2} = \sqrt{34} \approx$

5.83 newton. The direction angle is given by $\tan(\theta) = \frac{4\sqrt{2}}{-\sqrt{2}} = -4$. Since $\tan^{-1}(-4) \approx -76.0^\circ$ is in quadrant IV, and $\vec{F} = \langle -\sqrt{2}, 4\sqrt{2}\rangle$ is in quadrant II, we see that the direction angle of $\vec{F}$ is

$$\theta = 180^\circ + \tan^{-1}(-4) \approx 180^\circ - 76.0^\circ \approx 104^\circ.$$

b) We solve this example in much the same way we solved part (a). First, $\vec{F_1}$ and $\vec{F_2}$ in components is given by

$$\begin{aligned} \vec{F_1} &= \langle 7\cdot\cos\left(\frac{\pi}{6}\right), 7\cdot\sin\left(\frac{\pi}{6}\right)\rangle = \langle 7\cdot\frac{\sqrt{3}}{2}, 7\cdot\frac{1}{2}\rangle = \langle\frac{7\sqrt{3}}{2}, \frac{7}{2}\rangle \\ \vec{F_2} &= \langle 4\cdot\cos\left(\frac{5\pi}{3}\right), 4\cdot\sin\left(\frac{5\pi}{3}\right)\rangle = \langle 4\cdot\frac{1}{2}, 4\cdot\frac{-\sqrt{3}}{2}\rangle = \langle 2, -2\sqrt{3}\rangle \end{aligned}$$

The total force is therefore:

$$\begin{aligned} \vec{F} &= \vec{F_1} + \vec{F_2} = \langle\frac{7\sqrt{3}}{2}, \frac{7}{2}\rangle + \langle 2, -2\sqrt{3}\rangle = \langle\frac{7\sqrt{3}}{2} + 2, \frac{7}{2} - 2\sqrt{3}\rangle \\ &\approx \langle 8.06, 0.04\rangle \end{aligned}$$

The magnitude is approximately

$$||\vec{F}|| \approx \sqrt{(8.06)^2 + (0.04)^2} \approx 8.06 \text{ newton}.$$

The direction angle is given by $\tan(\theta) \approx \frac{0.04}{8.06}$. Since $\vec{F}$ is in quadrant I, we see that $\theta \approx \tan^{-1}(\frac{0.04}{8.06}) \approx 0.3^\circ$.

□

Note 22.19: Vector space

In general, a *vector space* is an abstract algebraic notion that is fundamental to many areas of mathematics. Although we do not explicitly use this structure in this text, we will state its definition. A **vector space** is a set V, with the following structures and properties. The elements of V are called vectors, denoted by the usual symbol $\vec{v}$. For any vectors $\vec{v}$ and $\vec{w}$ there is a vector $\vec{v}+\vec{w}$, called the vector addition. For any real number r and vector $\vec{v}$, there is a vector $r\cdot\vec{v}$ called the scalar product. These operations have to satisfy the following properties.

Associativity:	$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
Commutativity:	$\vec{v}+\vec{w}=\vec{w}+\vec{v}$
Zero element:	there is a vector $\vec{o}$ such that $\vec{o}+\vec{v}=\vec{v}$ and $\vec{v}+\vec{o}=\vec{v}$ for every vector $\vec{v}$
Negative element:	for every $\vec{v}$ there is a vector $-\vec{v}$ such that $\vec{v}+(-\vec{v})=\vec{o}$ and $(-\vec{v})+\vec{v}=\vec{o}$
Distributivity (1):	$r\cdot(\vec{v}+\vec{w})=r\cdot\vec{v}+r\cdot\vec{w}$
Distributivity (2):	$(r+s)\cdot\vec{v}=r\cdot\vec{v}+s\cdot\vec{v}$
Scalar compatibility:	$(r\cdot s)\cdot\vec{v}=r\cdot(s\cdot\vec{v})$
Identity:	$1\cdot\vec{v}=\vec{v}$

An important example of a vector space is the 2-dimensional plane $V=\mathbb{R}^2$ as it was discussed in this chapter. A thorough introduction to this topic will be provided in a course in linear algebra.

22.3 Exercises

Exercise 22.1

Graph the vectors in the plane.

a) $\overrightarrow{PQ}$ with $P(2,1)$ and $Q(4,7)$
b) $\overrightarrow{PQ}$ with $P(-3,3)$ and $Q(-5,-4)$
c) $\overrightarrow{PQ}$ with $P(0,-4)$ and $Q(6,0)$
d) $\langle -2,4\rangle$
e) $\langle -3,-3\rangle$
f) $\langle 5,5\sqrt{2}\rangle$

Exercise 22.2

Find the magnitude and direction angle of the vector.

a) $\langle 6,8\rangle$
b) $\langle -2,5\rangle$
c) $\langle -4,-4\rangle$
d) $\langle 3,-3\rangle$
e) $\langle 2,-2\rangle$
f) $\langle 4\sqrt{3},4\rangle$
g) $\langle -\sqrt{3},-1\rangle$
h) $\langle -4,4\sqrt{3}\rangle$
i) $\langle -2\sqrt{3},-2\rangle$

j) $\overrightarrow{PQ}$, where $P(3,1)$ and $Q(7,4)$
k) $\overrightarrow{PQ}$, where $P(4,-2)$ and $Q(-5,7)$

Exercise 22.3

Perform the operation on the vectors.

a) $5 \cdot \langle 3, 2\rangle$
b) $2 \cdot \langle -1, 4\rangle$
c) $(-10) \cdot \langle -\frac{3}{2}, -\frac{7}{5}\rangle$
d) $\langle 2, 3\rangle + \langle 6, 1\rangle$
e) $\langle 5, -4\rangle - \langle -8, -9\rangle$
f) $3 \cdot \langle 5, 3\rangle + 4 \cdot \langle 2, 8\rangle$
g) $(-2)\langle -5, -4\rangle - 6\langle -1, -2\rangle$
h) $\frac{2}{3}\langle -3, 6\rangle - \frac{7}{5}\langle 10, -15\rangle$
i) $\sqrt{2} \cdot \langle \frac{\sqrt{8}}{6}, \frac{-5\sqrt{2}}{12}\rangle - 2\langle \frac{2}{3}, \frac{5}{3}\rangle$
j) $6\vec{i} - 4\vec{j}$
k) $-5\vec{i} + \vec{j} + 3\vec{i}$
l) $3 \cdot \langle -4, 2\rangle - 8\vec{j} + 12\vec{i}$

m) find $4\vec{v} + 7\vec{w}$ for $\vec{v} = \langle 2, 3\rangle$ and $\vec{w} = \langle 5, 1\sqrt{3}\rangle$
n) find $\vec{v} - 2\vec{w}$ for $\vec{v} = \langle -11, -6\rangle$ and $\vec{w} = \langle -3, 2\rangle$
o) find $3\vec{v} - \vec{w}$ for $\vec{v} = -4\vec{i} + 7\vec{j}$ and $\vec{w} = 6\vec{i} + \vec{j}$
p) find $-\vec{v} - \sqrt{5}\vec{w}$ for $\vec{v} = 5\vec{j}$ and $\vec{w} = -8\vec{i} + \sqrt{5}\vec{j}$

Exercise 22.4

Find a unit vector in the direction of the given vector.

a) $\langle 8, -6\rangle$
b) $\langle -3, -\sqrt{7}\rangle$
c) $\langle 9, 2\rangle$
d) $\langle -\sqrt{5}, \sqrt{31}\rangle$
e) $\langle 5\sqrt{2}, 3\sqrt{10}\rangle$
f) $\langle 0, -\frac{3}{5}\rangle$

Exercise 22.5

Find the approximate magnitude and direction angle of sum $\vec{v} = \vec{v_1} + \vec{v_2}$ of the given vectors $\vec{v_1}$ and $\vec{v_2}$ (see Example 22.18).

a) $||\vec{v_1}|| = 6$, and $\theta_1 = 60°$, and
$||\vec{v_2}|| = 2$, and $\theta_2 = 180°$

b) $||\vec{v_1}|| = 3.7$, and $\theta_1 = 92°$, and
$||\vec{v_2}|| = 2.2$, and $\theta_2 = 253°$

c) $||\vec{v_1}|| = 8$, and $\theta_1 = \frac{3\pi}{4}$, and
$||\vec{v_2}|| = 8\sqrt{2}$, and $\theta_2 = \frac{3\pi}{2}$

Chapter 23

Complex numbers

We have already encountered complex numbers in the previous chapters. We now go a bit further by representing them in the plane and use the trigonometric functions to rewrite complex numbers in polar form. We will see that this can simplify the multiplication and division of complex numbers.

23.1 Polar form of complex numbers

We now recall the definition of complex numbers and show how to represent them in the complex plane.

Definition 23.1: Imaginary unit

We define the **imaginary unit** or **complex unit** to be

$$\text{imaginary unit:} \qquad \boxed{i = \sqrt{-1}} \tag{23.1}$$

In other words, i is a solution of the equation:

$$\boxed{i^2 = -1} \tag{23.2}$$

Using the imaginary unit, a complex number is defined as a number with a real part and an imaginary part.

Definition 23.2: Complex number

A **complex number** is a number of the form

$$a + bi$$

where a and b are any real numbers, and i is the complex unit. The number a is called the **real part** of $a+bi$, and b is called the **imaginary part** of $a + bi$.
The set of all complex numbers is denoted by $\mathbb{C}$.

Example 23.3

Here are some examples of complex numbers:

$$3 + 2i, \quad 1 - 1 \cdot i, \quad \sqrt{2} + \pi \cdot i, \quad 5 + 0 \cdot i, \quad 0 + 3 \cdot i$$

Note that we can write $5 + 0 \cdot i = 5$, which has an imaginary part of zero, and so we see that the real number 5 is also a complex number. Indeed, any real number $a = a + 0 \cdot i$ is also a complex number.
Note also that $0 + 3 \cdot i = 3i$ is a complex number, and similarly any multiple of i is a complex number; these numbers are called *purely imaginary*.

We briefly recall the usual operations on complex numbers.

Example 23.4

Perform the operation.

a) $(2 - 3i) + (-6 + 4i)$ b) $(3 + 5i) \cdot (-7 + i)$ c) $\dfrac{5 + 4i}{3 + 2i}$

Solution.

a) Adding real and imaginary parts, respectively, gives,

$$(2 - 3i) + (-6 + 4i) = 2 - 3i - 6 + 4i = -4 + i.$$

b) We multiply (using FOIL), and use that $i^2 = -1$.

$$\begin{aligned} (3 + 5i) \cdot (-7 + i) &= -21 + 3i - 35i + 5i^2 = -21 - 32i + 5 \cdot (-1) \\ &= -21 - 32i - 5 = -26 - 32i \end{aligned}$$

c) Recall that we may simplify a quotient of complex numbers by multiplying the complex conjugate of the denominator to both numerator and denominator.

$$\begin{aligned}\frac{5+4i}{3+2i} &= \frac{(5+4i)\cdot(3-2i)}{(3+2i)\cdot(3-2i)} = \frac{15-10i+12i-8i^2}{9-6i+6i-4i^2} \\ &= \frac{15+2i+8}{9+4} = \frac{23+2i}{13} = \frac{23}{13} + \frac{2}{13}i\end{aligned}$$

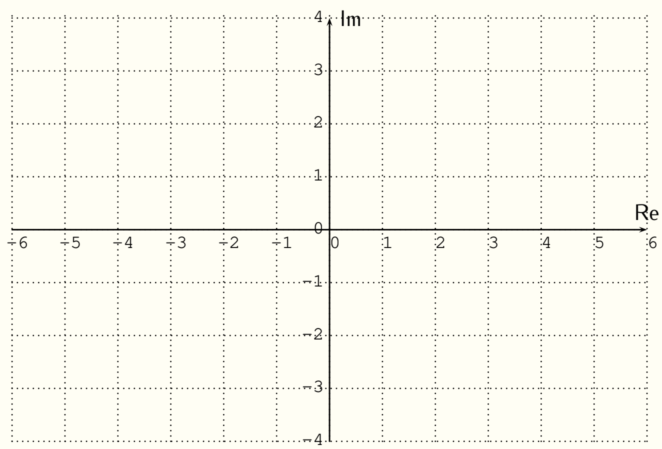
The real part of the solution is $\frac{23}{13}$; the imaginary part is $\frac{2}{13}$.

□

Complex numbers can be pictured as points in the plane.

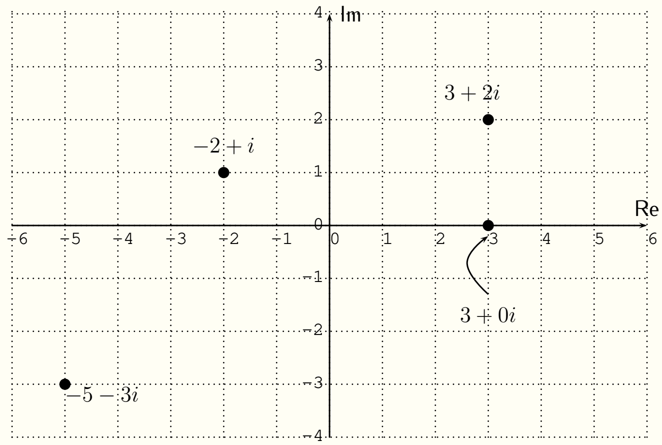
Observation 23.5: Complex plane

In analogy to Section 1.1, where we represented the real numbers on the number line, we can represent complex numbers in the *complex plane*:



The complex number $a + bi$ is represented as the point with coordinates

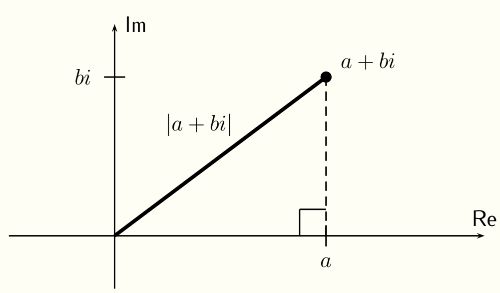
(a, b) in the complex plane.



Just as with the magnitude and the direction angle of vectors, we can use the planar representation of a complex number to study its distance from the origin, as well as its direction angle. We start with the distance of a complex number $a + bi$ to the origin 0, which is called the ***absolute value*** $|a + bi|$ of $a + bi$.

Observation 23.6: Absolute value or modulus

Let $a + bi$ be a complex number. The **absolute value** or **modulus** of $a+bi$, denoted by $|a+bi|$, is the length between the point $a+bi$ in the complex plane and the origin $(0, 0)$.



Just as in Observation 22.5, we can use the Pythagorean theorem to calculate $|a + bi|$ as $a^2 + b^2 = |a + bi|^2$, and so

$$\boxed{|a + bi| = \sqrt{a^2 + b^2}} \qquad (23.3)$$

Example 23.7

Find the absolute value of the complex numbers below.

a) $5-3i$ b) $-8-6i$ c) $-3+3i$ d) $4\sqrt{3}+4i$ e) $7i$

Solution.
The absolute values are calculated as follows.

a) $|5-3i| = \sqrt{5^2+(-3)^2} = \sqrt{25+9} = \sqrt{34}$

b) $|-8-6i| = \sqrt{(-8)^2+(-6)^2} = \sqrt{64+36} = \sqrt{100} = 10$

c) $|-3+3i| = \sqrt{(-3)^2+3^2} = \sqrt{9+9} = \sqrt{18} = \sqrt{9\cdot 2} = 3\cdot\sqrt{2}$

d) $\left|4\sqrt{3}+4i\right| = \sqrt{(4\sqrt{3})^2+4^2} = \sqrt{16\cdot 3+16} = \sqrt{64} = 8$

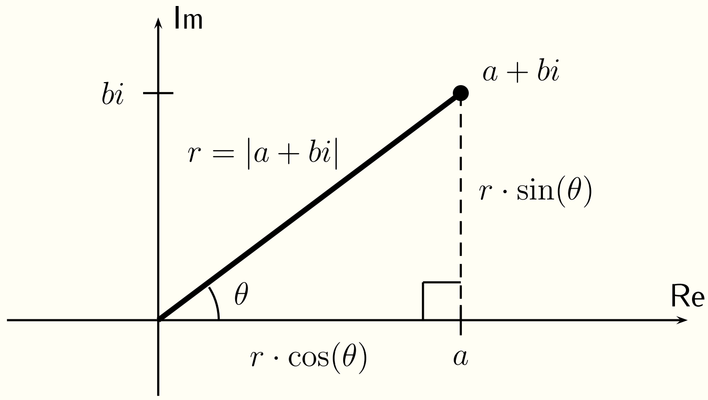
e) $|7i| = |0+7i| = \sqrt{0^2+(7)^2} = \sqrt{0+49} = 7$

□

Next, we apply the concept of the direction angle to a complex number.

Observation 23.8: Angle or argument

Let $a+bi$ be a complex number. Just as in Observation 22.7, we define the **angle** or **argument** of $a+bi$ to be the angle θ (read as "theta") determined by the line segment connecting the origin to $a+bi$.



Repeating the calculation from Observation 22.7, we write $r=|a+bi|$ for the absolute value, so that using (21.2), the coordinates a and b in the plane are given by $\sin(\theta) = \frac{b}{r}$ and $\cos(\theta) = \frac{a}{r}$. The angle is given by $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{b}{r}}{\frac{a}{r}} = \frac{b}{r} \cdot \frac{r}{a} = \frac{b}{a}$, which evaluates as in Note 22.9 to

$$\boxed{\theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & \text{if } a+bi \text{ is in quadrant I or IV} \\ \tan^{-1}(\frac{b}{a}) + 180° & \text{if } a+bi \text{ is in quadrant II or III} \end{cases}} \tag{23.4}$$

We can rewrite a complex number via its absolute value and angle.

Definition 23.9: Polar form

For a complex number $a+bi$, we write $r = |a+bi|$ for the absolute value, and θ for the angle (given by (23.4)). Then, using that $a = r \cdot \cos(\theta)$, and $b = r \cdot \sin(\theta)$), the real and imaginary parts of the complex number $a + bi$ can be rewritten as follows:

$$a + bi = r \cdot \cos(\theta) + r \cdot \sin(\theta) \cdot i$$

After factoring r, we get:

$$\boxed{a + bi = r \cdot \big(\cos(\theta) + i \cdot \sin(\theta)\big)} \tag{23.5}$$

We say a complex number is in **polar form** if it is written in the form $r \cdot (\cos(\theta) + i \cdot \sin(\theta))$.
We say a complex number is in **standard form** or **rectangular form** if it is written as $a + bi$.

We can convert a complex number from standard form to polar form and vice versa, which we do in the next two examples.

Example 23.10

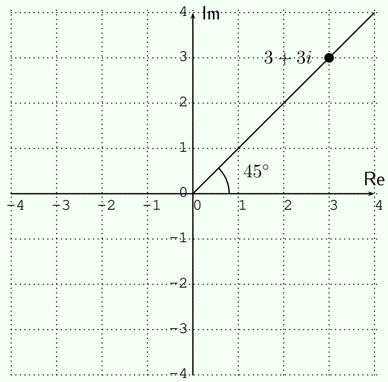
Convert the complex number to polar form.

a) $3 + 3i$ b) $-2 - 2\sqrt{3}i$ c) $-6\sqrt{3} + 6i$ d) $4 - 3i$ e) $-4i$

Solution.

a) First, the absolute value is $r = |3+3i| = \sqrt{3^2 + 3^2} = \sqrt{18} = \sqrt{9 \cdot 2} =$

$3\sqrt{2}$. Furthermore, since $a = 3$ and $b = 3$, we have $\tan(\theta) = \frac{b}{a} = \frac{3}{3} = 1$. To obtain θ, we calculate $\tan^{-1}(1) = 45°$. Note that $45°$ is in the first quadrant, and so is the complex number $2 + 3i$



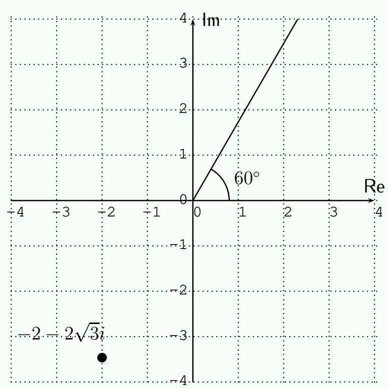
Therefore, $\theta = 45°$, and we obtain our answer:

$$2 + 3i = 3\sqrt{2} \cdot \big(\cos(45°) + i\sin(45°)\big).$$

b) For $-2 - 2\sqrt{3}i$, we first calculate the absolute value:

$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 4 \cdot 3} = \sqrt{4 + 12} = \sqrt{16} = 4.$$

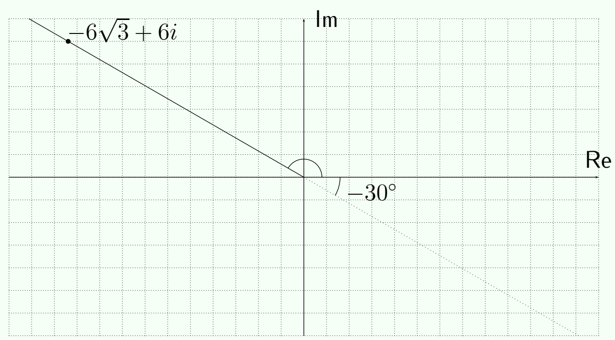
Furthermore, $\tan(\theta) = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$. We have that $\tan^{-1}(\sqrt{3}) = 60°$. However, graphing the angle $60°$ and the number $-2 - 2\sqrt{3}i$, we see that $60°$ is in the first quadrant, whereas $-2 - 2\sqrt{3}i$ is in the third quadrant.



Therefore, we have to add $180°$ to $60°$ to get the correct angle for $-2-2\sqrt{3}i$, that is, $\theta = 60° + 180° = 240°$. Our complex number in polar form is

$$-2-2\sqrt{3}i = 4\cdot\big(\cos\big(240°\big)+i\sin\big(240°\big)\big).$$

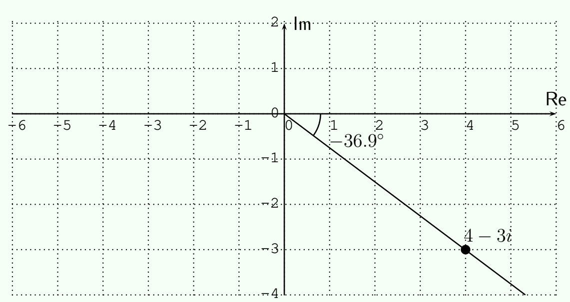
c) The absolute value of $-6\sqrt{3}+6i$ is $r = |-6\sqrt{3}+6i| = \sqrt{(-6\sqrt{3})^2+6^2} = \sqrt{36\cdot 3+36} = \sqrt{144} = 12$. The angle satisfies $\tan(\theta) = \frac{b}{a} = \frac{6}{-6\sqrt{3}} = \frac{1}{-\sqrt{3}}$, and $\tan^{-1}(\frac{1}{-\sqrt{3}}) = -30°$, which is in quadrant IV. Graphing $-6\sqrt{3}+6i$ in the complex plane shows it is in quadrant II.



Therefore, the angle is $\theta = -30° + 180° = 150°$, and so

$$-6\sqrt{3}+6i = 12\cdot\big(\cos\big(150°\big)+i\sin\big(150°\big)\big).$$

d) For $4-3i$ we calculate $r = \sqrt{4^2+(-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$. The angle $\tan^{-1}(\frac{-3}{4}) \approx -36.9°$ is in the fourth quadrant, and the complex number $4-3i$ is in the fourth quadrant as well.



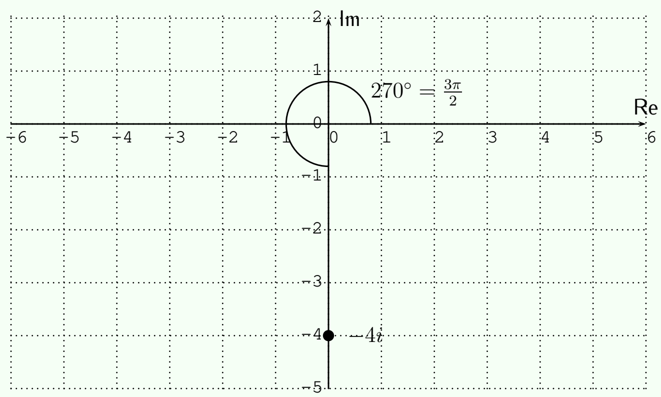
Therefore, $\theta \approx -36.9°$, and we write

$$4 - 3i \approx 5 \cdot \big(\cos(-36.9°) + i\sin(-36.9°)\big)$$

If we prefer an angle between $0°$ and $360°$, then we can also use the angle $-36.9° + 360° = 323.1°$, and write

$$4 - 3i \approx 5 \cdot \big(\cos(323.19°) + i\sin(323.1°)\big)$$

e) We calculate the absolute value of $0 - 4i$ as $r = \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4$. However, when calculating the angle θ of $0 - 4i$, we are led to consider $\tan^{-1}(\frac{-4}{0})$, which is *undefined*! The reason for this can be seen by plotting the number $-4i$ in the complex plane.



The angle $\theta = 270°$ (or alternatively $\theta = -90°$), so that the complex number is

$$\begin{aligned} -4i &= 4 \cdot (\cos(270°) + i\sin(270°)) \\ &= 4 \cdot \Big(\cos\big(\frac{3\pi}{2}\big) + i\sin\big(\frac{3\pi}{2}\big)\Big) \end{aligned}$$

Note that we can always write our answer with an angle either in degree or radian mode, as we did in the last equality.

□

Conversely, we can convert a complex number from polar form to standard form $a + bi$ by evaluation the $\sin$ and the $\cos$.

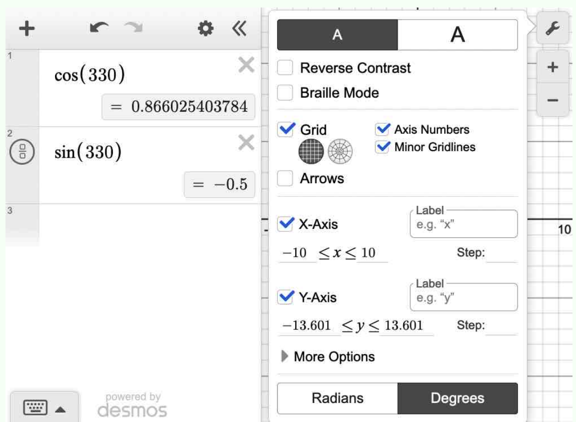
Example 23.11

Convert the number from polar form to standard form $a + bi$.

a) $4 \cdot (\cos(330°) + i\sin(330°))$ b) $3 \cdot (\cos(117°) + i\sin(117°))$

Solution.

a) We compute that $\cos(330°) = \frac{\sqrt{3}}{2}$ and $\sin(330°) = -\frac{1}{2}$, which is easily done with the calculator, since $\frac{\sqrt{3}}{2} \approx 0.866$ (review Example 17.10 if needed).



With this, we obtain the complex number in standard form.

$$\begin{aligned} 4 \cdot (\cos(330°) + i\sin(330°)) &= 4 \cdot \left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) \\ &= \frac{4\sqrt{3}}{2} - i \cdot \frac{4}{2} = 2\sqrt{3} - 2 \cdot i \end{aligned}$$

b) Since we do not have an exact formula for $\cos(117°)$ or $\sin(117°)$, we use the calculator to obtain approximate values.

$$3 \cdot (\cos(117°) + i\sin(117°)) \approx 3 \cdot (-0.454 + i \cdot 0.891) = -1.362 + 2.673i$$

□

23.2 Multiplication and division of complex numbers in polar form

It turns out that the angle of a product (or quotient) of complex numbers changes by adding (or subtracting) the angles. This is precisely stated in the following proposition.

Proposition 23.12

Let $r_1(\cos(\theta_1)+i\sin(\theta_1))$ and $r_2(\cos(\theta_2)+i\sin(\theta_2))$ be two complex numbers in polar form. Then the product and quotient of these are given by:

$$\boxed{\begin{aligned} &r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot r_2(\cos(\theta_2)+i\sin(\theta_2)) \\ &\qquad = r_1r_2\cdot(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\end{aligned}} \tag{23.6}$$

$$\boxed{\frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))}=\frac{r_1}{r_2}\cdot(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))} \tag{23.7}$$

Proof. The proof uses the addition formulas for trigonometric functions $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ from Proposition 17.11 on page 306.

$$\begin{aligned} &r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot r_2(\cos(\theta_2)+i\sin(\theta_2)) \\ &\quad = r_1r_2\cdot(\cos(\theta_1)\cos(\theta_2)+i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta_2)+i^2\sin(\theta_1)\sin(\theta_2)) \\ &\quad = r_1r_2\cdot((\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2))+i(\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2))) \\ &\quad = r_1r_2\cdot(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\end{aligned}$$

For the division formula, note that the multiplication formula (23.6) gives

$$\begin{aligned} r_2(\cos(\theta_2)+i\sin(\theta_2))\cdot\frac{1}{r_2}(\cos(-\theta_2)+i\sin(-\theta_2)) &= r_2\frac{1}{r_2}(\cos(\theta_2-\theta_2)+i\sin(\theta_2-\theta_2)) \\ &= 1\cdot(\cos 0+i\sin 0)=1\cdot(1+i\cdot 0)=1 \\ \implies \qquad \frac{1}{r_2(\cos(\theta_2)+i\sin(\theta_2))} &= \frac{1}{r_2}(\cos(-\theta_2)+i\sin(-\theta_2)),\end{aligned}$$

so that

$$\begin{aligned} &\frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))}=r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot\frac{1}{r_2(\cos(\theta_2)+i\sin(\theta_2))} \\ &\quad = r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot\frac{1}{r_2}(\cos(-\theta_2)+i\sin(-\theta_2))=\frac{r_1}{r_2}\cdot(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)).\end{aligned}$$

□

Example 23.13

Multiply or divide the complex numbers and write your answer in both polar form and standard form.

a) $5(\cos(48^\circ)+i\sin(48^\circ))\cdot 8(\cos(87^\circ)+i\sin(87^\circ))$

b) $3(\cos(\frac{5\pi}{8})+i\sin(\frac{5\pi}{8}))\cdot 12(\cos(\frac{7\pi}{8})+i\sin(\frac{7\pi}{8}))$

c) $\dfrac{8(\cos(257^\circ)+i\sin(257^\circ))}{6(\cos(47^\circ)+i\sin(47^\circ))}$

d) $\dfrac{32(\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4}))}{10(\cos(\frac{7\pi}{12})+i\sin(\frac{7\pi}{12}))}$

Solution.

We will multiply and divide the complex numbers using Equations (23.6) and (23.7), respectively, and then convert them to standard form.

a) For the product of the two complex numbers, we multiply the absolute values and add the angles.

$$5(\cos(48^\circ)+i\sin(48^\circ))\cdot 8(\cos(87^\circ)+i\sin(87^\circ))$$
$$=5\cdot 8\cdot(\cos(48^\circ+87^\circ)+i\sin(48^\circ+87^\circ)) = 40(\cos(135^\circ)+i\sin(135^\circ))$$

To write this in standard form, we evaluate $\cos(135^\circ)=-\frac{\sqrt{2}}{2}$ and $\sin(135^\circ)=\frac{\sqrt{2}}{2}$. Thus, we get

$$40\cdot\Big(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)=-\frac{40\sqrt{2}}{2}+i\frac{40\sqrt{2}}{2}=-20\sqrt{2}+20\sqrt{2}i$$

b) Similarly, we obtain the next product.

$$3\Big(\cos(\frac{5\pi}{8})+i\sin(\frac{5\pi}{8})\Big)\cdot 12\Big(\cos(\frac{7\pi}{8})+i\sin(\frac{7\pi}{8})\Big)$$
$$=36\Big(\cos(\frac{5\pi}{8}+\frac{7\pi}{8})+i\sin(\frac{5\pi}{8}+\frac{7\pi}{8})\Big)$$

Now, $\dfrac{5\pi}{8}+\dfrac{7\pi}{8}=\dfrac{5\pi+7\pi}{8}=\dfrac{12\pi}{8}=\dfrac{3\pi}{2}$, for which $\cos(\frac{3\pi}{2})=0$ and $\sin(\frac{3\pi}{2})=-1$. Therefore, we obtain that the product is

$$36\Big(\cos(\frac{3\pi}{2})+i\sin(\frac{3\pi}{2})\Big)=36(0+i\cdot(-1))=-36i$$

c) Next, we calculate

$$\frac{8(\cos(257°)+i\sin(257°))}{6(\cos(47°)+i\sin(47°))}=\frac{4}{3}\cdot\Big(\cos(210°)+i\sin(210°)\Big).$$

Computing $\cos(210°)=-\frac{\sqrt{3}}{2}$ and $\sin(210°)=-\frac{1}{2}$, we obtain

$$\begin{aligned}\frac{4}{3}\cdot\Big(\cos(210°)+i\sin(210°)\Big)&=\frac{4}{3}\cdot\Big(-\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\Big)\\&=-\frac{4\cdot\sqrt{3}}{3\cdot 2}-i\cdot\frac{4\cdot 1}{3\cdot 2}=-\frac{2\sqrt{3}}{3}-\frac{2}{3}\cdot i\end{aligned}$$

d) For the quotient, we use the subtraction formula (23.7).

$$\frac{32(\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4}))}{10(\cos(\frac{7\pi}{12})+i\sin(\frac{7\pi}{12}))}=\frac{32}{10}\Big(\cos(\frac{\pi}{4}-\frac{7\pi}{12})+i\sin(\frac{\pi}{4}-\frac{7\pi}{12})\Big)$$

The difference in the argument of $\cos$ and $\sin$ is given by

$$\frac{\pi}{4}-\frac{7\pi}{12}=\frac{3\pi-7\pi}{12}=\frac{-4\pi}{12}=-\frac{\pi}{3}$$

and $\cos(-\frac{\pi}{3})=\frac{1}{2}$ and $\sin(-\frac{\pi}{3})=-\frac{\sqrt{3}}{2}$. With this, we obtain

$$\begin{aligned}\frac{32(\cos(\frac{\pi}{4})+i\sin(\frac{\pi}{4}))}{10(\cos(\frac{7\pi}{12})+i\sin(\frac{7\pi}{12}))}&=\frac{32}{10}\Big(\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})\Big)\\&=\frac{16}{5}\cdot\Big(\frac{1}{2}-i\frac{\sqrt{3}}{2}\Big)=\frac{16}{10}-\frac{16\sqrt{3}}{10}\cdot i=\frac{8}{5}-\frac{8\sqrt{3}}{5}\cdot i\end{aligned}$$

□

23.3 Exercises

Exercise 23.1

Plot the complex numbers in the complex plane.

a) $4+2i$ b) $-3-5i$ c) $6-2i$ d) $-5+i$ e) $-2i$
f) $\sqrt{2}-\sqrt{2}i$ g) 7 h) i i) 0 j) $2i-\sqrt{3}$

Exercise 23.2

Add, subtract, multiply, and divide as indicated.

a) $(5-2i)+(-2+6i)$ b) $(-9-i)-(5-3i)$
c) $(3+2i)\cdot(4+3i)$ d) $(-2-i)\cdot(-1+4i)$
e) $\frac{2+3i}{2+i}$ f) $(5+5i)\div(2-4i)$

Exercise 23.3

Find the absolute value $|a+bi|$ of the given complex number, and simplify your answer as much as possible.

a) $|4+3i|$ b) $|6-6i|$ c) $|-3i|$ d) $|-2-6i|$
e) $|\sqrt{8}-i|$ f) $|-2\sqrt{3}-2i|$ g) $|-5|$ h) $|-\sqrt{17}+4\sqrt{2}i|$

Exercise 23.4

Convert the complex number into polar form $r(\cos(\theta)+i\sin(\theta))$.

a) $2+2i$ b) $4\sqrt{3}-4i$ c) $-7+7\sqrt{3}i$ d) $-5-5i$
e) $8-8i$ f) $-8+8i$ g) $-\sqrt{5}-\sqrt{15}i$ h) $\sqrt{7}-\sqrt{21}i$
i) $-5-12i$ j) $6i$ k) -10 l) $-\sqrt{3}+3i$

Exercise 23.5

Convert the complex number into the standard form $a+bi$.

a) $6(\cos(150°)+i\sin(150°))$ b) $10(\cos(315°)+i\sin(315°))$
c) $2(\cos(90°)+i\sin(90°))$ d) $\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})$
e) $\frac{1}{2}(\cos(\frac{7\pi}{6})+i\sin(\frac{7\pi}{6}))$ f) $6(\cos(-\frac{5\pi}{12})+i\sin(-\frac{5\pi}{12}))$

Exercise 23.6

Multiply the complex numbers and write the answer in standard form $a + bi$.

a) $4(\cos(27°) + i\sin(27°)) \cdot 10(\cos(123°) + i\sin(123°))$

b) $7(\cos(182°) + i\sin(182°)) \cdot 6(\cos(43°) + i\sin(43°))$

c) $(\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})) \cdot (\cos(\frac{7\pi}{12}) + i\sin(\frac{7\pi}{12}))$

d) $8(\cos(\frac{3\pi}{7}) + i\sin(\frac{3\pi}{7})) \cdot 1.5(\cos(\frac{4\pi}{7}) + i\sin(\frac{4\pi}{7}))$

e) $0.2(\cos(196°) + i\sin(196°)) \cdot 0.5(\cos(88°) + i\sin(88°))$

f) $4(\cos(\frac{7\pi}{8}) + i\sin(\frac{7\pi}{8})) \cdot 0.25(\cos(\frac{-5\pi}{24}) + i\sin(\frac{-5\pi}{24}))$

Exercise 23.7

Divide the complex numbers and write the answer in standard form $a+bi$.

a) $\dfrac{18(\cos(320°) + i\sin(320°))}{3(\cos(110°) + i\sin(110°))}$

b) $\dfrac{10(\cos(207°) + i\sin(207°))}{15(\cos(72°) + i\sin(72°))}$

c) $\dfrac{7(\cos(\frac{11\pi}{15}) + i\sin(\frac{11\pi}{15}))}{3(\cos(\frac{\pi}{15}) + i\sin(\frac{\pi}{15}))}$

d) $\dfrac{\cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5})}{2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10}))}$

e) $\dfrac{42(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4}))}{7(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12}))}$

f) $\dfrac{30(\cos(-175°) + i\sin(-175°))}{18(\cos(144°) + i\sin(144°))}$

Chapter 24

Sequences and series

In the next chapters, we will define and study sequences and series, which are concepts that are fundamental for calculus. Two specific types of sequences (arithmetic sequences and geometric sequences) will be studied in more detail.

24.1 Introduction to sequences and series

In this section, we define sequences and series.

Definition 24.1: Sequence

A **sequence** is an enumerated list of numbers. In other words, a sequence is a list of numbers

$$a_1, a_2, a_3, a_4, \ldots$$

where a_1 is the first number, a_2 is the second number, a_3 is the third number, etc.
We also denote the sequence by a or $\{a_n\}$ or $\{a_n\}_{n\geq 1}$.

Note that a sequence a is just an assignment, which assigns to each $n = 1, 2, 3, \ldots$ a number $a(n) = a_n$. In this sense, a sequence is just a function $a : \mathbb{N} \to R$ from the natural numbers $\mathbb{N}$ to a range R, which is a set of numbers such as, for example, the set of real or complex numbers.

Note 24.2

Here are some examples of sequences.

a) $4, 6, 8, 10, 12, 14, 16, 18, \ldots$

b) $1, 3, 9, 27, 81, 243, \ldots$

c) $+5, -5, +5, -5, +5, -5, \ldots$

d) $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

e) $5, 8, -12, 4, 5.3, 7, -3, \sqrt{2}, 18, \frac{2}{3}, 9, \ldots$

For many of these sequences we can find explicit rules that describe how to obtain the individual terms.

a) Note that in the sequence in (a), we always add the fixed number 2 to the previous number to obtain the next, starting from the first term 4. Assuming that the following numbers in this sequence continue in this pattern, this would be an example of an *arithmetic sequence*, and we will study those in more detail in Section 24.2 below. We note, however, that knowing only a few numbers in a sequence is not enough to conclude that all of the following terms will, indeed, follow a similar pattern. To understand the complete sequence, we must specify all terms of the sequence (as is done, for example, in Examples 24.3 and 24.4 below).

b) In (b), we start with the first element 1 and multiply by the fixed number 3 to obtain the next term. Assuming this rule persists for the whole sequence, this would be an example of a *geometric sequence*, and we will study those in more detail in Chapter 25 below.

c) The sequence in (c) alternates between $+5$ and -5, starting from $+5$. Note that we can get from one term to the next by multiplying (-1) to the term. Assuming this as the rule for the whole sequence, this is another example of a geometric sequence.

d) In (d) we wrote the first few terms of a sequence called the **Fibonacci sequence**. In the Fibonacci sequence, each term is calculated by adding the previous two terms, starting with the first two terms 1 and 1:

$$1+1=2, \quad 1+2=3, \quad 2+3=5, \quad 3+5=8, \quad 5+8=13, \quad \dots$$

e) Finally, the sequence in (e) does not seem to have any obvious rule by which the terms are generated.

To fully describe a sequence, we must specify every term of the sequence. This can be done, for example, by giving a formula for the nth term a_n of the sequence.

Note 24.3

Consider the sequence $\{a_n\}$ with $a_n = 4n+3$. We can calculate the individual terms of this sequence:

$$\begin{array}{lll} \text{first term:} & a_1 = & 4\cdot 1+3=7, \\ \text{second term:} & a_2 = & 4\cdot 2+3=11, \\ \text{third term:} & a_3 = & 4\cdot 3+3=15, \\ \text{fourth term:} & a_4 = & 4\cdot 4+3=19, \\ \text{fifth term:} & a_5 = & 4\cdot 5+3=23 \\ & \vdots & \end{array}$$

Thus, the sequence is: $7, 11, 15, 19, 23, 27, 31, 35, \dots$
Furthermore, from the formula, we can directly calculate any higher term, for example, the 200th term is given by:

$$200\text{th term:} \quad a_{200} = \; 4\cdot 200+3 = 803$$

Example 24.4

Find the first 6 terms of each sequence.

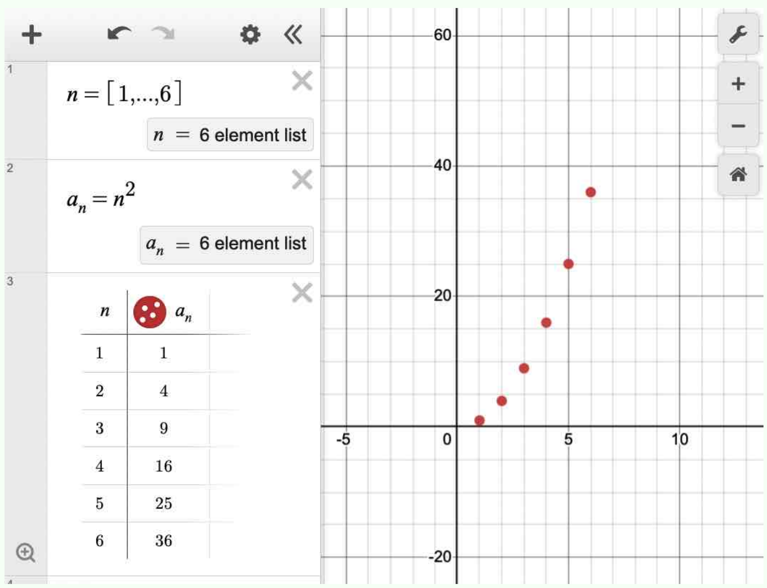
a) $a_n = n^2$ b) $a_n = \frac{n}{n+1}$ c) $a_n = (-1)^n$ d) $a_n = (-1)^{n+1}\cdot 2^n$

Solution.

a) We can easily calculate the first 6 terms of $a_n = n^2$ directly:

$$1, 4, 9, 16, 25, 36, \dots$$

We can also use the calculator to produce the terms of a sequence. To this end, we first need to specify a finite list of indices n that we want to consider. To define the list $1, 2, 3, 4, 5, 6$ for our indices n, we write $n = [1, \ldots, 6]$. With this, we can generate the induced list for a_n by writing $a_n = n^2$. To see the values in this list, we generate a table using the  button (on the top left). We remind the reader that generating a table was described in Example 4.7 on page 57. We replace the input and output values (x_1 and y_1) of the table with n and a_n, respectively, which then shows the first six numbers of our sequence.



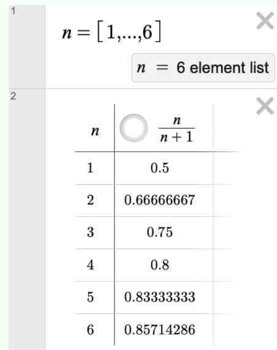
Note that we also get a graphical representation of the first six numbers in the sequence in the graph on the right.

b) We calculate the lowest terms of $a_n = \frac{n}{n+1}$:

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, \quad a_2 = \frac{2}{2+1} = \frac{2}{3}, \quad a_3 = \frac{3}{3+1} = \frac{3}{4}, \quad \ldots$$

Identifying the pattern, we can simply write $a_1, \ldots, a_6$ as follows:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$$



Note that in the table on the right, we directly specified the output values $\frac{n}{n+1}$ without first defining the list a_n.

c) Since $(-1)^n$ is $+1$ for even n, but is -1 for odd n, the sequence $a_n = (-1)^n$ is:

$$-1, +1, -1, +1, -1, +1$$

d) Similar to part (c), $(-1)^{n+1}$ is -1 for even n, and is $+1$ for odd n. This, together with the calculation $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16$, etc., we get the first six terms of the sequence:

$$+2, -4, +8, -16, +32, -64$$

□

Another way to describe a sequence is by giving a *recursive* formula for the nth term a_n in terms of the lower terms. Here are some examples.

Example 24.5

Find the first 6 terms in the sequence described below.

a) $a_1 = 4$, and $a_n = a_{n-1} + 5$ for $n > 1$
b) $a_1 = 3$, and $a_n = 2 \cdot a_{n-1}$ for $n > 1$
c) $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$

Solution.

a) The first term is explicitly given as $a_1 = 4$. Then we can calculate the following terms via $a_n = a_{n-1} + 5$:

$$\begin{aligned} a_2 &= a_1 + 5 = 4 + 5 = 9 \\ a_3 &= a_2 + 5 = 9 + 5 = 14 \\ a_4 &= a_3 + 5 = 14 + 5 = 19 \\ a_5 &= a_4 + 5 = 19 + 5 = 24 \\ &\vdots \end{aligned}$$

b) We start with $a_1 = 3$, and calculate $a_2 = 2 \cdot a_1 = 2 \cdot 3 = 6$, $a_3 = 2 \cdot a_2 = 2 \cdot 6 = 12$, $a_4 = 2 \cdot a_3 = 2 \cdot 12 = 24$, etc. We see that the effect of the recursive relation $a_n = 2 \cdot a_{n-1}$ is to double the previous number. The sequence is:

$$3, 6, 12, 24, 48, 96, 192, \ldots$$

c) Starting from $a_1 = 1$, and $a_2 = 1$, we can calculate the higher terms:

$$\begin{aligned} a_3 &= a_1 + a_2 = 1 + 1 = 2 \\ a_4 &= a_2 + a_3 = 1 + 2 = 3 \\ a_5 &= a_3 + a_4 = 2 + 3 = 5 \\ a_6 &= a_4 + a_5 = 3 + 5 = 8 \\ &\vdots \end{aligned}$$

Studying the sequence for a short while, we see that this is precisely the Fibonacci sequence from Example 24.2(d).

□

Note 24.6

There is no specific reason for using the indexing variable n in the sequence $\{a_n\}$. Indeed, we may as well use any other variable. For example, if the sequence $\{a_n\}_{n\geq 1}$ is given by the formula $a_n = 4n + 3$, then we can also write this as $a_k = 4k + 3$ or $a_i = 4i + 3$. In particular, the sequences $\{a_n\}_{n\geq 1} = \{a_k\}_{k\geq 1} = \{a_i\}_{i\geq 1} = \{a_j\}_{j\geq 1}$ are all identical as sequences.

We will be concerned with the task of adding terms in a sequence, such as $a_1 + a_2 + a_3 + \cdots + a_k$, for which we will use a standard summation notation.

Definition 24.7: Series

A **series** is a sum of terms in a sequence. We denote the sum of the first k terms in a sequence with the following notation:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k \tag{24.1}$$

The summation symbol "$\sum$" comes from the Greek letter Σ, pronounced "sigma," which is the Greek symbol for the /s/ sound.
More generally, we denote the sum from the jth to the kth term (where $j \leq k$) in a sequence with the following notion:

$$\sum_{n=j}^{k} a_n = a_j + a_{j+1} + \cdots + a_k$$

Furthermore, for typesetting reasons, $\sum\limits_{n=j}^{k} a_n$ is sometimes also written as $\sum_{n=j}^{k} a_n$, where indices are placed next to the summation symbol "$\sum$" instead of above and below.

Example 24.8

Find the sum.

a) $\displaystyle\sum_{n=1}^{4} a_n,$ for $a_n = 7n + 3$

b) $\displaystyle\sum_{j=1}^{6} a_j,$ for $a_n = (-2)^n$

c) $\displaystyle\sum_{k=1}^{5} \left(4 + k^2\right)$

Solution.

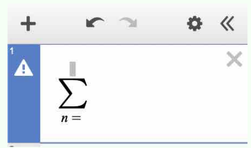
a) The first four terms a_1, a_2, a_3, a_4 of the sequence $\{a_n\}_{n\geq 1}$ are:

$$10, 17, 24, 31$$

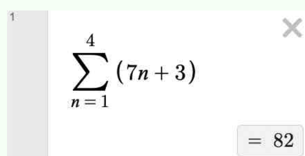
The sum is therefore:

$$\sum_{n=1}^{4} a_n = a_1 + a_2 + a_3 + a_4 = 10 + 17 + 24 + 31 = 82$$

We may also find the answer with the calculator. Typing the letters "sum" in the calculator will create the summation symbol:



Entering the specified values for the sequence, as well as where to begin and the end the sum, displays the answer.



Note that we need to place parentheses around the terms to be summed $7n+3$. (Confirm this by observing that you get a different result without the parentheses!)

b) The first six terms of $\{a_n\}$ with $a_n = (-2)^n$ are:

$$\begin{array}{lll} (-2)^1 = -2, & (-2)^2 = 4, & (-2)^3 = -8, \\ (-2)^4 = 16, & (-2)^5 = -32, & (-2)^6 = 64 \end{array}$$

We calculate $\sum_{j=1}^{6} a_j$ by adding these first six terms. (Note that the sum is independent of the index j appearing in the sum $\sum_{j=1}^{6} a_j$, which we could also replace by any other index $\sum_{j=1}^{6} a_j = \sum_{n=1}^{6} a_n$, etc.) We get:

$$\begin{aligned} \sum_{j=1}^{6} a_j &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= (-2) + 4 + (-8) + 16 + (-32) + 64 = 42 \end{aligned}$$

c) For the sum $\sum_{k=1}^{5}(4+k^2)$ we need to add the first five terms of the sequence $a_k = 4+k^2$. Calculating and adding the terms of this sequence, we obtain the sum:

$$\begin{aligned}\sum_{k=1}^{5}\left(4+k^2\right) &= (4+1^2)+(4+2^2)+(4+3^2)+(4+4^2)+(4+5^2)\\ &= (4+1)+(4+4)+(4+9)+(4+16)+(4+25)\\ &= 5+8+13+20+29\\ &= 75\end{aligned}$$

This answer can, of course, also be confirmed with the calculator.

□

24.2 The arithmetic sequence

We have already encountered examples of arithmetic sequences in the previous section. An arithmetic sequence is a sequence for which we add a constant number to get from one term to the next, for example:

$$8, \underbrace{\;}_{+3} 11, \underbrace{\;}_{+3} 14, \underbrace{\;}_{+3} 17, \underbrace{\;}_{+3} 20, \underbrace{\;}_{+3} 23, \ldots$$

Definition 24.9: Arithmetic sequence

A sequence $\{a_n\}$ is called an **arithmetic sequence** if any two consecutive terms have a *common difference* d. The arithmetic sequence is determined by d and the first value a_1. This can be written recursively as:

$$a_n = a_{n-1} + d \qquad \text{for } n \geq 2$$

Alternatively, we have the general formula for the nth term of the arithmetic sequence:

$$\boxed{a_n = a_1 + (n-1)\cdot d} \tag{24.2}$$

Example 24.10

Determine whether the terms below could be the terms of an arithmetic sequence. If so, then find the general formula for a_n in the form of Equation (24.2).

a) $7, 13, 19, 25, 31, \ldots$ b) $18, 13, 8, 3, -2, -7, \ldots$
c) $10, 13, 16, 20, 23, \ldots$ d) $a_n = 8 \cdot n + 3$

Solution.

a) Calculating the difference between two consecutive terms always gives the same answer: $13 - 7 = 6$, $19 - 13 = 6$, $25 - 19 = 6$, etc. Therefore, the common difference $d = 6$, which shows that these are the terms in an arithmetic sequence. Furthermore, the first term is $a_1 = 7$, so that the general formula for the nth term would be $a_n = 7 + (n - 1) \cdot 6$.

b) One checks that the common difference is $13-18 = -5$, $8-12 = -5$, etc., so that this could be an arithmetic sequence with $d = -5$. Since $a_1 = 18$, the general term would be $a_n = 18 + (n - 1) \cdot (-5)$ or $a_n = 18 - (n - 1) \cdot 5$.

c) We have $13 - 10 = 3$, but $20 - 16 = 4$, so that this *cannot* be an arithmetic sequence.

d) If we write out the first couple of terms of $a_n = 8n + 3$, we get the sequence:
$$11, 19, 27, 35, 43, 51, \ldots$$
From this, it seems reasonable to suspect that the above is an arithmetic sequence with common difference $d = 8$ and first term $a_1 = 11$. The general term of such an arithmetic sequence is
$$a_1 + (n - 1) \cdot d = 11 + (n - 1) \cdot 8 = 11 + 8n - 8 = 8n + 3 = a_n.$$
This shows that $a_n = 8n + 3 = 11 + (n - 1) \cdot 8$ is an arithmetic sequence.

□

Example 24.11

Find the formula of an arithmetic sequence $a_n = a_1 + (n-1) \cdot d$ with the properties given below.

a) $a_1 = -5$, and $a_9 = 27$
b) $d = 12$, and $a_6 = 68$
c) $a_5 = 38$, and $a_{16} = 115$

Solution.

a) We are given $a_1 = -5$, but we still need to find the common difference d. Plugging $a_9 = 27$ into $a_n = a_1 + (n-1) \cdot d$, we obtain

$$27 = a_9 = -5 + (9-1) \cdot d = -5 + 8d \overset{(+5)}{\Longrightarrow} 32 = 8d \overset{(\div 8)}{\Longrightarrow} 4 = d$$

The nth term is therefore, $a_n = -5 + (n-1) \cdot 4$.

b) We know that $d = 12$, so we only need to find a_1. Plugging $a_6 = 68$ into $a_n = a_1 + (n-1) \cdot d$, we can solve for a_1:

$$68 = a_6 = a_1 + (6-1) \cdot 12 = a_1 + 5 \cdot 12 = a_1 + 60 \overset{(-60)}{\Longrightarrow} a_1 = 68 - 60 = 8$$

The nth term is therefore, $a_n = 8 + (n-1) \cdot 12$.

c) In this case, neither a_1 nor d are given. However, the two conditions $a_5 = 38$ and $a_{16} = 115$ give two equations in the two unknowns a_1 and d.

$$\begin{cases} 38 = a_5 = a_1 + (5-1) \cdot d \\ 115 = a_{16} = a_1 + (16-1) \cdot d \end{cases} \Longrightarrow \begin{cases} 38 = a_1 + 4 \cdot d \\ 115 = a_1 + 15 \cdot d \end{cases}$$

To solve this system of equations, we need to recall the methods for doing so. One convenient method is the addition/subtraction method. For this, we subtract the top equation from the bottom equation:

$$\begin{array}{rrcrl} & 115 & = & a_1 & +15 \cdot d \\ -(& 38 & = & a_1 & +4 \cdot d \quad) \\ \hline & 77 & = & & +11 \cdot d \end{array} \quad \overset{(\div 11)}{\Longrightarrow} \quad 7 = d$$

Plugging $d = 7$ into either of the two equations gives a_1. We plug it into the first equation $38 = a_1 + 4d$:

$$38 = a_1 + 4 \cdot 7 \quad \Longrightarrow \quad 38 = a_1 + 28 \quad \overset{(-28)}{\Longrightarrow} \quad 10 = a_1$$

Therefore, the nth term is given by $a_n = 10 + (n-1) \cdot 7$.

□

We want to find a sum of terms in an arithmetic sequence. Since the arithmetic sequence is given by an easy and straightforward rule, it turns out that one can find a nice formula for the sum of the first k terms in the sequence, as well.

Note 24.12: Summing integers from 1 to 100

Find the sum of the first 100 integers, starting from 1. In other words, we want to find the sum of $1 + 2 + 3 + \cdots + 99 + 100$. Note that the sequence $1, 2, 3, \ldots$ is an arithmetic sequence. Instead of adding all these 100 numbers, we use a trick that will turn out to work for any arithmetic sequence:
In order to compute $S = 1 + 2 + 3 + \cdots + 98 + 99 + 100$, we write the sum for these numbers twice, once in ascending order, and once in descending order:

$$\begin{array}{rcccccccccccc} 2 \cdot S = & & 1 & + & 2 & + & 3 & \cdots & + & 98 & + & 99 & + & 100 \\ & + & 100 & + & 99 & + & 98 & \cdots & + & 3 & + & 2 & + & 1 \end{array}$$

Note that the second line is just S, but we are adding the terms in reverse order. Adding the terms vertically, that is, adding the two terms in each column, each sum is precisely 101:

$$2 \cdot S = 101 + 101 + 101 + \cdots + 101 + 101 + 101$$

Note that there are 100 terms of 101 on the right-hand side. So,

$$2S = 100 \cdot 101 \text{ and therefore } S = \frac{100 \cdot 101}{2} = 5050.$$

According to lore, this formula was discovered by the German mathematician Carl Friedrich Gauss as a child in primary school.

An appropriate generalization of the previous note yields a computation that applies to any arithmetic sequence.

Observation 24.13: Arithmetic series

Let $\{a_n\}$ be an arithmetic sequence whose nth term is given by the formula $a_n = a_1 + (n-1) \cdot d$. Then the sum $a_1 + a_2 + \cdots + a_{k-1} + a_k$ is given by adding $(a_1 + a_k)$ precisely $\frac{k}{2}$ times:

$$\boxed{\sum_{n=1}^{k} a_n = \frac{k}{2} \cdot (a_1 + a_k)} \tag{24.3}$$

Proof. For the proof of Equation (24.3), we write $S = a_1 + a_2 + \cdots + a_{k-1} + a_k$. We then add it to itself, but in reverse order:

$$\begin{array}{rcccccccccccc} 2 \cdot S = & & a_1 & + & a_2 & + & a_3 & \cdots & + & a_{k-2} & + & a_{k-1} & + & a_k \\ & + & a_k & + & a_{k-1} & + & a_{k-2} & \cdots & + & a_3 & + & a_2 & + & a_1 \end{array}.$$

Now note that in general, $a_\ell + a_m = a_1 + (\ell - 1) \cdot d + a_1 + (m-1) \cdot d = 2a_1 + (\ell + m - 2) \cdot d$. Since the vertical terms are always terms a_ℓ and a_m with $\ell + m = k + 1$, these add to $a_\ell + a_m = 2a_1 + (k-1) \cdot d$. We see that adding vertically gives

$$\begin{aligned} 2 \cdot S &= (2a_1 + (k-1) \cdot d) + \cdots + (2a_1 + (k-1) \cdot d) \\ &= k \cdot (2a_1 + (k-1) \cdot d) = k \cdot (a_1 + (a_1 + (k-1) \cdot d)) = k \cdot (a_1 + a_k). \end{aligned}$$

Dividing by 2 gives the desired result. □

Here are some examples in which we apply formula (24.3).

Example 24.14

Find the value of the arithmetic series.

a) Find the sum $a_1 + \cdots + a_{60}$ for the arithmetic sequence

$$a_n = 2 + (n-1) \cdot 13.$$

b) Determine the value of the sum:

$$\sum_{n=1}^{1001} (5 - 6n)$$

c) Find the sum of the first 333 terms of the sequence

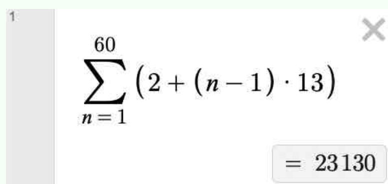
$$15, 11, 7, 3, -1, -5, -9, \ldots$$

Solution.

a) The sum is given by the formula (24.3): $\sum_{n=1}^{k} a_n = \frac{k}{2} \cdot (a_1 + a_k)$. Here, $k = 60$, and $a_1 = 2$ and $a_{60} = 2 + 13 \cdot (60 - 1) = 2 + 13 \cdot 59 = 2 + 767 = 769$. We obtain a sum of

$$a_1 + \cdots + a_{60} = \sum_{n=1}^{60} a_n = \frac{60}{2} \cdot (2 + 769) = 30 \cdot 771 = 23,130.$$

We may confirm this with the calculator as described in Example 24.8 (on page 419) of the previous section.



b) Again, we use the above formula $\sum_{n=1}^{k} a_n = \frac{k}{2} \cdot (a_1 + a_k)$, in which the arithmetic sequence is given by $a_n = 5 - 6n$ and $k = 1001$. Using the values $a_1 = 5 - 6 \cdot 1 = 5 - 6 = -1$ and $a_{1001} = 5 - 6 \cdot 1001 = 5 - 6006 = -6001$, we obtain:

$$\begin{aligned}\sum_{n=1}^{1001} (5 - 6n) &= \frac{1001}{2}(a_1 + a_{1001}) = \frac{1001}{2}((-1) + (-6001)) \\ &= \frac{1001}{2} \cdot (-6002) = 1001 \cdot (-3001) = -3,004,001\end{aligned}$$

c) First note that the given numbers $15, 11, 7, 3, -1, -5, -9, \ldots$ are the beginning of an arithmetic sequence. The sequence is determined by the first term $a_1 = 15$ and common difference $d = 11 - 15 = -4$. The nth term is given by $a_n = 15 - (n - 1) \cdot 4$, and summing the first $k = 333$ terms gives:

$$\sum_{n=1}^{333} a_n = \frac{333}{2} \cdot (a_1 + a_{333})$$

We still need to find a_{333} in the above formula:

$$a_{333} = 15 - (333-1)\cdot 4 = 15 - 332\cdot 4 = 15 - 1328 = -1313$$

This gives a total sum of

$$\sum_{n=1}^{333} a_n = \frac{333}{2}\cdot(15+(-1313)) = \frac{35}{2}\cdot(-1298) = -216,117.$$

□

24.3 Exercises

Exercise 24.1

Find the first seven terms of the sequence.

a) $a_n = 3n$ b) $a_n = 5n+3$ c) $a_n = n^2+2$
d) $a_n = n$ e) $a_n = (-1)^{n+1}$ f) $a_n = \frac{\sqrt{n+1}}{n}$
g) $a_k = 10^k$ h) $a_i = 5+(-1)^i$ i) $a_n = \sin(\frac{\pi}{2}\cdot n)$

Exercise 24.2

Find the first six terms of the sequence.

a) $a_1 = 5$, $a_n = a_{n-1}+3$ for $n \geq 2$
b) $a_1 = 7$, $a_n = 10\cdot a_{n-1}$ for $n \geq 2$
c) $a_1 = 1$, $a_n = 2\cdot a_{n-1}+1$ for $n \geq 2$
d) $a_1 = 6$, $a_2 = 4$, $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

Exercise 24.3

Find the value of the series.

a) $\sum_{n=1}^{4} a_n$, where $a_n = 5n$ b) $\sum_{k=1}^{5} a_k$, where $a_k = k$
c) $\sum_{i=1}^{4} a_i$, where $a_n = n^2$ d) $\sum_{n=1}^{6}(n-4)$
e) $\sum_{k=1}^{3}(k^2+4k-4)$ f) $\sum_{j=1}^{4}\frac{1}{j+1}$

Exercise 24.4

Is the sequence below part of an arithmetic sequence? If it is part of an arithmetic sequence, find the formula for the nth term a_n in the form $a_n = a_1 + (n-1) \cdot d$.

a)	$5, 8, 11, 14, 17, \ldots$	b)	$-10, -7, -4, -1, 2, \ldots$
c)	$-1, 1, -1, 1, -1, 1, \ldots$	d)	$18, 164, 310, 474, \ldots$
e)	$73.4, 51.7, 30, \ldots$	f)	$9, 3, -3, -8, -14, \ldots$
g)	$4, 4, 4, 4, 4, \ldots$	h)	$-2.72, -2.82, -2.92, -3.02, -3.12, \ldots$
i)	$\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{11}, \ldots$	j)	$\frac{-3}{5}, \frac{-1}{10}, \frac{2}{5}, \ldots$
k)	$a_n = 4 + 5 \cdot n$	l)	$a_j = 2 \cdot j - 5$
m)	$a_n = n^2 + 8n + 15$	n)	$a_k = 9 \cdot (k+5) + 7k - 1$

Exercise 24.5

Determine the general nth term a_n of an arithmetic sequence $\{a_n\}$ with the data given below.

a)	$d = 4$, and $a_8 = 57$	b)	$d = -3$, and $a_{99} = -70$
c)	$a_1 = 14$, and $a_7 = -16$	d)	$a_1 = -80$, and $a_5 = 224$
e)	$a_3 = 10$, and $a_{14} = -23$	f)	$a_{20} = 2$, and $a_{60} = 32$

Exercise 24.6

Determine the value of the indicated term of the given arithmetic sequence.

a)	if $a_1 = 8$, and $a_{15} = 92$,	find a_{19}
b)	if $d = -2$, and $a_3 = 31$,	find a_{81}
c)	if $a_1 = 0$, and $a_{17} = -102$,	find a_{73}
d)	if $a_7 = 128$, and $a_{37} = 38$,	find a_{26}

Exercise 24.7

Determine the sum of the arithmetic sequence.

a) Find the sum $a_1 + \cdots + a_{48}$ for the arithmetic sequence $a_n = 4n + 7$.

b) Find the sum $\sum_{n=1}^{21} a_n$ for the arithmetic sequence $a_n = 2 - 5n$.

c) Find the sum: $\sum_{n=1}^{99}(10 \cdot n + 1)$

d) Find the sum: $\sum_{n=1}^{200}(-9 - n)$

e) Find the sum of the first 100 terms of the arithmetic sequence:

$$2, 4, 6, 8, 10, 12, \ldots$$

f) Find the sum of the first 83 terms of the arithmetic sequence:

$$25, 21, 17, 13, 9, 5, \ldots$$

g) Find the sum of the first 75 terms of the arithmetic sequence:

$$2012, 2002, 1992, 1982, \ldots$$

h) Find the sum of the first 16 terms of the arithmetic sequence:

$$-11, -6, -1, \ldots$$

i) Find the sum of the first 99 terms of the arithmetic sequence:

$$-8, -8.2, -8.4, -8.6, -8.8, -9, -9.2, \ldots$$

j) Find the sum

$$7 + 8 + 9 + 10 + \cdots + 776 + 777$$

k) Find the sum of the first 40 terms of the arithmetic sequence:

$$5, 5, 5, 5, 5, \ldots$$

Chapter 25

The geometric series

We now study another sequence—the geometric sequence. Our analysis follows steps similar to the one of the arithmetic sequence in Section 24.2.

25.1 Finite geometric series

We have already encountered examples of geometric sequences in Example 24.2(b) and (c). A geometric sequence is a sequence for which we *multiply* a constant number to get from one term to the next, for example:

$$5, \underbrace{\quad}_{\times 4} 20, \underbrace{\quad}_{\times 4} 80, \underbrace{\quad}_{\times 4} 320, \underbrace{\quad}_{\times 4} 1280, \dots$$

Definition 25.1: Geometric sequence

A sequence $\{a_n\}$ is called a **geometric sequence** if any two consecutive terms have a *common ratio* r. The geometric sequence is determined by r and the first value a_1. This can be written recursively as:

$$a_n = a_{n-1} \cdot r \qquad \text{for } n \geq 2$$

Alternatively, we have the general formula for the nth term of the geometric sequence:

$$\boxed{a_n = a_1 \cdot r^{n-1}} \tag{25.1}$$

Example 25.2

Determine whether the terms below are the first terms of an arithmetic sequence, a geometric sequence, neither, or both. If they are the terms of an arithmetic or geometric sequence, then find the general formula a_n of the sequence in the form (24.2) or (25.1).

a)	$3, 6, 12, 24, 48, \ldots$	b)	$100, 50, 25, 12.5, \ldots$
c)	$2, 4, 16, 256, \ldots$	d)	$700, -70, 7, -0.7, 0.07, \ldots$
e)	$3, 10, 17, 24, \ldots$	f)	$-3, -3, -3, -3, -3, \ldots$
g)	$a_n = n^2$	h)	$a_n = \left(\frac{3}{7}\right)^n$

Solution.

a) First, the differences of two consecutive terms $6-3=3$ and $12-6=6$ are different. So, these are not the terms of an arithmetic sequence. On the other hand, the quotient of two consecutive terms always gives the same number $6 \div 3 = 2$, $12 \div 6 = 2$, $24 \div 12 = 2$, etc. Therefore, the common ratio is $r = 2$, which shows that these are the terms in a geometric sequence. Furthermore, the first term is $a_1 = 3$, so the general formula for the nth term is $a_n = 3 \cdot 2^{n-1}$.

b) Since the differences $50 - 100 = -50$ and $25 - 50 = -25$ are not the same, this is not an arithmetic sequence. We see that the common ratio between two terms is $r = \frac{50}{100} = \frac{1}{2}$, so that this is a geometric sequence. Since the first term is $a_1 = 100$, we have the general term $a_n = 100 \cdot (\frac{1}{2})^{n-1}$.

c) The difference between the first two terms is $4 - 2 = 2$, while the next two terms have a difference $16 - 4 = 12$. Therefore, this is also not an arithmetic sequence. Furthermore, the quotient of the first two terms is $4 \div 2 = 2$, whereas the quotient of the next two terms is $16 \div 4 = 4$. Since these quotients are not equal, this is not a geometric sequence.

d) This is not an arithmetic sequence, but these are terms of a geometric sequence. Two consecutive terms have a ratio of $r = -\frac{1}{10}$, and the first term is $a_1 = 700$. The general term of this geometric sequence is $a_n = 700 \cdot (-\frac{1}{10})^{n-1}$.

e) The quotient of the first couple of terms is not equal: $\frac{10}{3} \neq \frac{17}{10}$, so this is not a geometric sequence. The difference between any two terms is $7 = 10 - 3 = 17 - 10 = 24 - 17$, so this is part of an arithmetic sequence with common difference $d = 7$. The general formula is $a_n = a_1 + d \cdot (n-1) = 3 + (n-1) \cdot 7$.

f) The common ratio is $r = (-3) \div (-3) = 1$, so this is a geometric sequence with $a_n = (-3) \cdot 1^{n-1}$. On the other hand, the common difference is $(-3) - (-3) = 0$, so this is *also* an arithmetic sequence with $a_n = (-3) + (n-1) \cdot 0$. Of course, both formulas reduce to the simpler expression $a_n = -3$.

g) We write the first terms in the sequence $\{n^2\}_{n \geq 1}$:

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

Calculating the quotients of consecutive terms, we get $4 \div 1 = 4$ and $9 \div 4 = 2.25$, so this is not a geometric sequence. Also the difference of consecutive terms is $4 - 1 = 3$ and $9 - 4 = 5$, so this is also not an arithmetic sequence.

h) Writing the first couple of terms in the sequence $\{(\frac{3}{7})^n\}$, we obtain:

$$\left(\frac{3}{7}\right)^1, \left(\frac{3}{7}\right)^2, \left(\frac{3}{7}\right)^3, \left(\frac{3}{7}\right)^4, \left(\frac{3}{7}\right)^5, \ldots$$

Thus, we get from one term to the next by multiplying $r = \frac{3}{7}$, so this is a geometric sequence. The first term is $a_1 = \frac{3}{7}$, so $a_n = \frac{3}{7} \cdot \left(\frac{3}{7}\right)^{n-1}$. This is clearly the given sequence, since we may simplify this as

$$a_n = \frac{3}{7} \cdot \left(\frac{3}{7}\right)^{n-1} = \left(\frac{3}{7}\right)^{1+n-1} = \left(\frac{3}{7}\right)^n$$

We can also confirm that this is not an arithmetic sequence.

□

Example 25.3

Find the general formula $a_n = a_1 \cdot r^{n-1}$ of a geometric sequence with the given properties.

a) $r = 4$, and $a_5 = 6400$
b) $a_1 = \frac{2}{5}$, and $a_4 = -\frac{27}{20}$
c) $a_5 = 216$, $a_7 = 24$, and r is positive

Solution.

a) We know that $r = 4$, and we still need to find a_1. Using $a_5 = 64000$, we obtain:

$$6400 = a_5 = a_1 \cdot 4^{5-1} = a_1 \cdot 4^4 = 256 \cdot a_1 \overset{(\div 256)}{\Longrightarrow} a_1 = \frac{6400}{256} = 25$$

The sequence is therefore given by the formula, $a_n = 25 \cdot 4^{n-1}$.

b) The geometric sequence $a_n = a_1 \cdot r^{n-1}$ has $a_1 = \frac{2}{5}$. We calculate r using the second condition.

$$-\frac{27}{20} = a_4 = a_1 \cdot r^{4-1} = \frac{2}{5} \cdot r^3 \overset{(\times \frac{5}{2})}{\Longrightarrow} r^3 = -\frac{27}{20} \cdot \frac{5}{2} = -\frac{27}{4} \cdot \frac{1}{2} = \frac{-27}{8}$$
$$\overset{(\text{take } \sqrt[3]{\ })}{\Longrightarrow} r = \sqrt[3]{\frac{-27}{8}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{8}} = \frac{-3}{2}$$

Therefore, $a_n = \frac{2}{5} \cdot \left(\frac{-3}{2}\right)^{n-1}$.

c) The question provides neither a_1 nor r for our formula $a_n = a_1 \cdot r^{n-1}$. However, we obtain two equations in the two variables a_1 and r:

$$\begin{cases} 216 = a_5 = a_1 \cdot r^{5-1} \\ 24 = a_7 = a_1 \cdot r^{7-1} \end{cases} \Longrightarrow \begin{cases} 216 = a_1 \cdot r^4 \\ 24 = a_1 \cdot r^6 \end{cases}$$

In order to solve this, we need to eliminate one of the variables. Looking at the equations on the right, we see that dividing the top equation by the bottom equation cancels a_1.

$$\frac{216}{24} = \frac{a_1 \cdot r^4}{a_1 \cdot r^6} \Longrightarrow \frac{9}{1} = \frac{1}{r^2} \overset{(\text{take reciprocal})}{\Longrightarrow} \frac{1}{9} = \frac{r^2}{1} \Longrightarrow r^2 = \frac{1}{9}$$

To obtain r, we have to solve this quadratic equation. In general, there are, in fact, two solutions:

$$r = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$$

Since the problem states that r is positive, we see that we need to take the positive solution $r = \frac{1}{3}$. Plugging $r = \frac{1}{3}$ back into either of the two equations, we may solve for a_1. For example, using the first equation $a_5 = 216$, we obtain:

$$216 = a_5 = a_1 \cdot \left(\frac{1}{3}\right)^{5-1} = a_1 \cdot \left(\frac{1}{3}\right)^4 = a_1 \cdot \frac{1}{3^4} = a_1 \cdot \frac{1}{81}$$
$$\overset{(\times 81)}{\Longrightarrow} \quad a_1 = 81 \cdot 216 = 17,496$$

So, we finally arrive at the general formula for the nth term of the geometric sequence, $a_n = 17,496 \cdot (\frac{1}{3})^{n-1}$.

□

We can find the sum of the first k terms of a geometric sequence using another trick, which is very different from the one we used for the arithmetic sequence.

Note 25.4: Summing over terms in a geometric sequence

Consider the geometric sequence $a_n = 7 \cdot 10^{n-1}$, that is the sequence:

$$7, \quad 70, \quad 700, \quad 7000, \quad 70,000, \quad 700,000, \quad \dots$$

We want to add the first 5 terms of this sequence.

$$7 + 70 + 700 + 7000 + 70,000 = 77,777$$

The above example can, of course, easily be computed by hand. In general, however, much more work is necessary to find a sum of a geometric sequence, especially if the sequence is more complicated and we want to add a lot more terms. To get to a general formula, we will add the terms in the above sum in a different way, which may appear to be more complicated than necessary. However, the advantage of the following calculation is that it is an illustration for a general method, which allows us to find the sum of terms in any geometric sequence. To this end, we multiply $(1-10)$ to the sum $(7+70+700+7000+70,000)$, and simplify this using the distributive law:

$$\begin{aligned} & (1-10)\cdot(7+70+700+7000+70,000) \\ = \; & 7-70+70-700+700-7000 \\ & \qquad +7000-70,000+70,000-700,000 \\ = \; & 7-700,000 \end{aligned}$$

The sum in the second line above is called a *telescopic sum*, which is a sum where consecutive terms cancel each other. In the above sum the only remaining terms are the very first and last terms. Dividing by $(1-10)$, we obtain:

$$7+70+700+7000+70,000 = \frac{7-700,000}{1-10} = \frac{-699,993}{-9} = 77,777$$

An appropriate generalization of the previous note yields a computation that applies to any geometric sequence.

Observation 25.5: Geometric series

Let $\{a_n\}$ be a geometric sequence whose nth term is given by the formula $a_n = a_1 \cdot r^{n-1}$. We furthermore assume that $r \neq 1$. Then the sum $a_1 + a_2 + \cdots + a_{k-1} + a_k$ is given by:

$$\boxed{\sum_{i=1}^{k} a_i = a_1 \cdot \frac{1-r^k}{1-r}} \tag{25.2}$$

Proof. We multiply $(1-r)$ to the sum $(a_1 + a_2 + \cdots + a_{k-1} + a_k)$:

$$\begin{aligned} & (1-r)\cdot(a_1+a_2+\cdots+a_k) \\ = \; & (1-r)\cdot(a_1\cdot r^0 + a_1\cdot r^1 + \cdots + a_1\cdot r^{k-1}) \\ = \; & a_1\cdot r^0 - a_1\cdot r^1 + a_1\cdot r^1 - a_1\cdot r^2 + \cdots + a_1\cdot r^{k-1} - a_1\cdot r^k \\ = \; & a_1\cdot r^0 - a_1\cdot r^k = a_1\cdot(1-r^k) \end{aligned}$$

Dividing by $(1-r)$, we obtain

$$a_1 + a_2 + \cdots + a_k = \frac{a_1 \cdot (1-r^k)}{(1-r)} = a_1 \cdot \frac{1-r^k}{1-r}$$

This is the formula we wanted to prove. □

Example 25.6

Find the value of the geometric series.

a) Find the sum $\sum_{n=1}^{6} a_n$ for the geometric sequence $a_n = 10 \cdot 3^{n-1}$.

b) Determine the value of the geometric series: $\sum_{k=1}^{5} \left(-\frac{1}{2}\right)^k$

c) Find the sum of the first 12 terms of the geometric sequence

$$-3, -6, -12, -24, \ldots$$

Solution.

a) We need to find the sum $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$, and we do so by using the formula provided in Equation (25.2). Since $a_n = 10 \cdot 3^{n-1}$, we have $a_1 = 10$ and $r = 3$, so

$$\sum_{n=1}^{6} a_n = 10 \cdot \frac{1-3^6}{1-3} = 10 \cdot \frac{1-729}{1-3} = 10 \cdot \frac{-728}{-2} = 10 \cdot 364 = 3640$$

b) Again, we use the formula for the geometric series $\sum_{k=1}^{n} a_k = a_1 \cdot \frac{1-r^n}{1-r}$, since $a_k = (-\frac{1}{2})^k$ is a geometric series. We may calculate the first term $a_1 = -\frac{1}{2}$, and the common ratio is also $r = -\frac{1}{2}$. With this, we obtain:

$$\begin{aligned}\sum_{k=1}^{5} \left(-\frac{1}{2}\right)^k &= \left(-\frac{1}{2}\right) \cdot \frac{1-(-\frac{1}{2})^5}{1-(-\frac{1}{2})} = \left(-\frac{1}{2}\right) \cdot \frac{1-((-1)^5 \frac{1^5}{2^5})}{1-(-\frac{1}{2})} \\ &= \left(-\frac{1}{2}\right) \cdot \frac{1-(-\frac{1}{32})}{1-(-\frac{1}{2})} = \left(-\frac{1}{2}\right) \cdot \frac{1+\frac{1}{32}}{1+\frac{1}{2}} = \left(-\frac{1}{2}\right) \cdot \frac{\frac{32+1}{32}}{\frac{2+1}{2}} \\ &= \left(-\frac{1}{2}\right) \cdot \frac{\frac{33}{32}}{\frac{3}{2}} = \left(-\frac{1}{2}\right) \cdot \frac{33}{32} \cdot \frac{2}{3} = -\frac{1}{2} \cdot \frac{11}{16} = -\frac{11}{32}\end{aligned}$$

c) Our first task is to find the formula for the provided geometric series $-3, -6, -12, -24, \ldots$. The first term is $a_1 = -3$ and the common ratio is $r = 2$, so that $a_n = (-3) \cdot 2^{n-1}$. The sum of the first 12 terms of this sequence is again given by Equation (25.2):

$$\begin{aligned} \sum_{i=1}^{12} (-3) \cdot 2^{i-1} &= (-3) \cdot \frac{1 - 2^{12}}{1 - 2} = (-3) \cdot \frac{1 - 4096}{1 - 2} = (-3) \cdot \frac{-4095}{-1} \\ &= (-3) \cdot 4095 = -12,285 \end{aligned}$$

□

25.2 Infinite geometric series

In some cases, it makes sense to add not only finitely many terms of a geometric sequence, but all infinitely many terms of the sequence! An informal and intuitive infinite geometric series is exhibited in the next note.

Note 25.7: Summing over all terms in a geometric sequence

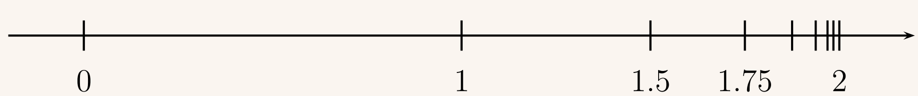
Consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Here, the common ratio is $r = \frac{1}{2}$, and the first term is $a_1 = 1$, so that the formula for a_n is $a_n = \left(\frac{1}{2}\right)^{n-1}$. We are interested in summing *all infinitely many* terms of this sequence:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

We add these terms one by one, and picture these sums on the number line:



$$\begin{aligned} 1 &= 1 \\ 1 + \frac{1}{2} &= 1.5 \end{aligned}$$

$$\begin{aligned} 1+\frac{1}{2}+\frac{1}{4} &= 1.75 \\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= 1.875 \\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= 1.9375 \end{aligned}$$

We see that adding each term takes the sum closer and closer to the number 2. More precisely, adding a term a_n to the partial sum $a_1 + \cdots + a_{n-1}$ decreases the distance between 2 and $a_1 + \cdots + a_{n-1}$ by half. For this reason, we can, in fact, get arbitrarily close to 2, so it is reasonable to expect that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2$$

In the next definition and observation, this equation will be justified and made more precise.

First, we give a definition of an infinite series.

Definition 25.8: Infinite series

An infinite series is given by adding infinitely many terms of a sequence. We write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \tag{25.3}$$

To be more precise, the infinite sum is defined as the limit $\sum_{n=1}^{\infty} a_n := \lim_{k\to\infty} \left(\sum_{n=1}^{k} a_n \right)$. Therefore, an infinite sum is defined precisely when this limit exists.

Observation 25.9: Infinite geometric series

Let $\{a_n\}$ be a geometric sequence with $a_n = a_1 \cdot r^{n-1}$. Then the infinite geometric series is defined whenever $-1 < r < 1$. In this case, we have:

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r}} \tag{25.4}$$

Proof. Informally, this follows from the formula $\sum_{n=1}^{k} a_n = a_1 \cdot \frac{1-r^k}{1-r}$ and the fact that, for $-1 < r < r$, the term r^k approaches zero when k increases without bound.

More formally, the proof uses the notion of limits, and proceeds as follows:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right) = \lim_{k\to\infty}\left(a_1 \cdot \frac{1-r^k}{1-r}\right) = a_1 \cdot \frac{1-\lim\limits_{k\to\infty}(r^k)}{1-r} = a_1 \cdot \frac{1}{1-r}$$

□

Example 25.10

Find the value of the infinite geometric series.

a) $\sum_{j=1}^{\infty} a_j$, for $a_j = 5 \cdot \left(\frac{1}{3}\right)^{j-1}$ b) $\sum_{n=1}^{\infty} 3 \cdot (0.71)^n$

c) $500 - 100 + 20 - 4 + \dots$ d) $3 + 6 + 12 + 24 + 48 + \dots$

Solution.

a) We use formula (25.4) for the geometric series $a_n = 5 \cdot (\frac{1}{3})^{n-1}$, that is $a_1 = 5 \cdot (\frac{1}{3})^{1-1} = 5 \cdot (\frac{1}{3})^0 = 5 \cdot 1 = 5$ and $r = \frac{1}{3}$. Therefore,

$$\sum_{j=1}^{\infty} a_j = a_1 \cdot \frac{1}{1-r} = 5 \cdot \frac{1}{1-\frac{1}{3}} = 5 \cdot \frac{1}{\frac{3-1}{3}} = 5 \cdot \frac{1}{\frac{2}{3}} = 5 \cdot \frac{3}{2} = \frac{15}{2}$$

b) In this case, $a_n = 3 \cdot (0.71)^n$, so that $a_1 = 3 \cdot 0.71^1 = 3 \cdot 0.71 = 2.13$ and $r = 0.71$. Again using formula (25.4), we can find the infinite geometric series as

$$\sum_{n=1}^{\infty} 3 \cdot (0.71)^n = a_1 \cdot \frac{1}{1-r} = 2.13 \cdot \frac{1}{1-0.71} = 2.13 \cdot \frac{1}{0.29} = \frac{2.13}{0.29} = \frac{213}{29}$$

In the last step, we simplified the fraction by multiplying 100 to both numerator and denominator, which had the effect of eliminating the decimals.

c) Our first task is to identify the given sequence as an infinite geometric sequence:

$$\{a_n\} \text{ is given by } 500, -100, 20, -4, \dots$$

Notice that the first term is 500, and each consecutive term is given by dividing by -5, or in other words, by multiplying by the common ratio $r = -\frac{1}{5}$. Therefore, this is an infinite geometric series, which can be evaluated as

$$500 - 100 + 20 - 4 + \cdots = \sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r} = 500 \cdot \frac{1}{1-(-\frac{1}{5})}$$
$$= 500 \cdot \frac{1}{1+\frac{1}{5}} = \frac{500}{\frac{5+1}{5}} = \frac{500}{\frac{6}{5}} = 500 \cdot \frac{5}{6} = \frac{2500}{6} = \frac{1250}{3}$$

d) We want to evaluate the infinite series $3+6+12+24+48+\dots$. The sequence $3, 6, 12, 24, 48, \dots$ is a geometric sequence with $a_1 = 3$ and common ratio $r = 2$. Since $r \geq 1$, we see that formula (25.4) *cannot* be applied, as (25.4) only applies to $-1 < r < 1$. However, since we add larger and larger terms, the series gets larger than any possible bound, so that the whole sum becomes infinite.

$$3 + 6 + 12 + 24 + 48 + \cdots = \infty$$

□

Example 25.11

The fraction $0.55555\dots$ may be written as:

$$0.55555 \cdots = 0.5 + 0.05 + 0.005 + 0.0005 + 0.00005 + \dots$$

Noting that the sequence

$$0.5, \underset{\times 0.1}{\curvearrowright} 0.05, \underset{\times 0.1}{\curvearrowright} 0.005, \underset{\times 0.1}{\curvearrowright} 0.0005, \underset{\times 0.1}{\curvearrowright} 0.00005, \dots$$

is a geometric sequence with $a_1 = 0.5$ and $r = 0.1$, we can calculate the infinite sum as:

$$0.55555 \cdots = \sum_{n=1}^{\infty} 0.5 \cdot (0.1)^{n-1} = 0.5 \cdot \frac{1}{1-0.1} = 0.5 \cdot \frac{1}{0.9} = \frac{0.5}{0.9} = \frac{5}{9},$$

Here we multiplied numerator and denominator by 10 in the last step in order to eliminate the decimals.

25.3 Exercises

Exercise 25.1

Which of these sequences is geometric, arithmetic, neither, or both. Write the sequence in the usual form $a_n = a_1 + (n-1)\cdot d$ if it is an arithmetic sequence, and $a_n = a_1 \cdot r^{n-1}$ if it is a geometric sequence.

a)	$7, 14, 28, 56, \dots$	b)	$3, -30, 300, -3000, \dots$
c)	$81, 27, 9, 3, 1, \frac{1}{3}, \dots$	d)	$-7, -5, -3, -1, 1, 3, 5, 7, \dots$
e)	$-6, 2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \dots$	f)	$-2, -2\cdot\frac{2}{3}, -2\cdot\left(\frac{2}{3}\right)^2, -2\cdot\left(\frac{2}{3}\right)^3, \dots$
g)	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$	h)	$2, 2, 2, 2, 2, \dots$
i)	$5, 1, 5, 1, 5, 1, 5, 1, \dots$	j)	$-2, 2, -2, 2, -2, 2, \dots$
k)	$0, 5, 10, 15, 20, \dots$	l)	$5, \frac{5}{3}, \frac{5}{3^2}, \frac{5}{3^3}, \frac{5}{3^4}, \dots$
m)	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$	n)	$\log(2), \log(4), \log(8), \log(16), \dots$
o)	$a_n = -4^n$	p)	$a_n = -4n$
q)	$a_n = 2\cdot(-9)^n$	r)	$a_n = \left(\frac{1}{3}\right)^n$
s)	$a_n = -\left(\frac{5}{7}\right)^n$	t)	$a_n = \left(-\frac{5}{7}\right)^n$
u)	$a_n = \frac{2}{n}$	v)	$a_n = 3n+1$

Exercise 25.2

A geometric sequence, $a_n = a_1 \cdot r^{n-1}$, has the given properties. Find the term a_n of the sequence.

a)	$a_1 = 3$, and $r = 5$,	find a_4
b)	$a_1 = 200$, and $r = -\frac{1}{2}$,	find a_6
c)	$a_1 = -7$, and $r = 2$,	find a_n (for all n)
d)	$r = 2$, and $a_4 = 48$,	find a_1
e)	$r = 100$, and $a_4 = 900,000$,	find a_n (for all n)
f)	$a_1 = 20$, $a_4 = 2500$,	find a_n (for all n)
g)	$a_1 = \frac{1}{8}$, and $a_6 = \frac{3^5}{8^6}$,	find a_n (for all n)
h)	$a_3 = 36$, and $a_6 = 972$,	find a_n (for all n)
i)	$a_8 = 4000$, $a_{10} = 40$, and r is negative,	find a_n (for all n)

Exercise 25.3

Find the value of the finite geometric series using formula (25.2). Confirm the formula either by adding the the summands directly, or alternatively by using the calculator.

a) Find the sum $\sum_{j=1}^{4} a_j$ for the geometric sequence $a_j = 5 \cdot 4^{j-1}$.

b) Find the sum $\sum_{i=1}^{7} a_i$ for the geometric sequence $a_n = \left(\frac{1}{2}\right)^n$.

c) Find: $\sum_{m=1}^{5} \left(-\frac{1}{5}\right)^m$

d) Find: $\sum_{k=1}^{6} 2.7 \cdot 10^k$

e) Find the sum of the first 5 terms of the geometric sequence:

$$2, 6, 18, 54, \dots$$

f) Find the sum of the first 6 terms of the geometric sequence:

$$-5, 15, -45, 135, \dots$$

g) Find the sum of the first 8 terms of the geometric sequence:

$$-1, -7, -49, -343, \dots$$

h) Find the sum of the first 10 terms of the geometric sequence:

$$600, -300, 150, -75, 37.5, \dots$$

i) Find the sum of the first 40 terms of the geometric sequence:

$$5, 5, 5, 5, 5, \dots$$

Exercise 25.4

Find the value of the infinite geometric series.

a) $\sum_{j=1}^{\infty} a_j$, for $a_j = 3 \cdot \left(\frac{2}{3}\right)^{j-1}$

b) $\sum_{j=1}^{\infty} 7 \cdot \left(-\frac{1}{5}\right)^j$

c) $\sum_{j=1}^{\infty} 6 \cdot \frac{1}{3^j}$

d) $\sum_{n=1}^{\infty} -2 \cdot (0.8)^n$

e) $\sum_{n=1}^{\infty} (0.99)^n$

f) $27 + 9 + 3 + 1 + \frac{1}{3} + \dots$

g) $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \dots$

h) $-6 - 2 - \frac{2}{3} - \frac{2}{9} - \dots$

i) $100 + 40 + 16 + 6.4 + \dots$

j) $-54 + 18 - 6 + 2 - \dots$

Exercise 25.5

Rewrite the decimal using an infinite geometric sequence, and then use the formula for the infinite geometric series to rewrite the decimal as a fraction (see Example 25.11).

a) $0.44444\dots$

b) $0.77777\dots$

c) $5.55555\dots$

d) $0.2323232323\dots$

e) $39.393939\dots$

f) $0.248248248\dots$

g) $20.02002\dots$

h) $0.5040504\dots$

Review of complex numbers, sequences, and the binomial theorem

Exercise V.1

Find the magnitude and direction angle of the vector

$$\vec{v} = \langle 7, -7\sqrt{3} \rangle.$$

Exercise V.2

For the vectors $\vec{v} = \langle 3, -2 \rangle$ and $\vec{w} = \langle -5, 6 \rangle$, evaluate the following expression:

$$7 \cdot \vec{v} - 3 \cdot \vec{w}$$

Exercise V.3

Convert the complex number to polar form:

a) $-3 - 3i$ b) $-5\sqrt{3} + 5i$

Exercise V.4

Multiply and write the answer in standard form:

$$4(\cos(207°) + i\sin(207°)) \cdot 2(\cos(108°) + i\sin(108°))$$

Exercise V.5

Divide and write the answer in standard form:

$$\frac{9(\cos(190°) + i \sin(190°))}{15(\cos(70°) + i \sin(70°))}$$

Exercise V.6

Evaluate the sum:

$$\sum_{n=1}^{7} (3n^2 + 4n)$$

Exercise V.7

Determine whether the sequence is an arithmetic sequence, geometric sequence, or neither. If it is one of these, then find the general formula for the nth term a_n of the sequence.

a) $54, -18, 6, -2, \frac{2}{3}, \dots$
b) $2, 4, 8, 10, \dots$
c) $9, 5, 1, -3, -7, \dots$

Exercise V.8

Find the sum of the first 75 terms of the arithmetic sequence:

$$-30, -22, -14, -6, 2, \dots$$

Exercise V.9

Find the sum of the first 8 terms of the geometric series:

$$-7, -14, -28, -56, -112, \dots$$

Exercise V.10

Find the value of the infinite geometric series:

$$80 - 20 + 5 - 1.25 + \dots$$

Appendix A

The binomial theorem

In this appendix, we discuss the generalized binomial theorem.

A.1 The binomial theorem

Recall the well-known binomial formula:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2} \tag{A.1}$$

which follows from a direct computation: $(a+b)^2 = (a+b)\cdot(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

In this section, we generalize (A.1) to find similar expressions for $(a+b)^n$ for any natural number n. This is the content of the (generalized) binomial theorem below. Before we can state the theorem, we need to define the notion of a factorial and combinations.

Definition A.1: Factorial

For a natural number $n = 1, 2, 3, \ldots$, we define $n!$ to be the number

$$\boxed{n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}$$

The number $n!$ is called n **factorial**.
To make the formulas below work nicely, we also define $0!$ to be $0! = 1$.

Example A.2

$$\begin{aligned} 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24 \\ 7! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040 \\ 2! &= 1 \cdot 2 = 2 \end{aligned}$$

Factorials also can be easily computed with the calculator.

Note that the factorial becomes very large even for relatively small integers. For example $18! \approx 6.402 \cdot 10^{15}$ as shown above.

The next concept that we introduce is that of the binomial coefficient.

Definition A.3: Binomial coefficient

Let $n = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots, n$ be natural numbers or zero, so that $0 \leq r \leq n$. Then we define the **binomial coefficient** as

$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

The binomial coefficient is also written as ${}_nC_r = \binom{n}{r}$, and we read them as "n-**choose**-r."

Note A.4

- The binomial coefficient $\binom{n}{r}$ should not be confused with the fraction $\left(\frac{n}{r}\right)$.
- A subset of the set $\{1, 2, \ldots, n\}$ with r elements is called an r-**combination**. The binomial coefficient can be interpreted as counting the number of distinct r-combinations. More precisely, there are exactly $\binom{n}{r}$ distinct r-combinations of the set $\{1, \ldots, n\}$.

Example A.5

Calculate the binomial coefficients.

a) $\binom{6}{4}$ b) $\binom{8}{5}$ c) $\binom{25}{23}$ d) $\binom{7}{1}$ e) $\binom{11}{11}$ f) $\binom{11}{0}$

Solution.

a) Many binomial coefficients may be calculated by hand, such as:

$$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2} = \frac{5 \cdot 6}{2} = 15$$

b) Again, we can calculate this by hand

$$\binom{8}{5} = \frac{8!}{5!3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 1 \cdot 2 \cdot 3} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = 7 \cdot 8 = 56$$

We can also use the calculator to find the answer. The binomial coefficients are computed with the command "nCr". (Note that this is case sensitive, that is: the letter "C" has to be capitalized, and "n" and "r" are lower case.)

We calculate the remaining binomial coefficients (c)–(f), which can also be confirmed with the calculator.

c) $\binom{25}{23} = \frac{25!}{23!\cdot 2!} = \frac{23!\cdot 24\cdot 25}{23!\cdot 1\cdot 2} = \frac{24\cdot 25}{2} = 300$

d) $\binom{7}{1} = \frac{7!}{1!\cdot 6!} = \frac{6!\cdot 7}{1\cdot 6!} = \frac{7}{1} = 7$

e) $\binom{11}{11} = \frac{11!}{11!\cdot 0!} = \frac{1}{1\cdot 1} = 1$

f) $\binom{11}{0} = \frac{11!}{0!\cdot 11!} = \frac{1}{1\cdot 1} = 1$

Note that in the last two equations we needed to use the fact that $0! = 1$.

□

We state some useful facts about the binomial coefficient that can already be seen in the previous example.

Observation A.6: Basic properties of the binomial coefficient

For all $n = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots, n$, we have:

$$\binom{n}{n-r} = \binom{n}{r} \qquad \binom{n}{0} = \binom{n}{n} = 1 \qquad \binom{n}{1} = \binom{n}{n-1} = n$$

Proof. We have:

$$\begin{aligned}
\binom{n}{n-r} &= \frac{n!}{(n-r)!\cdot(n-(n-r))!} = \frac{n!}{(n-r)!\cdot r!} = \binom{n}{r} \\
\binom{n}{0} &= \binom{n}{n} = \frac{n!}{0!\cdot n!} = \frac{1}{1} = 1 \\
\binom{n}{1} &= \binom{n}{n-1} = \frac{n!}{1!\cdot(n-1)!} = \frac{n}{1} = n
\end{aligned}$$

□

Note A.7: Binomial coefficients and Pascal's triangle

The binomial coefficients are found in what is known as Pascal's triangle. For this, calculate the lowest binomial coefficients and write them in a triangular arrangement:

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ & & & & & \vdots & & & & & \end{array} = \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

The triangle on the right is known as Pascal's triangle. Each entry in the triangle is obtained by adding the two entries right above it.

The binomial coefficients appear in the expressions for $(a+b)^n$, as we will see in the next example.

Example A.8

We now calculate some simple examples.

$$\begin{aligned} (a+b)^3 &= (a+b)\cdot(a+b)\cdot(a+b) \\ &= (a^2+2ab+b^2)\cdot(a+b) \\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3 \end{aligned}$$

Note that the numbers $1, 3, 3$, and 1 that appear as coefficients of a^3, a^2b, ab^2, and b^3, respectively, are precisely the binomial coefficients $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$.
We also calculate the fourth power.

$$\begin{aligned} (a+b)^4 &= (a+b)\cdot(a+b)\cdot(a+b)\cdot(a+b) \\ &= (a^3+3a^2b+3ab^2+b^3)\cdot(a+b) \\ &= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4 \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \end{aligned}$$

Again, the numbers $1, 4, 6, 4$, and 1 are precisely the binomial coefficients $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$.

We are now ready to state the general **binomial theorem**:

Proposition A.9: Binomial theorem

The nth power $(a+b)^n$ can be expanded as:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}b^n$$

Using the summation symbol, we may write this in short:

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} \cdot a^{n-r} \cdot b^r} \tag{A.2}$$

Example A.10

Expand $(a+b)^5$.

Solution.

$$\begin{aligned}
&(a+b)^5 \\
&\quad = \binom{5}{0}a^5 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}b^5 \\
&\quad = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

□

A.2 Binomial expansion

Using the binomial theorem, we can also expand more general powers of sums or differences.

Example A.11

Expand the expression.

a) $(x^2+2y^3)^5$ b) $(2xy^2-\frac{4}{y^2})^3$

c) $(\sqrt{2}+1)^6$ d) $(i-3)^4$

Solution.

a) We use the binomial theorem with $a=x^2$ and $b=2y^3$:

$$\begin{aligned}(x^2+2y^3)^5 &= (x^2)^5+\binom{5}{1}(x^2)^4(2y^3)+\binom{5}{2}(x^2)^3(2y^3)^2\\ &\quad +\binom{5}{3}(x^2)^2(2y^3)^3+\binom{5}{4}(x^2)(2y^3)^4+(2y^3)^5\\ &= x^{10}+5x^8\cdot 2y^3+10x^6\cdot 4y^6\\ &\quad +10x^4\cdot 2^3y^9+5x^2\cdot 2^4y^{12}+2^5y^{15}\\ &= x^{10}+10x^8y^3+40x^6y^6+80x^4y^9+80x^2y^{12}+32y^{15}\end{aligned}$$

b) For part (b), we use $a=2xy^2$ and $b=-\frac{4}{y^2}$.

$$\begin{aligned}&(2xy^2-\frac{4}{y^2})^3\\ &= (2xy^2)^3+\binom{3}{1}(2xy^2)^2(-\frac{4}{y^2})+\binom{3}{2}(2xy^2)(-\frac{4}{y^2})^2+(-\frac{4}{y^2})^3\\ &= 2^3x^3y^6+3\cdot 2^2x^2y^4(-\frac{4}{y^2})+3(2xy^2)(-1)^2\frac{4^2}{y^4}+(-1)^3\frac{4^3}{y^6}\\ &= 8x^3y^6-48x^2y^2+96x\cdot\frac{1}{y^2}-64\cdot\frac{1}{y^6}\end{aligned}$$

c) Similarly, for part (c), we now have $a=\sqrt{2}$ and $b=1$:

$$\begin{aligned}(\sqrt{2}+1)^6 &= (\sqrt{2})^6+\binom{6}{1}(\sqrt{2})^5\cdot 1+\binom{6}{2}(\sqrt{2})^4\cdot 1^2\\ &\quad +\binom{6}{3}(\sqrt{2})^3\cdot 1^3+\binom{6}{4}(\sqrt{2})^2\cdot 1^4+\binom{6}{5}(\sqrt{2})\cdot 1^5+1^6\\ &= \sqrt{64}+6\cdot\sqrt{32}+15\cdot\sqrt{16}\end{aligned}$$

$$\begin{aligned} & +20\cdot\sqrt{8}+15\cdot\sqrt{4}+6\cdot\sqrt{2}+1 \\ = \;& 8+6\cdot\sqrt{16\cdot 2}+15\cdot 4 \\ & +20\cdot\sqrt{4\cdot 2}+15\cdot 2+6\cdot\sqrt{2}+1 \\ = \;& 8+24\sqrt{2}+60+40\sqrt{2}+30+6\sqrt{2}+1 \\ = \;& 99+70\sqrt{2} \end{aligned}$$

Note that the last expression cannot be simplified any further (due to the order of operations).

d) Finally, we have $a = i$ and $b = -3$, and we use the fact that $i^2 = -1$, and therefore, $i^3 = -i$ and $i^4 = +1$:

$$\begin{aligned} (i-3)^4 &= i^4 + \binom{4}{1}\cdot i^3\cdot(-3) + \binom{4}{2}\cdot i^2\cdot(-3)^2 \\ &\quad + \binom{4}{3}\cdot i\cdot(-3)^3 + (-3)^4 \\ &= 1+4\cdot(-i)\cdot(-3)+6\cdot(-1)\cdot 9+4\cdot i\cdot(-27)+81 \\ &= 1+12i-54-108i+81 \\ &= 28-96i \end{aligned}$$

□

In some instances it is not necessary to write the full binomial expansion, but it may be enough to find a particular term, say the kth term of the expansion.

Observation A.12: Expanding $(a+b)^n$

Recall, for example, the binomial expansion of $(a+b)^6$:

$\binom{6}{0}a^6b^0$ +	$\binom{6}{1}a^5b^1$ +	$\binom{6}{2}a^4b^2$ +	$\binom{6}{3}a^3b^3$ +	$\binom{6}{4}a^2b^4$ +	$\binom{6}{5}a^1b^5$ +	$\binom{6}{6}a^0b^6$
first term	second term	third term	fourth term	fifth term	sixth term	seventh term

Note that the exponents of the a's and b's for each term always add up to 6, and that the exponents of a decrease from 6 to 0, and the exponents of b increase from 0 to 6. Furthermore, observe that in the

above expansion the fifth term is $\binom{6}{4}a^2b^4$.
In general, we *define* the kth term of the expansion $(a+b)^n$ to be given by:

$$\boxed{\binom{n}{k-1}a^{n-k+1}b^{k-1}} \tag{A.3}$$

Note in particular, that the kth term has a power of b given by b^{k-1} (and *not* b^k), it has a binomial coefficient $\binom{n}{k-1}$, and the exponents of a and b add up to n.

Example A.13

Determine:

a) the 4th term in the binomial expansion of $(p+3q)^5$
b) the 8th term in the binomial expansion of $(x^3y-2x^2)^{10}$
c) the 12th term in the binomial expansion of $(-\frac{5a}{b^7}-b)^{15}$

Solution.

a) We have $a=p$ and $b=3q$, and $n=5$ and $k=4$. Thus, the binomial coefficient of the 4th term is $\binom{5}{3}$, the b-term is $(3q)^3$, and the a-term is p^2. The 4th term is therefore given by

$$\binom{5}{3}\cdot p^2\cdot(3q)^3 = 10\cdot p^2\cdot 3^3q^3 = 270p^2q^3$$

b) In this case, $a=x^3y$ and $b=-2x^2$, and furthermore, $n=10$ and $k=8$. The binomial coefficient of the 8th term is $\binom{10}{7}$, the b-term is $(-2x^2)^7$, and the a-term is $(x^3y)^3$. Therefore, the 8th term is

$$\binom{10}{7}\cdot(x^3y)^3\cdot(-2x^2)^7 = 120\cdot x^9y^3\cdot(-128)x^{14} = -15,360\cdot x^{23}y^3$$

c) Similarly, we obtain the 12th term of $(-\frac{5a}{b^7}-b)^{15}$ as

$$\binom{15}{11}\cdot\Big(-\frac{5a}{b^7}\Big)^4\cdot(-b)^{11} = 1365\cdot\frac{5^4a^4}{b^{28}}\cdot(-b^{11})$$
$$= 1365\cdot\frac{625\cdot a^4\cdot(-b^{11})}{b^{28}} = -853,125\cdot\frac{a^4}{b^{17}}$$

□

The following is a variation of the above problem, in which we want to find the term for a specified power of some of the given variables.

Example A.14

Determine:

a) the x^4y^{12}–term in the binomial expansion of $(5x^2+2y^3)^6$
b) the x^{15}–term in the binomial expansion of $(x^3-x)^7$
c) the real part of the complex number $(3+2i)^4$

Solution.

a) In this case, we have $a=5x^2$ and $b=2y^3$. The term x^4y^{12} can be rewritten as $x^4y^{12}=(x^2)^2\cdot(y^3)^4$, so that the full term $\binom{n}{k-1}a^{n-k+1}b^{k-1}$ (including the coefficients) is

$$\binom{6}{4}\cdot(5x^2)^2\cdot(2y^3)^4=15\cdot 25x^4\cdot 16y^{12}=6000\cdot x^4y^{12}$$

b) The various powers of x in $(x^3-x)^7$ (in the order in which they appear in the binomial expansion) are:

$$(x^3)^7=x^{21},\ (x^3)^6\cdot x^1=x^{19},\ (x^3)^5\cdot x^2=x^{17},\ (x^3)^4\cdot x^3=x^{15},\ldots$$

The last term is precisely the x^{15}–term, that is, we take the fourth term, $k=4$. We obtain a total term (including the coefficients) of

$$\binom{7}{3}\cdot(x^3)^4\cdot(-x)^3=35\cdot x^{12}\cdot(-x)^3=-35\cdot x^{15}$$

c) Recall that i^n is real when n is even, and imaginary when n is odd:

$$\begin{aligned} i^1&=i\\ i^2&=-1\\ i^3&=-i\\ i^4&=1\\ i^5&=i\\ i^6&=-1\\ &\vdots \end{aligned}$$

The real part of $(3+2i)^4$ is therefore given by the first, third, and fifth term of the binomial expansion:

$$\begin{aligned}
\text{real part} &= \binom{4}{0}\cdot 3^4\cdot(2i)^0+\binom{4}{2}\cdot 3^2\cdot(2i)^2+\binom{4}{4}\cdot 3^0\cdot(2i)^4\\
&= 1\cdot 81\cdot 1+6\cdot 9\cdot 4i^2+1\cdot 1\cdot 16i^4\\
&= 81+216\cdot(-1)+16\cdot 1\\
&= 81-216+16\\
&= -119
\end{aligned}$$

The real part of $(3+2i)^4$ is -119.

□

A.3 Exercises

Exercise A.1

Find the value of the factorial or binomial coefficient.

a) $5!$ b) $3!$ c) $9!$ d) $2!$ e) $0!$ f) $1!$ g) $19!$ h) $64!$
i) $\binom{5}{2}$ j) $\binom{9}{6}$ k) $\binom{12}{1}$ l) $\binom{12}{0}$ m) $\binom{23}{22}$ n) $\binom{19}{12}$ o) $\binom{13}{11}$ p) $\binom{16}{5}$

Exercise A.2

Expand the expression via the binomial theorem.

a) $(m+n)^4$ b) $(x+2)^5$ c) $(x-y)^6$ d) $(-p-q)^5$

Exercise A.3

Expand the expression.

a) $(x-2y)^3$ b) $(x-10)^4$ c) $(x^2y+y^2)^5$ d) $(2y^2-5x^4)^4$
e) $(x+\sqrt{x})^3$ f) $(-2\frac{x^2}{y}-\frac{y^3}{x})^5$ g) $(\sqrt{2}-2\sqrt{3})^3$ h) $(1-i)^3$

Exercise A.4

Determine:

a) the first 3 terms in the binomial expansion of $(xy - 4x)^5$
b) the first 2 terms in the binomial expansion of $(2a^2 + b^3)^9$
c) the last 3 terms in the binomial expansion of $(3y^2 - x^2)^7$
d) the first 3 terms in the binomial expansion of $(\frac{x}{y} - \frac{y}{x})^{10}$
e) the last 4 terms in the binomial expansion of $(m^3n + \frac{1}{2}n^2)^6$

Exercise A.5

Determine:

a) the 5th term in the binomial expansion of $(x + y)^7$
b) the 3rd term in the binomial expansion of $(x^2 - y)^9$
c) the 10th term in the binomial expansion of $(2 - w)^{11}$
d) the 5th term in the binomial expansion of $(2x + xy)^7$
e) the 7th term in the binomial expansion of $(2a + 5b)^6$
f) the 6th term in the binomial expansion of $(3p^2 - q^3p)^7$
g) the 10th term in the binomial expansion of $(4 + \frac{b}{2})^{13}$

Exercise A.6

Determine:

a) the x^3y^6-term in the binomial expansion of $(x + y)^9$
b) the r^4s^4-term in the binomial expansion of $(r^2 - s)^6$
c) the x^4-term in the binomial expansion of $(x - 1)^{11}$
d) the x^3y^6-term in the binomial expansion of $(x^3 + 5y^2)^4$
e) the x^7-term in the binomial expansion of $(2x - x^2)^5$
f) the imaginary part of the number $(1 + i)^3$

Answers to exercises

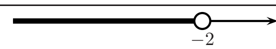
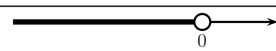
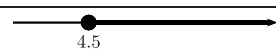
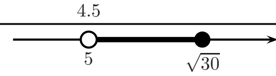
Here are the answers to the exercises given in each sections and in the reviews for each part.

Chapter 1 (exercises starting on page 14):

Exercise 1.1 Examples: a) $2, 3, 5$, b) $-3, 0, 6$, c) $-3, -4, 0$, d) $\frac{2}{3}, \frac{-4}{7}, 8$, e) $\sqrt{5}, \pi, \sqrt[3]{31}$, f) $\frac{1}{2}, \frac{2}{5}, 0.75$

Exercise 1.2 natural: $17000, \frac{12}{4}, \sqrt{25}$, integer: $-5, 0, 17000, \frac{12}{4}, \sqrt{25}$, rational: $\frac{7}{3}, -5, 0, 17000, \frac{12}{4}, \sqrt{25}$, real: all of the given numbers are real numbers, irrational: $\sqrt{7}$

Exercise 1.3

Inequality	Number line	Interval
$2 \leq x < 5$		$[2, 5)$
$x \leq 3$		$(-\infty, 3]$
$12 < x \leq 17$		$(12, 17]$
$x < -2$		$(-\infty, -2)$
$-2 \leq x \leq 6$		$[-2, 6]$
$x < 0$		$(-\infty, 0)$
$4.5 \leq x$		$[4.5, \infty)$
$5 < x \leq \sqrt{30}$		$(5, \sqrt{30}]$
$\frac{13}{7} < x < \pi$		$(\frac{13}{7}, \pi)$

Exercise 1.4 a) this is a function with domain $D = \{-5, -1, 0, 3, 6\}$ and range $R = \{2, 3, 5, 7, 8\}$, for example: the input $x = -5$ gives output $y = 5$, etc., b) not a function, since for $x = 4$ we have both $y = 0$ and $y = -1$, c) this is a function with $D = \{-11, -2, 3, 6, 7, 19\}, R = \{3\}$, d) this *is* a function with $D = \{1, 2, 3, 4, 5\}, R = \{\sqrt{19}, 5.33, 9, 13, 17\}$, e) this is not a function

Exercise 1.5 a) yes, b) no (assuming that there are two items in the store with the same price), c) the domain for the function in (a) is the set of all items in the store that are for sale, d) the range for the function in (a) is the set of all prices

Exercise 1.6 a) a given cash amount x determines the interest amount y, b) i) $\$0$, ii) $\$100,000$, iii) $\$0$, iv) $\$300,000$, v) $\$200,000$, vi) $\$40,000$

Exercise 1.7 a) $C(r) = 2\pi r$, b) $P(a) = 3a$, c) $P(a) = 2a + 6$, d) $V(a) = a^3$

Chapter 2 (exercises starting on page 26):

Exercise 2.1 a): i) 10, ii) 16, iii) -5, iv) 1, v) $3\sqrt{13} + 1$, vi) $3\sqrt{2} + 10$, vii) $-3x + 1$, viii) $3x + 7$, ix) $3x + 1 + h$, x) $3x + 3h + 1$

b): i) 6, ii) 20, iii) 6, iv) 0, v) $13 - \sqrt{13}$, vi) $8 + 5\sqrt{2}$, vii) $x^2 + x$, viii) $x^2 + 3x + 2$, ix) $x^2 - x + h$, x) $x^2 + 2xh + h^2 - x - h$

c): i) 0, ii) 4, iii) undefined, iv) undefined, v) 2, vi) $\sqrt{2 + 6\sqrt{2}}$, vii) $\sqrt{x^2 - 9}$, viii) $\sqrt{x^2 + 4x - 5}$, ix) $\sqrt{x^2 - 9} + h$, x) $\sqrt{x^2 + 2xh + h^2 - 9}$

d): i) $\frac{1}{3}$, ii) $\frac{1}{5}$, iii) $-\frac{1}{2}$, iv) undefined, v) $\frac{\sqrt{13}}{13}$, vi) $\frac{3-\sqrt{2}}{7}$, vii) $-\frac{1}{x}$, viii) $\frac{1}{x+2}$, ix) $\frac{1+xh}{x}$, x) $\frac{1}{x+h}$

e): i) $\frac{-2}{5}$, ii) 0, iii) undefined, iv) $\frac{-5}{2}$, v) $\frac{\sqrt{13}-5}{\sqrt{13}+2} = \frac{23-7\sqrt{13}}{9}$, vi) $\frac{\sqrt{2}-2}{\sqrt{2}+5} = \frac{-12+7\sqrt{2}}{23}$, vii) $\frac{-x-5}{-x+2}$, viii) $\frac{x-3}{x+4}$, ix) $\frac{x-5+hx+2h}{x+2}$, x) $\frac{x+h-5}{x+h+2}$

f): i) -27, ii) -125, iii) 8, iv) 0, v) $-\sqrt{2197}$, vi) $-45 - 29\sqrt{2}$, vii) x^3, viii) $f(x+2) = -(x+2)^3$ or in descending order $f(x+2) = -x^3 - 6x^2 - 12x - 8$, ix) $-x^3 + h$, x) $-(x+h)^3$ or $-x^3 - 3x^2h - 3xh^2 - h^3$

Exercise 2.2 a) $D = (-4, 6]$, b) -3, c) 25, d) -8, e) 9

Exercise 2.3 a) $D = (-\infty, 5) \cup (5, \infty)$ or, alternatively, $D = \mathbb{R} - \{5\}$, b) 0, c) -2, d) 7, e) 7, f) undefined, g) 22

Exercise 2.4 a) 5, b) 2, c) $2x + h$, d) $2x + 5 + h$, e) $2x + h$, f) $2x + 3 + h$, g) $2x + 4 + h$, h) $6x - 2 + 3h$, i) $8x + 6 + 4h$, j) $4x - 8 + 2h$, k) $-10x + 3 - 5h$, l) $3x^2 + 3xh + h^2$

Exercise 2.5 a) 3, b) 4, c) $x + a - 3$, d) $x + a + 4$, e) $7x + 7a + 2$, f) $\frac{-1}{ax}$

Exercise 2.6 a) $D = \mathbb{R}$ all real numbers, b) $D = \mathbb{R}$, c) $D = [2, \infty)$, d) $D = (-\infty, 4]$, e) $D = \mathbb{R}$, f) $D = \mathbb{R} - \{-6\}$, g) $D = \mathbb{R} - \{7\}$, h) $D = \mathbb{R} - \{2, 5\}$, i) $D = \mathbb{R} - \{2\}$, j) $D = (1, 2) \cup [3, \infty)$, k) $D = [0, 9) \cup (9, \infty)$, l) $D = (-4, \infty)$

Chapter 3 (exercises starting on page 43):

Exercise 3.1 a) $y = 2x - 4$, b) $y = -x + 3$, c) $y = -2x - 2$, d) $y = \frac{2}{5}x + 3$, e) $y = -x + 0$ or $y = -x$, f) $y = \frac{2}{3}x + 4$

Exercise 3.2 a) $y - 3 = \frac{1}{3} \cdot (x - 5)$, b) $y - 1 = -\frac{3}{2} \cdot (x - 4)$, c) $y + 2 = -\frac{1}{2} \cdot (x - 3)$, d) $y - 1 = 1 \cdot (x + 1)$, e) $y - 3 = -3 \cdot (x + 4)$, f) $y + 6 = -\frac{1}{2} \cdot (x + 5)$

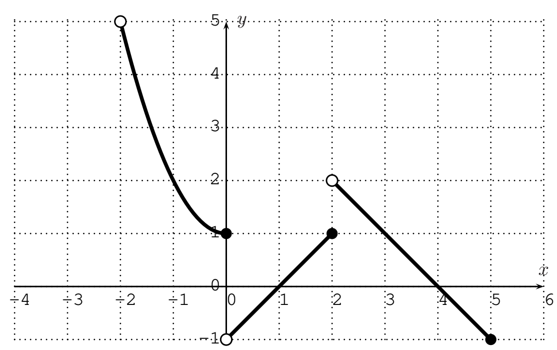
Exercise 3.3 a) domain $D_f = [1, 3) \cup [4, 6]$ and range $R_f = [1, 3]$, b) $D_g = \mathbb{R}$ and $R_g = [2, 3]$, c) $D_h = (-2, 0) \cup (0, 2) \cup (2, 3)$ and $R_h = \{-1\} \cup (0, 1]$, d) 1, e) 3, f) undefined, g) 2, h) 2, i) 3, j) undefined, k) 2, l) 2, m) 3, n) 2.5, o) 2, p) 2, q) 2, r) undefined, s) 1, t) undefined, u) -1, v) undefined, w) undefined, x) -1

Exercise 3.4 a) not a function, b) this *is* a function, c) not a function, d) not a function

Exercise 3.5 a) $D = (-3, 4) \cup (4, 7]$, b) $R = (-2, 2]$, c) $x = -2$ or $x = 0$ or $x = 7$, d) $x \in (4, 5]$, e) $x \in (-3, -1] \cup [0, 4) \cup [6, 7]$, f) $x \in (-2, 0) \cup (4, 7)$, g) $f(2) = -1$, $f(5) = 2$, h) $f(2)+f(5) = 1$, i) $f(2)+5 = 4$, j) $f(2+5) = 0$

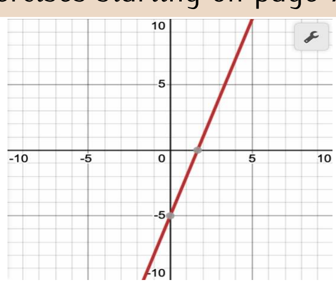
Exercise 3.6 a) Approximately 3900 students were admitted in the year 2000, b) The most students were admitted in 2007. c) In 2000, the number of admitted students rose fastest. d) In 2003 the number of admitted students declined.

Exercise 3.7 domain $D = (-2, 5]$, graph:

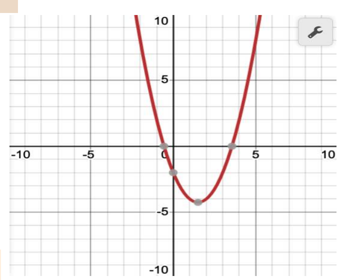


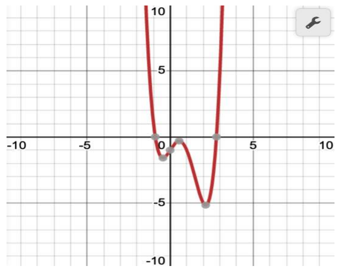
Chapter 4 (exercises starting on page 75):

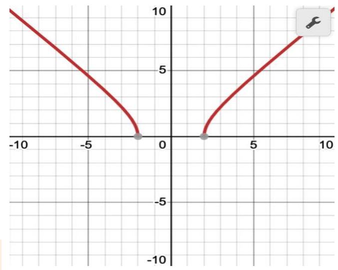
Exercise 4.1 a)

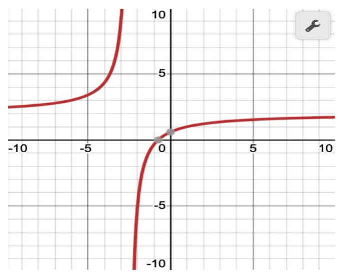


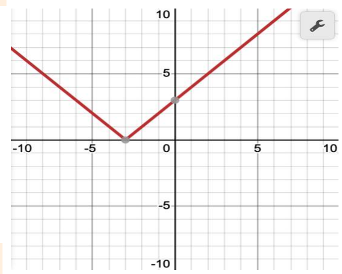
b)



c) 

d) 

e) 

f) 

Exercise 4.2

a) roots: $(x,y) \approx (-3.925, 0)$, $(x,y) \approx (-1.552, 0)$, $(x,y) \approx (1.477, 0)$, local max: $(x,y) \approx (-2.897, 6.051)$, local min: $(x,y) \approx (0.23, -9.236)$, y-intercept: $(x,y) = (0, -9)$,

b) roots: $(x,y) \approx (-0.414, 0)$, $(x,y) \approx (2.414, 0)$, $(x,y) = (4, 0)$, local max: $(x,y) \approx (0.709, 6.303)$, local min: $(x,y) \approx (3.291, -2.303)$, y-intercept: $(x,y) = (0, 4)$,

c) roots: $(x,y) \approx (-0.894, 0)$, $(x,y) \approx (-0.155, 0)$, $(x,y) \approx (1.8, 0)$, local max: $(x,y) \approx (1.054, 7.027)$, local min: $(x,y) \approx (-0.554, -1.277)$, y-intercept: $(x,y) = (0, 1)$,

d) roots: $(x,y) \approx (-0.4, 0)$, $(x,y) = (0, 0)$, local max: $(x,y) \approx (-0.267, 0.047)$, local min: $(x,y) \approx (0, 0)$, y-intercept: $(x,y) = (0, 0)$,

e) roots: $(x,y) \approx (-1.441, 0)$, $(x,y) \approx (-0.567, 0)$, $(x,y) \approx (0.485, 0)$, $(x,y) \approx (2.523, 0)$, local max: $(x,y) \approx (0, 1)$, local min: $(x,y) \approx (-1.088, -1.046)$, $(x,y) \approx (1.838, -7.31)$, y-intercept: $(x,y) = (0, 1)$,

f) roots: $(x,y) \approx (-1.061, 0)$, $(x,y) \approx (4.857, 0)$, local max: $(x,y) \approx (-0.486, 4.314)$, $(x,y) \approx (3.676, 54.064)$, local min: $(x,y) \approx (0.56, 1.54)$, y-intercept: $(x,y) = (0, 3)$,

g) roots: $(x,y) \approx (-1.802, 0)$, $(x,y) = (-1, 0)$, $(x,y) \approx (-0.445, 0)$, $(x,y) = (0, 0)$, $(x,y) \approx (1.247, 0)$, local max: $(x,y) \approx (-1.538, 0.665)$, $(x,y) \approx (-0.194, 0.091)$, local min: $(x,y) \approx (-0.755, -0.12)$, $(x,y) \approx (0.887, -2.158)$, y-intercept: $(x,y) = (0, 0)$,

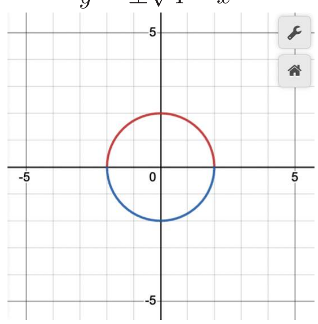
h) roots: $(x,y) \approx (1.574, 0)$, $(x,y) \approx (1.607, 0)$, $(x,y) = (2, 0)$, $(x,y) \approx (6.905, 0)$, local max: $(x,y) \approx (1.737, 0.103)$, local min: $(x,y) \approx (1.585, -0.17)$, $(x,y) \approx (4.905, -1.618)$, y-intercept: $(x,y) \approx (0, 4.414)$,

Exercise 4.3 a) $D = \mathbb{R}$, $R = [5, \infty)$, b) $D = \mathbb{R}$, $R = \mathbb{R}$, c) $D = \mathbb{R}$, $R = [-5, \infty)$, d) $D = \mathbb{R}$, $R = (-\infty, 19]$, e) $D = [-5, \infty)$, $R = [3, \infty)$, f) $D = (-\infty, 4]$, $R = [2, \infty)$, g) $D = \mathbb{R}$, $R = [-7, \infty)$, h) $D = \mathbb{R} - \{3\}$, $R = \mathbb{R} - \{1\}$

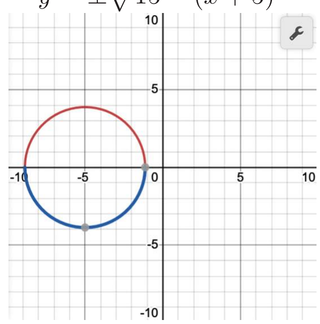
Exercise 4.4 a) function, b) not a function, c) not a function, d) function, e) function, f) not a function

Exercise 4.5

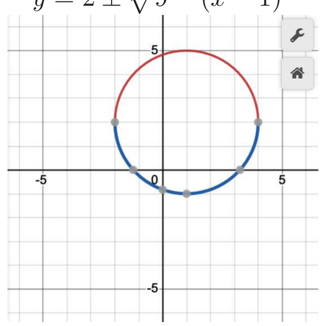
$y = \pm\sqrt{4 - x^2}$

a) 

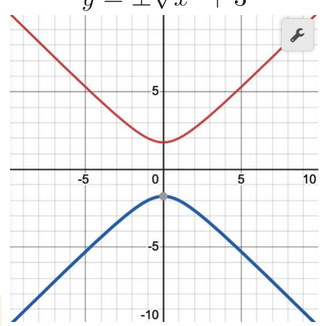
$y = \pm\sqrt{15 - (x+5)^2}$

b) 

$y = 2 \pm \sqrt{9 - (x-1)^2}$

c) 

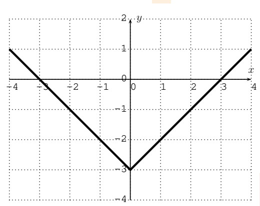
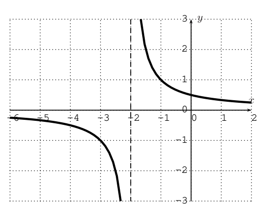
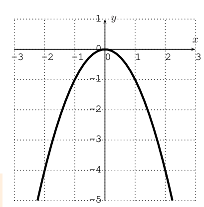
$y = \pm\sqrt{x^2 + 3}$

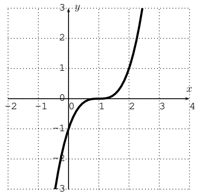
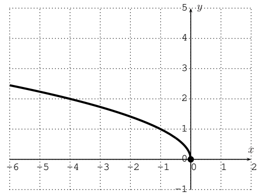
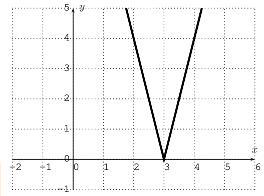
d) 

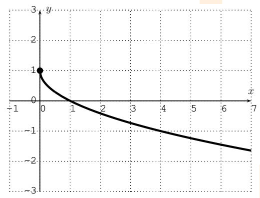
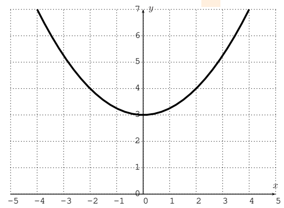
Exercise 4.6 $(x-4)^2 + (y+2)^2 = 9$

Exercise 4.7 a) $x \approx -1.488$, b) $x \approx -1.764$, $x \approx -0.416$, $x \approx -0.681$, c) $x \approx 5.22$, d) $x \approx -1.431$, $x \approx 0.038$, e) $x \approx -1.247$, $x = 0$, $x \approx 0.445$, $x \approx 1.802$, f) $x = 0$, $x = 1$, $x = 3$

Exercise 4.8 a) $y = |x| + 1$, b) $y = -\sqrt{x}$, c) $y = (x-2)^2 + 1$, d) $y = \frac{1}{x+2} + 2$, e) $y = -(x+2)^3$, f) $y = -|x-3| + 2$

Exercise 4.9 a)  b)  c) 

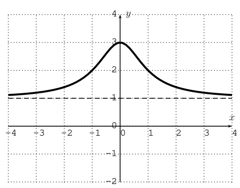
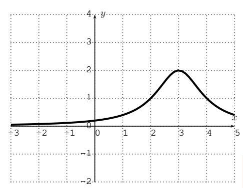
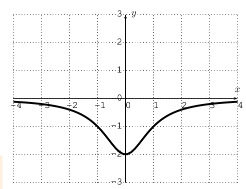
d)  e)  f) 

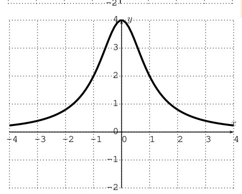
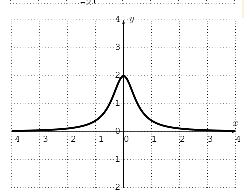
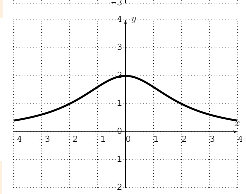
g)  h) 

Exercise 4.10 a) $y = x^2 - 7x - 3$, b) $y = (x+3)^2 - 7 \cdot (x+3) + 1 = x^2 - x - 11$, c) $y = -x^2 + 7x - 1$, d) $y = x^2 + 7x + 1$, e) $y = \frac{1}{9}x^2 - \frac{7}{3}x + 1$, f) $y = 4x^2 - 14x + 1$

Exercise 4.11 a) shift to the right by 5, b) stretched away from the x-axis by a factor 2, c) shift to the right by 4, d) compressed toward the y-axis by a factor 2, e) shifted to the right by 3 (to get the graph of $y = \frac{1}{x}$) and then reflected about the x-axis, f) compressed toward the x-axis by a factor 2 (you get $y = |x|$) then shifted to the left by 1 and up by 1

Exercise 4.12 a) odd, b) even, c) even, d) neither, e) even, f) odd, g) even, h) neither, i) odd

Exercise 4.13 a)  b)  c) 

d)  e)  f) 

Chapter 5 (exercises starting on page 89):

Exercise 5.1 a) $(f+g)(x) = x^2 + 9x - 5$ with domain $D_{f+g} = \mathbb{R}$, $(f-g)(x) = x^2 + 3x + 5$ with domain $D_{f-g} = \mathbb{R}$, $(f \cdot g)(x) = 3x^3 + 13x^2 - 30x$ with domain $D_{f \cdot g} = \mathbb{R}$

b) $(f+g)(x) = x^3 + 5x^2 + 12$, $D_{f+g} = \mathbb{R}$, $(f-g)(x) = x^3 - 5x^2 - 2$, $D_{f-g} = \mathbb{R}$, $(f \cdot g)(x) = 5x^5 + 7x^3 + 25x^2 + 35$, $D_{f \cdot g} = \mathbb{R}$

c) $(f+g)(x) = 2x^2+3x+12\sqrt{x}$, $D_{f+g} = [0,\infty)$, $(f-g)(x) = -2x^2+3x+2\sqrt{x}$, $D_{f-g} = [0,\infty)$, $(f\cdot g)(x) = 6x^3+14x^2\sqrt{x}+15x\sqrt{x}+35x$, $D_{f\cdot g} = [0,\infty)$

d) $(f+g)(x) = \frac{5x+1}{x+2}$, $D_{f+g} = \mathbb{R}-\{-2\}$, $(f-g)(x) = \frac{1-5x}{x+2}$, $D_{f-g} = \mathbb{R}-\{-2\}$, $(f\cdot g)(x) = \frac{5x}{(x+2)^2}$, $D_{f\cdot g} = \mathbb{R}-\{-2\}$

e) $(f+g)(x) = 3\sqrt{x-3}$, $D_{f+g} = [3,\infty)$, $(f-g)(x) = -\sqrt{x-3}$, $D_{f-g} = [3,\infty)$, $(f\cdot g)(x) = 2\cdot(\sqrt{x-3})^2 = 2\cdot(x-3)$, $D_{f\cdot g} = [3,\infty)$

f) $(f+g)(x) = x^2+5x-1$, $D_{f+g} = \mathbb{R}$, $(f-g)(x) = x^2-x+11$, $D_{f-g} = \mathbb{R}$, $(f\cdot g)(x) = 3x^3+3x-30$, $D_{f\cdot g} = \mathbb{R}$

g) $(f+g)(x) = 3x^2+6x+4$, $D_{f+g} = \mathbb{R}$, $(f-g)(x) = -x^2-4$, $D_{f-g} = \mathbb{R}$, $(f\cdot g)(x) = 2x^4+9x^3+13x^2+12x$, $D_{f\cdot g} = \mathbb{R}$

Exercise 5.2 a) $\left(\frac{f}{g}\right)(x) = \frac{3x+6}{2x-8}$ with domain $D_{\frac{f}{g}} = \mathbb{R}-\{4\}$, $\left(\frac{g}{f}\right)(x) = \frac{2x-8}{3x+6}$ with domain $D_{\frac{g}{f}} = \mathbb{R}-\{-2\}$, b) $\left(\frac{f}{g}\right)(x) = \frac{x+2}{x^2-5x+4} = \frac{x+2}{(x-4)(x-1)}$, $D_{\frac{f}{g}} = \mathbb{R}-\{1,4\}$, $\left(\frac{g}{f}\right)(x) = \frac{x^2-5x+4}{x+2}$, $D_{\frac{g}{f}} = \mathbb{R}-\{-2\}$, c) $\left(\frac{f}{g}\right)(x) = \frac{x+3}{(x-5)(x-2)}$, $D_{\frac{f}{g}} = \mathbb{R}-\{-3,2,5\}$, $\left(\frac{g}{f}\right)(x) = \frac{(x-5)(x-2)}{x+3}$, $D_{\frac{g}{f}} = \mathbb{R}-\{-3,5\}$, d) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+6}}{2x+5}$, $D_{\frac{f}{g}} = [-6,-\frac{5}{2})\cup(-\frac{5}{2},\infty)$, $\left(\frac{g}{f}\right)(x) = \frac{2x+5}{\sqrt{x+6}}$, $D_{\frac{g}{f}} = (-6,\infty)$, e) $\left(\frac{f}{g}\right)(x) = \frac{x^2+8x-33}{\sqrt{x}}$, $D_{\frac{f}{g}} = (0,\infty)$, $\left(\frac{g}{f}\right)(x) = \frac{\sqrt{x}}{x^2+8x-33}$, $D_{\frac{g}{f}} = [0,3)\cup(3,\infty)$

Exercise 5.3 a) 37, b) 7, c) 11, d) 147, e) -1, f) 81, g) $12x^2+20x+7$, h) $-4x-9$, i) -141, j) -5, k) $2x+2h-3$, l) $3x^2+6xh+3h^2+4x+4h$

Exercise 5.4 a) $(f\circ g)(x) = 6x+4$, b) $(f\circ g)(x) = x^2+6x+11$, c) $(f\circ g)(x) = 4x^2-2x$, d) $(f\circ g)(x) = x^4+4x^3+4x^2+\sqrt{x^2+2x+3}$, e) $(f\circ g)(x) = \frac{2}{x+h+4}$, f) $(f\circ g)(x) = x^2+2xh+h^2+4x+4h+3$

Exercise 5.5 a) $(f\circ g)(x) = 2x-6$, $(g\circ f)(x) = 2x-1$, $(f\circ f)(x) = 4x+12$, $(g\circ g)(x) = x-10$

b) $(f\circ g)(x) = x^2-2x+3$, $(g\circ f)(x) = x^2+4x+3$, $(f\circ f)(x) = x+6$, $(g\circ g)(x) = x^4-4x^3+2x^2+4x$

c) $(f\circ g)(x) = 6x-2-\sqrt{3x+2}$, $(g\circ f)(x) = \sqrt{6x^2-3x-16}$, $(f\circ f)(x) = 8x^4-8x^3-48x^2+25x+72$, $(g\circ g)(x) = \sqrt{3\sqrt{3x+2}+2}$

d) $(f\circ g)(x) = x$, $(g\circ f)(x) = x$, $(f\circ f)(x) = \frac{x+3}{3x+10}$, $(g\circ g)(x) = \frac{10x-3}{1-3x}$

e) $(f\circ g)(x) = x$, $(g\circ f)(x) = x$, $(f\circ f)(x) = (2(2x-7)^2-7)^2$ or expanded in descending degrees: $(f\circ f)(x) = 64x^4-896x^3+4592x^2-10192x+8281$, $(g\circ g)(x) = \frac{\sqrt{\frac{\sqrt{x}+7}{2}+7}}{2} = \frac{14+\sqrt{14+2\sqrt{x}}}{4}$

Exercise 5.6

x	1	2	3	4	5	6	7
$f(x)$	4	5	7	0	-2	6	4
$g(x)$	6	-8	5	2	9	11	2
$f(x)+3$	7	8	10	3	1	9	7
$4g(x)+5$	29	-28	25	13	41	49	13
$g(x)-2f(x)$	-2	-18	-9	2	13	-1	-6
$f(x+3)$	0	-2	6	4	undef.	undef.	undef.

Note, however, that the complete table for $y = f(x+3)$ is given by:

x	-2	-1	0	1	2	3	4
$f(x+3)$	4	5	7	0	-2	6	4

Exercise 5.7

x	1	2	3	4	5	6
$f(x)$	3	1	2	5	6	3
$g(x)$	5	2	6	1	2	4
$(g \circ f)(x)$	6	5	2	2	4	6
$(f \circ g)(x)$	6	1	3	3	1	5
$(f \circ f)(x)$	2	3	1	6	3	2
$(g \circ g)(x)$	2	2	4	5	2	1

Exercise 5.8

x	0	2	4	6	8	10	12
$f(x)$	4	8	5	6	12	-1	10
$g(x)$	10	2	0	-6	7	2	8
$(g \circ f)(x)$	0	7	undef.	-6	8	undef.	2
$(f \circ g)(x)$	-1	8	4	undef.	undef.	8	12
$(f \circ f)(x)$	5	12	undef.	6	10	undef.	-1
$(g \circ g)(x)$	2	2	10	undef.	undef.	2	7

Chapter 6 (exercises starting on page 103):

Exercise 6.1 a) no (that is: the function is not one-to-one), b) yes, c) no, d) no, e) no, f) no, g) yes, h) no

Exercise 6.2 a) $f^{-1}(x) = \frac{x-9}{4}$, b) $f^{-1}(x) = -\frac{x+3}{8}$, c) $f^{-1}(x) = x^2-8$, d) $f^{-1}(x) = \frac{x^2-7}{3}$, e) $f^{-1}(x) = -\left(\frac{x}{6}\right)^2 - 2 = \frac{-x^2-72}{36}$, f) $f^{-1}(x) = \sqrt[3]{x}$, g) $f^{-1}(x) = \frac{\sqrt[3]{x}-5}{2}$, h) $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$, i) $f^{-1}(x) = \frac{1}{x}$, j) $f^{-1}(x) = \frac{1}{x} + 1 = \frac{1+x}{x}$,

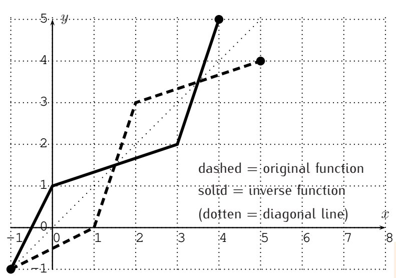
k) $f^{-1}(x) = \left(\frac{1}{x}\right)^2 + 2 = \frac{1+2x^2}{x^2}$, l) $f^{-1}(x) = \frac{5}{y} + 4 = \frac{5+4y}{y}$, m) $f^{-1}(x) = \frac{2x}{1-x}$, n) $f^{-1}(x) = \frac{6x}{x-3}$, o) $f^{-1}(x) = \frac{2-3x}{x-1}$, p) $f^{-1}(x) = \frac{5x+7}{x+1}$

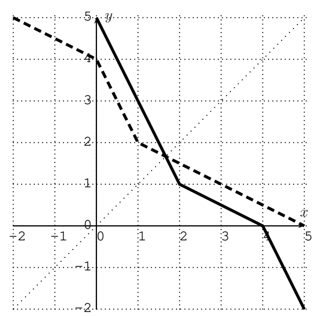
q)

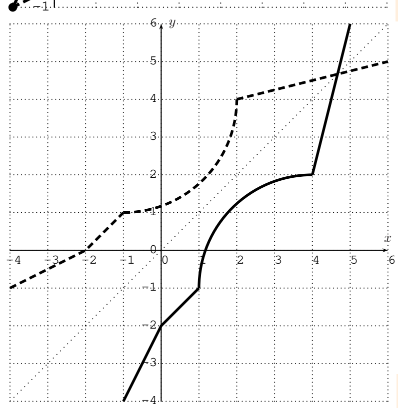
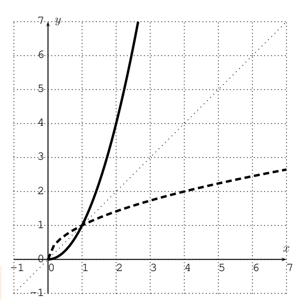
x	3	7	1	8	5	2
$f^{-1}(x)$	2	4	6	8	10	12

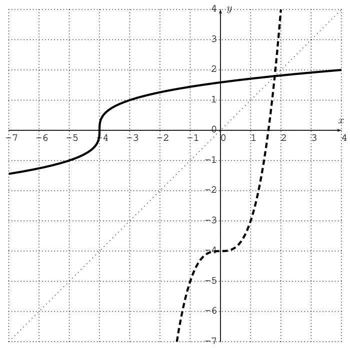
Exercise 6.3 a) restricting to the domain $D = [0, \infty)$ gives the inverse $f^{-1}(x) = \sqrt{x}$, b) restricting to the domain $D = [-5, \infty)$ gives the inverse $f^{-1}(x) = \sqrt{x-1} - 5$, c) restricting to the domain $D = [0, \infty)$ gives the inverse $f^{-1}(x) = x$, d) restricting to the domain $D = [4, \infty)$ gives the inverse $f^{-1}(x) = x + 6$, e) restricting to the domain $D = (0, \infty)$ gives the inverse $f^{-1}(x) = \sqrt{\frac{1}{x}}$, f) restricting to the domain $D = (-7, \infty)$ gives the inverse $f^{-1}(x) = \sqrt{-\frac{3}{x} - 7}$, g) restricting to the domain $D = [0, \infty)$ gives the inverse $f^{-1}(x) = \sqrt[4]{x}$, h) restricting to the domain $D = [3, \infty)$ gives the inverse $f^{-1}(x) = 3 + \sqrt[4]{10x}$

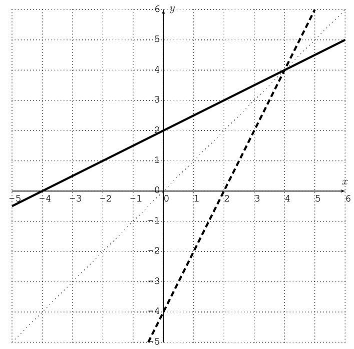
Exercise 6.4 a) yes (that is: the functions f and g are inverses of each other), b) no, c) no, d) yes, e) no, f) yes

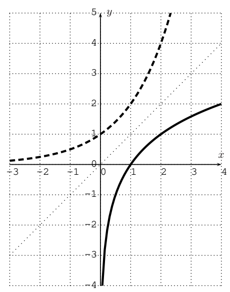
Exercise 7.5 a) 

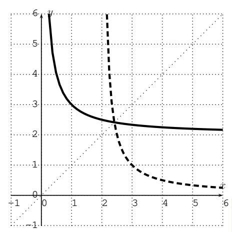
b) 

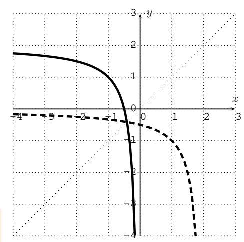
c) 
d) 


e)


f)


g)


h)


i)

Review of Part I (exercises starting on page 106):

Exercises I 1. $f(6) = 66$, $f(2)$ is undefined, $f(-6) = -9$, domain $D = (-8, 4] \cup (-3, 2) \cup [4, \infty)$, 2. $y = -\frac{1}{2}x + 1$, 3. $x \approx -0.481$, $x \approx 1.311$, $x \approx 3.170$, 4. $2x - 2 + 2h$, 5. domain $D = [2, 7]$, range $R = (1, 4]$, $f(3) = 2$, $f(5) = 2$, $f(7) = 4$, $f(9)$ is undefined, 6. $f(x) = -x^2 + 2$, 7. $(\frac{f}{g})(x) = \frac{5x+4}{x^2+8x+7} = \frac{5x+4}{(x+7)(x+1)}$ has domain $D = \mathbb{R} - \{-7, -1\}$, 8. $(f \circ g)(x) = 4x^2 - 12x + 9 + \sqrt{2x - 6}$ has domain $D = [3, \infty)$, 9. f and g are both functions, and the composition is given by the table:

x	2	3	4	5	6
$f(x)$	5	0	2	4	2
$g(x)$	6	2	3	4	1
$(f \circ g)(x)$	2	5	0	2	undef.

10. $f^{-1}(x) = \frac{1}{2x} - \frac{5}{2}$

Chapter 7 (exercises starting on page 123):

Exercise 7.1 a) $x^2 - 2x - 2 - \frac{3}{x-2}$, b) $x^2 + 3x - 2 + \frac{4}{x+3}$, c) $x + 6 - \frac{10}{x+1}$, d) $x^2 + x + \frac{5}{x+2}$, e) $2x^2 + 3x + 6 + \frac{11}{x-1}$, f) $2x^3 - 3x^2 + 15x - 74 + \frac{373}{x+5}$, g) $2x^3 + 8x^2 + x + 4 + \frac{3}{x-4}$, h) $x^2 - 3x + 9$, i) $x^3 + 2x^2 + x + 2 + \frac{2}{3x+1}$, j) $4x^2 + 3x + 6$, k) $x + 1 - \frac{7x+6}{x^2+2x+1}$, l) $x^3 + 3x^2 - 3x - 9 + \frac{9x+7}{x^2+3}$

Exercise 7.2 a) remainder $r = 15$, b) $r = 20$, c) $r = -2$, d) $r = 12$

Exercise 7.3 a) yes, $g(x)$ is a factor of $f(x)$, the root of $f(x)$ is $x=-3$, b) $g(x)$ is not a factor of $f(x)$, c) $g(x)$ is a factor of $f(x)$, the root of $f(x)$ is $x=-7$, d) $g(x)$ is a factor of $f(x)$, the root of $f(x)$ is $x=-1$

Exercise 7.4 a) $f(x)=(x-2)(x-1)(x+1)$, b) $f(x)=(x-1)(x-2)(x-3)$, c) $f(x)=(x-3)(x-i)(x+i)$, d) $f(x)=(x+2)^3$, e) $f(x)=(x+2)(x+4)(x+7)$, f) $f(x)=(x-4)(x+3)(x+4)$, g) $f(x)=(x-2)(x-1)(x+1)(x+2)(x+5)$

Exercise 7.5 a) $2x^2+7x+9+\frac{25}{x-2}$, b) $4x^2-9x+12-\frac{18}{x+3}$, c) $x^2+2x-7+\frac{15}{x+2}$, d) $x^3+2x^2+2x+2+\frac{3}{x-1}$, e) $x^4-2x^3+4x^2-8x+16$, f) $x^2-3+\frac{5}{x+5}$.

Chapter 8 (exercises starting on page 144):

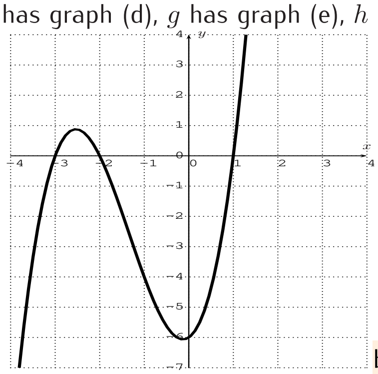
Exercise 8.1 a) yes, b) no (due to the discontinuity), c) no (due to horizontal asymptote), d) no (due to corner), e) yes (polynomial of degree 1), f) yes

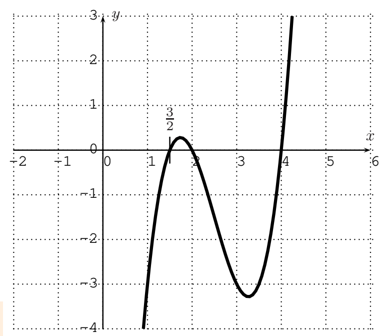
Exercise 8.2 f has graph (e), g has graph (c), h has graph (a), k has graph (f)

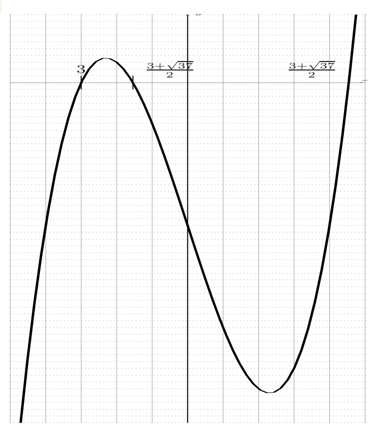
Exercise 8.3 f has graph (c), g has graph (f), h has graph (d), k has graph (b)

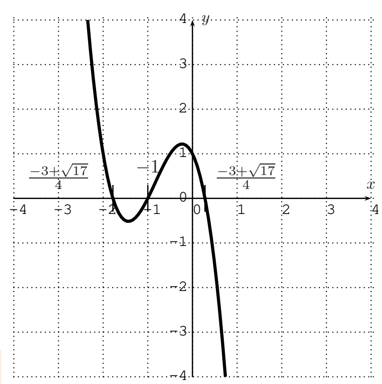
Exercise 8.4 f has graph (d), g has graph (e), h has graph (c), k has graph (b)

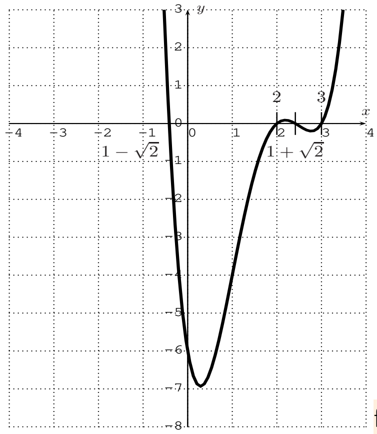
Exercise 8.5

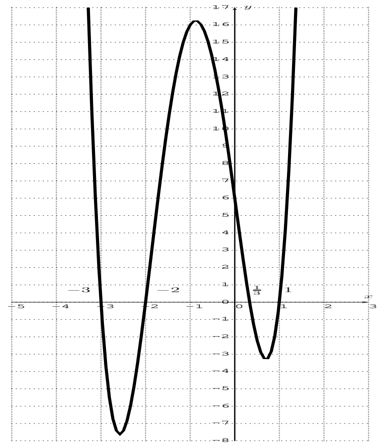
a) 

b) 

c) 

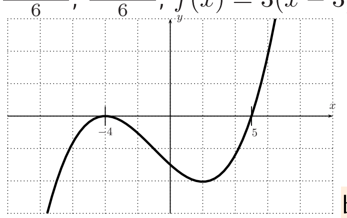
d) 

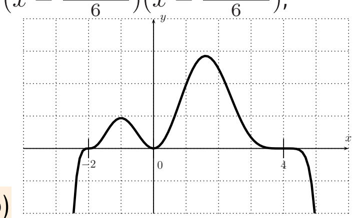
e) 

f) 

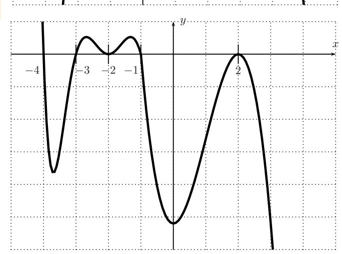
Exercise 8.6 a) $x = 2$, $x = 3$, or $x = 5$, b) $x = -1$, c) $x = 11$, d) $x = -8$, $x = -3$, $x = -2$, or $x = 4$, e) $x = -3 \pm \sqrt{6}$ (use the quadratic formula), f) $x = 0$

Exercise 8.7 a) roots: -1, 2, 4, $f(x) = (x-2)(x+1)(x-4)$, b) roots: 1, -3, -5, $f(x) = (x+3)(x-1)(x+5)$, c) roots: -2, -3, -4, $f(x) = (x+3)(x+2)(x+4)$, d) roots: 1 (of multiplicity 2), -6, $f(x) = (x+6)(x-1)^2$, e) roots: $\frac{2}{3}$, 2, -7, $f(x) = 3(x-\frac{2}{3})(x-2)(x+7)$, f) roots: $-\frac{1}{6}$, -1, 2, $f(x) = 6(x+\frac{1}{6})(x+1)(x-2)$, g) roots: $\frac{5}{2}$, $-\frac{1}{3}$, -2, $f(x) = 6(x-\frac{5}{2})(x+\frac{1}{3})(x+2)$, h) roots: 3, $2+\sqrt{3}$, $2-\sqrt{3}$, $f(x) = (x-3)(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))$, i) roots: 1, $\frac{-3+\sqrt{41}}{2}$, $\frac{-3-\sqrt{41}}{2}$, $f(x) = (x-1)(x-\frac{-3+\sqrt{41}}{2})(x-\frac{-3-\sqrt{41}}{2})$, j) roots: -2, $\frac{-3+\sqrt{17}}{4}$, $\frac{-3-\sqrt{17}}{4}$, $f(x) = 2(x+2)(x-\frac{-3+\sqrt{17}}{4})(x-\frac{-3-\sqrt{17}}{4})$, k) roots: 3, $\frac{1+\sqrt{85}}{6}$, $\frac{1-\sqrt{85}}{6}$, $f(x) = 3(x-3)(x-\frac{1+\sqrt{85}}{6})(x-\frac{1-\sqrt{85}}{6})$,

Exercise 8.8 a) 

b) 

c) 

d) 

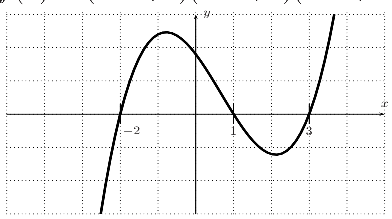
Chapter 9 (exercises starting on page 166):

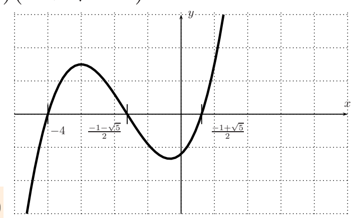
Exercise 9.1 a) $x = -1$, $x = 2$, $x = \frac{1}{2}$, b) $x = \frac{1}{3}$, c) $x = \frac{-3}{2}$, $x = -1$, $x = \frac{4}{3}$, d) $x = \frac{1}{2}$, $x = \frac{-2}{3}$, $x = -2 + \sqrt{3}$, $x = -2 - \sqrt{3}$, e) $x = -\frac{1}{4}$

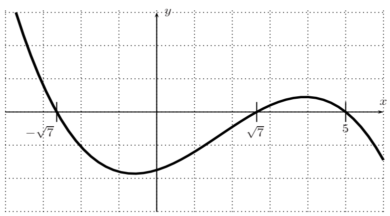
Exercise 9.2 a) $x = 1$, b) $x = 1$ or $x = -1$, c) $x = 3$, d) $x = -10$, e) $x = 3$ or $x = -3$, f) $x = 5$, g) $x = -2$, h) $x = 1$, i) $x = 8i$ or $x = -8i$

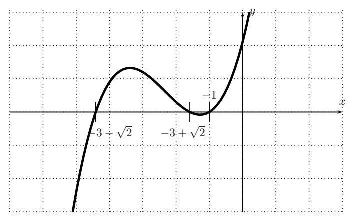
Exercise 9.3 a) $f(x) = (x-2)(x-1)(x+3)$, b) $f(x) = (x-5)(x+2)^2$, c) $f(x) = (x-4)(x-\frac{3+i\sqrt{11}}{2}))(x-\frac{3-i\sqrt{11}}{2})$, d) $f(x) = (x-2)(x-\frac{-3+\sqrt{5}}{2})(x-\frac{-3-\sqrt{5}}{2})$, e) $f(x) = 2(x+\frac{3}{2})(x+1)(x-2)$, f) $f(x) = 12(x-\frac{2}{3})(x+\frac{3}{4})(x+4)$, g) $f(x) = (x+1)(x-(2+3i))(x-(2-3i))$, h) $f(x) = (x-1)(x+1)(x-2)(x+2)$, i) $f(x) = (x-1)(x+1)(x-i)(x+i)$, j) $f(x) = x(x-1)(x+1)(x-3)^2$, k) $f(x) = (x-3)(x-\frac{-3+3\sqrt{3}\cdot i}{2})(x-\frac{-3-3\sqrt{3}\cdot i}{2})$, l) $f(x) = (x-\sqrt{3})(x+\sqrt{3})(x-\sqrt{5}\cdot i)(x+\sqrt{5}\cdot i)$

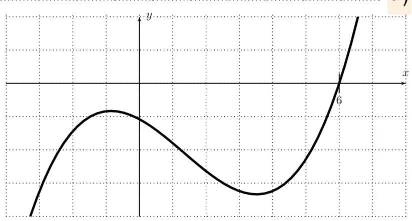
Exercise 9.4

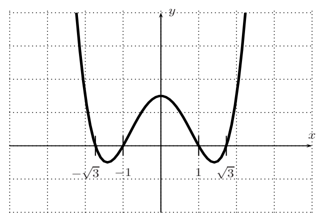
a) 

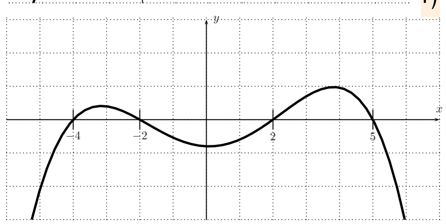
b) 

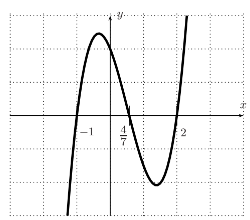
c) 

d) 

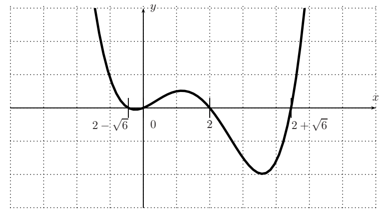
e) 

f) 

g) 

h) 

i) 

j) 

Exercise 9.5 a) $C = -12$, roots: $x = 1$, $x = -3$, $x = -4$, b) $C = 20$, roots: $x = -2$, $x = 3+i$, $x = 3-i$, c) $C = 9$, roots: $x = 3$, $x = 1$, $x = -3$, d) $C = -2$, roots: $x = -1$, $x = \frac{-7+\sqrt{57}}{2}$, $x = \frac{-7-\sqrt{57}}{2}$, e) $C = -18$, roots: $x = 2$, $x = \frac{3+3i\sqrt{3}}{2}$, $x = \frac{3-3i\sqrt{3}}{2}$

Exercise 9.6 a) $f(x) = 2(x-2)(x-3)(x-4)$, b) $f(x) = (-1) \cdot x(x-2)(x+1)(x+3)$, c) $f(x) = (-\frac{5}{2}) \cdot (x-2)(x+2)(x+1)$, d) $f(x) = -2 \cdot x(x-2)(x+1)(x+4)$, e) $f(x) = 3(x-7)(x-(2+5i))(x-(2-5i))$, f) $f(x) = (-2) \cdot (x-i)(x+i)(x-3)$, g) $f(x) = \frac{7}{4} \cdot (x-(5+i))(x-(5-i))(x-3)^2$, h) $f(x) = (x-i)(x+i)(x-(3+2i))(x-(3-2i))$ (other correct answers are possible, depending on the choice of the first coefficient), i) $f(x) = (x-(1+i))(x-(2+i))(x-(4-3i))(x+2)^3$ (other correct answers are possible, depending on the choice of the first coefficient), j) $f(x) = (x-i)(x-3)(x+7)^2$ (other correct answers are possible, depending on the choice of the first coefficient and the fourth root), k) $f(x) = (x-2)(x-3)(x-4)$ (other correct answers are possible, depending on the choice of the first coefficient), l) $f(x) = (x-1)^2(x-3)^2$, m) $f(x) = -x(x-2)(x-3)(x-4)$ (other correct answers are possible, depending on the choice of the first coefficient)

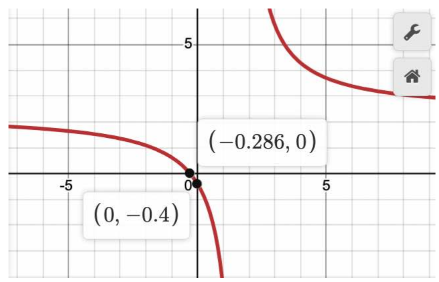
Chapter 10 (exercises starting on page 192):

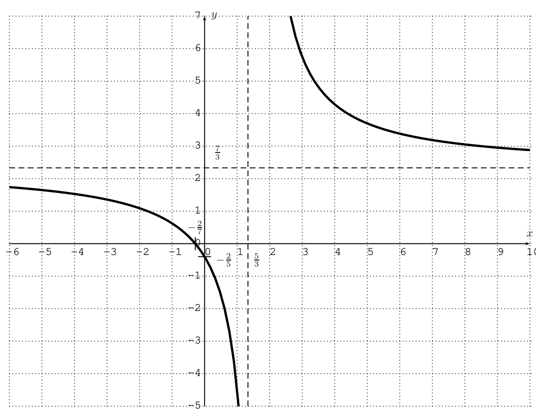
Exercise 10.1 a) domain $D = \mathbb{R} - \{2\}$, vertical asymptote at $x = 2$, no removable discontinuities, b) $D = \mathbb{R} - \{2, 4\}$, vertical asympt. at $x = 2$ and $x = 4$, no removable discont., c) $D = \mathbb{R} - \{-2, 0, 2\}$, vertical asympt. at $x = 0$ and $x = 2$, removable discont. at $x = -2$, d) $D = \mathbb{R} - \{-3, 2, 5\}$, vertical asympt. at $x = 2$ and $x = 5$, removable discont. at $x = -3$, e) $D = \mathbb{R} - \{1\}$, no vertical asympt., removable discont. at $x = 1$, f) $D = \mathbb{R} - \{-1, 1, 2\}$, vertical asympt. at $x = -1$ and $x = 1$ and $x = 2$, no removable discont.

Exercise 10.2 a) $y = 4$, b) $y = 0$, c) no horizontal asymptote (asymptotic behavior $y = x + 4$), d) $y = -4$

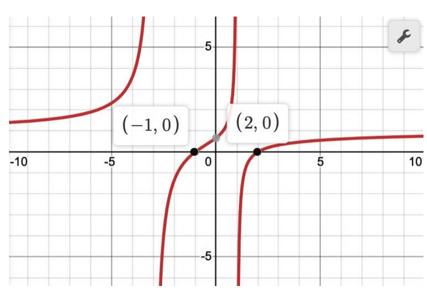
Exercise 10.3 a) x-intercept at $x = 3$, y-intercept at $y = 3$, b) x-intercepts at $x = 0$ and $x = -2$ and $x = 2$, y-intercept at $y = 0$, c) x-intercepts at $x = -4$ and $x = 1$ and $x = 3$, y-intercept at $y = \frac{6}{5}$, d) x-intercept at $x = -3$ (but not at $x = -2$ since $f(-2)$ is undefined), no y-intercept since $f(0)$ is undefined

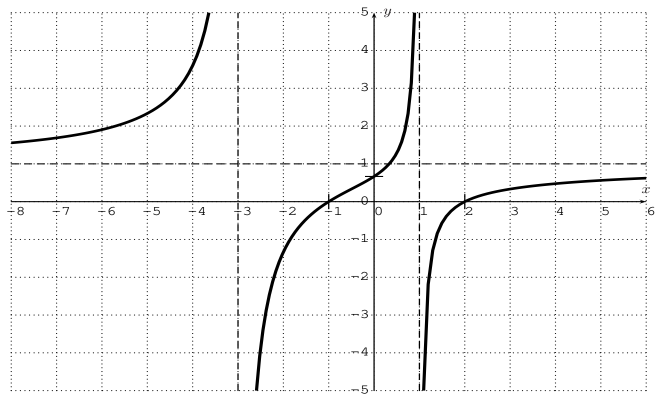
Exercise 10.4 a) $D = \mathbb{R} - \{\frac{5}{3}\}$, horizontal asympt. $y = \frac{7}{3}$, vertical asympt. $x = \frac{5}{3}$, no removable discont., x-intercept at $x = -\frac{2}{7}$, y-intercept at $y = -\frac{2}{5}$, graph:



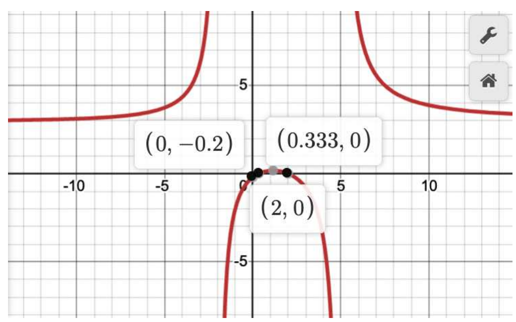


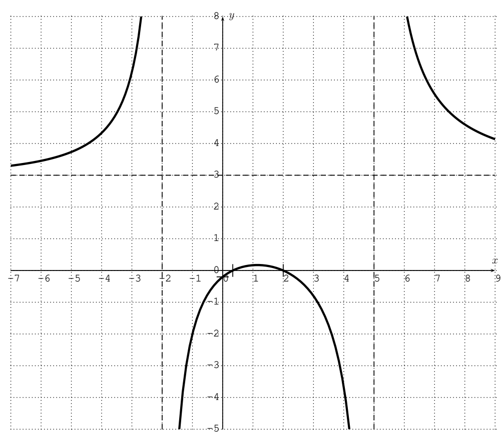
b) $f(x) = \frac{(x-2)(x+1)}{(x+3)(x-1)}$ has domain $D = \mathbb{R} - \{-3, 1\}$, horizontal asympt. $y = 1$, vertical asympt. $x = -3$ and $x = 1$, no removable discont., x-intercept at $x = 2$ and $x = -1$, y-intercept at $y = \frac{-2}{-3} = \frac{2}{3}$, graph:



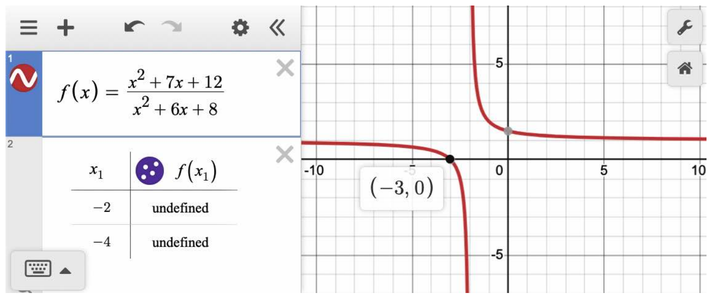


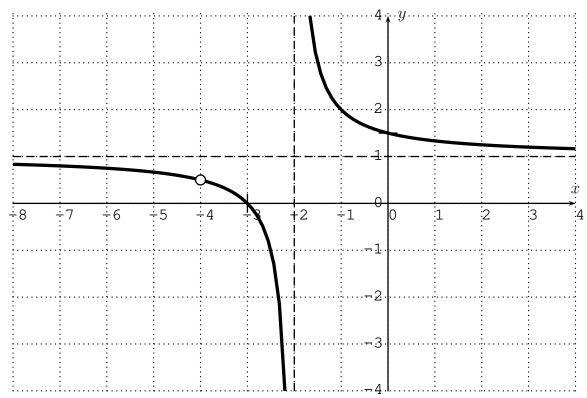
c) $f(x) = \frac{(3x-1)(x-2)}{(x+2)(x-5)} = \frac{3(x-\frac{1}{3})(x-2)}{(x+2)(x-5)}$ has domain $D = \mathbb{R} - \{-2, 5\}$, horizontal asympt. $y = 3$, vertical asympt. $x = -2$ and $x = 5$, no removable discont., x-intercept at $x = 2$ and $x = \frac{1}{3}$, y-intercept at $y = \frac{-1}{5} = -0.2$, graph:



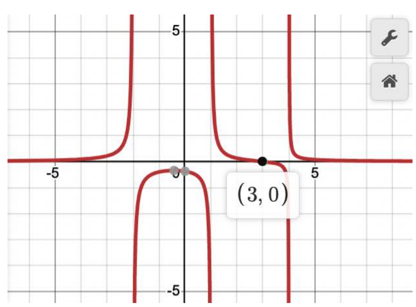


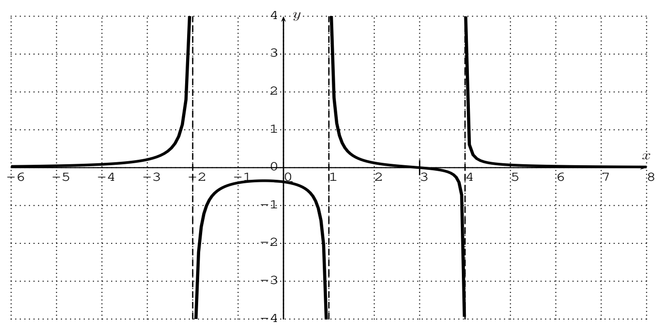
d) $f(x) = \frac{(x+3)(x+4)}{(x+2)(x+4)}$ has domain $D = \mathbb{R} - \{-2, -4\}$, horizontal asympt. $y = 1$, vertical asympt. $x = -2$, removable discont. at $x = -4$, x-intercepts at $x = 3$, y-intercept at $y = \frac{3}{2} = 1.5$, graph:



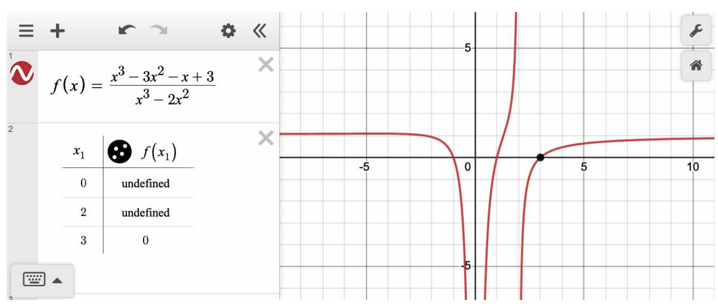


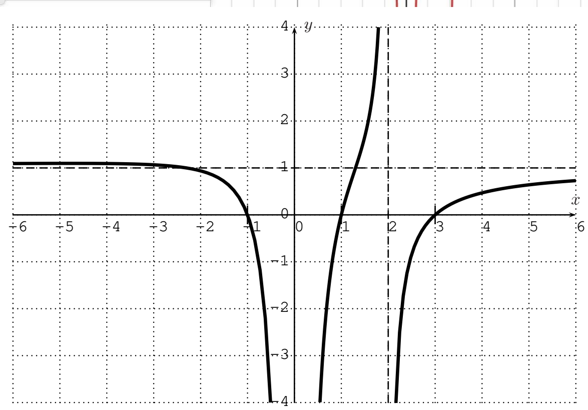
e) $f(x) = \frac{x-3}{(x-4)(x-1)(x+2)}$ has domain $D = \mathbb{R} - \{-2, 1, 4\}$, horizontal asympt. $y = 0$, vertical asympt. $x = -2$ and $x = 1$ and $x = 4$, no removable discont., x-intercept at $x = 3$, y-intercept at $y = \frac{-3}{8} = -0.375$, graph:





f) $f(x) = \frac{(x-3)(x-1)(x+1)}{x^2(x-2)}$ has domain $D = \mathbb{R} - \{0, 2\}$, horizontal asympt. $y = 1$, vertical asympt. $x = 0$ and $x = 2$, no removable discont., x-intercepts at $x = -1$ and $x = 1$ and $x = 3$, no y-intercept since $f(0)$ is undefined, graph:





Note that the graph intersects the horizontal asymptote $y = 1$ at approximately $x \approx -2.3$ and approaches the asymptote from above.

Exercise 10.5 a) for example $f(x) = \frac{1}{x-4}$, b) for example $f(x) = \frac{5x^2}{x^2-5x+6}$, c) for example $f(x) = \frac{x^2-x}{x-1}$

Chapter 11 (exercises starting on page 206):

Exercise 11.1 a) x-intercepts: $(-3,0)$, $(2,0)$, y-intercept: $(0,2)$, vertical asympt.: $x=-2$, $x=3$, horizontal asympt.: $y=2$, $f(x)=2\cdot\frac{(x-2)(x+3)}{(x+2)(x-3)}$, b) x-intercepts: $(-1,0)$, $(1,0)$, y-intercept: $(0,-1)$, vertical asympt.: $x=-2$, $x=2$, horizontal asympt.: $y=-4$, $f(x)=-4\cdot\frac{(x-1)(x+1)}{(x+2)(x-2)}$, c) x-intercepts: $(-2,0)$, $(1,0)$, y-intercept: $(0,-2)$, vertical asympt.: $x=-3$, $x=-1$, horizontal asympt.: $y=3$, $f(x)=3\cdot\frac{(x-1)(x+2)}{(x+1)(x+3)}$

Exercise 11.2 a) x-intercept: $(-2,0)$, y-intercept: $(0,2)$, vertical asympt.: $x=-3$, $x=-1$, $x=2$, horizontal asympt.: $y=0$, $f(x)=-6\cdot\frac{(x+2)}{(x+3)(x+1)(x-2)}$, b) x-intercept: $(4,0)$, y-intercept: $(0,1)$, vertical asympt.: $x=-2$, $x=1$, $x=6$, horizontal asympt.: $y=0$, $f(x)=-3\cdot\frac{(x-4)}{(x+2)(x-1)(x-6)}$, c) x-intercept: $(2,0)$, y-intercept: $(0,-4)$, vertical asympt.: $x=-1$, $x=1$, $x=4$, horizontal asympt.: $y=0$, $f(x)=8\cdot\frac{(x-2)}{(x-1)(x+1)(x-4)}$

Exercise 11.3 a) $D=\mathbb{R}-\{4,-5\}$, hole $(x,y)=(4,\frac{1}{9})$, b) $D=\mathbb{R}-\{-3,5\}$, hole $(x,y)=(5,\frac{21}{8})$, c) $D=\mathbb{R}-\{-3,2,6\}$, hole $(x,y)=(2,-\frac{7}{20})$, d) $D=\mathbb{R}-\{3,-4\}$, hole $(x,y)=(-4,\frac{2}{7})$, e) $D=\mathbb{R}-\{3,-2\}$, hole $(x,y)=(3,\frac{6}{5})$, f) $D=\mathbb{R}-\{0,1,-2\}$, hole $(x,y)=(1,-\frac{2}{3})$

Exercise 11.4 a) $y=x+6$, b) $y=2x-5$, c) $y=4x+2$, d) $y=-3x+2$

Exercise 11.5 a) $f(x)\to 1$, b) $f(x)\to 3$, c) the limit does not exist, since limits from the right and from the left do not coincide, d) $f(x)\to 2$, e) $f(x)\to 2$, f) $f(x)\to 2$, g) $f(x)\to 4$, h) $f(x)\to 4$, i) $f(x)\to 4$, j) $f(x)\to 4$, k) $f(x)\to -2$, l) the limit does not exist

Exercise 11.6 a)

x	$\frac{x-3}{\lvert x-3\rvert}$
2.9	−1
2.99	−1
2.999	−1

$\lim_{x\to 3^-}\frac{x-3}{\lvert x-3\rvert}=-1$, b)

x	$\frac{x^3-1}{\sqrt{x}-1}$
1.1	6.7815573
1.01	6.0753128
1.001	6.0075031
1.0001	6.00075

$\lim_{x\to 1^+}\frac{x^3-1}{\sqrt{x}-1}=6$, c)

x	$\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$
1.9	−0.26315789
1.99	−0.25125628
1.999	−0.25012506
1.9999	−0.2500125

$\lim_{x\to 2^-}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=-0.25$, d)

x	$\frac{x^3+5x^2}{\lvert x+5\rvert}$
−4.9	24.01
−4.99	24.9001
−4.999	24.990001
−4.9999	24.999

$\lim_{x\to -5^+}\frac{x^3+5x^2}{\lvert x+5\rvert}=25$, e)

x	$\frac{x^3+5x^2}{\lvert x+5\rvert}$
−5.1	−26.01
−5.01	−25.1001
−5.001	−25.010001
−5.0001	−25.001

$\lim_{x\to -5^-}\frac{x^3+5x^2}{\lvert x+5\rvert}=-25$, f)

x	$\frac{x-1}{x-4}$
3.9	−29
3.99	−299
3.999	−2999
3.9999	−29999
3.99999	−299999

$\lim_{x\to 4^-}\frac{x-1}{x-4}=-\infty$

Chapter 12 (exercises starting on page 227):

Exercise 12.1 a) $x \leq 3$, b) $\frac{1}{2} > x$, c) $-4 \geq x$, d) $x > \frac{22}{5}$, e) $-5 \leq x \leq 7$, f) $-4 < x \leq 3$, g) $x \geq 2$ (this then also implies $x \geq -\frac{7}{2}$), h) no solution

Exercise 12.2 a) $(-\infty, -2) \cup (7, \infty)$, b) $(-\infty, -5] \cup [7, \infty)$, c) $[-2, 2]$, d) $(\frac{-3-\sqrt{21}}{2}, \frac{-3+\sqrt{21}}{2})$, e) $[-3, 2]$, f) $(-\frac{1}{3}, 1)$, g) $\mathbb{R}-\{2\}$, h) $[-2, 1]\cup[3, \infty)$, i) $(-\infty, -4)$, j) $(0, 2) \cup (2, \infty)$, k) $[-3, -1] \cup [1, 3]$, l) $(-1, 1) \cup (2, 3)$, m) $(-\infty, 0) \cup (0, 2) \cup (3, \infty)$, n) $(-\infty, -2] \cup [0, 1] \cup [2, 5]$, o) $[1, 2] \cup [3, 4] \cup [5, \infty)$, p) $(-\infty, 1]$

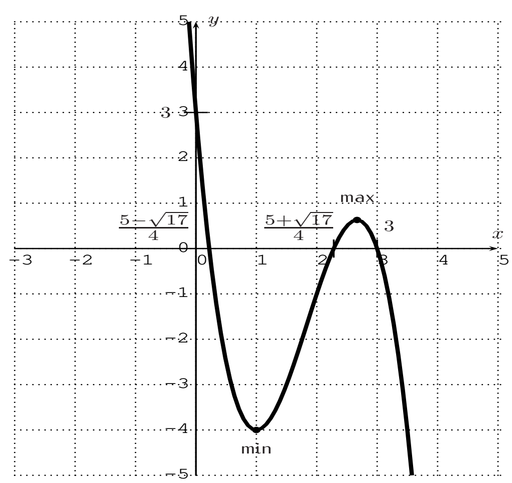
Exercise 12.3 a) $D = (-\infty, 3] \cup [5, \infty)$, b) $D = (-\infty, -3] \cup [0, 3]$, c) $D = [1, 4]$, d) $D = [2, 5] \cup [6, \infty)$, e) $D = (-\infty, 3)$, f) $D = (-\infty, -1) \cup (7, \infty)$

Exercise 12.4 a) $(-\infty, -8) \cup (1, \infty)$, b) $(\frac{-5}{6}, \frac{1}{6})$, c) $(-\infty, \frac{1}{3}] \cup [3, \infty)$, d) $[-12, -2]$, e) $(-\infty, -\frac{1}{4}] \cup [\frac{1}{2}, \infty)$, f) $(-15, -5)$

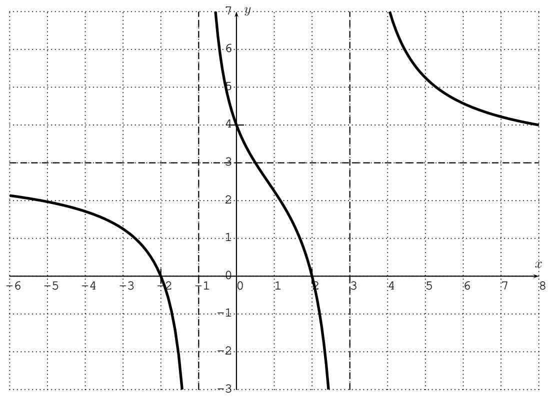
Exercise 12.5 a) $(-\infty, -4) \cup [-2, \infty)$, b) $(2, 5)$, c) $(-\frac{15}{7}, \frac{11}{9}]$, d) $(-\infty, -\frac{4}{13}] \cup (\frac{1}{6}, \infty)$, e) $(-\frac{8}{3}, \frac{2}{7})$, f) $(-2, 1] \cup (2, \infty)$, g) $(-\infty, -1) \cup (2, 5)$, h) $(-\infty, -3] \cup (-2, 2) \cup [3, \infty)$, i) $(-\infty, -5] \cup (-3, \infty)$, j) $(-10, -9.8)$, k) $(-1, 2) \cup [4, \infty)$, l) $(-\infty, -4) \cup [0, \infty)$

Review of Part II (exercises starting on page 229):

Exercises II 1. $x^2 - x - 3 + \frac{1}{2x+3}$, 2. 21, 3. $x - 1$ is a factor, $x + 1$ is not a factor, $x - 0$ is not a factor, 4. $a) \leftrightarrow$ iii), $b) \leftrightarrow$ iv), $c) \leftrightarrow$ i), $d) \leftrightarrow$ ii),
5. x-intercepts: $x = 3$, $x = \frac{5+\sqrt{17}}{4}$, $x = \frac{5-\sqrt{17}}{4}$, y-intercept: $y = 3$, local max $(x, y) \approx (2.667, 0.630)$, local min $(x, y) \approx (1, -4)$, graph:



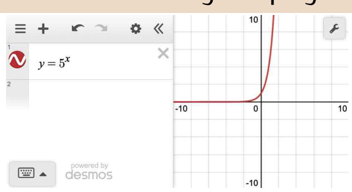
6. $f(x) = (x-1)(x+3)(x+4)$, 7. $f(x) = (-5)\cdot x(x-1)(x-3)$, 8. $f(x) = (x+2)(x-5)(x-(3-2i))(x-(3+2i))$ (other correct answers are possible, depending on the choice of the first coefficient), 9. $f(x) = \frac{3(x-2)(x+2)}{(x-3)(x+1)}$ has domain $D = \mathbb{R} - \{-1, 3\}$, horizontal asympt. $y = 3$, vertical asympt. $x = -1$ and $x = 3$, no removable discont., x-intercepts at $x = -2$ and $x = 2$ and $x = 3$, y-intercept at $y = 4$, graph:

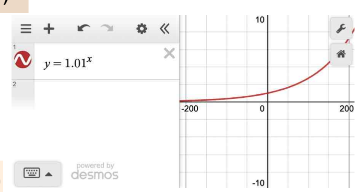


10. a) $(-\infty, -5) \cup (1, \infty)$, b) $\left(-\infty, -\frac{3}{2}\right] \cup \left[-\frac{5}{6}, \infty\right)$, c) $\left(-\frac{8}{3}, \frac{2}{5}\right]$

Chapter 13 (exercises starting on page 250):

Exercise 13.1

a) 

, b) 

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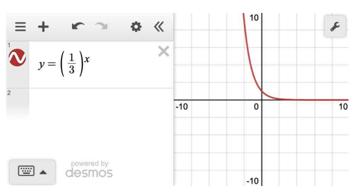
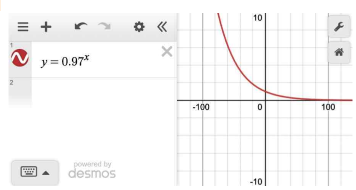
c) 

, d) 

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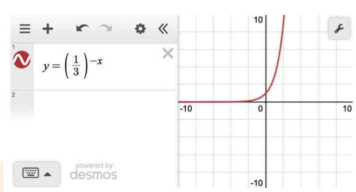
e) same as c), since $y = (\frac{1}{3})^x = 3^{-x}$, f) 

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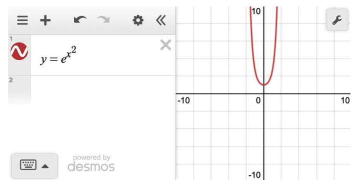
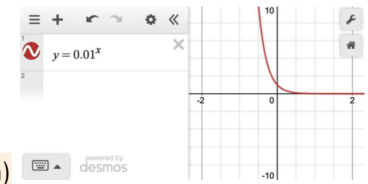
g) 

, h) 

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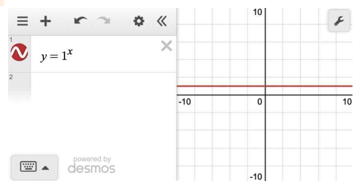
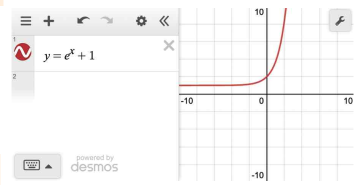
i) 

, j) 

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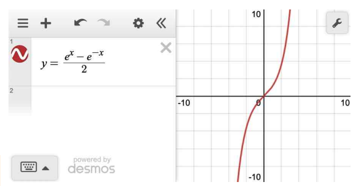
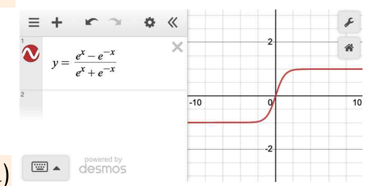
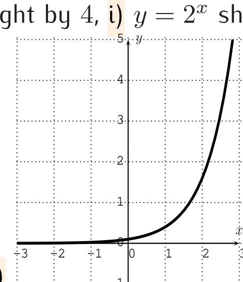
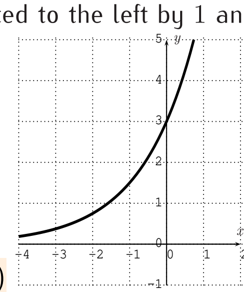
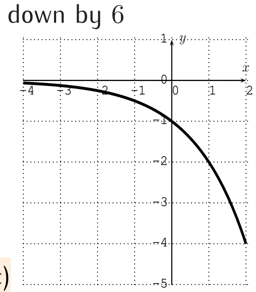
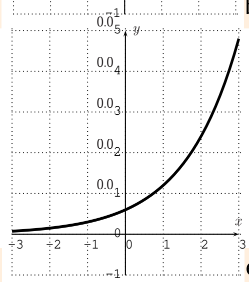
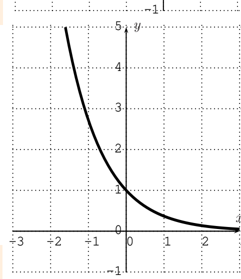
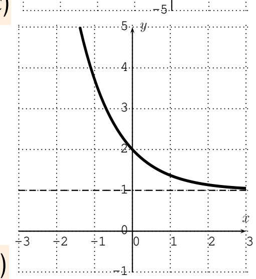
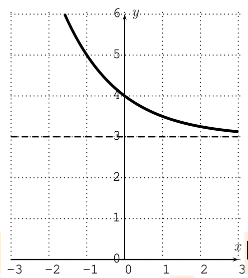
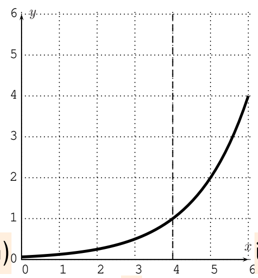
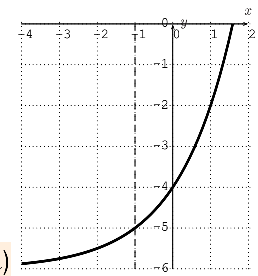
k) 

, l) 

Exercise 13.2 a) $y = 4^x$ is compressed toward the x-axis by the factor 0.1 (graph below), b) $y = 2^x$ stretched away from x-axis, c) $y = 2^x$ reflected about the x-axis, d) $y = 2^x$ compressed toward the x-axis, e) $y = e^x$ reflected about the y-axis, f) $y = e^x$ reflected about the y-axis and shifted up by 1, g) $y = (\frac{1}{2})^x$ shifted up by 3, h) $y = 2^x$ shifted to the right by 4, i) $y = 2^x$ shifted to the left by 1 and down by 6

a) 
b) 
c) 

d) 
e) 
f) 

g)  h)  i) 

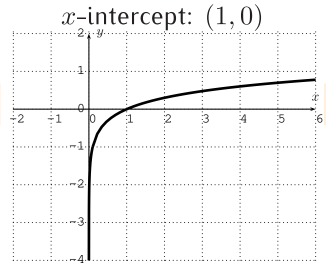
Exercise 13.3 a) $\log_4(16) = 2$, b) $\log_2(256) = 8$, c) $\ln(7) = x$, d) $\log(0.1) = -1$, e) $\log_3(12) = x$, f) $\log_5(12) = 7x$, g) $\log_3(44) = 2a+1$, h) $\log_{\frac{1}{2}}(30) = \frac{x}{h}$

Exercise 13.4 a) 2, b) 4, c) 6, d) 2, e) -2, f) 3, g) 4, h) 1, i) -1, j) -2, k) 0, l) -3

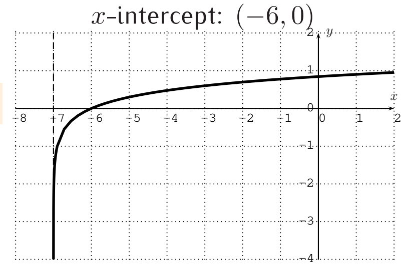
Exercise 13.5 a) 3.561, b) 2.262, c) -0.290, d) -4.911

Exercise 13.6

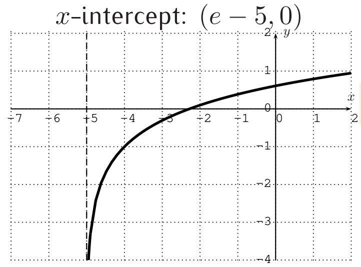
a) $D = (0, \infty)$
vertical aympt.: $x = 0$
x-intercept: $(1, 0)$



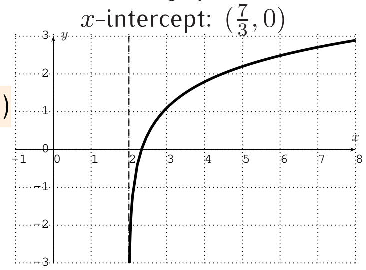
b) $D = (-7, \infty)$
vertical aympt.: $x = -7$
x-intercept: $(-6, 0)$



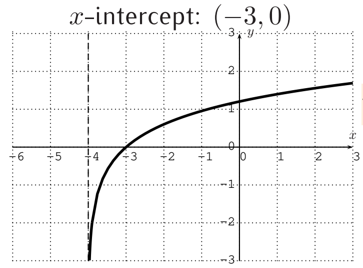
c) $D = (-5, \infty)$
vertical aympt.: $x = -5$
x-intercept: $(e-5, 0)$



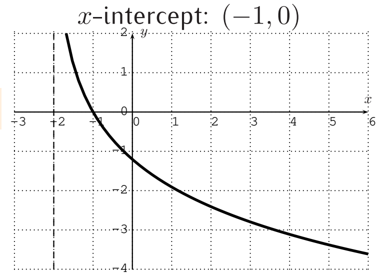
d) $D = (2, \infty)$
vertical aympt.: $x = 2$
x-intercept: $(\frac{7}{3}, 0)$



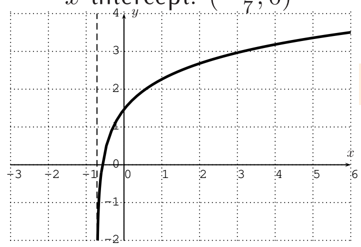
e) $D = (-4, \infty)$
vertical aympt.: $x = -4$
x-intercept: $(-3, 0)$



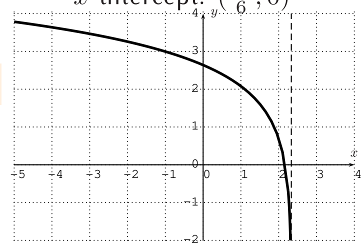
f) $D = (-2, \infty)$
vertical aympt.: $x = -2$
x-intercept: $(-1, 0)$



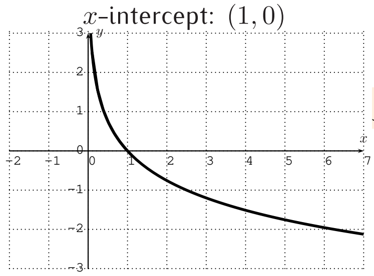
g) $D = (-\frac{5}{7}, \infty)$
vertical aympt.: $x = -\frac{5}{7}$
x–intercept: $(-\frac{4}{7}, 0)$



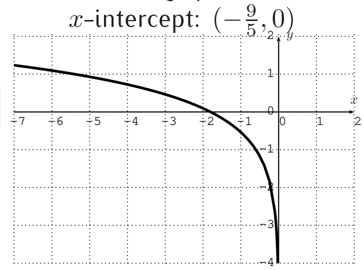
h) $D = (-\infty, \frac{7}{3})$
vertical aympt.: $x = \frac{7}{3}$
x–intercept: $(\frac{13}{6}, 0)$



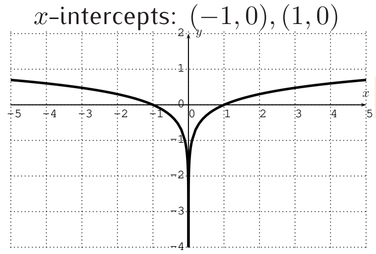
i) $D = (0, \infty)$
vertical aympt.: $x = 0$
x–intercept: $(1, 0)$



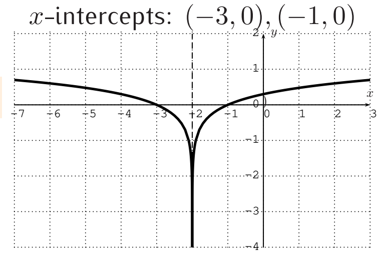
j) $D = (-\infty, 0)$
vertical aympt.: $x = 0$
x–intercept: $(-\frac{9}{5}, 0)$



k) $\mathbb{R} - \{0\}$
vertical aympt.: $x = 0$
x–intercepts: $(-1, 0), (1, 0)$



l) $\mathbb{R} - \{-2\}$
vertical aympt.: $x = -2$
x–intercepts: $(-3, 0), (-1, 0)$



Chapter 14 (exercises starting on page 260):

Exercise 14.1 a) $\ln(x^3 \cdot y)$, b) $\log\left(\frac{x}{y^{\frac{2}{3}}}\right) = \log\left(\frac{x}{\sqrt[3]{y^2}}\right)$, c) $\log\left(\frac{\sqrt[3]{x} z^4}{y}\right)$, d) $\log\left(\frac{z}{x^3 y}\right)$, e) $\ln\left(\frac{\sqrt[4]{x}\,\sqrt[3]{z^2}}{\sqrt{y}}\right)$, f) $\ln\left(\frac{1}{x+1}\right)$, g) $\ln(x^{10})$, h) $\log_5\left(25 \cdot (a+1)\right)$

Exercise 14.2 a) $3u + v$, b) $\frac{2}{3}u + \frac{7}{4}v$, c) $\frac{1}{2}u + \frac{1}{6}v$, d) $3u - 4v$, e) $2u - \frac{1}{2}v - 2w$, f) $\frac{1}{2}u + \frac{3}{2}v - \frac{1}{4}w$, g) $\frac{3}{4}u - 3v + \frac{1}{4}w$, h) $2 - 2v + \frac{1}{5}w$, i) $\frac{1}{6}v + \frac{4}{3}w - \frac{2}{3}$

Exercise 14.3 a) $x = 3$, b) $x = 6$, c) $x = \frac{5}{31}$, d) $x = 76$, e) $x = 1$, f) $x = 8$, g) $x = 18$, h) $x = 25$, i) $x = 2$, j) $x = -1$

Chapter 15 (exercises starting on page 273):

Exercise 15.1 a) $x = 4$, b) $x = 4$, c) $x = 9$, d) $x = -2$, e) $x = -\frac{6}{5}$, f) $x = 2$, g) $x = -2$, h) $x = \frac{11}{2}$, i) $x = \frac{10}{17}$, j) $x = \frac{7}{6}$

Exercise 15.2 a) $x = \frac{\log 57}{\log 4} \approx 2.92$, b) $x = \frac{\log 7}{\log 9} + 2 \approx 2.89$, c) $x = \frac{\log 31}{\log 2} - 1 \approx 3.94$, d) $x = \frac{\log(63)-7\log(3.8)}{2\log(3.8)} \approx -1.95$, e) $x = \frac{5\cdot\log(5)}{\log(8)-\log(5)} \approx 17.12$, f) $x = \frac{2\cdot\log(3)}{\log(0.4)-\log(3)} \approx -1.09$, g) $x = \frac{9\log(1.02)}{2\log(1.02)-\log(4.35)} \approx -0.12$, h) $x = \frac{\log(4)-2\log(5)}{\log(5)-\log(4)} \approx -8.21$, i) $x = \frac{3\log(9)+6\log(4)}{\log(9)+\log(4)} \approx 4.16$, j) $x = \frac{7\log(2.4)-4\log(3.8)}{2\log(2.4)+3\log(3.8)} \approx 0.14$, k) $x = \frac{4\log(9)-2\log(4)}{2\log(9)-9\log(4)} \approx -0.74$, l) $x = \frac{4\log(1.2)+4\log(1.95)}{7\log(1.2)-3\log(1.95)} \approx -4.68$

Exercise 15.3 a) $f(x) = 4\cdot 3^x$, b) $f(x) = 5\cdot 2^x$, c) $f(x) = 3200\cdot 0.1^x$, d) $f(x) = 1.5\cdot 2^x$, e) $f(x) = 20 \cdot 5^x$, f) $f(x) = \frac{3}{5} \cdot \sqrt{5}^x$

Exercise 15.4 a) $y = 8.4\cdot 1.101^t$ with $t = 0$ corresponding to the year 2017, b) approx. 20.0 million, c) It will reach 25 million in the year 2028.

Exercise 15.5 a) $y = 79,000 \cdot 1.0369^t$ with $t = 0$ corresponding to the year 1998, b) approx. 252,000, c) approx. 302,000, d) The city will reach maximum capacity in the year 2068.

Exercise 15.6 The city will be at 90% of its current size after approximately 4.6 years.

Exercise 15.7 It will take the company 4.76 years.

Exercise 15.8 The ant colony has doubled its population after approximately 23.1 weeks.

Exercise 15.9 It will take 4.62 months for the beehive to have decreased to half its current size.

Exercise 15.10 It will take 138.6 years until the world population has doubled.

Chapter 16 (exercises starting on page 287):

Exercise 16.1 a) \$7346.64, b) It takes approximately 18 years.

Exercise 16.2 a) \$862.90, b) \$1564.75, c) \$1566.70, d) \$541.46, e) \$6242.86, f) \$1654.22, g) \$910.24

Exercise 16.3 a) $P = \$1484.39$, b) $P = \$2938.67$, c) $P = \$709.64$, d) $r = 4.23\%$, e) $r = 4.31\%$, f) $t \approx 1.69$ years, g) $t \approx 3.81$ years, h) $t \approx 10.27$ years, i) $t \approx 13.73$ years

Exercise 16.4 It takes 49.262 minutes until 2 mg are left of the element.

Exercise 16.5 2.29 grams are left after 1 year.

Exercise 16.6 The half-life of fermium-252 is 25.38 minutes.

Exercise 16.7 You have to wait approximately 101.3 days.

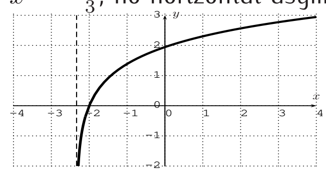
Exercise 16.8 67.8% of the carbon-14 is left in the year 2000.

Exercise 16.9 The wood is approximately 3323 years old.

Exercise 16.10 The bone is approximately 3952 years old.

Review of Part III (exercises starting on page 290):

Exercises III 1. $D = (-\frac{7}{3}, \infty)$, vertical asymptote: $x = -\frac{7}{3}$, no horizontal asymptote, x-intercept: $x = -\frac{6}{3} = -2$, graph:



2. a) $\ln\left(\sqrt[3]{x^2}y^4\right)$, b) $\log_2\left(\frac{\sqrt{xz^3}}{\sqrt[4]{y^3}}\right)$, 3. a) $\frac{4}{3}u - 2v$, b) $u + \frac{5}{2}v$, c) $\frac{1}{5}u + \frac{4}{5}v$, 4. $x = 9$, 5. a) $x = \frac{2\log 6}{\log 7 - \log 6}$, b) $x \approx 23.25$, 6. 42.8 million, 7. 49.5 months, 8. $\$1828.48$, 9. 1.82 hours, 10. 3561 years

Chapter 17 (exercises starting on page 310):

Exercise 17.1 a) $45°$, b) $120°$, c) $150°$, d) $315°$, e) $270°$, f) $225°$, g) $390°$, h) $-300°$

Exercise 17.2 a) $\frac{2\pi}{3}$, b) $\frac{\pi}{6}$, c) $\frac{5\pi}{3}$, d) $\frac{3\pi}{4}$, e) $\frac{\pi}{2}$, f) $\frac{5\pi}{4}$, g) $\frac{8\pi}{3}$, h) $-\frac{5\pi}{6}$,

Exercise 17.3 a) $\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$, $\tan(150°) = -\frac{\sqrt{3}}{3}$, b) $\sin(45°) = \frac{\sqrt{2}}{2}$, $\cos(45°) = \frac{\sqrt{2}}{2}$, $\tan(45°) = 1$, c) $\sin(210°) = -\frac{1}{2}$, $\cos(210°) = -\frac{\sqrt{3}}{2}$, $\tan(210°) = \frac{\sqrt{3}}{3}$, d) $\sin(60°) = \frac{\sqrt{3}}{2}$, $\cos(60°) = \frac{1}{2}$, $\tan(60°) = \sqrt{3}$, e) $\sin(30°) = \frac{1}{2}$, $\cos(30°) = \frac{\sqrt{3}}{2}$, $\tan(30°) = \frac{\sqrt{3}}{3}$, f) $\sin(300°) = -\frac{\sqrt{3}}{2}$, $\cos(300°) = \frac{1}{2}$, $\tan(300°) = -\sqrt{3}$, g) $\sin(90°) = 1$, $\cos(90°) = 0$, $\tan(90°)$ is undefined, h) $\sin(315°) = -\frac{\sqrt{2}}{2}$, $\cos(315°) = \frac{\sqrt{2}}{2}$, $\tan(315°) = -1$, i) $\sin(225°) = -\frac{\sqrt{2}}{2}$, $\cos(225°) = -\frac{\sqrt{2}}{2}$, $\tan(225°) = 1$, j) $\sin(180°) = 0$, $\cos(180°) = -1$, $\tan(180°) = 0$, k) $\sin(120°) = \frac{\sqrt{3}}{2}$, $\cos(120°) = -\frac{1}{2}$, $\tan(120°) = -\sqrt{3}$, l) $\sin(270°) = -1$, $\cos(270°) = 0$, $\tan(270°)$ is undefined, m) $\sin(405°) = \frac{\sqrt{2}}{2}$, $\cos(405°) = \frac{\sqrt{2}}{2}$, $\tan(405°) = 1$, n) $\sin(-135°) = -\frac{\sqrt{2}}{2}$, $\cos(-135°) = -\frac{\sqrt{2}}{2}$, $\tan(-135°) = 1$, o) $\sin(-240°) = \frac{\sqrt{3}}{2}$, $\cos(-240°) = -\frac{1}{2}$, $\tan(-240°) = -\sqrt{3}$, p) $\sin(690°) = -\frac{1}{2}$, $\cos(690°) = \frac{\sqrt{3}}{2}$, $\tan(690°) = -\frac{\sqrt{3}}{3}$, q) $\sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}$, $\cos(\frac{5\pi}{3}) = \frac{1}{2}$, $\tan(\frac{5\pi}{3}) = -\sqrt{3}$, r) $\sin(\frac{\pi}{6}) = \frac{1}{2}$, $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, $\tan(\frac{\pi}{6}) = \frac{\sqrt{3}}{3}$, s) $\sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$, $\cos(\frac{4\pi}{3}) = -\frac{1}{2}$, $\tan(\frac{4\pi}{3}) = \sqrt{3}$, t) $\sin(\frac{5\pi}{6}) = \frac{1}{2}$, $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, $\tan(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{3}$, u) $\sin(\frac{7\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{7\pi}{3}) = \frac{1}{2}$, $\tan(\frac{7\pi}{3}) = \sqrt{3}$, v) $\sin(\frac{7\pi}{4}) = -\frac{\sqrt{2}}{2}$, $\cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$, $\tan(\frac{7\pi}{4}) = -1$, w) $\sin(-\frac{\pi}{2}) = -1$, $\cos(-\frac{\pi}{2}) = 0$, $\tan(-\frac{\pi}{2})$ is undefined, x) $\sin(\frac{13\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{13\pi}{3}) = \frac{1}{2}$, $\tan(\frac{13\pi}{3}) = \sqrt{3}$,

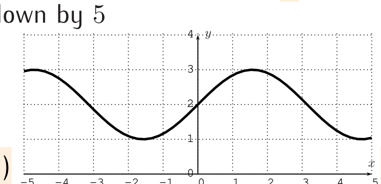
Exercise 17.4 a) $\frac{\sqrt{2}+\sqrt{6}}{4}$, b) $\frac{\sqrt{2}+\sqrt{6}}{4}$, c) $\frac{\sqrt{3}+1}{1-\sqrt{3}} = -2-\sqrt{3}$, d) $\frac{\sqrt{2}-\sqrt{6}}{4}$, e) $\frac{\sqrt{2}+\sqrt{6}}{4}$, f) $\frac{\sqrt{6}-\sqrt{2}}{4}$, g) $\frac{\sqrt{6}-\sqrt{2}}{4}$, h) $\sqrt{3}-2$, i) $\frac{-\sqrt{2}-\sqrt{6}}{4}$, j) $\frac{\sqrt{6}-\sqrt{2}}{4}$, k) $2-\sqrt{3}$, l) $\frac{\sqrt{2}-\sqrt{6}}{4}$,

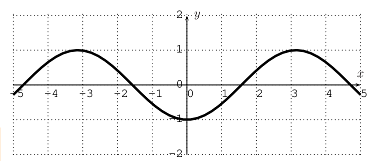
Exercise 17.5 a) $\frac{\sqrt{2+\sqrt{2}}}{2}$, b) $\frac{\sqrt{2-\sqrt{3}}}{2}$, c) $\frac{\sqrt{2+\sqrt{3}}}{2}$, d) $2-\sqrt{3}$, e) $\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}$, f) $-2-\sqrt{3}$, g) $\frac{\sqrt{2+\sqrt{2}}}{2}$, h) $-\frac{\sqrt{2+\sqrt{3}}}{2}$

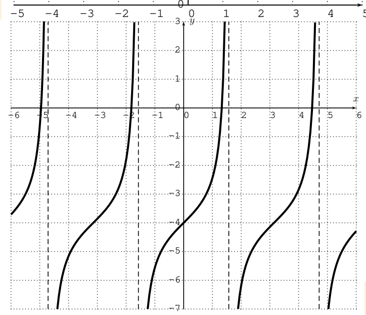
Exercise 17.6 a) $\sin(x+\frac{\pi}{2}) = \cos(x)$, b) $\cos\left(x-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin(x)+\cos(x))$, c) $\tan(\pi - x) = -\tan(x)$, d) $\sin\left(\frac{\pi}{6}-x\right) = \frac{1}{2}\cos(x) - \frac{\sqrt{3}}{2}\sin(x)$, e) $\cos\left(\frac{2\pi}{3}-x\right) = \frac{-1}{2}\cdot\cos(x) + \frac{\sqrt{3}}{2}\cdot\sin(x)$, f) $\cos\left(x+\frac{11\pi}{12}\right) = \left(\frac{-(\sqrt{2}+\sqrt{6})}{4}\right)\cdot\cos(x) - \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\cdot\sin(x)$

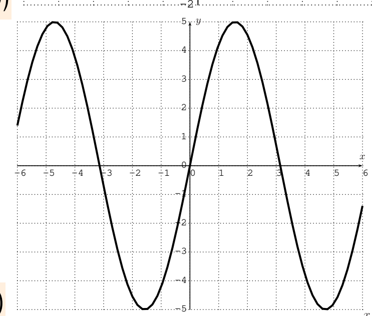
Chapter 18 (exercises starting on page 329):

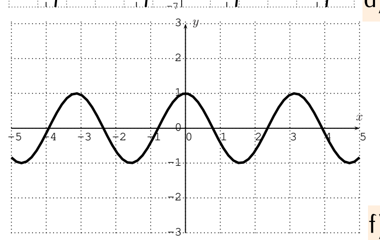
Exercise 18.1 a) shift $y=\sin(x)$ up by 2 (see graph below), b) $y=\cos(x)$ shifted to the right by π, c) $y=\tan(x)$ shifted down by 4, d) $y=\sin(x)$ stretched away from the x-axis by a factor 5, e) $y=\cos(x)$ compressed toward the y-axis by a factor 2, f) $y=\sin(x)$ shifted to the right by 2 and down by 5

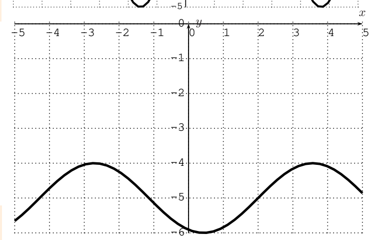
a) 

b) 

c) 

d) 

e) 

f) 

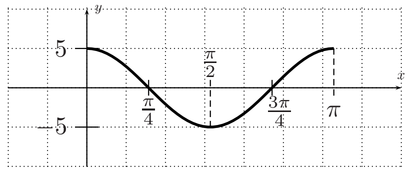
Exercise 18.2 a) $g(x)$, b) $h(x)$, c) $j(x)$, d) $k(x)$, e) $i(x)$, f) $f(x)$

Exercise 18.3 a) $y = 5\cos(x)$, b) $y = -5\cos(x)$, c) $y = -5\sin(x)$, d) $y = \cos(x) + 5$, e) $y = \sin(x) + 5$, f) $y = 2\sin(x) + 3$

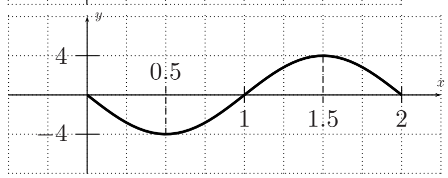
Exercise 18.4 a) amplitude $A = 5$, period $P = \pi$, phase shift $S = \frac{-\pi}{2}$, b) amplitude $A = 3$, period $v\frac{\pi}{2}$, phase shift $S = \frac{\pi}{8}$, c) amplitude $A = 4$, period $P = \frac{\pi}{3}$, phase shift $S = 0$, d) amplitude $A = 2$, period $P = \frac{2\pi}{7}$, phase shift $S = \frac{-\pi}{28}$, e) amplitude $A = 8$, period $P = \pi$, phase shift $S = \frac{3\pi}{2}$, f) amplitude $A = 3$, period $P = 8\pi$, phase shift $S = 0$, g) amplitude $A = 4$, period $P = \frac{2\pi}{5}$, phase shift $S = \frac{-\pi}{15}$, h) amplitude $A = 7$, period $P = 4\pi$, phase shift $S = \frac{12\pi}{5}$, i) amplitude $A = 1$, period $P = \pi$, phase shift $S = 0$, j) amplitude $A = 6$, period $P = 2$, phase shift $S = 1$

Exercise 18.5

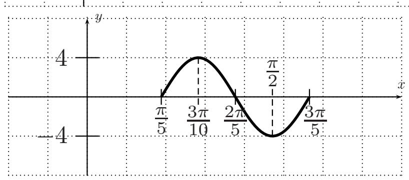
a) amplitude $A = 5$, period $P = \pi$, phase shift $S = 0$



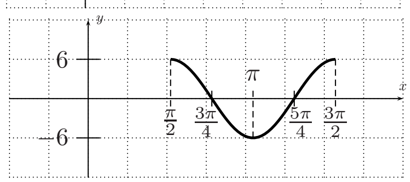
b) amplitude $A = 4$, period $P = 2$, phase shift $S = 0$



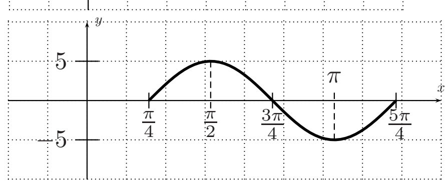
c) amplitude $A = 4$, period $P = \frac{2\pi}{5}$, phase shift $S = \frac{\pi}{5}$



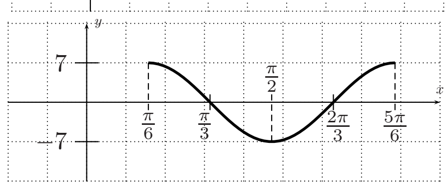
d) amplitude $A = 6$, period $P = \pi$, phase shift $S = \frac{\pi}{2}$



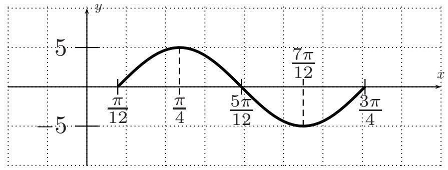
e) amplitude $A = 5$, period $P = \pi$, phase shift $S = \frac{\pi}{4}$



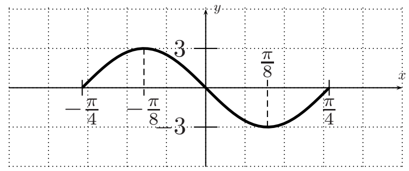
f) amplitude $A = 7$, period $P = \frac{2\pi}{3}$, phase shift $S = \frac{\pi}{6}$



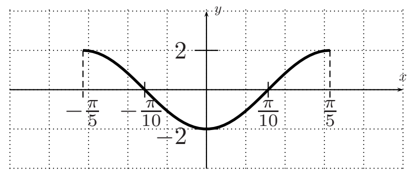
g) $\begin{cases} \text{amplitude } A = 5 \\ \text{period } P = \frac{2\pi}{3} \\ \text{phase shift } S = \frac{\pi}{12} \end{cases}$



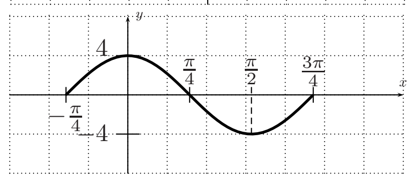
h) $\begin{cases} \text{amplitude } A = 3 \\ \text{period } P = \frac{\pi}{2} \\ \text{phase shift } S = -\frac{\pi}{4} \end{cases}$



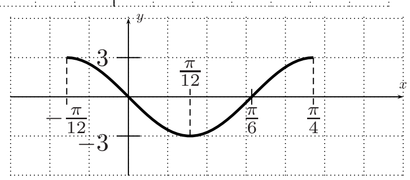
i) $\begin{cases} \text{amplitude } A = 2 \\ \text{period } P = \frac{2\pi}{5} \\ \text{phase shift } S = -\frac{\pi}{5} \end{cases}$



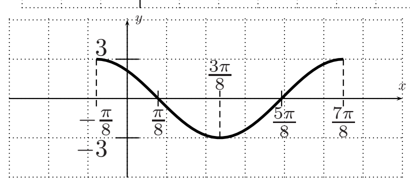
j) $\begin{cases} \text{amplitude } A = 4 \\ \text{period } P = \pi \\ \text{phase shift } S = -\frac{\pi}{4} \end{cases}$



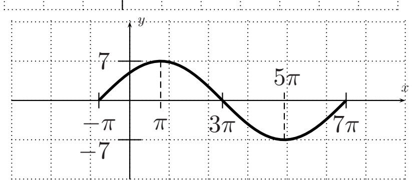
k) $\begin{cases} \text{amplitude } A = 3 \\ \text{period } P = \frac{\pi}{3} \\ \text{phase shift } S = -\frac{\pi}{12} \end{cases}$



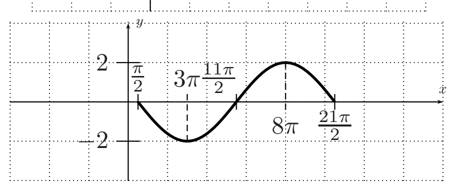
l) $\begin{cases} \text{amplitude } A = 3 \\ \text{period } P = \pi \\ \text{phase shift } S = -\frac{\pi}{8} \end{cases}$

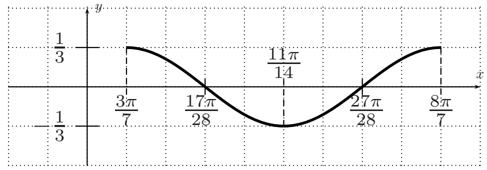


m) $\begin{cases} \text{amplitude } A = 7 \\ \text{period } P = 8\pi \\ \text{phase shift } S = -\pi \end{cases}$

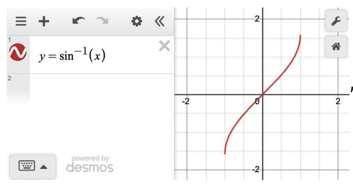


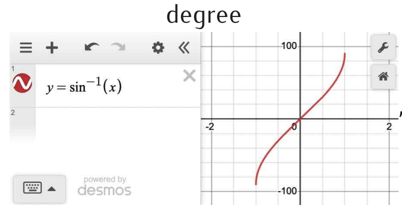
n) $\begin{cases} \text{amplitude } A = 2 \\ \text{period } P = 10\pi \\ \text{phase shift } S = \frac{\pi}{2} \end{cases}$



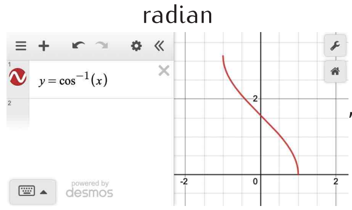
o) $\begin{cases} \text{amplitude } A = \frac{1}{3} \\ \text{period } P = \frac{5\pi}{7} \\ \text{phase shift } S = \frac{3\pi}{7} \end{cases}$ 

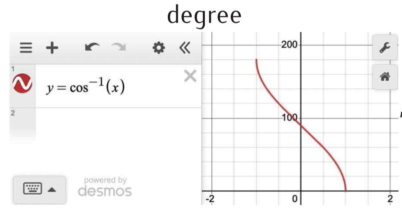
Chapter 19 (exercises starting on page 341):

Exercise 19.1 a) radian 

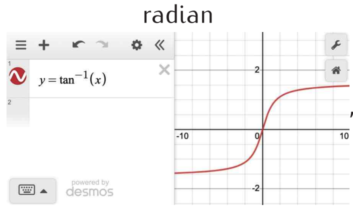
, degree 

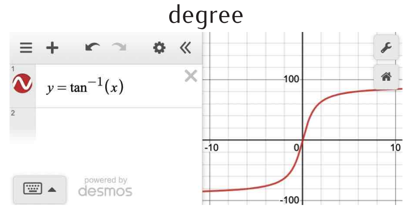
,

b) radian 

, degree 

,

c) radian 

, degree 

Exercise 19.2 a) $\frac{\pi}{3}$, b) $\frac{\pi}{6}$, c) $\frac{\pi}{3}$, d) 0, e) $\frac{\pi}{4}$, f) $\frac{3\pi}{4}$, g) $-\frac{\pi}{2}$, h) $-\frac{\pi}{3}$, i) $\frac{5\pi}{6}$, j) $-\frac{\pi}{4}$, k) $-\frac{\pi}{3}$, l) $-\frac{\pi}{6}$

Exercise 19.3 a) 1.37, b) -0.85, c) 1.23, d) 1.57, e) -1.11, f) undefined, g) $47.16°$, h) $-45°$, i) $75°$, j) $90.00°$, k) $67.5°$, l) $-7.5°$

Chapter 20 (exercises starting on page 358):

Exercise 20.1 a) $x = \frac{\pi}{6} + n\pi$, where $n = 0, \pm1, \dots$, b) $x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{2\pi}{3} + 2n\pi$, where $n = 0, \pm1, \dots$, c) $x = -\frac{\pi}{4} + 2n\pi$ or $x = \frac{5\pi}{4} + 2n\pi$, where $n = 0, \pm1, \dots$, d) $x = \pm\frac{\pi}{6} + 2n\pi$, where $n = 0, \pm1, \dots$, e) $x = \pm\frac{\pi}{2} + 2n\pi$, where $n = 0, \pm1, \dots$, f) $x = \pm\frac{2\pi}{3} + 2n\pi$, where $n = 0, \pm1, \dots$, g) $x = 2n\pi$, where $n = 0, \pm1, \dots$, h) there is no solution (since $-1 \leq \sin(x) \leq 1$), i) $x = 2n\pi$ or $x = \pi + 2n\pi$, where $n = 0, \pm1, \dots$, j) $x = -\frac{\pi}{2} + 2n\pi$, where $n = 0, \pm1, \dots$, (the solutions $x = \frac{3\pi}{2} + 2n\pi$ coincide with these and do not need to be stated separately), k) $x = -\frac{\pi}{3} + n\pi$, where $n = 0, \pm1, \dots$, l) $x = \pm\cos^{-1}(0.2) + 2n\pi$, where $n = 0, \pm1, \dots$

Exercise 20.2 a) $x \approx 1.411 + n\pi$, where $n = 0, \pm1, \ldots$, b) $x \approx \pm 1.104 + 2n\pi$, where $n = 0, \pm1, \ldots$, c) $x \approx 1.143 + 2n\pi$ or $x \approx 1.998 + 2n\pi$, where $n = 0, \pm1, \ldots$, d) $x \approx \pm 2.453 + 2n\pi$, where $n = 0, \pm1, \ldots$, e) $x \approx -0.197 + n\pi$, where $n = 0, \pm1, \ldots$ f) $x \approx -0.06 + 2n\pi$ or $x \approx 3.082 + 2n\pi$, where $n = 0, \pm1, \ldots$,

Exercise 20.3 a) $\frac{-\pi}{4}$, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, $\frac{-5\pi}{4}$, $\frac{-9\pi}{4}$, b) $\frac{\pi}{4}$, $\frac{-\pi}{4}$, $\frac{9\pi}{4}$, $\frac{-9\pi}{4}$, $\frac{17\pi}{4}$, $\frac{-17\pi}{4}$, c) $\frac{-\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{-2\pi}{3}$, $\frac{-7\pi}{3}$, d) 0, π, 2π, $-\pi$, -2π, e) $\frac{\pi}{2}$, $\frac{-\pi}{2}$, $\frac{3\pi}{2}$, $\frac{-3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{-5\pi}{2}$, f) $\cos^{-1}(0.3)$, $-\cos^{-1}(0.3)$, $\cos^{-1}(0.3) + 2\pi$, $-\cos^{-1}(0.3) + 2\pi$, $\cos^{-1}(0.3) - 2\pi$, $-\cos^{-1}(0.3) - 2\pi$, g) $\sin^{-1}(0.4)$, $-\sin^{-1}(0.4) + \pi$, $-\sin^{-1}(0.4) - \pi$, $\sin^{-1}(0.4) + 2\pi$, $\sin^{-1}(0.4) - 2\pi$, h) $\frac{3\pi}{2}$, $\frac{7\pi}{2}$, $\frac{11\pi}{2}$, $\frac{-\pi}{2}$, $\frac{-5\pi}{2}$

Exercise 20.4 a) $x = \frac{\pi}{4} + n\pi$, where $n = 0, \pm1, \ldots$, b) $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, where $n = 0, \pm1, \ldots$, c) $x = \pm\frac{5\pi}{6} + 2n\pi$, where $n = 0, \pm1, \ldots$, d) $x = \pm\frac{\pi}{4} + 2n\pi$, where $n = 0, \pm1, \ldots$, e) $x = \pm\frac{2\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, f) $x = \frac{\pi}{6} + n\pi$, where $n = 0, \pm1, \ldots$

Exercise 20.5 a) $x = 2n\pi$, $x = \pi + 2n\pi$, $x = \frac{\pi}{4} + 2n\pi$, or $x = \frac{3\pi}{4} + 2n\pi$, where $n = 0, \pm1, \ldots$, b) $x = n\pi$ or $x = -\frac{\pi}{4} + n\pi$, where $n = 0, \pm1, \ldots$, c) $x = \pm\frac{\pi}{2} + 2n\pi$ or $x = \pm\frac{5\pi}{6} + 2n\pi$, where $n = 0, \pm1, \ldots$, d) $x = 2n\pi$, $x = \pi + 2n\pi$, or $x = \frac{3\pi}{2} + 2n\pi$, where $n = 0, \pm1, \ldots$, e) $x = \pm\frac{\pi}{3} + n\pi$, where $n = 0, \pm1, \ldots$, f) $x = \pm\frac{\pi}{3} + 2n\pi$ or $x = \pm\frac{2\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, g) $x = \frac{\pi}{3} + 2n\pi$, $x = \frac{2\pi}{3} + 2n\pi$, $x = -\frac{\pi}{3} + 2n\pi$, or $x = \frac{4\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, h) $x = 2n\pi$ or $x = \pi + 2n\pi$, where $n = 0, \pm1, \ldots$ (Note: the solutions of $\cos(x) + 1 = 0$ given by the formula on page 351 are $\pm\pi + 2n\pi$ with $n = 0, \pm1, \ldots$. Since every solution appears twice in this expression, we can reduce this to $x = \pi + 2n\pi$.), i) $x = \pm\frac{\pi}{2} + 2n\pi$ or $x = -\frac{\pi}{3} + n\pi$, where $n = 0, \pm1, \ldots$, j) $x = \pi + 2n\pi$, where $n = 0, \pm1, \ldots$, k) $x = \pm\frac{\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, l) $x = -\frac{\pi}{6} + 2n\pi$ or $x = \frac{7\pi}{6} + 2n\pi$, where $n = 0, \pm1, \ldots$, m) $x = -\frac{3\pi}{2} + 2n\pi$, $x = \frac{\pi}{6} + 2n\pi$, or $x = \frac{5\pi}{6} + 2n\pi$, where $n = 0, \pm1, \ldots$, n) $x = 2n\pi$, or $x = \pm\frac{\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, o) $x = \pm\frac{\pi}{3} + 2n\pi$, where $n = 0, \pm1, \ldots$, p) $x = \pm\frac{\pi}{4} + n\pi$, or $x = n\pi$, where $n = 0, \pm1, \ldots$,

Exercise 20.6 a) $x \approx -1.995 + 2n\pi$, or $x \approx 0.424 + 2n\pi$, where $n = 0, \pm1, \ldots$, b) $x \approx -0.848 + n\pi$, or $x \approx 0.148 + n\pi$, or $x \approx 0.700 + n\pi$, where $n = 0, \pm1, \ldots$, c) $x \approx 0.262 + n\frac{2\pi}{3}$, or $x \approx 0.906 + n\frac{2\pi}{3}$, or $x \approx 1.309 + n\frac{2\pi}{3}$, or $x \approx 1.712 + n\frac{2\pi}{3}$, where $n = 0, \pm1, \ldots$, d) $x \approx 0.443 + 2n\pi$, or $x \approx 2.193 + 2n\pi$, where $n = 0, \pm1, \ldots$

Chapter 21 (exercises starting on page 374):

Exercise 21.1 a) $\sin(x)$, b) $\csc(x)$, c) $\cot(x)$, d) $\csc(x)$, e) $\cos(x)$, f) $\sec(x)$

Exercise 21.2 a) False, b) True: $\frac{\sin(x)}{\cot(x)} = \frac{\sin(x)}{\frac{1}{\tan(x)}} = \sin(x) \cdot \frac{\tan(x)}{1} = \sin(x) \cdot \tan(x)$ and $\frac{\tan(x)}{\csc(x)} = \frac{\tan(x)}{\frac{1}{\sin(x)}} = \tan(x)\frac{\sin(x)}{1} = \sin(x) \cdot \tan(x)$, c) False, d) True: $\sin(x) \cdot \cos(x) \cdot \csc^2(x) = \sin(x) \cdot \cos(x) \cdot \frac{1}{\sin^2(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$ and $\frac{\csc(x)}{\sec(x)} = \frac{\frac{1}{\sin(x)}}{\frac{1}{\cos(x)}} = \frac{1}{\sin(x)} \cdot \frac{\cos(x)}{1} = \frac{\cos(x)}{\sin(x)} = \cot(x)$

Exercise 21.3 a) $-\sin(x)$, b) $\sin(x)\cos(x)$, c) $\csc^2(x)$, d) $-\cos^2(x)$, e) $\sec(x)$, f) $\cos(x)$, g) $\cos^2(x)$, h) $-\tan^2(x)$, i) $\cot^2(x)$, j) $\cos(x)\cot(x)$, k) $\cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$, l) $-\tan^2(x) - \sec^2(x) = 1 - 2\sec^2(x)$

Exercise 21.4 a) True: $\sin(x) - \sin(x)\cos^2(x) = \sin(x) \cdot (1 - \cos^2(x)) = \sin(x) \cdot \sin^2(x) = \sin^3(x)$ b) True: $\cot^2(x) - \csc^2(x) = -1 = \tan^2(x) - \sec^2(x)$ c) False, d) True: $\sin^3(x) - \sin(x) = \sin(x) \cdot (\sin^2(x) - 1) = \sin(x) \cdot (-\cos^2(x)) = -\sin(x) \cdot \cos^2(x)$, e) False, f) True: $(\sin(x) - \cos(x))^2 = \sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x) = 1 - 2\sin(x)\cos(x)$

Exercise 21.5 a) $-\sin(x)$, b) $-\tan(x)$, c) $-\tan(x)$, d) $\sin(x)$,

Exercise 21.6 a) $\sin(\frac{\alpha}{2}) = \frac{\sqrt{5}}{5}$, $\cos(\frac{\alpha}{2}) = \frac{2\sqrt{5}}{5}$, $\tan(\frac{\alpha}{2}) = \frac{1}{2}$, $\sin(2\alpha) = \frac{24}{25}$, $\cos(2\alpha) = \frac{-7}{25}$, $\tan(2\alpha) = \frac{-24}{7}$, b) $\sin(\frac{\alpha}{2}) = \frac{\sqrt{39}}{13}$, $\cos(\frac{\alpha}{2}) = \frac{-\sqrt{130}}{13}$, $\tan(\frac{\alpha}{2}) = \frac{-\sqrt{30}}{10}$, $\sin(2\alpha) = \frac{-28\sqrt{30}}{169}$, $\cos(2\alpha) = \frac{-71}{169}$, $\tan(2\alpha) = \frac{28\sqrt{30}}{71}$, c) $\sin(\frac{\alpha}{2}) = \frac{3\sqrt{10}}{10}$, $\cos(\frac{\alpha}{2}) = \frac{-\sqrt{10}}{10}$, $\tan(\frac{\alpha}{2}) = -3$, $\sin(2\alpha) = \frac{24}{25}$, $\cos(2\alpha) = \frac{7}{25}$, $\tan(2\alpha) = \frac{24}{7}$, d) $\sin(\frac{\alpha}{2}) = \frac{2\sqrt{5}}{5}$, $\cos(\frac{\alpha}{2}) = \frac{-\sqrt{5}}{5}$, $\tan(\frac{\alpha}{2}) = -2$, $\sin(2\alpha) = \frac{24}{25}$, $\cos(2\alpha) = \frac{-7}{25}$, $\tan(2\alpha) = \frac{-24}{7}$, e) $\sin(\frac{\alpha}{2}) = \frac{5\sqrt{26}}{26}$, $\cos(\frac{\alpha}{2}) = \frac{\sqrt{26}}{26}$, $\tan(\frac{\alpha}{2}) = 5$, $\sin(2\alpha) = \frac{-120}{169}$, $\cos(2\alpha) = \frac{119}{169}$, $\tan(2\alpha) = \frac{-120}{119}$, f) $\sin(\frac{\alpha}{2}) = \frac{\sqrt{30}}{6}$, $\cos(\frac{\alpha}{2}) = \frac{\sqrt{6}}{6}$, $\tan(\frac{\alpha}{2}) = \sqrt{5}$, $\sin(2\alpha) = \frac{-4\sqrt{5}}{9}$, $\cos(2\alpha) = \frac{-1}{9}$, $\tan(2\alpha) = 4\sqrt{5}$

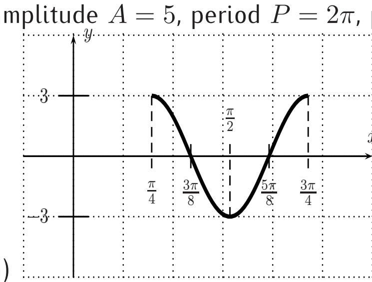
Review of Part IV (exercises starting on page 376):

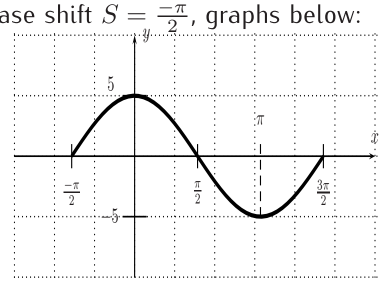
Exercises IV 1. a) $240°$, b) $\frac{7\pi}{4}$,

2.

x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

3. a) $-\frac{\sqrt{2}}{2}$, b) $\frac{\sqrt{3}}{2}$, c) $\frac{1}{2}$, d) $\frac{\sqrt{3}}{3}$, e) $-\sqrt{3}$, f) undefined, 4. a) $\frac{\sqrt{2}+\sqrt{6}}{4}$, b) $\frac{\sqrt{2-\sqrt{2}}}{2}$, 5. a) amplitude $A = 3$, period $P = \frac{\pi}{2}$, phase shift $S = \frac{\pi}{4}$, b) amplitude $A = 5$, period $P = 2\pi$, phase shift $S = \frac{-\pi}{2}$, graphs below:

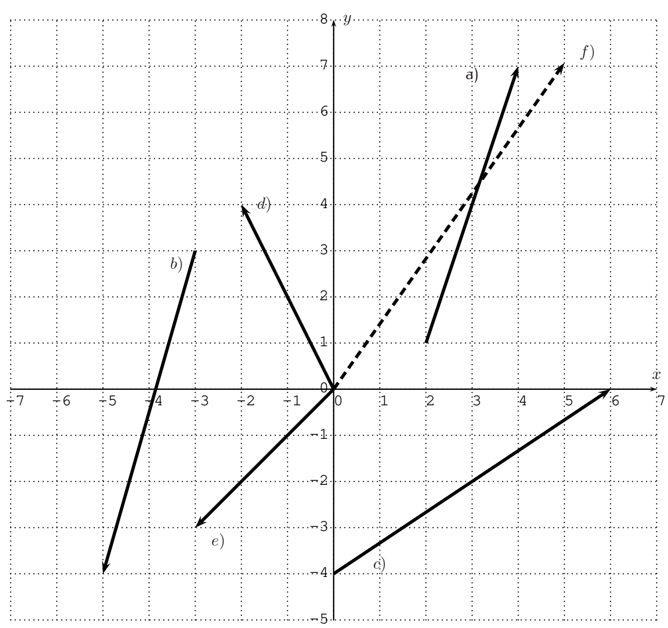
a) 

b) 

6. a) $\frac{\pi}{6}$, b) $\frac{5\pi}{6}$, c) $\frac{-\pi}{6}$, 7. a) $-\frac{\pi}{3}+2n\pi$ or $\frac{4\pi}{3}+2n\pi$ where $n = 0, \pm 1, \ldots$ b) $\frac{\pi}{6} + n\pi$ where $n = 0, \pm 1, \ldots$, 8. a) $\frac{\pi}{3} + n\pi$ or $-\frac{\pi}{3} + n\pi$ where $n = 0, \pm 1, \ldots$, b) $\pm\frac{\pi}{3} + 2n\pi$ or $\pm\frac{2\pi}{3} + 2n\pi$ where $n = 0, \pm 1, \ldots$, 9. a) $0 + 2n\pi$ or $\pi + 2n\pi$ or $-\frac{\pi}{6} + 2n\pi$ or $\frac{7\pi}{6} + 2n\pi$ where $n = 0, \pm 1, \ldots$, b) $\pm\frac{\pi}{2} + 2n\pi$ or $\pm\frac{3\pi}{4} + 2n\pi$ where $n = 0, \pm 1, \ldots$, 10. $\tan^2(x)\cos(x) - \sec(x) = \frac{\sin^2(x)}{\cos^2(x)}\cos(x) - \frac{1}{\cos(x)} = \frac{\sin^2(x)}{\cos(x)} - \frac{1}{\cos(x)} = \frac{\sin^2(x)-1}{\cos(x)} = \frac{-\cos^2(x)}{\cos(x)} = -\cos(x)$

Chapter 22 (exercises starting on page 395):

Exercise 22.1



Exercise 22.2

a) $10, 53°$, b) $\sqrt{29}, 112°$, c) $4\sqrt{2}, 225°$, d) $3\sqrt{2}, -45°$, e) $2\sqrt{2}, -45°$, f) $8, 30°$, g) $2, 210°$, h) $8, 120°$, i) $4, 210°$, j) $5, 37°$, k) $9\sqrt{2}, 135°$

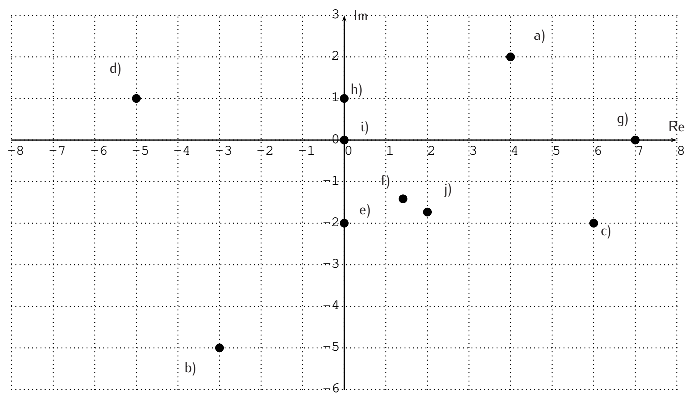
Exercise 22.3 a) $\langle 15, 10\rangle$, b)$\langle -2, 8\rangle$, c) $\langle 15, 14\rangle$, d) $\langle 8, 4\rangle$, e) $\langle 13, 5\rangle$, f) $\langle 23, 41\rangle$, g) $\langle 16, 20\rangle$, h) $\langle -16, 25\rangle$, i) $\langle -\frac{2}{3}, -\frac{25}{6}\rangle$, j) $\langle 6, -4\rangle$, k) $\langle -2, 1\rangle$, l) $\langle 0, -2\rangle$, m) $\langle 43, 12 + 7\sqrt{3}\rangle$, n) $\langle -5, -10\rangle$, o) $\langle -18, 20\rangle$, p) $\langle 8\sqrt{5}, -10\rangle$

Exercise 22.4 a) $\langle \frac{4}{5}, -\frac{3}{5}\rangle$, b) $\langle -\frac{3}{4}, -\frac{\sqrt{7}}{4}\rangle$, c) $\langle \frac{9\sqrt{85}}{85}, \frac{2\sqrt{85}}{85}\rangle$, d) $\langle -\frac{\sqrt{5}}{6}, \frac{\sqrt{31}}{6}\rangle$, e) $\langle \frac{5\sqrt{70}}{70}, \frac{3\sqrt{14}}{14}\rangle$, f) $\langle 0, -1\rangle$

Exercise 22.5 a) $\vec{v} = \langle 1, 3\sqrt{3}\rangle$, $\|\vec{v}\| = 2\sqrt{7}, \theta \approx 79°$ b) $\vec{v} \approx \langle -.772, 1.594\rangle$, $\|\vec{v}\| \approx 7.63, \theta \approx 116°$ c) $\vec{v} = \langle -4\sqrt{2}, -4\sqrt{2}\rangle$, $\|v\| = 8, \theta = 225° = \frac{5\pi}{4}$

Chapter 23 (exercises starting on page 410):

Exercise 23.1



Exercise 23.2 a) $3 + 4i$, b) $-14 + 2i$, c) $6 + 17i$, d) $6 - 7i$, e) $\frac{7}{5} + \frac{4}{5}i$, f) $-\frac{1}{2} + \frac{3}{2}i$

Exercise 23.3 a) 5, b) $6\sqrt{2}$, c) 3, d) $2\sqrt{10}$, e) 3, f) 4, g) 5, h) 7

Exercise 23.4 a) $2\sqrt{2}(\cos(45°) + i\sin(45°))$, b) $8(\cos(330°) + i\sin(330°))$, c) $14(\cos(120°) + i\sin(120°))$, d) $5\sqrt{2}(\cos(225°) + i\sin(225°))$, e) $8\sqrt{2}(\cos(315°) + i\sin(315°))$, f) $8\sqrt{2}(\cos(135°) + i\sin(135°))$, g) $2\sqrt{5}(\cos(240°) + i\sin(240°))$, h) $2\sqrt{7}(\cos(300°) + i\sin(300°))$ i) approximately $13(\cos(247.38°) + i\sin(247.38°))$, j) $6(\cos(90°) + i\sin(90°))$, k) $10(\cos(180°) + i\sin(180°))$, l) $2\sqrt{3}(\cos(120°) + i\sin(120°))$

Exercise 23.5 a) $-3\sqrt{3} + 3i$, b) $5\sqrt{2} - 5\sqrt{2}i$, c) $2i$, d) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, e)$-\frac{\sqrt{3}}{4} - \frac{1}{4}i$, f) approximately $1.553 - 5.796i$

Exercise 23.6 a) $40(\cos(150°) + i\sin(150°)) = -20\sqrt{3} + 20i$, b) $42(\cos(225°) + i\sin(225°)) = -21\sqrt{2} - 21\sqrt{2}i$, c) $\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, d) $12(\cos(\pi) + i\sin(\pi)) = -12$, e) $.1(\cos(284°) + i\sin(284°)) \approx .0242 - .0970i$, f) $\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Exercise 23.7 a) $6(\cos(210°)+i\sin(210°)) = -3\sqrt{3}-3i$, b) $\frac{2}{3}(\cos(135°)+i\sin(135°)) = -\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{3}i$, c) $\frac{7}{3}(\cos(\frac{2\pi}{3})+i\sin(\frac{2\pi}{3})) = -\frac{7}{6}+\frac{7\sqrt{3}}{6}i$, d) $\frac{1}{2}(\cos(\frac{3\pi}{2})+i\sin(\frac{3\pi}{2})) = -\frac{1}{2}i$, e) $6(\cos(\frac{4\pi}{3})+i\sin(\frac{4\pi}{3})) = -3-3\sqrt{3}i$, f) $\frac{5}{3}(\cos(-319°)+i\sin(-319°)) \approx 1.258+1.093i$

Chapter 24 (exercises starting on page 426):

Exercise 24.1 a) $3, 6, 9, 12, 15, 18, 21$, b) $8, 13, 18, 23, 28, 33, 38$, c) $3, 6, 11, 18, 27, 38, 51$, d) $1, 2, 3, 4, 5,, 6, 7$, e) $1, -1, 1, -1, 1, -1, 1$, f) $\sqrt{2}, \frac{\sqrt{3}}{2}, \frac{2}{3}, \frac{\sqrt{5}}{4}, \frac{\sqrt{6}}{5}, \frac{\sqrt{7}}{6}, \frac{\sqrt{8}}{7}$, g) $10, 100, 1000, 10,000, 100,000, 1,000,000, 10,000,000$, h) $4, 6, 4, 6, 4, 6, 4$, i) $0, 1, 0, -1, 0, 1, 0$

Exercise 24.2 a) $5, 8, 11, 14, 17$, b) $7, 70, 700, 7000, 70,000$, c) $1, 3, 7, 15, 31$, d) $6, 4, -2, -6, -4$,

Exercise 24.3 a) 50, b) 15, c) 30, d) -3, e) 26, f) $\frac{77}{60}$

Exercise 24.4 a) $5+(n-1)\cdot 3$, b) $-10+(n-1)\cdot 3$ c) no, d) no, e) $73.4-(n-1)\cdot 21.7$, f) no, g) $4+(n-1)\cdot 0$, h) $-2.72-(n-1)\cdot 0.1$, i) no, j) $-\frac{3}{5}+(n-1)\cdot\frac{1}{2}$, k) $9+(n-1)\cdot 5$, l) $-3+(j-1)\cdot 2$, m) no, n) $29+(k-1)\cdot 16$

Exercise 24.5 a) $57+(n-8)\cdot 4 = 29+4(n-1)\cdot 4$, b) $-70-(n-99)\cdot 3 = 224-(n-1)\cdot 3$, c) $14-(n-1)\cdot 5$, d) $-80+76(n-1)$, e) $10-(n-3)\cdot 3 = 16-(n-1)\cdot 3$, f) $2+(n-2)\cdot\frac{3}{4} = -49/4+(n-1)\cdot\frac{3}{4}$

Exercise 24.6 a) 116, b) 187, c) $-\frac{3621}{8}$, d) 71

Exercise 24.7 a) 5040, b) -1113, c) $49,599$, d) $-21,900$, e) $10,100$, f) $-11,537$, g) $123,150$, h) 424, i) -1762.2, j) $302,232$, k) 200

Chapter 25 (exercises starting on page 440):

Exercise 25.1 a) geometric, $7\cdot 2^{n-1}$, b) geometric, $3\cdot(-10)^{n-1}$, c) geometric, $81(\frac{1}{3})^{n-1}$, d) arithmetic, $-7+(n-1)\cdot 2$, e) geometric, $-6(-\frac{1}{3})^{n-1}$, f) geometric, $-2(\frac{2}{3})^{n-1}$, g) geometric, $\frac{1}{2}(\frac{1}{2})^{n-1}$, h) both, $2 = 2+(n-1)\cdot 0 = 2(1)^{n-1}$, i) neither, j) geometric, $-2(-1)^{n-1}$, k) arithmetic, $0+(n-1)\cdot 5$, l) geometric, $5(\frac{1}{3})^{n-1}$, m) geometric, $\frac{1}{2}(\frac{1}{2})^{n-1}$, n) neither, o) geometric, $-4(4)^{n-1}$, p) arithmetic, $-4+(n-1)\cdot(-4)$, q) geometric, $-18(-9)^{n-1}$, r) geometric, $\frac{1}{3}(\frac{1}{3})^{n-1}$, s) geometric, $-\frac{5}{7}(\frac{5}{7})^{n-1}$, t) geometric, $-\frac{5}{7}(-\frac{5}{7})^{n-1}$, u) neither, v) arithmetic, $4+(n-1)\cdot 3$

Exercise 25.2 a) 375, b) -6.25, c) $-7\cdot 2^{n-1}$, d) 6, e) $\frac{9}{10}(100)^{n-1}$, f) $20\cdot(5)^{n-1}$, g) $\frac{1}{8}(\frac{3}{8})^{n-1}$, h) $4\cdot 3^{n-1}$, i) $-40,000,000,000(-\frac{1}{10})^{n-1}$

Exercise 25.3 a) 425, b) $\frac{127}{128}$, c) $-\frac{521}{3125}$, d) $2,999,997$, e) 242, f) 910, g) $-960,800$, h) $\frac{25,575}{64}$, i) 200

Exercise 25.4 a) 9, b) $-\frac{7}{6}$, c) 3, d) -8, e) 99, f) $\frac{81}{2}$, g) $-\frac{4}{3}$, h) -9, i) $\frac{500}{3}$, j) $-\frac{81}{2}$

Exercise 25.5 a) $\frac{4}{9}$, b) $\frac{7}{9}$, c) $\frac{50}{9}$, d) $\frac{23}{99}$, e) $\frac{1300}{33}$, f) $\frac{248}{999}$, g) $\frac{20{,}000}{999}$, h) $\frac{560}{1111}$

Review of Part V (exercises starting on page 443):

Exercises V 1. magnitude $||\langle 7,-7\sqrt{3}\rangle|| = 14$, direction angle $\theta = 300°$, 2. $\langle 36,-32\rangle$, 3. a) $3\sqrt{2}\cdot(\cos(225°)+i\sin(225°))$, b) $10\cdot(\cos(150°)+i\sin(150°))$, 4. $4\sqrt{2}-4\sqrt{2}\cdot i$, 5. $-\frac{3}{10}+\frac{3\sqrt{3}}{10}\cdot i$, 6. 532, 7. a) geometric $a_n = 54\cdot(-\frac{1}{3})^{n-1}$, b) neither, c) arithmetic $a_n = 9-2\cdot(n-1)$, 8. $19,950$, 9. -1785, 10. 64

Appendix A (exercises starting on page 455):

Exercise A.1 a) 120, b) 6, c) $3,628,80$, d) 2, e) 1, f) 1, g) $\approx 1.216\cdot 10^{17}$, h) $\approx 1.269\cdot 10^{89}$, i) 10, j) 84, k) 12, l) 1, m) 23, n) $50,388$, o) 78, p) 4368

Exercise A.2 a) $m^4+4m^3n+6m^2n^2+4mn^3+n^4$, b) $x^5+10x^4+40x^3+80x^2+80x+32$, c) $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$, d) $-p^5-5p^4q-10p^3q^2-10p^2q^3-5pq^4-q^5$

Exercise A.3 a) $x^3-6x^2y+12xy^2-8y^3$, b) $x^4-40x^3+600x^2-4000x+10,000$, c) $x^{10}y^5+5x^8y^6+10x^6y^7+10x^4y^8+5x^2y^9+y^{10}$, d) $16y^8-160x^4y^6+600x^8y^4-1000x^{12}y^2+625x^{16}$, e) $x^3+3x^{\frac{5}{2}}+3x^2+x^{\frac{3}{2}}$, f) $-32\frac{x^{10}}{y^5}-80\frac{x^7}{y}-80x^4y^3-40xy^7-10\frac{y^{11}}{x^2}-\frac{y^{15}}{x^5}$, g) $38\sqrt{2}-36\sqrt{3}$, h) $-2-2i$

Exercise A.4 a) $x^5y^5-20x^5y^4+160x^5y^3$, b) $512a^{18}+2304a^{16}b^3$, d) $-189x^{10}y^4+21x^{12}y^2-x^{14}$, d) $\frac{x^{10}}{y^{10}}-10\frac{x^8}{y^8}+45\frac{x^6}{y^6}$, e) $\frac{5}{2}m^9n^9+\frac{15}{16}m^6n^{10}+\frac{3}{16}m^3n^{11}+\frac{1}{64}n^{12}$

Exercise A.5 a) $35x^3y^4$, b) $36x^{14}y^2$, c) $-220w^9$, d) $280x^7y^4$, e) $15,625b^6$, f) $-189p^9q^{15}$, g) $\frac{715}{2}b^9$

Exercise A.6 a) $84x^3y^6$, b) $15r^4s^4$, c) $-330x^4$, c) $500x^3y^6$, e) $80x^7$, f) 2

References

The topics in this book are all standard and can be found in many precalculus textbooks. In particular, the precalculus textbooks below can be consulted for further reading.

Open-source precalculus textbooks:

Abramson, Jay, Valeree Falduto, Rachael Gross, David Lippman, Melonie Rasmussen, Rick Norwood, Nicholas Belloit, Harold Whipple, Jean-Marie Magnier, and Christina Fernandez. *Precalculus*. 2nd ed. Houston: Rice University, Openstax, 2023. https://openstax.org/details/books/precalculus-2e.

Lippman, David, and Melonie Rasmussen. *Precalculus: An Investigation of Functions*. 2nd ed. Creative Commons, 2022. https://www.opentextbookstore.com/precalc/.

Further precalculus textbooks:

Blitzer, Robert F. *Precalculus*. 5th ed. Upper Saddle River, NJ: Pearson, 2013.

Hungerford, Thomas W., and Douglas J. Shaw. *Contemporary Precalculus: A Graphing Approach*. 5th ed. Belmont, CA: Brooks Cole, 2009.

Narasimhan, Revathi. *Precalculus: Building Concepts and Connections: Intructor's Edition*. Boston: Houghton Mifflin, 2009.

Safier, Fred.*Schaum's Outlines: Precalculus*. 4th ed. New York: McGraw-Hill, 2020.

Smith, Karl J. *Precalculus: A Functional Approach to Graphing and Problem Solving*. 6th ed. Burlington, MA: Jones & Bartlett, 2011.

Stewart, James, Lothar Redlin, and Saleem Watson. *Precalculus: Mathematics for Calculus*. 7th ed. Boston: Cengage, 2011.

Important formulas used in precalculus

Algebraic theorems and formulas

Quadratic formula: The solutions of $ax^2 + bx + c = 0$ are:

$$\boxed{x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Remainder theorem and factor theorem:

$$\boxed{\text{Dividing a polynomial } f(x) \text{ by } (x-c) \text{ has a remainder of } r = f(c).}$$

In particular: $\boxed{g(x) = x - c \text{ is a factor of } f(x) \quad \Longleftrightarrow \quad f(c) = 0}$

Rational root theorem:
The rational solutions of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ with integer coefficients $a_0, \dots, a_n$ (where $a_0 \neq 0$ and $a_n \neq 0$) are of the form $x = \frac{p}{q}$ where p is a factor of a_0, and q is a factor of a_n.

Fundamental theorem of algebra:

$$\boxed{\text{Every non-constant polynomial has a root.}}$$

Exponential and logarithmic formulas:

$$\boxed{\begin{array}{c} b^x \cdot b^y = b^{x+y} \\ \frac{b^x}{b^y} = b^{x-y} \\ (b^x)^n = b^{n \cdot x} \end{array}} \qquad \boxed{\begin{array}{c} \log_b(x \cdot y) = \log_b(x) + \log_b(y) \\ \log_b(\frac{x}{y}) = \log_b(x) - \log_b(y) \\ \log_b(x^n) = n \cdot \log_b(x) \\ \log_b(x) = \frac{\log(x)}{\log(b)} \end{array}}$$

Applications of exponential and logarithmic functions:

Rate of growth:	$y = c \cdot b^x$	where $b = e^r$
Half-life:	$y = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}}$	where h is the half-life
Compound interest:	$A = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$	(compounded n times per year)
Compound interest:	$A = P \cdot e^{r \cdot t}$	(continuous compounding)

Vectors $\vec{v} = \langle a, b \rangle$:

Magnitude:	$\|\|\langle a, b \rangle\|\| = \sqrt{a^2 + b^2}$
Direction angle:	$\tan(\theta) = \frac{b}{a}$
Scalar multiplication:	$r \cdot \langle a, b \rangle = \langle r \cdot a, r \cdot b \rangle$
Vector addition:	$\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$

Complex numbers $z = a + bi$: $\boxed{i^2 = -1}$

Absolute value: $|a + bi| = \sqrt{a^2 + b^2}$

Polar form: $\boxed{a + bi = r \cdot (\cos(\theta) + i \cdot \sin(\theta))}$

where $\boxed{r = \sqrt{a^2 + b^2}}$ and $\boxed{\tan(\theta) = \dfrac{b}{a}}$

Multiplication: $r_1(\cos(\theta_1) + i\sin(\theta_1)) \cdot r_2(\cos(\theta_2) + i\sin(\theta_2))$
$= r_1 r_2 \cdot (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Division: $\frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} = \frac{r_1}{r_2} \cdot (\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

Arithmetic and geometric series:

	arithmetic sequence	geometric sequence
nth term	$a_n = a_1 + (n-1) \cdot d$	$a_n = a_1 \cdot r^{n-1}$
series $a_1 + \cdots + a_k$	$\sum_{i=1}^{k} a_i = \frac{k}{2} \cdot (a_1 + a_k)$	$\sum_{i=1}^{k} a_i = a_1 \cdot \frac{1-r^k}{1-r}$
infinite series	–	$\sum_{i=1}^{\infty} a_i = a_1 \cdot \frac{1}{1-r}$

Binomial formula:

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} \cdot a^{n-r} \cdot b^r}$$

where $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$,
and $k! = 1 \cdot 2 \cdot \dots \cdot k$

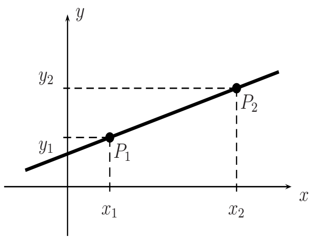
The kth term is $\binom{n}{k-1} a^{n-k+1} b^{k-1}$.

Graphs of functions

Lines:

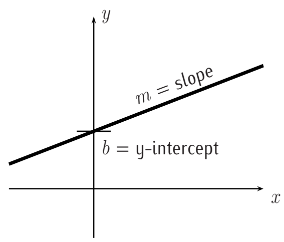
Slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



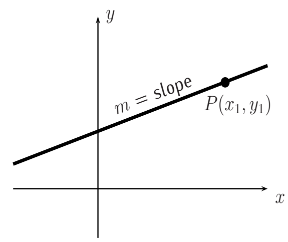
Slope-intercept form of the line:

$$y = m \cdot x + b$$

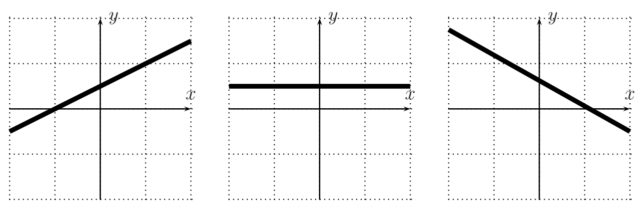


Point-slope form of the line:

$$y - y_1 = m \cdot (x - x_1)$$



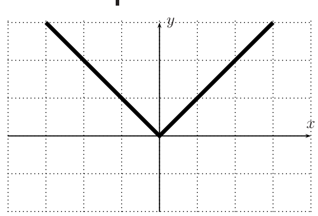
Line $y = mx + b$:



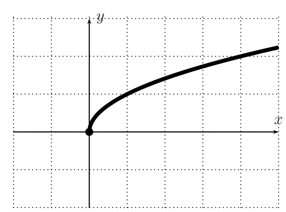
$m > 0$ $\quad$ $m = 0$ $\quad$ $m < 0$

Absolute value and square root:

$y = |x|$

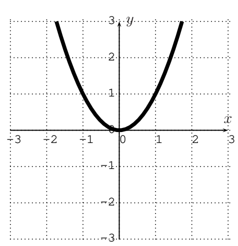


$y = \sqrt{x}$

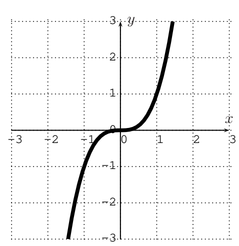


Polynomials and rational functions:

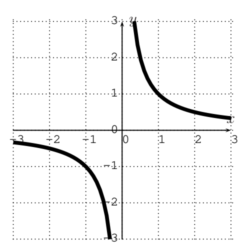
$y = x^2$



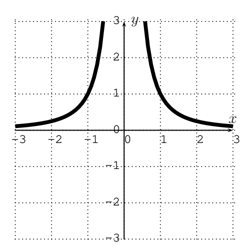
$y = x^3$



$y = \frac{1}{x}$

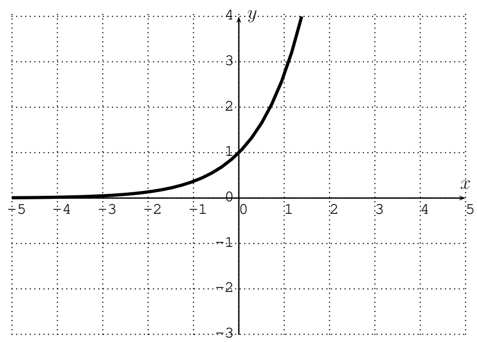


$y = \frac{1}{x^2}$

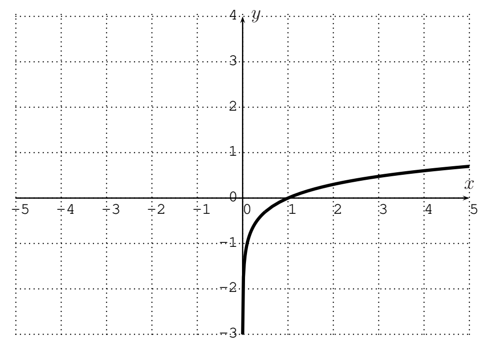


Exponential and logarithmic functions:

$y = e^x$

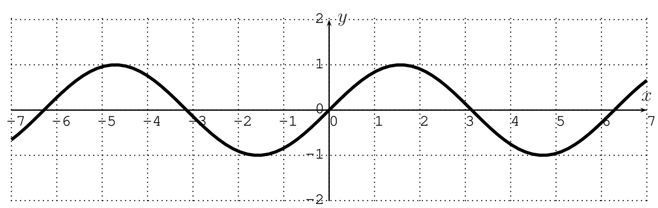


$y = \log(x)$

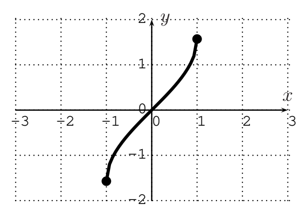


Trigonometric and inverse trigonometric functions:

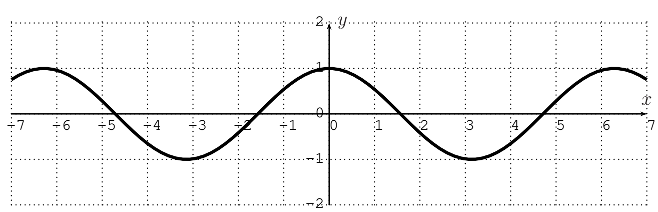
$y = \sin(x)$



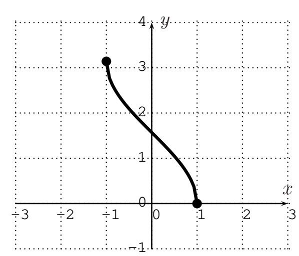
$y = \sin^{-1}(x)$



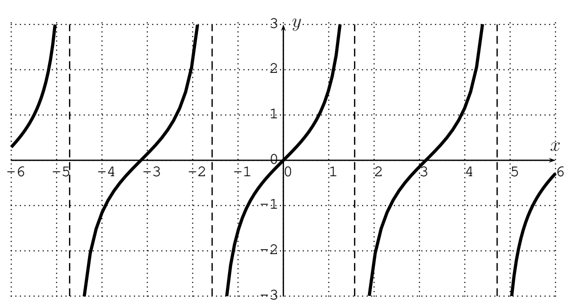
$y = \cos(x)$



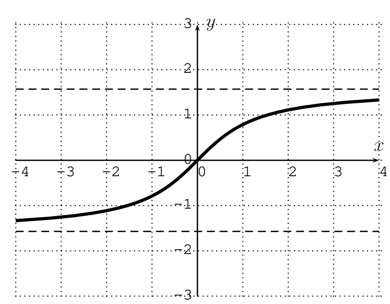
$y = \cos^{-1}(x)$



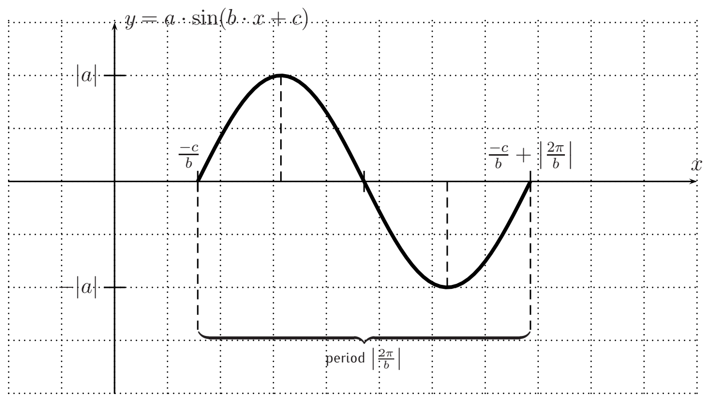
$y = \tan(x)$



$y = \tan^{-1}(x)$



$$f(x) = a \cdot \sin(b \cdot x + c) \quad \text{or} \quad f(x) = a \cdot \cos(b \cdot x + c)$$



amplitude $A = |a|$

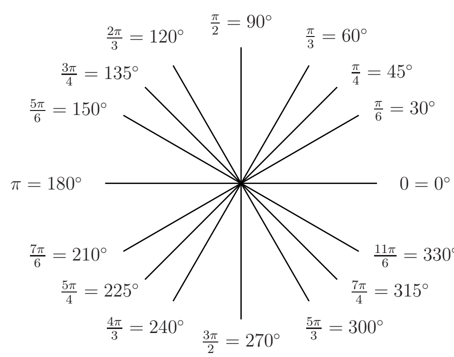
period $P = \left|\frac{2\pi}{b}\right|$

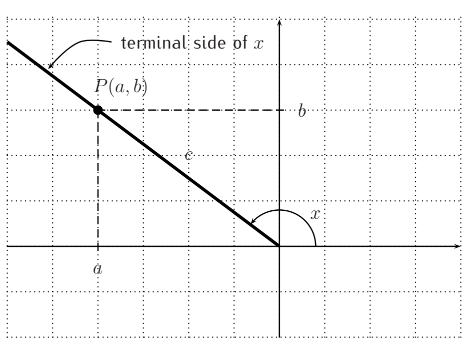
phase shift $S = \frac{-c}{b}$

Trigonometric formulas

Basic facts:

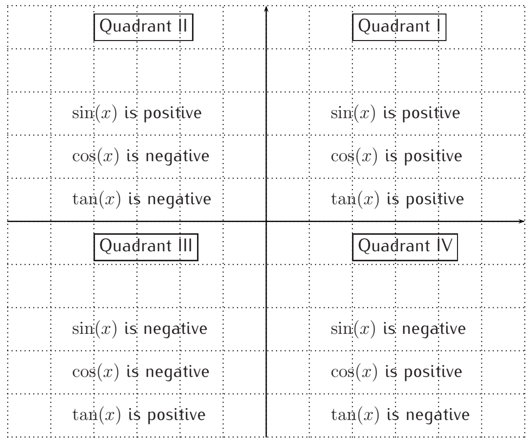
$$\boxed{\pi = 180^\circ}$$

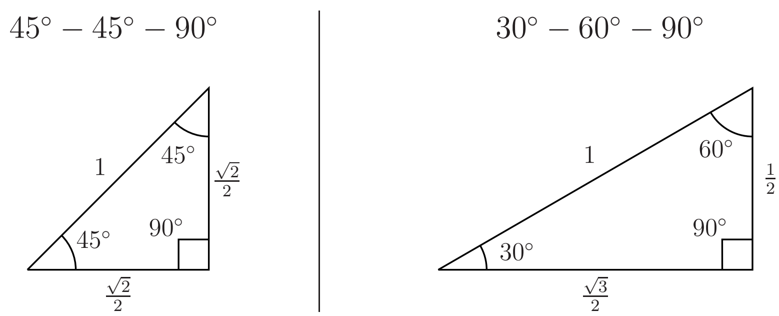




$$a^2 + b^2 = c^2$$
$$\implies c = \sqrt{a^2 + b^2}$$

$\sin(x) = \frac{b}{c}$	$\csc(x) = \frac{c}{b}$
$\cos(x) = \frac{a}{c}$	$\sec(x) = \frac{c}{a}$
$\tan(x) = \frac{b}{a}$	$\cot(x) = \frac{a}{b}$





x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

Solving trigonometric equations:

Solve: $\sin(x) = c$	Solve: $\cos(x) = c$	Solve: $\tan(x) = c$
The general solution is: $x = \sin^{-1}(c) + 2n\pi$, $x = (\pi - \sin^{-1}(c)) + 2n\pi$ where $n = 0, \pm1, \pm2, \ldots$	The general solution is: $x = \cos^{-1}(c) + 2n\pi$, $x = -\cos^{-1}(c) + 2n\pi$ where $n = 0, \pm1, \pm2, \ldots$	The general solution is: $x = \tan^{-1}(c) + n\pi$ where $n = 0, \pm1, \pm2, \ldots$

Trigonometric identities:

$$\csc(x) = \frac{1}{\sin(x)},\ \sec(x) = \frac{1}{\cos(x)},\ \tan(x) = \frac{\sin(x)}{\cos(x)},\ \cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1} \quad \boxed{\sec^2(x) = 1 + \tan^2(x)} \quad \boxed{\csc^2(x) = 1 + \cot^2(x)}$$

$$\sin(-x) = -\sin(x), \quad \cos(x + 2\pi) = \cos(x), \quad \tan(-x) = -\tan(x)$$

$$\sin(x + 2\pi) = \sin(x), \quad \sin(\pi - x) = \sin(x), \quad \sin(x \pm \tfrac{\pi}{2}) = \pm\cos(x)$$

$$\cos(x + 2\pi) = \cos(x), \quad \cos(\pi - x) = -\cos(x), \quad \cos(x \pm \tfrac{\pi}{2}) = \mp\sin(x)$$

Addition and subtraction of angles:

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

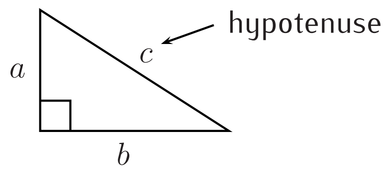
Half-angles and multiple angles:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1 - \cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1 + \cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} \\
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1 - \tan^2\alpha}
\end{aligned}$$

The signs "±" in the half-angle formulas above are determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies.

Geometric formulas

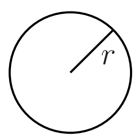
Pythagorean Theorem:



$$a^2 + b^2 = c^2$$

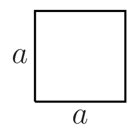
2-dimensional (planar) geometric shapes:

Circle



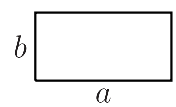
Area $A = \pi r^2$
Circumference $C = 2\pi r$

Square



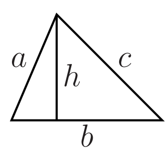
Area $A = a^2$
Perimeter $P = 4a$

Rectangle



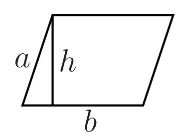
Area $A = a \cdot b$
Perimeter $P = 2a + 2b$

Triangle



Area $A = \frac{1}{2}bh$
Perimeter $P = a + b + c$

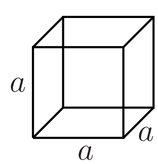
Parallelogram



Area $A = bh$
Perimeter $P = 2a + 2b$

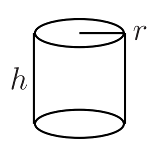
3-dimensional geometric shapes:

Cube



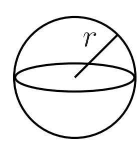
Volume $V = a^3$
Surface area $A = 6a^2$

Cylinder



Volume $V = \pi r^2 h$
Surface area
$A = 2\pi r^2 + 2\pi rh$

Sphere



Volume $V = \frac{4}{3}\pi r^3$
Surface area $A = 4\pi r^2$

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