

Complex numbers

Lesson #23

MAT 1375 Precalculus

New York City College of Technology CUNY



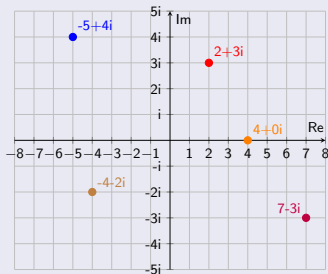
Complex numbers

A **complex number** is a number

$$a + b \cdot i, \quad \text{where } i^2 = -1$$

and a = real part, b = imaginary part

Complex plane:



Plot in the complex plane:

$$2 + 3i, \quad -5 + 4i, \quad 7 - 3i, \quad -4 - 2i, \quad 4 = 4 + 0i$$

Perform the indicated operation:

$$① \quad (2 + 3i) + (-6 + 5i) = -4 + 8i$$

$$\begin{aligned} ② \quad (2 + 3i) - (-6 + 5i) \\ = 2 + 3i + 6 - 5i \\ = 8 - 2i \end{aligned}$$

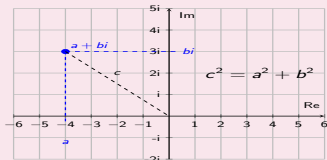
$$\begin{aligned} ③ \quad (2 + 3i) \cdot (-6 + 5i) \\ = -12 + 10i - 18i + 15i^2 \\ = -12 - 8i - 15 \\ = -27 - 8i \end{aligned}$$

$$\begin{aligned} ④ \quad \frac{2+3i}{-6+5i} \\ = \frac{(2+3i)(-6-5i)}{(-6+5i)(-6-5i)} \\ = \frac{-12-10i-18i-15i^2}{36+30i-30i-25i^2} \\ = \frac{-12-28i+15}{36+25} \\ = \frac{3-28i}{61} \\ = \frac{3}{61} - \frac{28}{61}i \end{aligned}$$

The modulus (or absolute value) of a complex number

Modulus of a complex numbers

The **modulus** or **absolute value** of $a + bi$ is



$$|a + bi| = \sqrt{a^2 + b^2}$$

Find the modulus of the complex number.

1 $3 + 7i$
 $|3 + 7i| = \sqrt{3^2 + 7^2} = \sqrt{9 + 49} = \sqrt{58}$

2 $4 + 2i$
 $|4 + 2i| = \sqrt{4^2 + 2^2} = \sqrt{16 + 4}$
 $= \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$

3 $-3 + 4i$
 $|-3 + 4i| = \sqrt{(-3)^2 + 4^2}$
 $= \sqrt{9 + 16} = \sqrt{25} = 5$

4 $5 - 5i$
 $|5 - 5i| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25}$
 $= \sqrt{50} = 5\sqrt{2}$

5 $-4 - 4i$
 $|-4 - 4i| = \sqrt{(-4)^2 + (-4)^2}$
 $= \sqrt{16 + 16} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

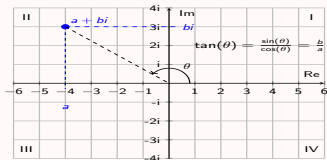
6 $2 - 2\sqrt{3}i$
 $|2 - 2\sqrt{3}i| = \sqrt{2^2 + (-2\sqrt{3})^2}$
 $= \sqrt{4 + 4 \cdot 3} = \sqrt{4 + 12} = \sqrt{16} = 4$

7 $5\sqrt{3} + 5i$
 $|5\sqrt{3} + 5i| = \sqrt{(5\sqrt{3})^2 + 5^2}$
 $= \sqrt{25 \cdot 3 + 25} = \sqrt{100} = 10$

The argument (or angle) of a complex number

Argument of a complex numbers

The **argument** or **angle** of $a + bi$ is



$$\theta = \arctan\left(\frac{b}{a}\right) \quad \text{if in I or IV}$$

$$\theta = \arctan\left(\frac{b}{a}\right) + 180^\circ \quad \text{if in II or III}$$

Find the argument of the complex number.

① $5 + 2i$

$$\theta = \arctan\left(\frac{2}{5}\right) \approx 21.8^\circ$$

② $-5 - 2i$ in quadrant III

$$\begin{aligned} \theta &= \arctan\left(\frac{-2}{-5}\right) + 180^\circ \\ &\approx 21.8^\circ + 180^\circ \approx 201.8^\circ \end{aligned}$$

③ $3 + 5i$ in quadrant I
 $\theta = \arctan\left(\frac{5}{3}\right) \approx 59.04^\circ$

④ $-2 + 7i$ in quadrant II
 $\theta = \arctan\left(\frac{7}{-2}\right) + 180^\circ$
 $\approx -74.05^\circ + 180^\circ \approx 105.95^\circ$

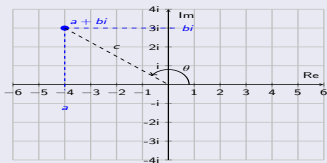
⑤ $4 - 4i$ in quadrant IV
 $\theta = \arctan\left(\frac{-4}{4}\right) = -45^\circ$
or: $\theta = -45^\circ + 360^\circ = 315^\circ$

⑥ $-2 - 2\sqrt{3}i$ in quadrant III
 $\theta = \arctan\left(\frac{-2\sqrt{3}}{-2}\right) + 180^\circ$
 $= 60^\circ + 180^\circ = 240^\circ$

⑦ $-4\sqrt{3} + 4i$ in quadrant II
 $\theta = \arctan\left(\frac{4}{-4\sqrt{3}}\right) + 180^\circ$
 $= -30^\circ + 180^\circ = 150^\circ$

Polar form of a complex number

Polar form and standard form of a complex number



Standard form:

$$a + bi$$

Since $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, we get:

$a = c \cdot \cos(\theta)$ and $b = c \cdot \sin(\theta)$ and thus:

$$a + bi = c \cos(\theta) + c \sin(\theta)i = c(\cos(\theta) + i \sin(\theta))$$

Polar form:

$$c \cdot (\cos(\theta) + i \sin(\theta))$$

Convert to polar form:

① $-3 - 3i$

$$c = |-3 - 3i| = \sqrt{(-3)^2 + (-3)^2} \\ = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \arctan\left(\frac{-3}{-3}\right) + 180^\circ \\ = 45^\circ + 180^\circ = 225^\circ$$

Therefore:

$$-3 - 3i = 3\sqrt{2} \cdot (\cos(225^\circ) + i \sin(225^\circ))$$

② $4\sqrt{3} - 4i$

$$c = |4\sqrt{3} - 4i| = \sqrt{(4\sqrt{3})^2 + (-4)^2} \\ = \sqrt{16 \cdot 3 + 16} = \sqrt{64} = 8$$

$$\theta = \arctan\left(\frac{-4}{4\sqrt{3}}\right) = -30^\circ$$

Therefore:

$$4 - 4\sqrt{3}i = 8 \cdot (\cos(-30^\circ) + i \sin(-30^\circ))$$

Converting polar form \leftrightarrow standard form - exercises

Convert to polar form:

$$\textcircled{3} -7 + 7\sqrt{3}i$$

$$c = |-7\sqrt{3} + 7i| = \sqrt{(-7)^2 + (7\sqrt{3})^2}$$
$$= \sqrt{49 + 49 \cdot 3} = \sqrt{196} = 14$$

$$\theta = \arctan\left(\frac{7\sqrt{3}}{-7}\right) + 180^\circ$$
$$= -60^\circ + 180^\circ = 120^\circ$$

Therefore:

$$-7 + 7\sqrt{3}i$$
$$= 14 \cdot (\cos(120^\circ) + i \sin(120^\circ))$$

Convert to standard form:

$$\textcircled{4} 4 \cdot (\cos(210^\circ) + i \sin(210^\circ))$$

$$= 4 \cdot \left(\frac{-\sqrt{3}}{2} + i \frac{-1}{2}\right)$$

$$= 4 \cdot \frac{-\sqrt{3}}{2} + 4 \cdot \frac{-1}{2} \cdot i$$

$$= -2\sqrt{3} - 2i$$

Convert to standard form:

$$\textcircled{5} 6 \cdot (\cos(300^\circ) + i \sin(300^\circ))$$

$$= 6 \cdot \left(\frac{1}{2} + i \frac{-\sqrt{3}}{2}\right)$$

$$= 6 \cdot \frac{1}{2} + 6 \cdot \frac{-\sqrt{3}}{2} \cdot i$$

$$= 3 - 3\sqrt{3}i$$

$$\textcircled{6} 7 \cdot (\cos(135^\circ) + i \sin(135^\circ))$$

$$= 7 \cdot \left(\frac{-\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right)$$

$$= \frac{-7\sqrt{2}}{2} + \frac{7\sqrt{2}}{2} \cdot i$$

$$\textcircled{7} 5 \cdot (\cos(270^\circ) + i \sin(270^\circ))$$

$$= 5 \cdot (0 + i(-1))$$

$$= 0 - 5i$$

$$= -5i$$

Multiplying and dividing complex numbers in polar form

Multiplying and dividing complex numbers in polar form

Multiplication in polar form:

$$c_1(\cos \theta_1 + i \sin \theta_1) \cdot c_2(\cos \theta_2 + i \sin \theta_2) = c_1 c_2 \cdot (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

Division in polar form:

$$\frac{c_1(\cos \theta_1 + i \sin \theta_1)}{c_2(\cos \theta_2 + i \sin \theta_2)} = \frac{c_1}{c_2} \cdot (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Perform the multiplication or division in polar form:

1

$$\begin{aligned} & 3(\cos 77^\circ + i \sin 77^\circ) \cdot 4(\cos 54^\circ + i \sin 54^\circ) \\ &= 12 \cdot (\cos 131^\circ + i \sin 131^\circ) \end{aligned}$$

2

$$\begin{aligned} & 6(\cos 99^\circ + i \sin 99^\circ) \cdot 10(\cos 183^\circ + i \sin 183^\circ) \\ &= 60 \cdot (\cos 282^\circ + i \sin 282^\circ) \end{aligned}$$

3

$$\begin{aligned} & \frac{8(\cos 149^\circ + i \sin 149^\circ)}{6(\cos 36^\circ + i \sin 36^\circ)} \\ &= \frac{4}{3} \cdot (\cos 113^\circ + i \sin 113^\circ) \end{aligned}$$

4

$$\begin{aligned} & \frac{20(\cos 50^\circ + i \sin 50^\circ)}{4(\cos 70^\circ + i \sin 70^\circ)} \\ &= 5 \cdot (\cos(-20^\circ) + i \sin(-20^\circ)) \end{aligned}$$

Multiplying and dividing complex numbers in polar form - exercises

Multiply or divide and write your answer in $a + bi$ form without approximation:

$$\begin{aligned}1 \quad & 2(\cos 35^\circ + i \sin 35^\circ) \cdot 7(\cos 100^\circ + i \sin 100^\circ) \\ &= 14 \cdot (\cos 135^\circ + i \sin 135^\circ) \\ &= 14 \cdot \left(\frac{-\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \\ &= -7\sqrt{2} + 7\sqrt{2}i\end{aligned}$$

$$\begin{aligned}2 \quad & 3(\cos 123^\circ + i \sin 123^\circ) \cdot 5(\cos 177^\circ + i \sin 177^\circ) \\ &= 15 \cdot (\cos 300^\circ + i \sin 300^\circ) \\ &= 15 \cdot \left(\frac{1}{2} + \frac{-\sqrt{3}}{2}i\right) \\ &= \frac{15}{2} - \frac{15\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}3 \quad & 8\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \cdot 4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \\ &= 32 \cdot \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\ &= 32 \cdot (0 + i \cdot 1) \\ &= 0 + 32i \\ &= 32i\end{aligned}$$

$$\begin{aligned}4 \quad & \frac{5(\cos 217^\circ + i \sin 217^\circ)}{9(\cos 67^\circ + i \sin 67^\circ)} \\ &= \frac{5}{9} \cdot (\cos 150^\circ + i \sin 150^\circ) \\ &= \frac{5}{9} \cdot \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= -\frac{5\sqrt{3}}{18} + \frac{5}{18}i\end{aligned}$$

$$\begin{aligned}5 \quad & \frac{6(\cos 339^\circ + i \sin 339^\circ)}{7(\cos 114^\circ + i \sin 114^\circ)} \\ &= \frac{6}{7} \cdot (\cos 225^\circ + i \sin 225^\circ) \\ &= \frac{6}{7} \cdot \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \\ &= -\frac{6\sqrt{2}}{14} - \frac{6\sqrt{2}}{14}i \\ &= -\frac{3\sqrt{2}}{7} - \frac{3\sqrt{2}}{7}i\end{aligned}$$

$$\begin{aligned}6 \quad & \frac{10(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})}{25(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})} \\ &= \frac{2}{5} \cdot \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)\right) \\ &= \frac{2}{5} \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -\frac{1}{5} - \frac{\sqrt{3}}{5}i\end{aligned}$$

