Precalculus

Third Edition (3.0)

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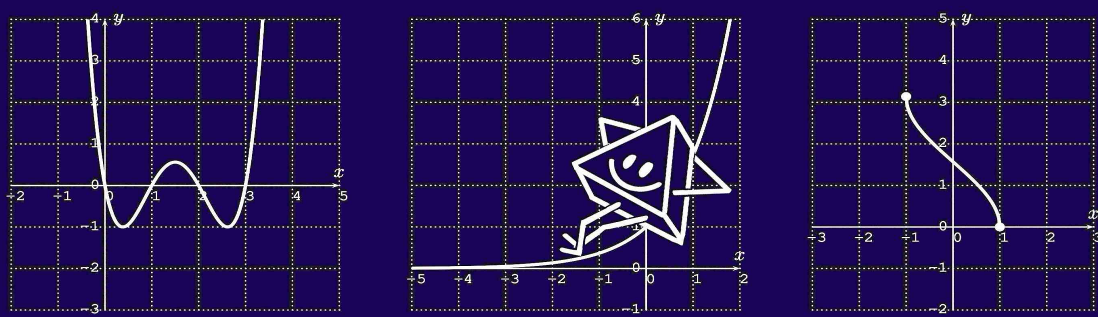

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Chapter 9

Roots of polynomials

We have seen in Observation 7.10 on page 117 that every root c of a polynomial $f(x)$ gives a factor $(x-c)$ of $f(x)$. As we would like to use this to factor polynomials, it will be helpful to know more about the nature of roots of polynomials. In Section 9.1, we will discuss a statement concerning roots that are rational numbers (the rational root theorem), while in Section 9.2 we give a general statement about the existence of roots (the fundamental theorem of algebra).

9.1 *Optional section:* The rational root theorem

Our first comment concerns rational roots for a polynomial with integer coefficients.

Note 9.1

Consider, for example, the equation $10x^3-6x^2+5x-3=0$. Let x be a rational solution of this equation, that is $x=\frac{p}{q}$ is a rational number such that

$$10\cdot\left(\frac{p}{q}\right)^3-6\cdot\left(\frac{p}{q}\right)^2+5\cdot\frac{p}{q}-3=0.$$

We assume that $x=\frac{p}{q}$ is *completely reduced*, that is, p and q have no common factors that can be used to cancel the numerator and denominator of the fraction $\frac{p}{q}$. Now, simplifying the above equation, and combining terms, we obtain:

$$\begin{aligned}
& & 10\cdot\frac{p^3}{q^3}-6\cdot\frac{p^2}{q^2}+5\cdot\frac{p}{q}-3=0 \\
\text{(multiply by } q^3\text{)} &\implies & 10p^3-6p^2q+5pq^2-3q^3=0 \\
\text{(add } 3q^3\text{)} &\implies & 10p^3-6p^2q+5pq^2=3q^3 \\
\text{(factor } p \text{ on the left)} &\implies & p\cdot(10p^2-6pq+5q^2)=3q^3.
\end{aligned}$$

Therefore, p is a factor of $3q^3$ (with the other factor being $(10p^2-6pq+5q^2)$). Since p and q have no common factors, p must be a factor of 3. That is, p is one of the following integers: $p=+1,+3,-1,-3$.
Similarly, starting from $10p^3-6p^2q+5pq^2-3q^3=0$, we can write

$$\begin{aligned}
\text{(add } +6p^2q-5pq^2+3q^3\text{)} &\implies & 10p^3=6p^2q-5pq^2+3q^3 \\
\text{(factor } q \text{ on the right)} &\implies & 10p^3=(6p^2-5pq+3q^2)\cdot q.
\end{aligned}$$

Now, q must be a factor of $10p^3$. Since q and p have no common factors, q must be a factor of 10. In other words, q is one of the following numbers: $q=\pm1,\pm2,\pm5,\pm10$. Putting this together with the possibilities for $p=\pm1,\pm3$, we see that all possible rational roots are the following:

$$\pm\frac{1}{1},\quad \pm\frac{1}{2},\quad \pm\frac{1}{5},\quad \pm\frac{1}{10},\quad \pm\frac{3}{1},\quad \pm\frac{3}{2},\quad \pm\frac{3}{5},\quad \pm\frac{3}{10}.$$

The observation in the previous example holds for a general polynomial equation with integer coefficients.

Observation 9.2: Rational root theorem

Consider the equation

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0, \tag{9.1}$$

where every coefficient $a_n, a_{n-1},\dots,a_0$ is an integer and $a_0\neq 0$, $a_n\neq 0$. Assume that $x=\frac{p}{q}$ is a rational solution of (9.1) and the fraction $x=\frac{p}{q}$ is completely reduced. Then a_0 is an integer multiple of p, and a_n is an integer multiple of q. In particular, if x is an integer root of (9.1), then a_0 is an integer multiple of x (which follows if we apply the above to the case $x=\frac{p}{1}$).

In other words:

- Any rational solution of (9.1) can be written as a fraction $x = \frac{p}{q}$ where p is a factor of a_0 and q is a factor of a_n.
- Any integer solution x of (9.1) is a factor of a_0.

We can use this observation to find good candidates for the roots of a given polynomial.

Example 9.3

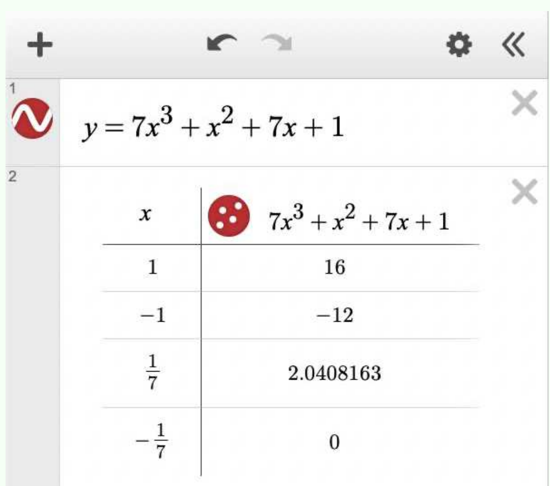
a) Find all *rational* roots of $f(x) = 7x^3 + x^2 + 7x + 1$.

b) Find all *real* roots of $f(x) = 2x^3 + 11x^2 - 2x - 2$.

c) Find all *real* roots of $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$.

Solution.

a) If $x = \frac{p}{q}$ is a rational root, then p is a factor of 1, that is $p = \pm 1$; and q is a factor of 7, that is $q = \pm 1, \pm 7$. The candidates for rational roots are therefore $x = \pm\frac{1}{1}, \pm\frac{1}{7}$. To see which of these candidates are indeed roots of f we plug these numbers into f via the table function on the graphing calculator (see Example 4.7). We obtain the following:



The only root among $\pm 1, \pm\frac{1}{7}$ is $x = -\frac{1}{7}$.

b) We need to identify all real roots of $f(x) = 2x^3 + 11x^2 - 2x - 2$. In general, it is a quite difficult task to find a root of a polynomial of degree 3, so that it will be helpful if we can find the rational roots first. If $x = \frac{p}{q}$ is a rational root, then p is a factor of -2, that is $p = \pm 1, \pm 2$; and q is a factor of 2, that is $q = \pm 1, \pm 2$. The possible rational roots $x = \frac{p}{q}$ of f are:

$$\pm 1, \quad \pm 2, \quad \pm \frac{1}{2}$$

Using the calculator, we see that the only rational root is $x = \frac{1}{2}$.

x	$2x^3 + 11x^2 - 2x - 2$
1	9
-1	9
2	54
-2	30
$\frac{1}{2}$	0
$-\frac{1}{2}$	1.5

Therefore, by the factor theorem (Observation 7.10), we see that $(x - \frac{1}{2})$ is a factor of f, that is $f(x) = q(x) \cdot (x - \frac{1}{2})$. To avoid fractions in the long division, we rewrite this as

$$f(x) = q(x) \cdot (x - \frac{1}{2}) = q(x) \cdot \frac{2x - 1}{2} = \frac{q(x)}{2} \cdot (2x - 1),$$

so that we may divide $f(x)$ by $(2x - 1)$ instead of $(x - \frac{1}{2})$ (note that this cannot be done with synthetic division). We obtain the following quotient:

$$\begin{array}{r|rrrr} & & x^2 & +6x & +2 \\ \hline 2x-1 & 2x^3 & +11x^2 & -2x & -2 \\ & -(2x^3 & -x^2) & & \\ \hline & & 12x^2 & -2x & -2 \\ & & -(12x^2 & -6x) & \\ \hline & & & 4x & -2 \\ & & & -(4x & -2) \\ \hline & & & & 0 \end{array}$$

Therefore, $f(x) = (x^2 + 6x + 2)(2x - 1)$, and any root of f is either a root of $x^2 + 6x + 2$ or of $2x - 1$. We know that the root of $2x - 1$ is $x = \frac{1}{2}$, and that x^2+6x+2 has no other rational roots. Nevertheless, we can identify all other real roots of $x^2 + 6x + 2$ via the quadratic formula, (see Proposition 8.14).

$$\begin{aligned} & x^2 + 6x + 2 = 0 \\ \Longrightarrow \quad x_{1/2} &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 2}}{2} \\ &= \frac{-6 \pm \sqrt{36 - 8}}{2} = \frac{-6 \pm \sqrt{28}}{2} \\ &= \frac{-6 \pm \sqrt{4 \cdot 7}}{2} = \frac{-6 \pm 2\sqrt{7}}{2} \\ &= -3 \pm \sqrt{7} \end{aligned}$$

Therefore, the roots of f are precisely the following:

$$x_1 = -3 + \sqrt{7}, \quad x_2 = -3 - \sqrt{7}, \quad x_3 = \frac{1}{2}.$$

c) First we find the rational roots $x = \frac{p}{q}$ of $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$. Since p is a factor of -6 it must be $p = \pm 1, \pm 2, \pm 3, \pm 6$, and since q is a factor of 4 it must be $q = \pm 1, \pm 2, \pm 4$. All candidates for rational roots $x = \frac{p}{q}$ are the following (where we excluded repeated ways of writing x):

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{1}{4}, \quad \pm\frac{3}{4}$$

Checking all these candidates with the calculator produces exactly two rational roots: $x = 6$ and $x = -\frac{1}{4}$. Therefore, we may divide $f(x)$ by both $(x-6)$ and by $(x+\frac{1}{4})$ without remainder. To avoid fractions, we use the term $4 \cdot (x+\frac{1}{4}) = (4x+1)$ instead of $(x+\frac{1}{4})$ for our factor of f. Therefore, $f(x) = q(x) \cdot (x-6) \cdot (4x+1)$. The quotient $q(x)$ is determined by performing a long division by $(x-6)$ and then another long division by $(4x+1)$, or alternatively by only one long division by

$$(x-6) \cdot (4x+1) = 4x^2 + x - 24x - 6 = 4x^2 - 23x - 6.$$

Dividing $f(x) = 4x^4 - 23x^3 - 2x^2 - 23x - 6$ by $4x^2 - 23x - 6$ produces the quotient $q(x)$:

$$\begin{array}{r|rrrrr}
 & & & x^2 & & +1 \\ \hline
4x^2-23x-6 & 4x^4 & -23x^3 & -2x^2 & -23x & -6 \\
 & -(4x^4 & -23x^3 & -6x^2) & & \\ \hline
 & & & 4x^2 & -23x & -6 \\
 & & & -(4x^2 & -23x & -6) \\ \hline
 & & & & & 0
\end{array}$$

We obtain the factored expression for $f(x)$ as $f(x) = (x^2+1)(4x+1)(x-6)$. The only remaining real roots we need to find are those of x^2+1. However,

$$x^2 + 1 = 0 \quad \Longrightarrow \quad x^2 = -1$$

has no real solution. In other words, there are only complex solutions of $x^2 = -1$, which are $x = i$ and $x = -i$ (we will discuss complex solutions in more detail in the next section). Since the problem requires us to find the *real* roots of f, our answer is that the only real roots are $x_1 = 6$ and $x_2 = -\frac{1}{4}$.

□

9.2 The fundamental theorem of algebra

There is a general theorem which tells us when a polynomial has a root. This theorem is called the *fundamental theorem of algebra*. Since complex numbers play a crucial role in this theorem, we briefly recall the basic notations concerning complex numbers. A more thorough discussion of complex numbers will be given in Chapter 23.

Review 9.4: Complex numbers

There is no real number whose square is minus 1, that is, there is no x with $x^2 = -1$. So we *denote* by i a solution of this equation. This i is not a real number but a new kind of number called a *complex* number. We can think of i as $i = \sqrt{-1}$.
We can then consider numbers of the form $a + bi$ where a and b are real numbers. Numbers of this form constitute the set of complex numbers, denoted by $\mathbb{C}$. a is called the *real part* and bi is called the *imaginary part* of the complex number $a + bi$.
We can add two complex numbers by adding their real and imaginary parts to form the real and imaginary parts of the sum. We can multiply two complex numbers by ordinary distribution (FOIL) then use the property that $i^2 = -1$.

Example 9.5

Here is an example for the subtraction and multiplication of two complex numbers.

$$(2-3i) - (4+3i) = (2-4) + (-3-3)i = -2-6i,$$
$$(2-3i)\cdot(4+3i) = 8+6i-12i-9i^2 = 8-6i-9(-1) = 17-6i.$$

We can see that these numbers arise naturally as roots of quadratic equations, such as, for example $x^2+6 = 0$, which can be written as $x^2 = -6$ and has a solution given by $x = \sqrt{-6} = \sqrt{-1}\cdot\sqrt{6} = i\sqrt{6}$. The following fundamental theorem of algebra guarantees the existence of a root of *any* polynomial of degree ≥ 1, as long as we allow complex numbers for our roots.

Theorem 9.6: Fundamental theorem of algebra

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree ≥ 1. Then there exists a complex number c which is a root of f.

Let us make two remarks about the fundamental theorem of algebra to clarify the statement of the theorem.

Note 9.7

- In the above Theorem 9.6, we did not specify what kind of coefficients $a_0, \ldots a_n$ are allowed for the theorem to hold. In fact, to be precise, the fundamental theorem of algebra states that for any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree ≥ 1 where $a_0, \ldots a_n$ are *complex* numbers, the polynomial f has a root c (which is also a complex number).

- The theorem states that a polynomial f of degree ≥ 1 always has a *complex* root c, but, in general, f may not have any *real* roots. For example, consider $f(x) = x^2 + 1$, and consider a root c of f, that is $c^2 + 1 = 0$. Since, for any real number c, we always have $c^2 \geq 0$, so that $f(c) = c^2 + 1 \geq 1$, this shows that there cannot be a real root c of f. However, we can easily check that the complex number i is a root of f:

  $$f(i) = i^2 + 1 = -1 + 1 = 0$$

  Indeed $f(x)$ has the roots i and $-i$, and can be factored as

  $$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1.$$

Now, while the fundamental theorem of algebra guarantees a root c of a polynomial f, we can use the remainder theorem from Observation 7.10 together with the calculator (and also the rational root theorem) to check possible candidates c for the roots. Once we found a root, we can use the factor theorem (also from Observation 7.10) to factor $f(x) = q(x) \cdot (x - c)$.

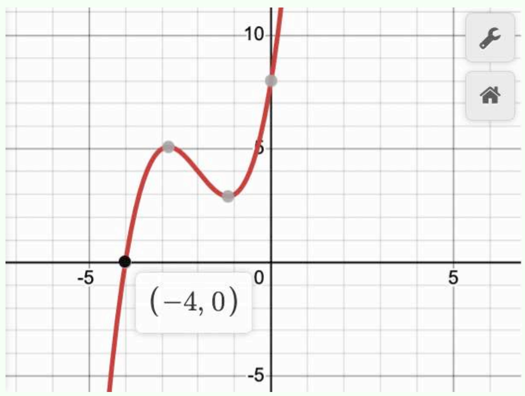
Example 9.8

Find all (real and complex) roots of the polynomial. Sketch a complete graph and label all roots.

a) $f(x) = x^3 + 6x^2 + 10x + 8$
b) $g(x) = x^3 - 6x^2 + 10x - 4$
c) $h(x) = x^4 + 2x^3 - 6x^2 - 3x + 18$

Solution.

a) In order to find a root of f, we use the graph to make a guess for one of the roots.



The graph suggests that the root may be at $x = -4$, which is also easily confirmed by plugging -4 into the function:

$$\begin{aligned} f(-4) &= (-4)^3 + 6 \cdot (-4)^2 + 10 \cdot (-4) + 8 \\ &= -64 + 96 - 40 + 8 = 0 \end{aligned}$$

Next, we divide $f(x) = x^3 + 6x^2 + 10x + 8$ by $(x + 4)$.

$$\begin{array}{r|rrrr} & & x^2 & +2x & +2 \\ \hline x+4 & x^3 & +6x^2 & +10x & +8 \\ & -(x^3 & +4x^2) & & \\ \hline & & 2x^2 & +10x & +8 \\ & & -(2x^2 & +8x) & \\ \hline & & & 2x & +8 \\ & & & -(2x & +8) \\ \hline & & & & 0 \end{array}$$

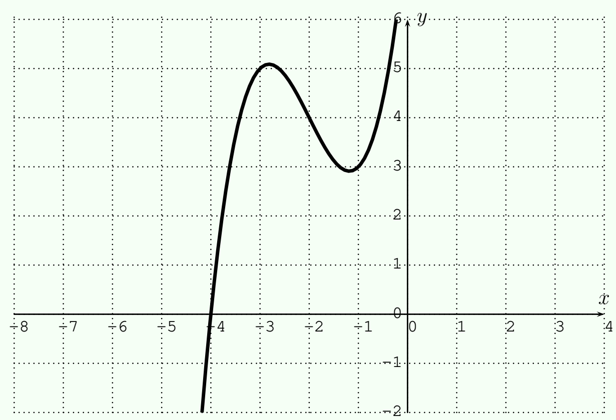
Therefore, $f(x) = (x+4)(x^2+2x+2)$. To find the remaining roots of f, we use the quadratic formula for the second polynomial x^2+2x+2:

$$x^2+2x+2 = 0 \implies x = \frac{-2\pm\sqrt{2^2-4\cdot1\cdot2}}{2} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2}$$
$$= \frac{-2\pm\sqrt{-1}\sqrt{4}}{2} = \frac{-2\pm i\cdot 2}{2} = \frac{2(-1\pm i)}{2} = -1\pm i$$

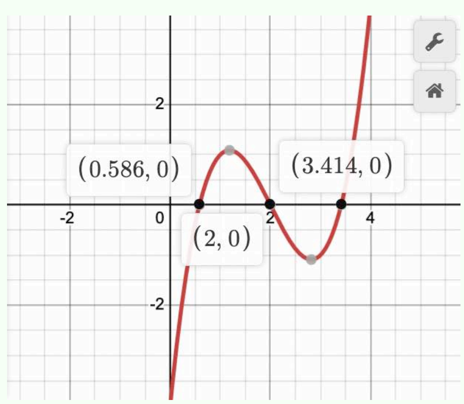
Therefore, there is only one real root -4, and two complex roots $-1+i$ and $-1-i$. The polynomial can be factored as

$$f(x) = (x+4)\cdot(x-(-1+i))\cdot(x-(-1-i))$$

The complete graph is displayed below. The only real root is shown at -4.



b) We first check the graph of $g(x) = x^3 - 6x^2 + 10x - 4$.



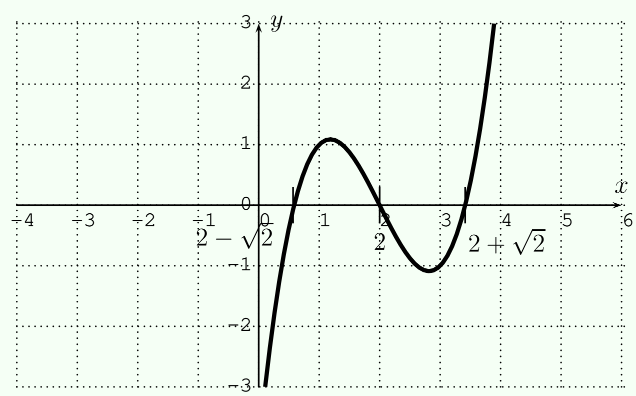
From the graph we suspect that $x = 2$ is a root, while there are two more real roots which are not at integer values. We confirm the root at 2 by direct computation, or by performing a long division by $x - 2$.

$$
\begin{array}{r|rrrr}
 & & x^2 & -4x & +2 \\
\hline
x-2 & x^3 & -6x^2 & +10x & -4 \\
 & -(x^3 & -2x^2) & & \\
\cline{2-3}
 & & -4x^2 & +10x & -4 \\
 & & -(-4x^2 & +8x) & \\
\cline{3-4}
 & & & 2x & -4 \\
 & & & -(2x & -4) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

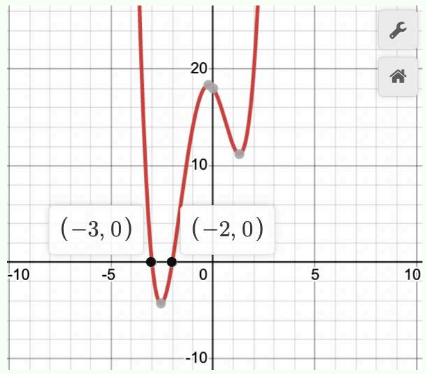
We find the remaining roots via the quadratic formula. Setting $x^2 - 4x + 2 = 0$ gives

$$
\begin{aligned}
x &= \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 1\cdot 2}}{2} = \frac{4\pm\sqrt{16-8}}{2} = \frac{4\pm\sqrt{8}}{2} \\
&= \frac{4\pm\sqrt{4}\sqrt{2}}{2} = \frac{4\pm 2\cdot\sqrt{2}}{2} = \frac{2\cdot(2\pm\sqrt{2})}{2} = 2\pm\sqrt{2}
\end{aligned}
$$

Therefore, $g(x) = (x-2)\cdot(x-(2+\sqrt{2}))\cdot(x-(2-\sqrt{2}))$. The roots of g are 2, $2+\sqrt{2}$, $2-\sqrt{2}$. The complete graph of g is drawn below.



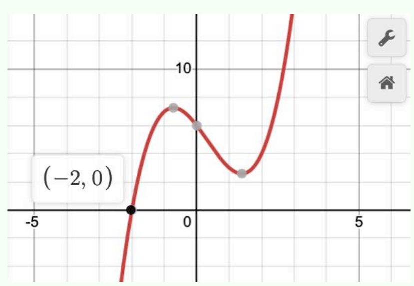
c) We first graph $h(x) = x^4 + 2x^3 - 6x^2 - 3x + 18$. Note that if we want to see a complete graph, we need to zoom out vertically.



Two integer roots of h appear to be at $x = -2$ and $x = -3$. Dividing by, say, $x + 3$, gives

$$
\begin{array}{r|rrrrr}
 & & x^3 & -x^2 & -3x & +6 \\
\hline
x+3 & x^4 & +2x^3 & -6x^2 & -3x & +18 \\
 & -(x^4 & +3x^3) & & & \\
\cline{2-3}
 & & -x^3 & -6x^2 & -3x & +18 \\
 & & -(-x^3 & -3x^2) & & \\
\cline{3-4}
 & & & -3x^2 & -3x & +18 \\
 & & & -(-3x^2 & -9x) & \\
\cline{4-5}
 & & & & 6x & +18 \\
 & & & & -(6x & +18) \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

Therefore, $h(x) = (x+3)(x^3 - x^2 - 3x + 6)$. To factor $x^3 - x^2 - 3x + 6$, we graph it to find possible roots.



There is a root at $x = -2$, as we might have already suspected, since we noted that -2 is a root of h. Next, we performing another long division.

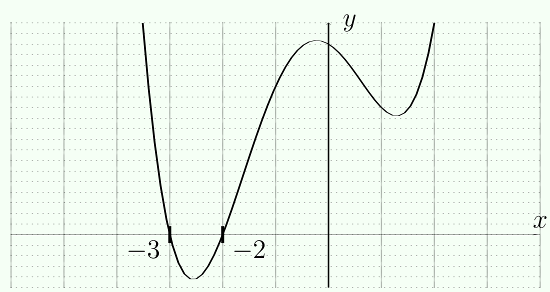
$$
\begin{array}{r|rrrr}
 & x^2 & -3x & +3 & \\
\hline
x+2 & x^3 & -x^2 & -3x & +6 \\
 & -(x^3 & +2x^2) & & \\
\hline
 & & -3x^2 & -3x & +6 \\
 & & -(-3x^2 & -6x) & \\
\hline
 & & & 3x & +6 \\
 & & & -(3x & +6) \\
\hline
 & & & & 0
\end{array}
$$

With this, we have $h(x) = (x+3)(x+2)(x^2 - 3x + 3)$. To find the roots of the last factor, apply the quadratic formula to $x^2 - 3x + 3 = 0$.

$$
\begin{aligned}
x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{3 \pm \sqrt{9-12}}{2} \\
&= \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm \sqrt{-1}\sqrt{3}}{2} = \frac{3 \pm i \cdot \sqrt{3}}{2}
\end{aligned}
$$

Thus, $h(x) = (x+3) \cdot (x+2) \cdot \left(x - \frac{3 + i \cdot \sqrt{3}}{2}\right) \cdot \left(x - \frac{3 - i \cdot \sqrt{3}}{2}\right)$.

The complete graph is shown below.



$h(x)$ has roots at -3, -2, $\frac{3+i\cdot\sqrt{3}}{2}$, $\frac{3-i\cdot\sqrt{3}}{2}$.

□

As we have seen in the last example, we can use the roots to factor a polynomial completely so that all factors are polynomials of degree 1. Furthermore, in this example, we had a complex root, $a + ib$, and its complex conjugate $a - ib$ was also a root. These observations hold more generally, as we state now.

Observation 9.9: Factors and roots of polynomials

(1) Every polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree n can be factored as

$$f(x) = a_n \cdot (x - c_1) \cdot (x - c_2) \cdot \cdots \cdot (x - c_n). \qquad (9.2)$$

This follows, since we can find a root c_1 of f (as guaranteed by the fundamental theorem of algebra), and use it to factor $f(x) = (x - c_1) \cdot g(x)$. We do the same for $g(x)$ and repeat until we arrive at (9.2).

(2) In particular, every polynomial of degree n has at most n roots. (However, these roots may be real or complex.)

(3) The factor $(x - c)$ for a root c could appear multiple times in (9.2), that is, we may have $(x - c)^k$ as a factor of f. The **multiplicity** of a root c is the number of times k that a root appears in the factored expression for f, as in (9.2).

(4) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has only *real* coefficients $a_0, \ldots, a_n$, and $c = a + bi$ is a *complex* root of f, then the complex conjugate $\bar{c} = a - bi$ is also a root of f.

Proof. If x is any root, then $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Applying the complex conjugate to this and using that $\overline{u \cdot v} = \bar{u} \cdot \bar{v}$ gives $\overline{a_n}\bar{x}^n + \overline{a_{n-1}}\bar{x}^{n-1} + \cdots + \overline{a_1}\bar{x} + \overline{a_0} = 0$. Since the coefficients a_j are real, we have that $\overline{a_j} = a_j$, so that $a_n\bar{x}^n + a_{n-1}\bar{x}^{n-1} + \cdots + a_1\bar{x} + a_0 = 0$. This shows that the complex conjugate $\bar{x}$ is a root of f as well. □

Example 9.10

For a chosen real number C, let f be the function (dependent on the C):

$$f(x) = 4x^3 - 16x^2 + 9x + C$$

a) Find the number C so that the polynomial $f(x)$ has a root at 3.

b) Find all remaining roots of $f(x)$ and write them in simplest radical form.

Solution.

a) For 3 to be a root of f, we know that $x-3$ has to be a factor of $f(x)$. We therefore perform a long division by $f(x) \div (x-3)$.

$$
\begin{array}{r|rrrr}
 & 4x^2 & -4x & -3 & \\
\hline
x-3 & 4x^3 & -16x^2 & +9x & +C \\
 & -(4x^3 & -12x^2) & & \\
\hline
 & & -4x^2 & +9x & +C \\
 & & -(-4x^2 & +12x) & \\
\hline
 & & & -3x & +C \\
 & & & -(-3x & +9) \\
\hline
 & & & & C-9
\end{array}
$$

Thus, $x-3$ is a factor of $f(x)$ exactly when the remainder $C-9$ is zero, that is, $C=9$. We thus have that

$$f(x) = 4x^3 - 16x^2 + 9x + 9$$

b) From (a), we know that f factors as $f(x) = (x-3)(4x^2-4x-3)$. We can use the quadratic formula to find the remaining roots of f by setting $4x^2-4x-3=0$.

$$
\begin{aligned}
\implies \quad x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4} \\
&= \frac{4 \pm \sqrt{16+48}}{8} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} \\
\implies x_1 &= \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2}, \quad x_2 = \frac{4-8}{8} = \frac{-4}{8} = -\frac{1}{2}
\end{aligned}
$$

We get that the roots of f are 3, $\frac{3}{2}$ and $-\frac{1}{2}$.

Note that, alternatively, we could have factored $4x^2-4x-3 = (2x-3)(2x+1) = 4(x-\frac{3}{2})(x+\frac{1}{2})$, resulting in the same roots $x_1 = \frac{3}{2}$ and $x_2 = -\frac{1}{2}$.

□

Example 9.11

Find a polynomial f with the following properties:

a) f has degree 3; the roots of f are precisely 4, 5, 6; and the leading coefficient of f is 7

b) f has degree 3 with real coefficients; f has roots $3i$, -5 (and possibly other roots as well); and $f(0) = 90$

c) f has degree 4 with complex coefficients; f has roots $i+1$, $2i$, 3

d) f has roots that are determined by the following graph of f:

Solution.

a) In general a polynomial f of degree 3 is of the form $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3)$. Identifying the roots and the leading coefficient, we obtain the polynomial

$$f(x) = 7 \cdot (x-4) \cdot (x-5) \cdot (x-6)$$

b) A polynomial f of degree 3 is of the form $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3)$. Roots of f are $3i$ and -5, and since the coefficients of f are real, it follows from Observation 9.9(4) that the complex conjugate $-3i$ is also a root of f. Therefore, $f(x) = m \cdot (x+5) \cdot (x-3i) \cdot (x+3i)$. To identify m, we use the last condition $f(0) = 90$.

$$90 = m \cdot (0+5) \cdot (0-3i) \cdot (0+3i) = m \cdot 5 \cdot (-9)i^2 = m \cdot 5 \cdot 9 = 45m$$

Dividing by 45, we obtain $m = 2$, so that

$$f(x) = 2 \cdot (x+5) \cdot (x-3i) \cdot (x+3i) = 2 \cdot (x+5) \cdot (x^2+9),$$

which clearly has real coefficients.

c) Since f is of degree 4, it can be written as $f(x) = m \cdot (x - c_1) \cdot (x - c_2) \cdot (x - c_3) \cdot (x - c_4)$. Three of the roots are identified as $i + 1$, $2i$, and 3:

$$f(x) = m \cdot (x - (1 + i)) \cdot (x - 2i) \cdot (x - 3) \cdot (x - c_4)$$

However, we have no further information on the fourth root c_4 or the leading coefficient m. (Note that Observation 9.9(4) cannot be used here, since we are *not* assuming that the polynomial has *real* coefficients. Indeed f *cannot* have real coefficients since then, besides the complex roots $1 + i$ and $2i$, their complex conjugates $1 - i$ and $-2i$ would also be roots of f, giving us 5 roots of f. However, a polynomial of degree 4 cannot have 5 roots.) We can therefore *choose any number* for these remaining variables. For example, a possible solution to the problem is given by choosing $m = 3$ and $c_4 = 2$, for which we obtain:

$$f(x) = 3 \cdot (x - (1 + i)) \cdot (x - 2i) \cdot (x - 3) \cdot (x - 2)$$

d) f is of degree 5, and we know that the leading coefficient is 1. The graph is zero at $x = 1, 2, 3$, and 4, so that the roots are 1, 2, 3, and 4. Moreover, since the graph just touches the root $x = 4$, this must be a multiple root, that is, it must occur more than once (see Section 8.3 for a discussion of multiple roots and their graphical consequences). We obtain the following solution:

$$f(x) = (x - 1)(x - 2)(x - 3)(x - 4)^2$$

Note that the root $x = 4$ is a root of multiplicity 2.

□

Note 9.12

By Observation 9.9(4), polynomials with real coefficients have complex roots that come in complex conjugate pairs. To find the product of the corresponding factors, an appropriate grouping may help to simplify the computation.

For example, when multiplying $(x-(2+3i))(x-(2-3i))$, we can group the x and 2, and then use the binomial formula $(a+b)(a-b)=a^2-b^2$ to evaluate:

$$\begin{aligned}(x-(2+3i))(x-(2-3i)) &= ((x-2)-3i)((x-2)+3i)\\ &= (x-2)^2-9i^2=(x-2)^2+9\end{aligned}$$

9.3 Exercises

Exercise 9.1

a) Find all rational roots of $f(x)=2x^3-3x^2-3x+2$.
b) Find all rational roots of $f(x)=3x^3-x^2+15x-5$.
c) Find all rational roots of $f(x)=6x^3+7x^2-11x-12$.
d) Find all real roots of $f(x)=6x^4+25x^3+8x^2-7x-2$.
e) Find all real roots of $f(x)=4x^3+9x^2+26x+6$.

Exercise 9.2

Find a root of the polynomial by guessing possible candidates of the root.

a) $f(x)=x^5-1$ b) $f(x)=x^4-1$ c) $f(x)=x^3-27$
d) $f(x)=x^3+1000$ e) $f(x)=x^4-81$ f) $f(x)=x^3-125$
g) $f(x)=x^5+32$ h) $f(x)=x^{777}-1$ i) $f(x)=x^2+64$

Exercise 9.3

Find the roots of the polynomial and use it to factor the polynomial completely.

a) $f(x)=x^3-7x+6$ b) $f(x)=x^3-x^2-16x-20$
c) $f(x)=x^3-7x^2+17x-20$ d) $f(x)=x^3+x^2-5x-2$
e) $f(x)=2x^3+x^2-7x-6$ f) $f(x)=12x^3+49x^2-2x-24$
g) $f(x)=x^3-3x^2+9x+13$ h) $f(x)=x^4-5x^2+4$
i) $f(x)=x^4-1$ j) $f(x)=x^5-6x^4+8x^3+6x^2-9x$
k) $f(x)=x^3-27$ l) $f(x)=x^4+2x^2-15$

Exercise 9.4

Find the exact roots of the polynomial; write the roots in simplest radical form, if necessary. Sketch a graph of the polynomial with all roots clearly marked.

a) $f(x) = x^3 - 2x^2 - 5x + 6$
b) $f(x) = x^3 + 5x^2 + 3x - 4$
c) $f(x) = -x^3 + 5x^2 + 7x - 35$
d) $f(x) = x^3 + 7x^2 + 13x + 7$
e) $f(x) = 2x^3 - 8x^2 - 18x - 36$
f) $f(x) = x^4 - 4x^2 + 3$
g) $f(x) = -x^4 + x^3 + 24x^2 - 4x - 80$
h) $f(x) = 7x^3 - 11x^2 - 10x + 8$
i) $f(x) = -15x^3 + 41x^2 + 15x - 9$
j) $f(x) = x^4 - 6x^3 + 6x^2 + 4x$

Exercise 9.5

Find a real number C so that the polynomial has a root as indicated. Then, for this choice of C, find all remaining roots of the polynomial.

a) $f(x) = x^3 + 6x^2 + 5x + C$ has root at $x = 1$
b) $f(x) = x^3 - 4x^2 - 2x + C$ has root at $x = -2$
c) $f(x) = x^3 - x^2 - 9x + C$ has root at $x = 3$
d) $f(x) = x^3 + 8x^2 + 5x + C$ has root at $x = -1$
e) $f(x) = x^3 - 5x^2 + 15x + C$ has root at $x = 2$

Exercise 9.6

Find a polynomial f that fits the given data.

a) f has degree 3. The roots of f are precisely 2, 3, 4. The leading coefficient of f is 2.

b) f has degree 4. The roots of f are precisely -1, 2, 0, -3. The leading coefficient of f is -1.

c) f has degree 3. f has roots -2, -1, 2, and $f(0) = 10$.

d) f has degree 4. f has roots 0, 2, -1, -4, and $f(1) = 20$.

e) f has degree 3. The coefficients of f are all real. The roots of f are precisely $2 + 5i$, $2 - 5i$, 7. The leading coefficient of f is 3.

f) f has degree 3. The coefficients of f are all real. f has roots i, 3, and $f(0) = 6$.

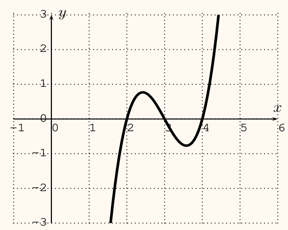
g) f has degree 4. The coefficients of f are all real. f has roots $5+i$ and $5-i$ of multiplicity 1, the root 3 of multiplicity 2, and $f(5)=7$.

h) f has degree 4. The coefficients of f are all real. f has roots i and $3+2i$.

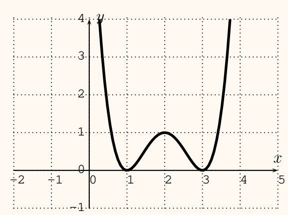
i) f has degree 6. f has complex coefficients. f has roots $1+i$, $2+i$, $4-3i$ of multiplicity 1 and the root -2 of multiplicity 3.

j) f has degree 5. f has complex coefficients. f has roots i, 3, -7 (and possibly other roots).

k) f has degree 3. The roots of f are determined by its graph:



l) f has degree 4. The coefficients of f are all real. The leading coefficient of f is 1. The roots of f are determined by its graph:



m) f has degree 4. The coefficients of f are all real. f has the following graph: