Precalculus

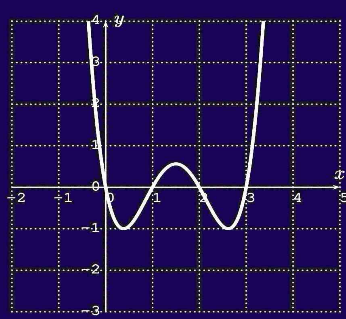
Third Edition (3.0)

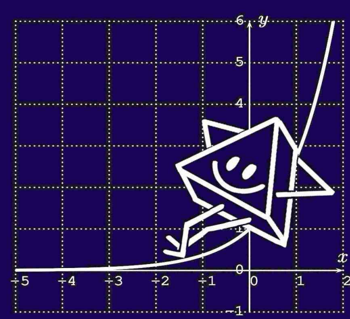
Thomas Tradler

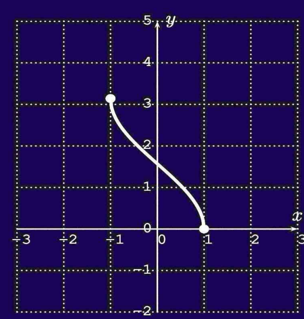
Holly Carley

The pages below contain a single chapter from the Precalculus textbook. The full textbook can be downloaded here:

Precalculus Textbook Download Page







This document was created with LaTeX.

The images of the Desmos graphing calculator were generated from the Desmos website at **https://www.desmos.com/calculator**

Chapter 6

The inverse of a function

For some functions, we can reverse the meaning of input and output. We can do this when each output of the function comes from exactly one input (the function is one-to-one). The function resulting from switching inputs and outputs is called the inverse of the function.

6.1 One-to-one functions

We have seen that some functions f may have the *same* outputs for *different* inputs. For example for $f(x) = x^2$, the inputs $x = 2$ and $x = -2$ have the same output $f(2) = 4$ and $f(-2) = 4$. A function is one-to-one, precisely when this is *not* the case.

Definition 6.1: One-to-one function

A function f is called **one-to-one** (or **injective**), if any two different inputs $x_1 \neq x_2$ always have different outputs $f(x_1) \neq f(x_2)$.

We now give a graphical interpretation for when a function is one-to-one.

Note 6.2

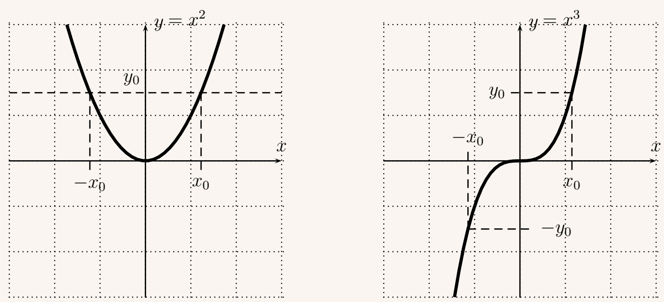
As was noted above, the function $f(x) = x^2$ is not one-to-one, because, for example, for inputs 2 and -2, we have the same output

$$f(-2) = (-2)^2 = 4, \qquad f(2) = 2^2 = 4.$$

On the other hand, $g(x) = x^3$ is one-to-one, since, for example, for inputs -2 and 2, we have different outputs:

$$g(-2) = (-2)^3 = -8, \qquad g(2) = 2^3 = 8.$$

The difference between the functions f and g can be seen from their graphs.

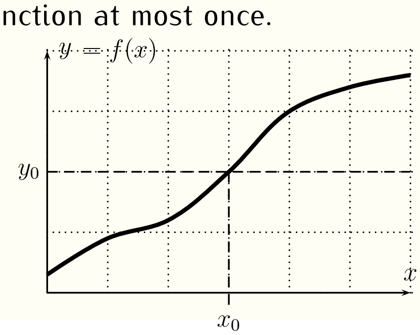


The graph of $f(x) = x^2$ on the left has for different inputs (x_0 and $-x_0$) the same output ($y_0 = (x_0)^2 = (-x_0)^2$). This is shown in the graph since the horizontal line at y_0 intersects the graph at two different points. In general, two inputs that have the same output y_0 give two points on the graph which also lie on the horizontal line at y_0.
Now, the graph of $g(x) = x^3$ on the right intersects with a horizontal line at some y_0 only once. This shows that for two different inputs, we can never have the same output y_0, so that the function g is one-to-one.

We summarize the above in the following observation.

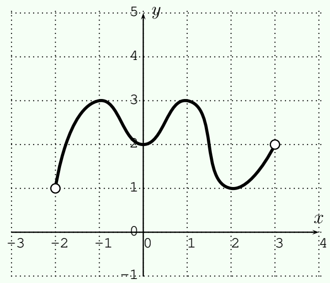
Observation 6.3: Horizontal Line Test

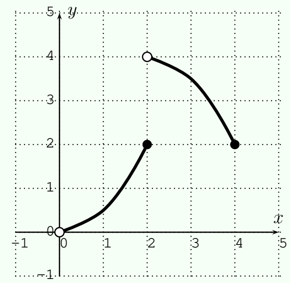
A function is one-to-one exactly when every horizontal line intersects the graph of the function at most once.



Example 6.4

Which of the following are, or represent, one-to-one functions?

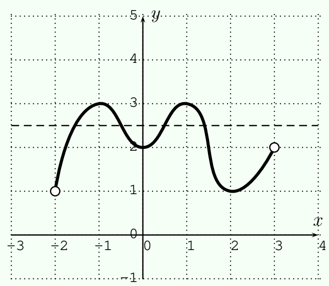
a) 

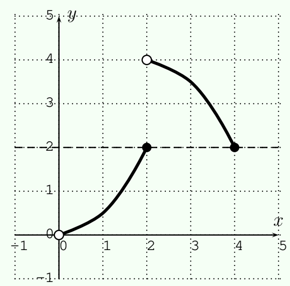
b) 

c) $f(x) = -x^3 + 6x^2 - 13x + 12$ d) $f(x) = x^3 - 2x^2 + 3$

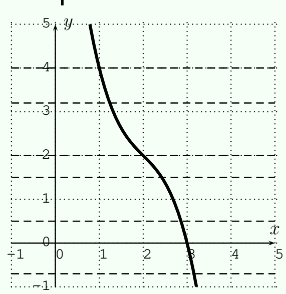
Solution.

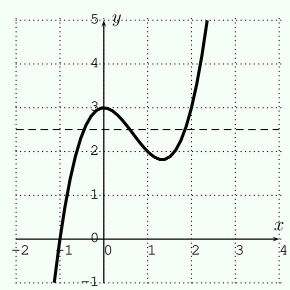
We use the horizontal line test to see which functions are one-to-one. For (a) and (b), we see that the functions are *not* one-to-one since there is a horizontal line that intersects with the graph more than once:

a) 

b) 

For (c), using the graphing calculator to graph the function $f(x) = -x^3 + 6x^2 - 13x + 12$, we see that all horizontal lines intersect the graph exactly once. Therefore, the function in part (c) is one-to-one. The function in part (d) however has a graph that intersects some horizontal line in several points. Therefore, $f(x) = x^3 - 2x^2 + 3$ is not one-to-one:

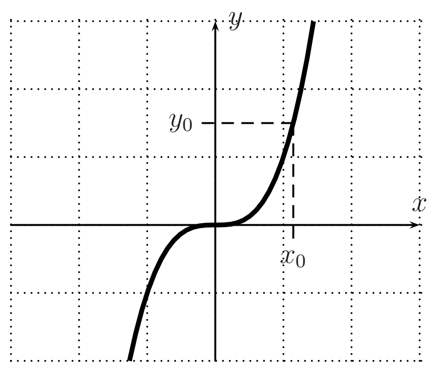
c) 

d) 

□

6.2 Inverse function

A function is one-to-one when each output is determined by exactly one input. Therefore, we can construct a new function called the inverse function, in which we reverse the roles of inputs and outputs. For example, when $y = x^3$, each y_0 comes from exactly one x_0 as shown in the picture below:

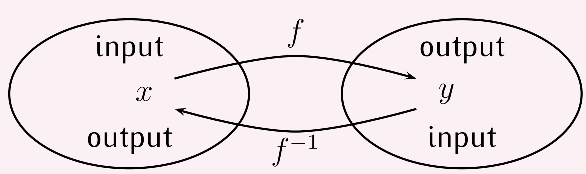


The inverse function assigns to the *input* y_0 the *output* x_0.

Definition 6.5: Inverse function

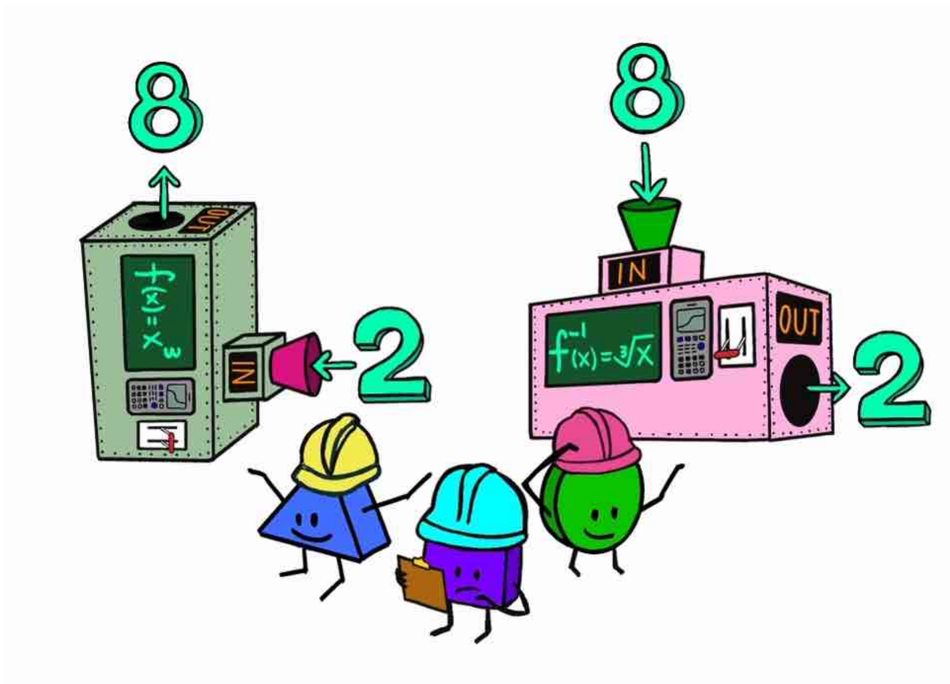
Let f be a function with domain D_f and range R_f, and assume that f is one-to-one. The **inverse** of f is the function f^{-1}, determined by:

$$f(x) = y \quad \text{means precisely that} \quad f^{-1}(y) = x.$$



Here the outputs of f are the inputs of f^{-1}, and the inputs of f are the outputs of f^{-1}. Therefore, the inverse function f^{-1} has a domain equal to the range of f, $D_{f^{-1}} = R_f$; and f^{-1} has a range equal to the domain of f, $R_{f^{-1}} = D_f$. In short, when f is a function $f : D_f \to R_f$, then the inverse function f^{-1} is a function $f^{-1} : R_f \to D_f$.

The inverse function reverses the roles of inputs and outputs.



Example 6.6

Find the inverse of the following functions.

a) $f(x) = 2x - 7$
b) $g(x) = \sqrt{x+2}$
c) $h(x) = \frac{1}{x+4}$
d) $j(x) = \frac{x+1}{x+2}$
e) $k(x) = (x-2)^2 + 3$ for $x \geq 2$

Solution.

a) First, reverse the role of input and output in $y = 2x - 7$ by exchanging the variables x and y. That is, we write $x = 2y - 7$. We need to solve this for y:

$$\overset{\text{(add 7)}}{\Longrightarrow} \quad x + 7 = 2y \implies \quad y = \frac{x+7}{2}$$

Therefore, we obtain that the inverse of f is $f^{-1}(x) = \frac{x+7}{2}$.

For the other parts, we always exchange x and y and solve for y:

b) Write $y = \sqrt{x+2}$ and exchange x and y:

$$x = \sqrt{y+2} \implies x^2 = y+2 \implies y = x^2 - 2$$
$$\implies g^{-1}(x) = x^2 - 2$$

c) Write $y = \frac{1}{x+4}$ and exchange x and y:

$$x = \frac{1}{y+4} \implies y+4 = \frac{1}{x} \implies y = \frac{1}{x} - 4$$
$$\implies h^{-1}(x) = \frac{1}{x} - 4$$

d) Write $y = \frac{x+1}{x+2}$ and exchange x and y:

$$x = \frac{y+1}{y+2} \overset{\times(y+2)}{\implies} x(y+2) = y+1 \implies xy + 2x = y+1$$
$$\implies xy - y = 1 - 2x \implies y(x-1) = 1-2x$$
$$\implies y = \frac{1-2x}{x-1} \implies j^{-1}(x) = \frac{1-2x}{x-1}$$

e) Write $y = (x-2)^2 + 3$ and exchange x and y:

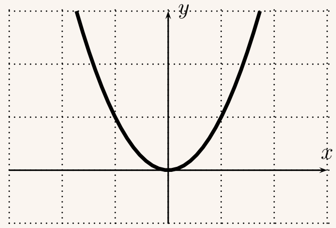
$$x = (y-2)^2 + 3 \implies x - 3 = (y-2)^2 \implies \sqrt{x-3} = y - 2$$
$$\implies y = 2 + \sqrt{x-3} \implies k^{-1}(x) = 2 + \sqrt{x-3}$$

□

The function in the last example is not one-to-one when allowing x to be any real number. This is why we had to restrict the example to the inputs $x \geq 2$. We exemplify the situation in the following note.

Note 6.7

Note that the function $y = x^2$ can be restricted to a one-to-one function by choosing the domain to be all non-negative numbers $[0, \infty)$, or by choosing the domain to be all non-positive numbers $(-\infty, 0]$.



Let $f : [0, \infty) \to [0, \infty)$ be the function $f(x) = x^2$, so that f has a domain of all non-negative numbers. Then, the inverse is the function $f^{-1}(x) = \sqrt{x}$.
On the other hand, we can take $g(x) = x^2$, whose domain consists of all non-positive numbers $(-\infty, 0]$, that is $g : (-\infty, 0] \to [0, \infty)$. Then, the inverse function g^{-1} must reverse domain and range, that is $g^{-1} : [0, \infty) \to (-\infty, 0]$. The inverse is obtained by exchanging x and y in $y = x^2$ as follows:

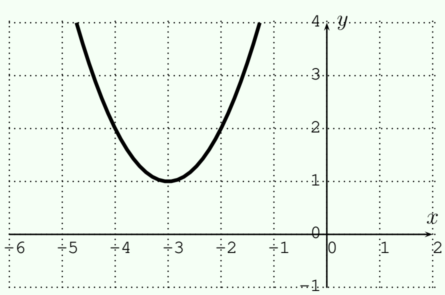
$$x = y^2 \quad \implies y = \pm\sqrt{x} \quad \implies g^{-1}(x) = -\sqrt{x}.$$

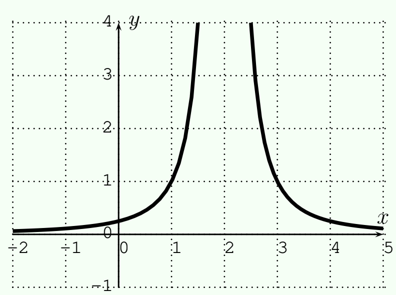
Example 6.8

Restrict the function to a one-to-one function. Find the inverse function, if possible.

a) $f(x) = (x+3)^2 + 1$ b) $g(x) = \dfrac{1}{(x-2)^2}$ c) $h(x) = x^3 - 3x^2$

Solution.
The graphs of f and g are displayed below.

a) 

b) 

a) The graph shows that f is one-to-one when restricted to all numbers $x \geq -3$, which is the choice we make to find an inverse function. Next, we replace x and y in $y = (x+3)^2 + 1$ to give $x = (y+3)^2 + 1$. When solving this for y, we must now remember that our choice of $x \geq -3$ becomes $y \geq -3$, after replacing x with y. We now solve for y.

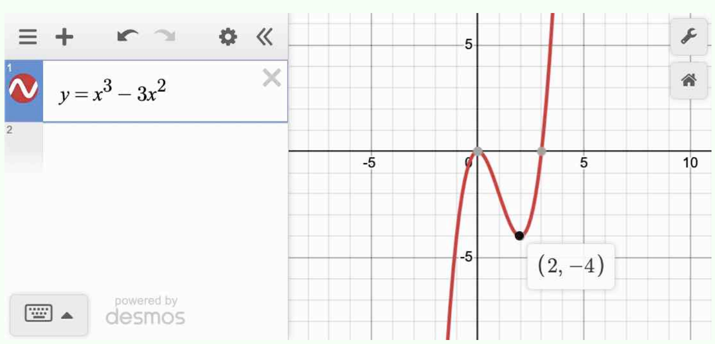
$$x = (y+3)^2 + 1 \implies x - 1 = (y+3)^2 \implies y + 3 = \pm\sqrt{x-1}$$
$$\implies y = -3 \pm \sqrt{x-1}$$

Since we have chosen the restriction of $y \geq -3$, we use the expression with the positive sign, $y = -3 + \sqrt{x-1}$, so that the inverse function is $f^{-1}(x) = -3 + \sqrt{x-1}$.

b) For the function g, the graph shows that we can restrict g to $x > 2$ to obtain a one-to-one function. The inverse for this choice is given as follows. Replacing x and y in $y = \frac{1}{(x-2)^2}$ gives $x = \frac{1}{(y-2)^2}$, which we solve for y under the condition $y > 2$.

$$x = \frac{1}{(y-2)^2} \implies (y-2)^2 = \frac{1}{x} \implies y - 2 = \pm\frac{1}{\sqrt{x}}$$
$$\implies y = 2 \pm \frac{1}{\sqrt{x}} \implies g^{-1}(x) = 2 + \frac{1}{\sqrt{x}}$$

c) Finally, $h(x) = x^3 - 3x^2$ can be graphed as follows:

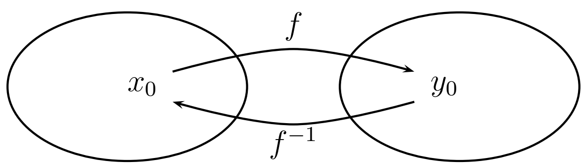


The above picture shows that the local minimum is at (approximately) $x = 2$. Therefore, if we restrict h to all $x \geq 2$, we obtain a one-to-one function. We replace x and y in $y = x^3 - 3x^2$, so that the inverse function is obtained by solving the equation $x = y^3 - 3y^2$ for y. However, this equation is quite complicated and solving it is beyond our capabilities at this time. Therefore, we simply say that $h^{-1}(x)$ is that $y \geq 2$ for which $y^3 - 3y^2 = x$, and leave the example with this.

□

Let f be a one-to-one function. If f maps x_0 to $y_0 = f(x_0)$, then f^{-1} maps y_0 to x_0. In other words, the inverse function is precisely the function for which

$$\boxed{f^{-1}(f(x_0)) = f^{-1}(y_0) = x_0} \quad \text{and} \quad \boxed{f(f^{-1}(y_0)) = f(x_0) = y_0}.$$



We therefore have the following observation.

Observation 6.9: Inverses compose to the identity

Let f and g be two one-to-one functions. Then f and g are inverses of each other exactly when

$$\boxed{f(g(x)) = x} \quad \text{and} \quad \boxed{g(f(x)) = x} \quad \text{for all } x. \tag{6.1}$$

In this case we write that $g = f^{-1}$ and $f = g^{-1}$.

Example 6.10

Are the following functions inverse to each other?

a) $f(x) = 5x + 7,\quad g(x) = \frac{x-7}{5}$
b) $f(x) = \frac{3}{x-6},\quad g(x) = \frac{3}{x} + 6$
c) $f(x) = \sqrt{x} - 3,\quad g(x) = x^2 + 3$

Solution.
We calculate the compositions $f(g(x))$ and $g(f(x))$.

a) $$f(g(x)) = f(\frac{x-7}{5}) = 5 \cdot \frac{x-7}{5} + 7 = (x-7) + 7 = x$$

$$g(f(x)) = g(5x+7) = \frac{(5x+7)-7}{5} = \frac{5x}{5} = x$$

b) $$f(g(x)) = f(\frac{3}{x}+6) = \frac{3}{(\frac{3}{x}+6)-6} = \frac{3}{\frac{3}{x}} = 3\cdot\frac{x}{3} = x$$

$$\begin{aligned} g(f(x)) &= g(\frac{3}{x-6}) = \frac{3}{\frac{3}{x-6}} + 6 = 3\cdot\frac{x-6}{3} + 6 \\ &= (x-6)+6 = x \end{aligned}$$

Using the Observation 6.9, we see that in both part (a) and (b) the functions are inverse to each other. For part (c), we calculate for a general x in the domain of g:

$$f(g(x)) = \sqrt{x^2+3} - 3 \neq x.$$

It is enough to show that for one composition $(f \circ g)(x)$ does not equal x to conclude that f and g are not inverses. (It is not necessary to also calculate the other composition $g(f(x))$.) □

Be careful!

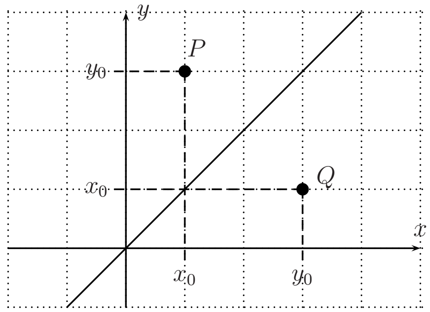
If f and g are functions such that the range of f is the domain of g, and the range of g is the domain of f, then one of the two equations in (6.1) also implies the validity of the other equation in (6.1). In other words, if, for example, we know that $f(g(x)) = x$ is true, then $g(f(x)) = x$ is also true. Nevertheless, we recommend to always check *both* equations. The reason for this is that it is easy to mistake one of the relations when we are not careful about the domain and range.

For example, let $f(x) = x^2$ and $g(x) = -\sqrt{x}$. Then, naively, we would calculate $f(g(x)) = (-\sqrt{x})^2 = x$ and $g(f(x)) = -\sqrt{x^2} = -|x|$, so that the first equation would say f and g are inverses, whereas the second equation may lead us to think they are not inverses.

We can resolve this apparent contradiction by being precise about the domain that we consider for f. Note that we can only find an inverse for f if we choose a domain that makes f into a one-to-one function. For example, if we take the domain of f to be all positive numbers and zero, $D_f = [0,\infty)$, then $f(g(x)) = f(-\sqrt{x})$ which is undefined, since f only takes non-negative inputs. Also, we have $g(f(x)) = -\sqrt{x^2} = -x$. Therefore, neither $f(g(x))$ equals x, nor $(g(f(x))$ equals x. The functions f and g *are not* inverse to each other!

On the other hand, if we restrict the function $f(x) = x^2$ to all negative numbers and zero, $D_f = (-\infty, 0]$, then $f(g(x)) = (-\sqrt{x})^2 = x$, since now f is defined for the negative input $-\sqrt{x}$. Also, for a negative number $x < 0$, we have $g(f(x)) = -\sqrt{x^2} = -|x| = x$. So, in this case, f and g *are* inverse to each other!

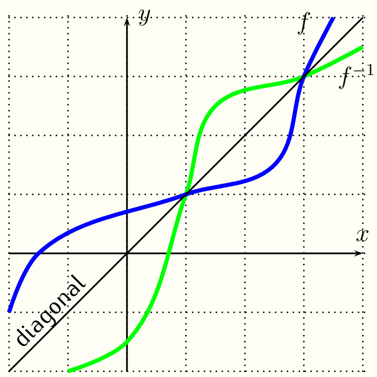
Our last observation in this chapter concerns the graph of inverse functions. If $f(x_0) = y_0$ then $f^{-1}(y_0) = x_0$, and the point $P(x_0, y_0)$ is on the graph of f, whereas the point $Q(y_0, x_0)$ is on the graph of f^{-1}.



We see that Q is the reflection of P along the diagonal $y = x$. Since this is true for any point on the graph of f and f^{-1}, we have the following general observation.

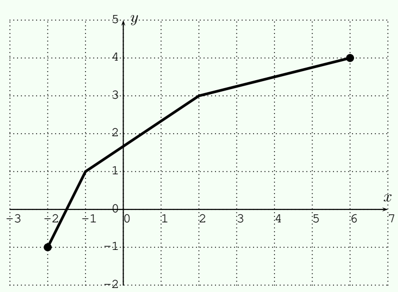
Observation 6.11: Graph of an inverse function

The graph of f^{-1} is the graph of f reflected about the diagonal.



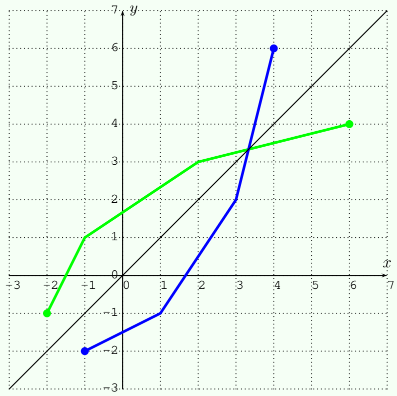
Example 6.12

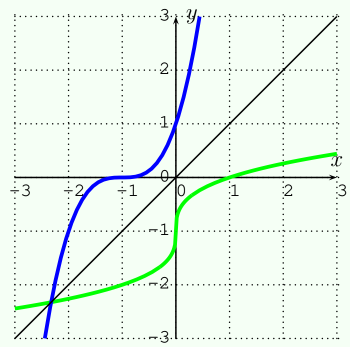
Find the graph of the inverse function of the function given below.

a) 

b) $f(x) = (x+1)^3$

Solution.
Carefully reflecting the graphs given in part (a) and (b) gives the following solution. The function $f(x) = (x+1)^3$ in part (b) can be graphed with a graphing calculator first.

a) 

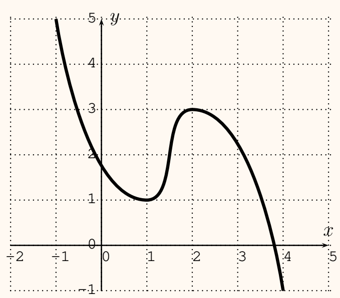
b) 

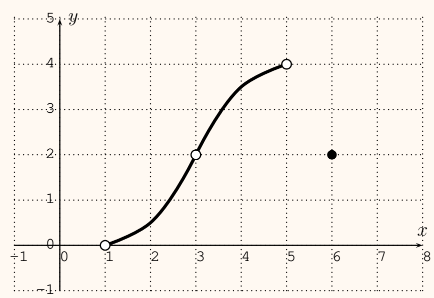
□

6.3 Exercises

Exercise 6.1

Use the horizontal line test to determine whether the function is one-to-one.

a) 

b) 

c) $f(x) = x^2 + 2x + 5$

d) $f(x) = x^2 - 14x + 29$

e) $f(x) = x^3 - 5x^2$

f) $f(x) = \frac{x^2}{x^2-3}$

g) $f(x) = \sqrt{x+2}$

h) $f(x) = \sqrt{|x+2|}$

Exercise 6.2

Find the inverse of the function f and check your solution.

a) $f(x) = 4x + 9$ b) $f(x) = -8x - 3$
c) $f(x) = \sqrt{x+8}$ d) $f(x) = \sqrt{3x+7}$
e) $f(x) = 6 \cdot \sqrt{-x-2}$ f) $f(x) = x^3$
g) $f(x) = (2x+5)^3$ h) $f(x) = 2 \cdot x^3 + 5$

i) $f(x) = \frac{1}{x}$ j) $f(x) = \frac{1}{x-1}$ k) $f(x) = \frac{1}{\sqrt{x-2}}$
l) $f(x) = \frac{-5}{4-x}$ m) $f(x) = \frac{x}{x+2}$ n) $f(x) = \frac{3x}{x-6}$
o) $f(x) = \frac{x+2}{x+3}$ p) $f(x) = \frac{7-x}{x-5}$ q) f given by the table below:

x	2	4	6	8	10	12
$f(x)$	3	7	1	8	5	2

Exercise 6.3

Restrict the domain of the function f in such a way that f becomes a one-to-one function. Find the inverse of f with the restricted domain.

a) $f(x) = x^2$ b) $f(x) = (x+5)^2 + 1$
c) $f(x) = |x|$ d) $f(x) = |x-4| - 2$
e) $f(x) = \frac{1}{x^2}$ f) $f(x) = \frac{-3}{(x+7)^2}$
g) $f(x) = x^4$ h) $f(x) = \frac{(x-3)^4}{10}$

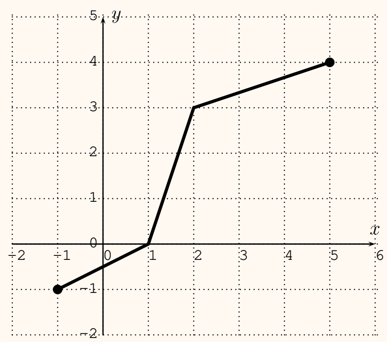
Exercise 6.4

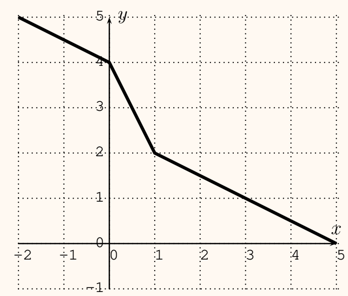
Determine whether the following functions f and g are inverse to each other.

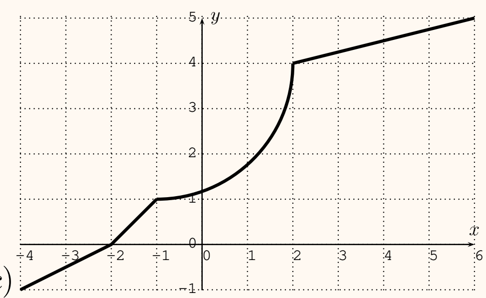
a) $f(x) = x + 3$ and $g(x) = x - 3$
b) $f(x) = -x - 4$ and $g(x) = 4 - x$
c) $f(x) = 2x + 3$ and $g(x) = x - \frac{3}{2}$
d) $f(x) = 6x - 1$ and $g(x) = \frac{x+1}{6}$
e) $f(x) = x^3 - 5$ and $g(x) = 5 + \sqrt[3]{x}$
f) $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{1}{x} + 2$

Exercise 6.5

Draw the graph of the inverse of the function given below.

a) 

b) 

c) 

d) $f(x) = \sqrt{x}$

e) $f(x) = x^3 - 4$

f) $f(x) = 2x - 4$

g) $f(x) = 2^x$

h) $f(x) = \frac{1}{x-2}$ for $x > 2$

i) $f(x) = \frac{1}{x-2}$ for $x < 2$.