Precalculus

Third Edition (3.0)

Thomas Tradler

Holly Carley

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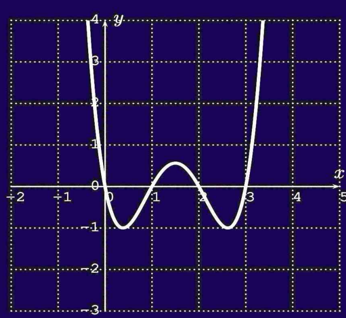
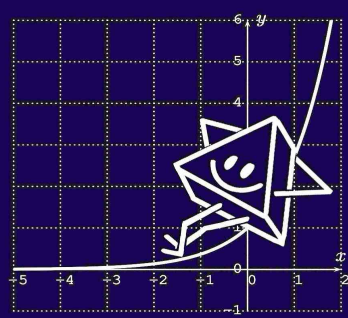
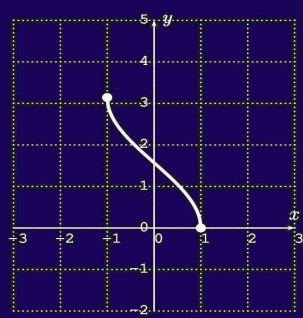

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Chapter 5

Operations on functions

We can combine two functions in many different ways, for example, by combining their output values (such as adding or multiplying them), or by composing the functions (that is, using the output of one as the input of the next function). In this chapter, we define and study these operations.

5.1 Operations on functions given by formulas

In the first example of this section, we show how to add, subtract, multiply, and divide functions that are given by a formula.

Example 5.1

Let $f(x) = x^2 + 5x$ and $g(x) = 7x - 3$. Find the following functions, and state their domain.

$$(f+g)(x), (f-g)(x), (f \cdot g)(x), \text{ and } \left(\frac{f}{g}\right)(x).$$

Solution.

The functions are calculated by adding them (or subtracting, multiplying, or dividing them).

$$\begin{aligned}
(f+g)(x) &= (x^2+5x)+(7x-3) = x^2+12x-3, \\
(f-g)(x) &= (x^2+5x)-(7x-3) \\
&= x^2+5x-7x+3 = x^2-2x+3,
\end{aligned}$$

$$\begin{aligned}(f\cdot g)(x) &= (x^2+5x)\cdot(7x-3)\\ &= 7x^3-3x^2+35x^2-15x = 7x^3+32x^2-15x,\\ \left(\frac{f}{g}\right)(x) &= \frac{x^2+5x}{7x-3}.\end{aligned}$$

The calculation of these functions was straightforward. To state their domain is also straightforward, except for the domain of the quotient $\frac{f}{g}$. Note that $f+g$, $f-g$, and $f\cdot g$ are all polynomials. According to the standard convention (Convention 2.7 on page 24), all these functions have the domain $\mathbb{R}$; that is, their domain is all real numbers.

Now, for the domain of $\frac{f}{g}$, we have to be a bit more careful, since the denominator of a fraction cannot be zero. The denominator of $\frac{f}{g}(x) = \frac{x^2+5x}{7x-3}$ is zero, exactly when

$$7x-3=0 \quad\Longrightarrow\quad 7x=3 \quad\Longrightarrow\quad x=\frac{3}{7}.$$

We have to exclude $\frac{3}{7}$ from the domain. The domain of the quotient $\frac{f}{g}$ is therefore $\mathbb{R}-\{\frac{3}{7}\}$. □

We can formally state the observation we made in the previous example.

Observation 5.2: Domain when adding, multiplying, dividing

Let f be a function with domain D_f, and let g be a function with domain D_g. A value x can be used as an input of $f+g$, $f-g$, and $f\cdot g$, exactly when x is an input of both f and g. Therefore, the domains of the combined functions are the intersection of the domains D_f and D_g:

$$\begin{aligned}D_{f+g} &= D_f\cap D_g = \{x \mid x\in D_f \text{ and } x\in D_g\},\\ D_{f-g} &= D_f\cap D_g,\\ D_{f\cdot g} &= D_f\cap D_g.\end{aligned}$$

For the quotient $\frac{f}{g}$, we also have to make sure that the denominator $g(x)$ is not zero.

$$D_{\frac{f}{g}} = \{x \mid x\in D_f, x\in D_g, \text{ and } g(x)\neq 0\}.$$

Example 5.3

Let $f(x) = \sqrt{x+2}$, and let $g(x) = x^2 - 5x + 4$. Find the functions $\frac{f}{g}$ and $\frac{g}{f}$ and state their domains.

Solution.

First, the domain of f consists of those numbers x for which the square root is defined. In other words, we need $x + 2 \geq 0$, which means that $x \geq -2$, and so the domain of f is $D_f = [-2, \infty)$. On the other hand, the domain of g is all real numbers, $D_g = \mathbb{R}$. Now, we have the quotients

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+2}}{x^2 - 5x + 4} \qquad \text{and} \qquad \left(\frac{g}{f}\right)(x) = \frac{x^2 - 5x + 4}{\sqrt{x+2}}.$$

For the domain of $\frac{f}{g}$, we need to exclude those numbers x for which $x^2 - 5x + 4 = 0$. Thus, factoring $x^2 - 5x + 4 = 0$ gives

$$(x-1)(x-4) = 0 \quad \Longrightarrow \quad x = 1 \text{ or } x = 4$$

We obtain the domain for $\frac{f}{g}$ as the combined domain for f and g, and exclude 1 and 4. Therefore, $D_{\frac{f}{g}} = [-2, \infty) - \{1, 4\}$.

Now, for $\frac{g}{f}(x) = \frac{x^2-5x+4}{\sqrt{x+2}}$, the denominator becomes zero exactly when

$$x + 2 = 0 \quad \Longrightarrow \quad x = -2$$

Therefore, we need to exclude -2 from the domain, that is

$$D_{\frac{g}{f}} = [-2, \infty) - \{-2\} = (-2, \infty).$$

□

Note 5.4

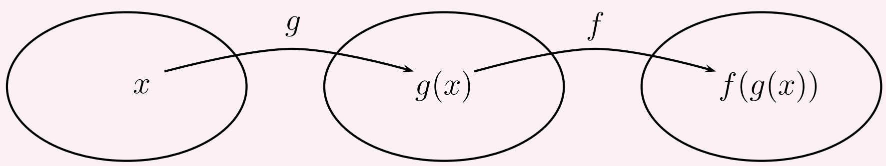
To form the quotient $\frac{f}{g}(x)$, where $f(x) = x^2 - 1$ and $g(x) = x + 1$, we write $\frac{f}{g}(x) = \frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x - 1$. One might be tempted to say that the domain is all real numbers. But it is not! The domain is all real numbers except -1, and the last step of the simplification performed above is only valid for $x \neq -1$.

Another operation we can perform is the composition of two functions.

Definition 5.5: Composition of functions

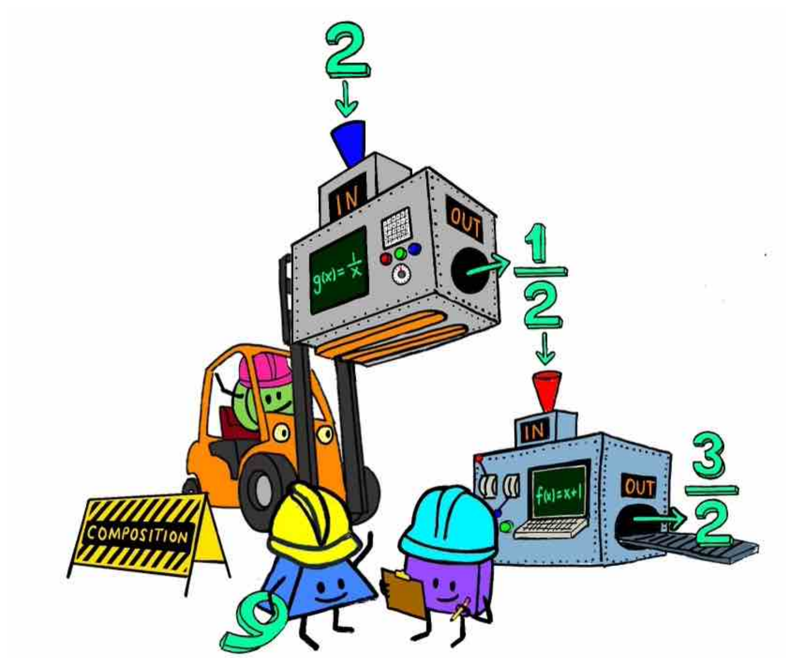
Let f and g be functions, and assume that $g(x)$ is in the domain of f. Then define the **composition of** f **and** g at x to be

$$(f \circ g)(x) := f(g(x)).$$



We can take any x as an input of $f \circ g$ which is an input of g and for which $g(x)$ is an input of f. Therefore, if D_f is the domain of f and D_g is the domain of g, the domain of $D_{f \circ g}$ is

$$D_{f \circ g} = \{\ x \mid x \in D_g,\ g(x) \in D_f\ \}.$$



Example 5.6

Let $f(x) = 2x^2 + 5x$ and $g(x) = 2 - x$. Find the following compositions:

a) $f(g(3))$ b) $g(f(3))$ c) $f(f(1))$ d) $f(2 \cdot g(5))$ e) $g(g(4) + 5)$

Solution.

We evaluate the expressions as follows:

$$\begin{array}{lrcl}
\text{a)} & f(g(3)) & = & f(2-3) = f(-1) = 2 \cdot (-1)^2 + 5 \cdot (-1) \\
 & & = & 2 - 5 = -3, \\
\text{b)} & g(f(3)) & = & g(2 \cdot 3^2 + 5 \cdot 3) = g(18 + 15) = g(33) \\
 & & = & 2 - 33 = -31, \\
\text{c)} & f(f(1)) & = & f(2 \cdot 1^2 + 5 \cdot 1) = f(2 + 5) = f(7) \\
 & & = & 2 \cdot 7^2 + 5 \cdot 7 = 98 + 35 = 133, \\
\text{d)} & f(2 \cdot g(5)) & = & f(2 \cdot (2 - 5)) = f(2 \cdot (-3)) = f(-6) \\
 & & = & 2 \cdot (-6)^2 + 5 \cdot (-6) = 72 - 30 = 42, \\
\text{e)} & g(g(4) + 5) & = & g((2 - 4) + 5) = g((-2) + 5) \\
 & & = & g(3) = 2 - 3 = -1.
\end{array}$$

□

We can also calculate composite functions for arbitrary x in the domain.

Example 5.7

Let $f(x) = x^2 + 1$ and $g(x) = x + 3$. Find the following compositions:

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$

Solution.

a) There are essentially two ways to evaluate $(f \circ g)(x) = f(g(x))$. We can either first use the explicit formula for $f(x)$ and then the one for $g(x)$, or vice versa. We will evaluate $f(g(x))$ by substituting $g(x)$ into the formula for $f(x)$:

$$\begin{array}{rcl}
(f \circ g)(x) & = & f(g(x)) = (g(x))^2 + 1 = (x+3)^2 + 1 \\
 & = & x^2 + 6x + 9 + 1 = x^2 + 6x + 10.
\end{array}$$

Similarly, we evaluate the other expressions (b)-(d):

b) $(g \circ f)(x) = g(f(x)) = f(x) + 3 = x^2 + 1 + 3 = x^2 + 4$

c) $(f \circ f)(x) = f(f(x)) = (f(x))^2 + 1 = (x^2+1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2$

d) $(g \circ g)(x) = g(g(x)) = g(x) + 3 = x + 3 + 3 = x + 6$

□

Example 5.8

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the following functions, and state their domains.

a) $f(x) = \frac{3}{x+2}$ and $g(x) = x^2 - 3x$
b) $f(x) = |3x - 2| - 6x + 4$ and $g(x) = 5x + 1$
c) $f(x) = \sqrt{\frac{1}{2} \cdot (x - 4)}$ and $g(x) = 2x^2 + 4$

Solution.

a) Composing $f \circ g$, we obtain

$$(f \circ g)(x) = f(g(x)) = \frac{3}{g(x) + 2} = \frac{3}{x^2 - 3x + 2}.$$

The domain is the set of numbers x for which the denominator is non-zero.

$$\begin{aligned} x^2 - 3x + 2 = 0 &\implies (x-2)(x-1) = 0 \\ &\implies x = 2 \text{ or } x = 1 \\ &\implies D_{f \circ g} = \mathbb{R} - \{1, 2\}. \end{aligned}$$

Similarly,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = f(x)^2 - 3f(x) = \left(\frac{3}{x+2}\right)^2 - 3\frac{3}{x+2} \\ &= \frac{9}{(x+2)^2} - \frac{9}{x+2} = \frac{9 - 9(x+2)}{(x+2)^2} \\ &= \frac{9 - 9x - 18}{(x+2)^2} = \frac{-9x - 9}{(x+2)^2} = \frac{-9 \cdot (x+1)}{(x+2)^2} \end{aligned}$$

The domain is all real numbers except $x = -2$, that is $D_{g\circ f} = \mathbb{R} - \{-2\}$.

b) We calculate the compositions as follows:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = |3g(x) - 2| - 6g(x) + 4 \\
&= |3(5x+1) - 2| - 6(5x+1) + 4 \\
&= |15x+1| - 30x - 2 \\
(g \circ f)(x) &= g(f(x)) = 5f(x) + 1 = 5 \cdot (|3x-2| - 6x + 4) + 1 \\
&= 5 \cdot |3x-2| - 30x + 20 + 1 = 5 \cdot |3x-2| - 30x + 21
\end{aligned}$$

Since the domains of f and g are all real numbers, so are also the domains for both $f \circ g$ and $g \circ f$.

c) Again we calculate the compositions.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = \sqrt{\frac{1}{2} \cdot (g(x) - 4)} = \sqrt{\frac{1}{2} \cdot (2x^2 + 4 - 4)} \\
&= \sqrt{\frac{1}{2} \cdot 2x^2} = \sqrt{x^2} = |x|.
\end{aligned}$$

The domain of g is all real numbers, and the outputs $g(x) = 2x^2 + 4$ are all ≥ 4, (since $2x^2 \geq 0$). Therefore, $g(x)$ is in the domain of f, and we have a combined domain of $f \circ g$ of $D_{f\circ g} = \mathbb{R}$. On the other hand,

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) = 2(f(x))^2 + 4 = 2 \cdot \left(\sqrt{\frac{1}{2} \cdot (x-4)}\right)^2 + 4 \\
&= 2 \cdot \left(\frac{1}{2} \cdot (x-4)\right) + 4 = (x-4) + 4 = x.
\end{aligned}$$

The domain of $g \circ f$ consists of all numbers x which are in the domain of f and for which $f(x)$ is in the domain of g. Now, the domain of f consists of all real numbers x that give a non-negative argument in the square-root, that is: $\frac{1}{2}(x-4) \geq 0$. Therefore, we must have $x - 4 \geq 0$, so that $x \geq 4$, and we obtain the domain $D_f = [4, \infty)$. Since the domain $D_g = \mathbb{R}$, the composition $g \circ f$ has the same domain as f:

$$D_{g\circ f} = D_f = [4, \infty).$$

We remark that at a first glance, we might have expected that $(g \circ f) = x$ has a domain of all real numbers. However, the composition $g(f(x))$ can only have those inputs that are also allowed inputs of f. We see that the domain of a composition is sometimes smaller than the domain that we use via the standard convention (Convention 2.7).

□

5.2 Operations on functions given by tables

We now show how to combine two functions that are given by tables.

Example 5.9

Let f and g be the functions defined by the following table.

x	1	2	3	4	5	6	7
$f(x)$	6	3	1	4	0	7	6
$g(x)$	4	0	2	5	-2	3	1

Describe the following functions via a table:

a) $2 \cdot f(x) + 3$ b) $f(x) - g(x)$ c) $f(x+2)$ d) $g(-x)$

Solution.
For (a) and (b) we obtain by immediate calculation:

x	1	2	3	4	5	6	7
$2 \cdot f(x) + 3$	15	9	5	11	3	17	15
$f(x) - g(x)$	2	3	-1	-1	2	4	5

For example, for $x = 3$, we obtain $2 \cdot f(x) + 3 = 2 \cdot f(3) + 3 = 2 \cdot 1 + 3 = 5$ and $f(x) - g(x) = f(3) - g(3) = 1 - 2 = -1$.
For part (c), we have a similar calculation of $f(x+2)$. For example, for $x = 1$, we get $f(1+2) = f(1+2) = f(3) = 1$.

x	1	2	3	4	5	6	7	-1	0
$f(x+2)$	1	4	0	7	6	undef.	undef.	6	3

Note that for the last two inputs $x = 6$ and $x = 7$ the expression $f(x+2)$ is undefined, since, for example for $x = 6$, we have $f(x+2) = f(6+2) = f(8)$ which is undefined. However, for $x = -1$, we obtain $f(x+2) = f(-1+2) = f(1) = 6$. If we define $h(x) = f(x+2)$, then the domain of h is therefore $D_h = \{-1, 0, 1, 2, 3, 4, 5\}$.
Finally, for part (d), we need to take x as inputs, for which $g(-x)$ is defined via the table for g. We obtain the following answer:

x	-1	-2	-3	-4	-5	-6	-7
$g(-x)$	4	0	2	5	-2	3	1

□

Example 5.10

Let f and g be the functions defined by the following table.

x	1	3	5	7	9	11
$f(x)$	3	5	11	4	9	7
$g(x)$	7	-6	9	11	9	5

Describe the following functions via a table:

a) $f \circ g$ b) $g \circ f$ c) $f \circ f$ d) $g \circ g$

Solution.

The compositions are calculated by repeated evaluation. For example,

$$(f \circ g)(1) = f(g(1)) = f(7) = 4.$$

The complete answer is displayed below.

x	1	3	5	7	9	11
$(f \circ g)(x)$	4	undef.	9	7	9	11
$(g \circ f)(x)$	-6	9	5	undef.	9	11
$(f \circ f)(x)$	5	11	7	undef.	9	4
$(g \circ g)(x)$	11	undef.	9	5	9	9

□

5.3 Exercises

Exercise 5.1

Find $f+g$, $f-g$, $f \cdot g$ for the functions below. State their domain.

a) $f(x) = x^2 + 6x$ and $g(x) = 3x - 5$
b) $f(x) = x^3 + 5$ and $g(x) = 5x^2 + 7$
c) $f(x) = 3x + 7\sqrt{x}$ and $g(x) = 2x^2 + 5\sqrt{x}$
d) $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{5x}{x+2}$
e) $f(x) = \sqrt{x-3}$ and $g(x) = 2\sqrt{x-3}$
f) $f(x) = x^2 + 2x + 5$ and $g(x) = 3x - 6$
g) $f(x) = x^2 + 3x$ and $g(x) = 2x^2 + 3x + 4$

Exercise 5.2

Find $\frac{f}{g}$, and $\frac{g}{f}$ for the functions below. State their domain.

a) $f(x) = 3x + 6$ and $g(x) = 2x - 8$
b) $f(x) = x + 2$ and $g(x) = x^2 - 5x + 4$
c) $f(x) = \frac{1}{x-5}$ and $g(x) = \frac{x-2}{x+3}$
d) $f(x) = \sqrt{x+6}$ and $g(x) = 2x + 5$
e) $f(x) = x^2 + 8x - 33$ and $g(x) = \sqrt{x}$

Exercise 5.3

Let $f(x) = 2x-3$ and $g(x) = 3x^2+4x$. Find the following compositions:

a) $f(g(2))$ b) $g(f(2))$ c) $f(f(5))$
d) $f(5g(-3))$ e) $g(f(2)-2)$ f) $f(f(3)+g(3))$
g) $g(f(2+x))$ h) $f(f(-x))$ i) $f(f(-3)-3g(2))$
j) $f(f(f(2)))$ k) $f(x+h)$ l) $g(x+h)$

Exercise 5.4

Find the composition $(f \circ g)(x)$ for the following functions:

a) $f(x) = 3x - 5$ and $g(x) = 2x + 3$
b) $f(x) = x^2 + 2$ and $g(x) = x + 3$
c) $f(x) = x^2 - 3x + 2$ and $g(x) = 2x + 1$
d) $f(x) = x^2 + \sqrt{x+3}$ and $g(x) = x^2 + 2x$
e) $f(x) = \frac{2}{x+4}$ and $g(x) = x + h$
f) $f(x) = x^2 + 4x + 3$ and $g(x) = x + h$

Exercise 5.5

Find the compositions

$$(f \circ g)(x), \quad (g \circ f)(x), \quad (f \circ f)(x), \quad (g \circ g)(x)$$

for the following functions:

a) $f(x) = 2x + 4$ and $g(x) = x - 5$
b) $f(x) = x + 3$ and $g(x) = x^2 - 2x$
c) $f(x) = 2x^2 - x - 6$ and $g(x) = \sqrt{3x+2}$
d) $f(x) = \frac{1}{x+3}$ and $g(x) = \frac{1}{x} - 3$
e) $f(x) = (2x - 7)^2$ and $g(x) = \frac{\sqrt{x}+7}{2}$

Exercise 5.6

Let f and g be the functions defined by the table below. Complete the table by performing the indicated operations.

x	1	2	3	4	5	6	7
$f(x)$	4	5	7	0	-2	6	4
$g(x)$	6	-8	5	2	9	11	2
$f(x) + 3$							
$4g(x) + 5$							
$g(x) - 2f(x)$							
$f(x + 3)$							

Exercise 5.7

Let f and g be the functions defined by the table below. Complete the table by composing the given functions.

x	1	2	3	4	5	6
$f(x)$	3	1	2	5	6	3
$g(x)$	5	2	6	1	2	4
$(g \circ f)(x)$						
$(f \circ g)(x)$						
$(f \circ f)(x)$						
$(g \circ g)(x)$						

Exercise 5.8

Let f and g be the functions defined by the table below. Complete the table by composing the given functions.

x	0	2	4	6	8	10	12
$f(x)$	4	8	5	6	12	-1	10
$g(x)$	10	2	0	-6	7	2	8
$(g \circ f)(x)$							
$(f \circ g)(x)$							
$(f \circ f)(x)$							
$(g \circ g)(x)$							