Precalculus

Third Edition (3.0)

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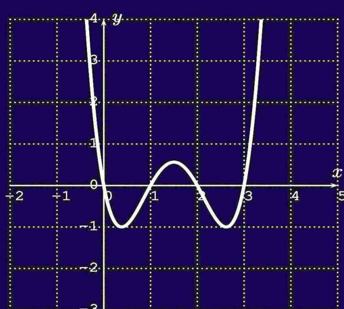

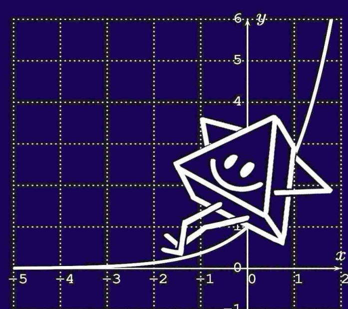

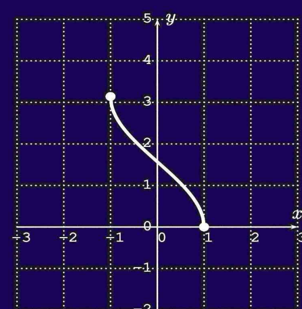

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Chapter 3

Functions via graphs

Another way to represent a function is via a graph. Before discussing graphs in general, we first review a familiar kind of graph, namely the graph of a linear function.

3.1 Review of graphs of linear functions

We recall the equation of a linear function.

Review 3.1: Linear function

A **linear function** is a function of the form

$$f(x) = a \cdot x + b$$

for some real numbers a and b where $a \neq 0$. By the standard convention of the domain, the domain of f consists of all real numbers, $D = \mathbb{R}$.

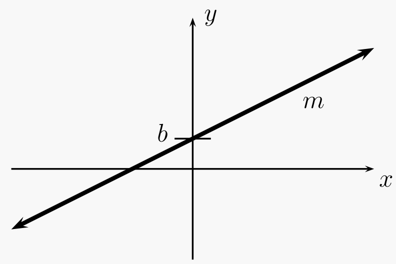
Recall that linear functions can be graphed in the x-y plane as a straight line. In this case, the coefficient a is also denoted by m and has the interpretation of the slope of the line. We review this now.

Review 3.2: Slope-intercept form of the line

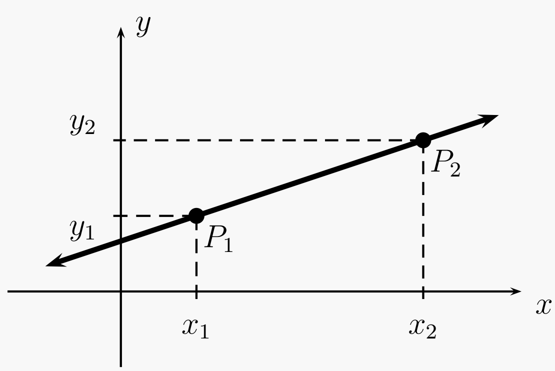
The **slope-intercept form of the line** is the equation

$$\boxed{y = m \cdot x + b} \tag{3.1}$$

Here, m is the **slope** and $(0, b)$ is the y**-intercept** of the line.



Generally, the slope describes how fast the line grows toward the right. For any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line L, the slope m is given by the following formula (which is $m = \frac{\text{rise}}{\text{run}}$):

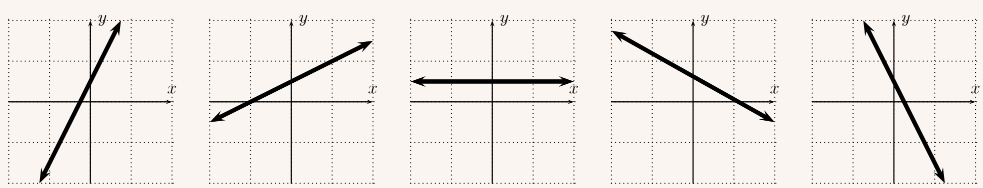


Slope:

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}} \tag{3.2}$$

Note 3.3: Sign of the slope

When the slope m is positive, the line rises toward the right. When the slope m is negative, the line declines toward the right.



$m = 2$ $\quad m = \frac{1}{2}$ $\quad m = 0$ $\quad m = -\frac{1}{2}$ $\quad m = -2$

Below is an example of the graph of a line in slope-intercept form.

Example 3.4

Graph the line $y = 2x + 3$.

Solution.

We calculate the output values y for various input values x. For example, when x is $-2, -1, 0, 1, 2$, or 3, we compute

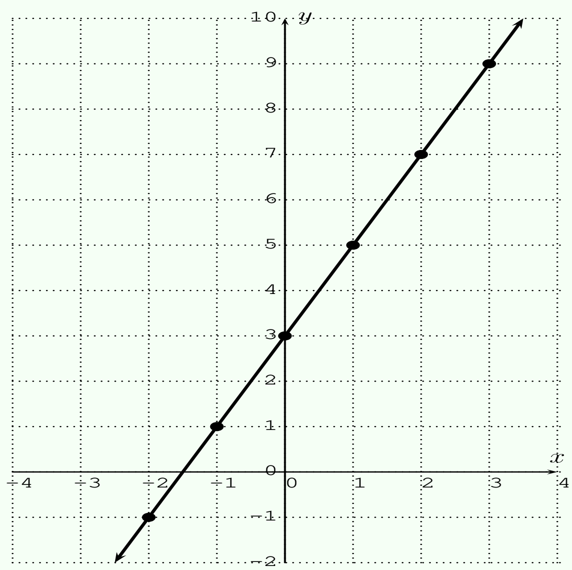
x	-2	-1	0	1	2	3
y	-1	1	3	5	7	9

In the above table each y value is calculated by substituting the corresponding x value into our equation $y = 2x + 3$:

$$\begin{aligned}
x &= -2 &\implies&\quad y = 2\cdot(-2)+3 = -4+3 = -1\\
x &= -1 &\implies&\quad y = 2\cdot(-1)+3 = -2+3 = 1\\
x &= 0 &\implies&\quad y = 2\cdot(0)+3 = 0+3 = 3\\
x &= 1 &\implies&\quad y = 2\cdot(1)+3 = 2+3 = 5\\
x &= 2 &\implies&\quad y = 2\cdot(2)+3 = 4+3 = 7\\
x &= 3 &\implies&\quad y = 2\cdot(3)+3 = 6+3 = 9
\end{aligned}$$

In the above calculation, the values for x were arbitrarily chosen. Since a line is completely determined by knowing two points on it, any two values for x would have worked for the purpose of graphing the line.

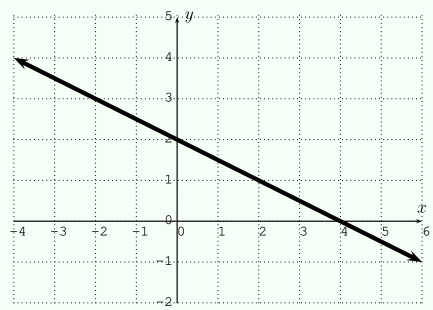
Drawing the above points in the coordinate plane and connecting them gives the graph of the line $y = 2x + 3$:



Alternatively, note that the y-intercept is $(0, 3)$ (3 is the additive constant in our initial equation $y = 2x+3$) and the slope $m = 2$ determines the rate at which the line grows: for each step to the right, we have to move two steps up. □

Example 3.5

Find the equation of the line in slope-intercept form.



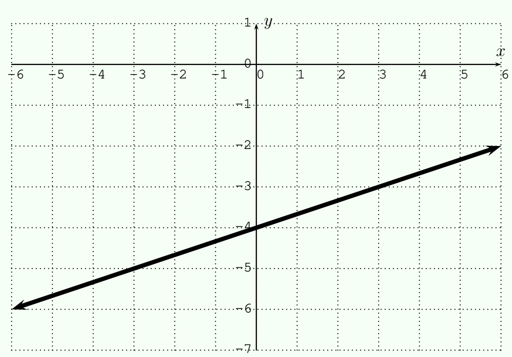
Solution. The y-intercept can be read off the graph giving us that $b = 2$. As for the slope, we use formula (3.2) and the two points on the line $P_1(0, 2)$ and $P_2(4, 0)$. We obtain

$$m = \frac{0-2}{4-0} = \frac{-2}{4} = -\frac{1}{2}.$$

Thus, the line has the slope-intercept form $y = -\frac{1}{2}x + 2$. □

Example 3.6

Find the equation of the line in slope-intercept form.



Solution.
The y-intercept is $b = -4$. To obtain the slope we can again use the y-intercept $P_1(0, -4)$. To use (3.2), we need another point P_2 on the line. We may pick any second point on the line, for example, $P_2(3, -3)$. With this, we obtain

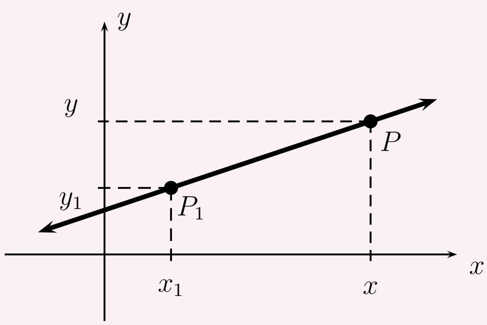
$$m = \frac{(-3) - (-4)}{3 - 0} = \frac{-3 + 4}{3} = \frac{1}{3}.$$

Thus, the line has the slope-intercept form $y = \frac{1}{3}x - 4$. □

There is another important way in which we can write the equation of a line.

Definition 3.7: Point-slope form of the line

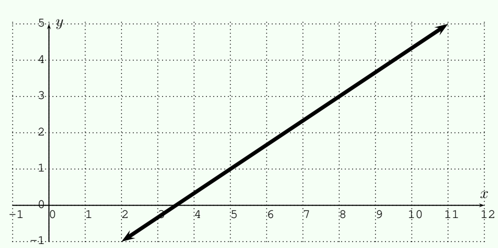
From Equation (3.2), we see that for a given slope m and a point $P_1(x_1, y_1)$ on the line, any other point (x, y) on the line satisfies $m = \frac{y-y_1}{x-x_1}$. Multiplying $(x - x_1)$ on both sides gives what is called the **point-slope form of the line**:



$$\boxed{y - y_1 = m \cdot (x - x_1)} \qquad (3.3)$$

Example 3.8

Find the equation of the line in point-slope form (3.3).



Solution.
We need to identify one point (x_1, y_1) on the line together with the slope m of the line so that we can write the line in point-slope form: $y - y_1 = m(x - x_1)$. By direct inspection, we identify the two points $P_1(5, 1)$ and $P_2(8, 3)$ on the line, and with this we calculate the slope as:

$$m = \frac{3-1}{8-5} = \frac{2}{3}$$

Using the point $(5, 1)$ we write the line in point-slope form as follows:

$$y - 1 = \frac{2}{3}(x - 5)$$

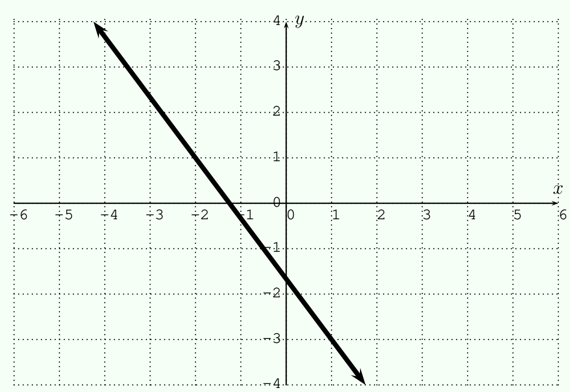
Note that our answer depends on the chosen point $(5, 1)$ on the line. Indeed, if we choose a different point on the line, such as $(8, 3)$, we obtain a different equation, (which nevertheless represents the same line):

$$y - 3 = \frac{2}{3}(x - 8)$$

Note that we do *not* need to solve this for y, since we are looking for an answer in point-slope form. □

Example 3.9

Find the equation of the line in point-slope form (3.3).



Solution.
We identify two points on the line, $P_1(1, -3)$ and $P_2(-2, 1)$. Therefore

the slope is $m = \frac{1-(-3)}{(-2)-1} = \frac{4}{-3} = -\frac{4}{3}$. Using, for example, the point $(1, -3)$, we write the line in point-slope form as follows:

$$y - (-3) = -\frac{4}{3}(x - 1)$$

Alternatively, we can also write this as $y + 3 = -\frac{4}{3}(x - 1)$. □

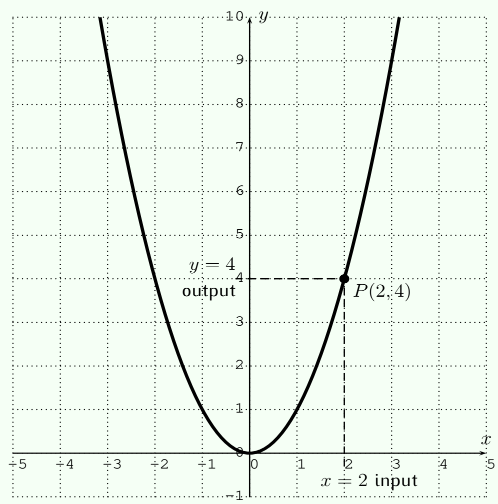
3.2 Functions given by graphs

Next, we study graphs more generally. Recall from the above examples that the graph of a function f is the set of all points (in the coordinate plane) of the form $(x, f(x))$, where x is in the domain of f. Here is another example that shows how we may obtain the graph of a function by computing sample points and plotting and connecting them in the plane.

Example 3.10

Let $y = x^2$ with domain $D = \mathbb{R}$ being the set of all real numbers. We can graph this after calculating a table as follows:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

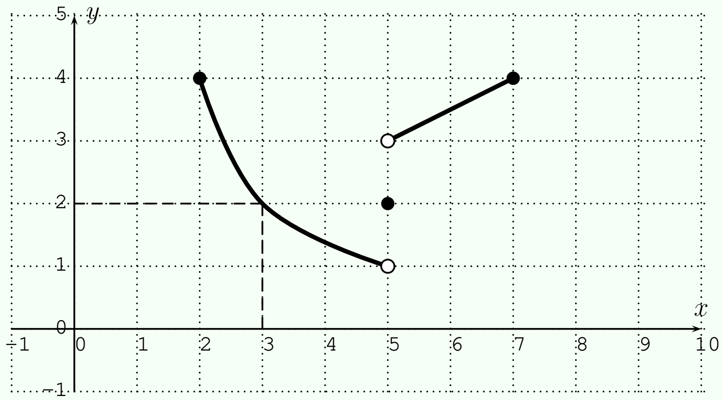


Many function values can be read from this graph. For example, for the input $x = 2$, we obtain the output $y = 4$. This corresponds to the point $P(2, 4)$ on the graph as depicted above.

In general, an input (placed on the x-axis) gets assigned to an output (placed on the y-axis) according to where the vertical line at x intersects with the given graph. This is used in the next example.

Example 3.11

Let f be the function given by the following graph.



Here, the dashed lines show that the input $x = 3$ gives an output of $y = 2$. Similarly, we can obtain other output values from the graph:

$$f(2) = 4, \quad f(3) = 2, \quad f(5) = 2, \quad f(7) = 4.$$

Note that, in the above graph, a closed point means that the point is part of the graph, whereas an open point means that it is not part of the graph.

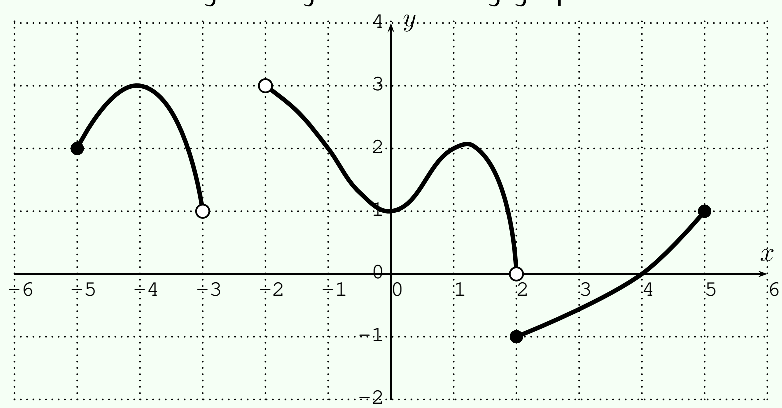
The domain is the set of all possible input values on the x-axis. Since we can take any number $2 \leq x \leq 7$ as an input, the domain is the interval $D = [2, 7]$. The range is the set of all possible output values on the y-axis. Since any number $1 < y \leq 4$ is obtained as an output, the range is $R = (1, 4]$. Note in particular that $y = 1$ is *not* an output, since $f(5) = 2$.

Note 3.12

In the above example we evaluated a function that was given by a graph. Looking at a drawn graph is by its nature an imprecise representation of the function. Indeed, it might be possible that, for example, there are hidden features of the graph that only become apparent after sufficiently zooming in on the graph. So, when studying the above example, we implicitly assumed that there are no hidden features that are not shown in the graph. An accurate evaluation would require more information regarding the function, such as, for example, a precise formula of the function.

Example 3.13

Let f be the function given by the following graph.



Here are some function values that can be read from the graph:

$$f(-5) = 2, \quad f(-4) = 3, \quad f(-3) \text{ and } f(-2) \text{ are undefined,}$$
$$f(-1) = 2, \quad f(0) = 1, \quad f(1) = 2, \quad f(2) = -1, \quad f(4) = 0, \quad f(5) = 1.$$

Note that the output value $f(3)$ is somewhere between -1 and 0, but we can only read off an approximation of $f(3)$ from the graph.
To find the domain of the function, we need to determine all possible x-coordinates (or in other words, we need to project the graph onto the x-axis). The possible x-coordinates are from the interval $[-5, -3)$ together with the intervals $(-2, 2)$ and $[2, 5]$. The last two intervals may be combined. We get the domain:

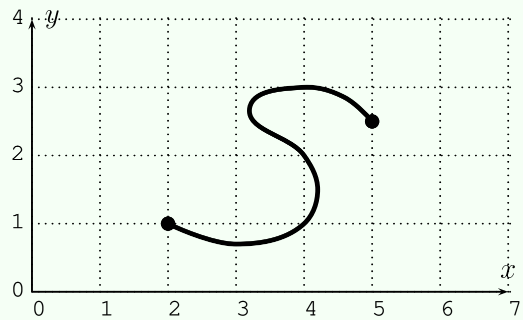
$$D = [-5, -3) \cup (-2, 5].$$

For the range, we look at all possible y-values. These are given by the intervals $(1, 3]$ and $(0, 3)$ and $[-1, 1]$. Combining these three intervals, we obtain the range

$$R = [-1, 3].$$

Example 3.14

Consider the following graph.

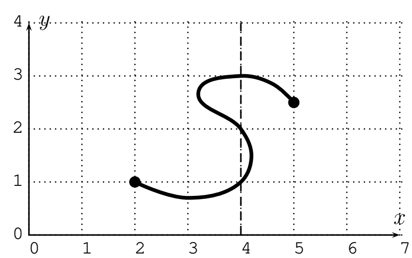


Consider the input $x = 4$. There are several outputs that we get for $x = 4$ from this graph:

$$f(4) = 1, \quad f(4) = 2, \quad f(4) = 3.$$

However, in a function, obtaining more than one output from one input is not allowed! Therefore, this graph is *not* the graph of a function!

The reason why the previous example is not a function is due to some input having more than one output: $f(4) = 1, f(4) = 2, f(4) = 3$.



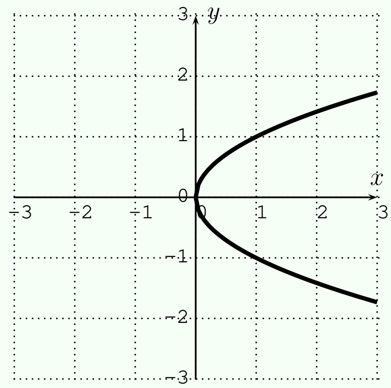
In other words, there is a vertical line ($x = 4$) which intersects the graph in more than one point. This observation is generalized in the following vertical line test.

Observation 3.15: Vertical Line Test

A graph is the graph of a function precisely when every vertical line intersects the graph at most once.

Example 3.16

Consider the graph of the equation $x = y^2$:



This does not pass the vertical line test, so y is not a function of x. However, x is a function of y since, if you consider y to be the input, each input has exactly one output (it passes the 'horizontal line' test).

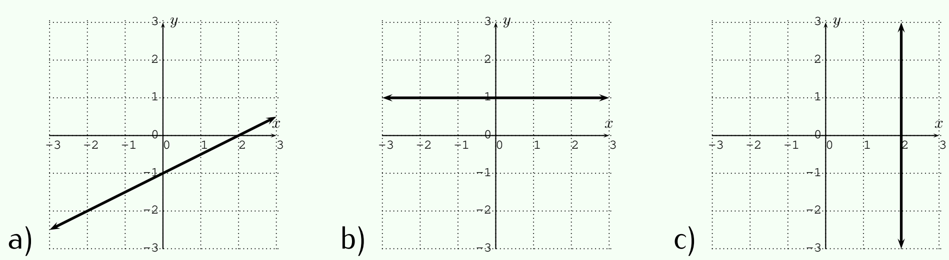
Example 3.17

Which of the following equations constitute functions of the form $y = f(x)$?

$$\text{a) } y = \frac{1}{2}x - 1 \qquad \text{b) } y = 1 \qquad \text{c) } x = 2$$

Solution.

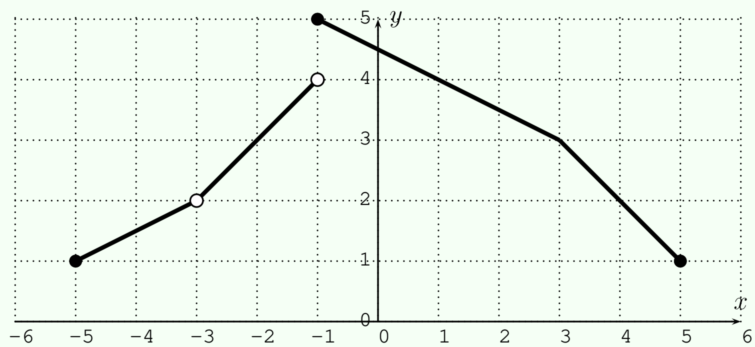
We graph each of the three equations.



From the vertical line test, we see that (a) and (b) are graphs of functions, while (c) is a vertical line, which is not the graph of a function. □

Example 3.18

Let f be the function given by the following graph.



a) What is the domain of f?
b) What is the range of f?
c) For which x is $f(x) = 3$?
d) For which x is $f(x) = 2$?
e) For which x is $f(x) > 2$?
f) For which x is $f(x) \leq 4$?
g) Find $f(1)$ and $f(4)$.
h) Find $f(1) + f(4)$.
i) Find $f(1) + 4$.
j) Find $f(1 + 4)$.

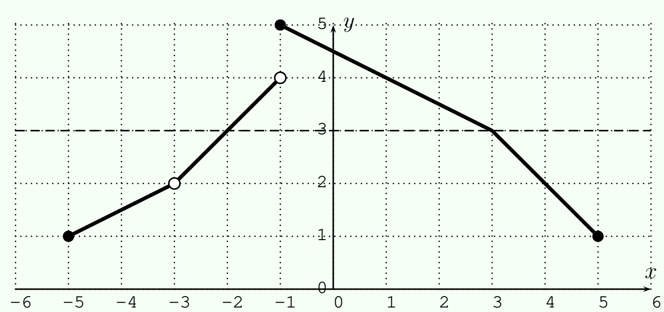
Solution.

Most of the answers can be read immediately from the graph.

a) For the domain, we project the graph to the x-axis. The domain consists of all numbers from -5 to 5 without -3, that is $D = [-5, -3) \cup (-3, 5]$.

b) For the range, we project the graph to the y-axis. The domain consists of all numbers from 1 to 5, that is $R = [1, 5]$.

c) To find x with $f(x) = 3$ we look at the horizontal line at $y = 3$:



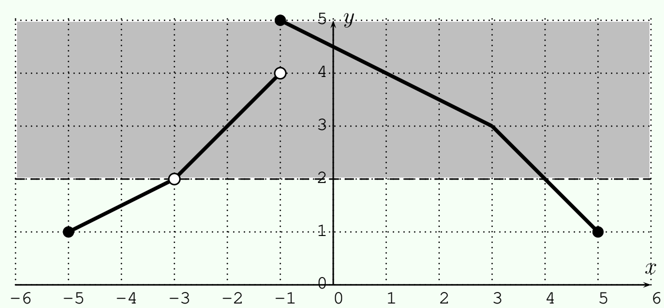
We see that there are two numbers x with $f(x) = 3$:

$$f(-2) = 3, \qquad f(3) = 3.$$

Therefore, the answer is $x = -2$ or $x = 3$.

d) Looking at the horizontal line $y = 2$, we see that there is only one x with $f(x) = 2$; namely $f(4) = 2$. Note that $x = -3$ does not solve the problem, since $f(-3)$ is undefined. The answer is $x = 4$.

e) To find x with $f(x) > 2$, the graph has to lie above the line $y = 2$.



We see that the answer is those numbers x greater than -3 and less than 4. The answer is therefore the interval $(-3, 4)$.

f) For $f(x) \leq 4$, we obtain all numbers x from the domain that are less than -1 or greater than or equal to 1. The answer is therefore,

$$[-5, -3) \cup (-3, -1) \cup [1, 5].$$

Note that -3 is not part of the answer, since $f(-3)$ is undefined.

g) $f(1) = 4$, and $f(4) = 2$

h) $f(1) + f(4) = 4 + 2 = 6$

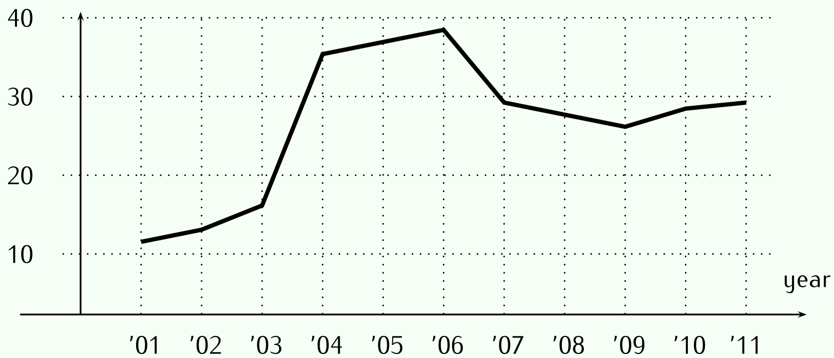
i) $f(1) + 4 = 4 + 4 = 8$

j) $f(1 + 4) = f(5) = 1$

□

Example 3.19

The following graph shows the population size in a small city from the years 2001–2011 in thousands of people.



a) What was the population size in the years 2004 and 2009?
b) In what years did the city have the most population?
c) In what year did the population grow the fastest?
d) In what year did the population decline the fastest?

Solution.

The population size in the year 2004 was approximately $36,000$. In the year 2009, it was approximately $26,000$. The largest population was in the year 2006, where the graph has its maximum. The fastest growth in the population was between the years 2003 and 2004. That is when the graph has the largest slope. Finally, the fastest decline happened from the years 2006–2007. □

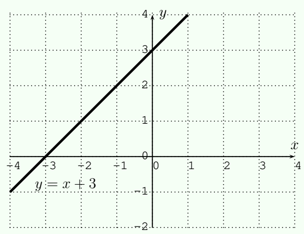
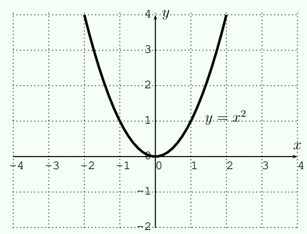
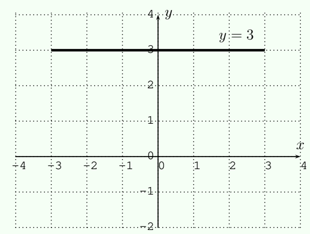
Example 3.20

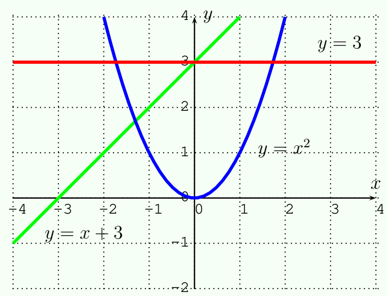
Graph the piecewise defined function described by the following formula:

$$f(x) = \begin{cases} x+3 & \text{, for } \quad -3 \leq x < -1 \\ x^2 & \text{, for } \quad -1 < x < 1 \\ 3 & \text{, for } \quad 2 < x \leq 3 \end{cases}$$

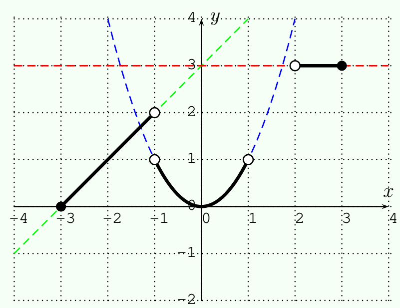
Solution.

We have to graph all three functions $y = x + 3$, $y = x^2$, and $y = 3$, and then restrict them to their respective domain. Graphing the three functions, we obtain the following tables and associated graphs, which we draw in one x-y plane:

$y = x + 3$	$y = x^2$	$y = 3$
		



However, we need to cut off the functions according to their specific input domain that is given by the original function.

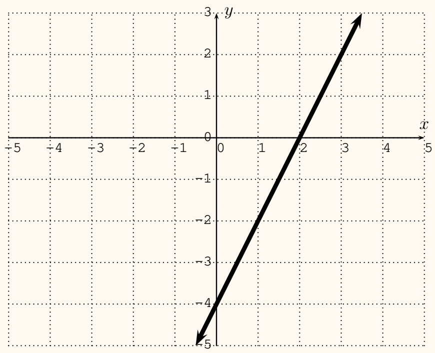


Note that the open and closed circles at the endpoints of each branch correspond to the "$<$" and "$\leq$" rules in the original description of the function. □

3.3 Exercises

Exercise 3.1

Find the slope and y-intercept of the line with the given data. Using the slope and y-intercept, write the equation of the line in slope-intercept form.

a)


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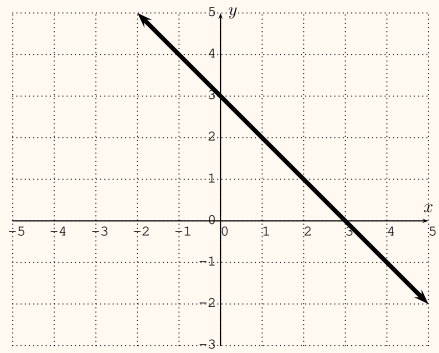
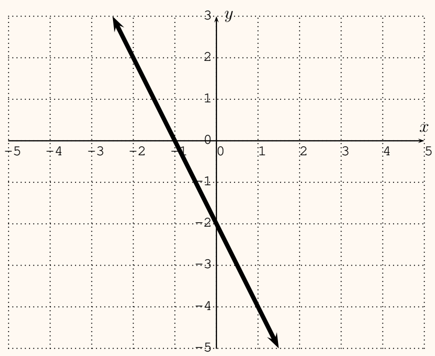
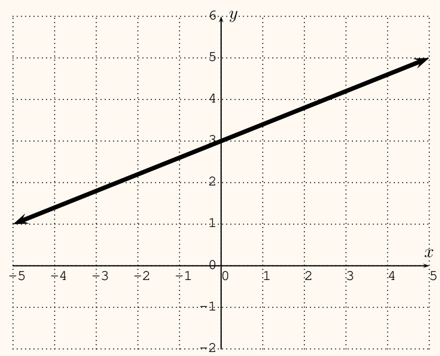
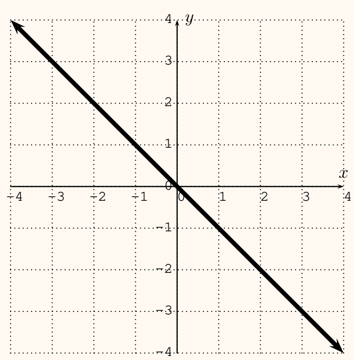
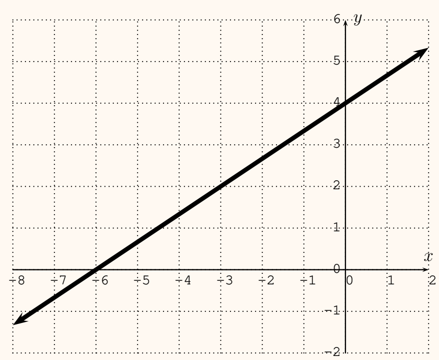


c)


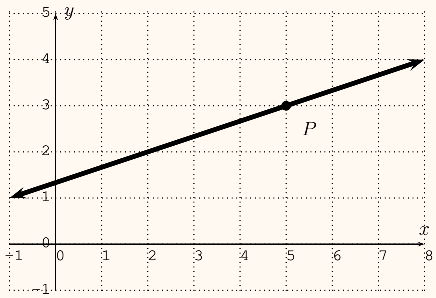
d)


e)


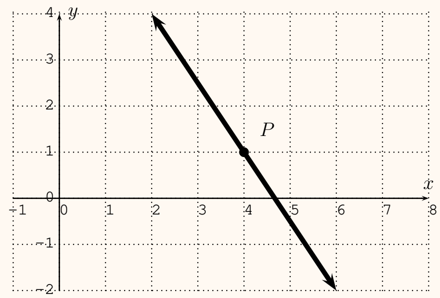
f)


Exercise 3.2

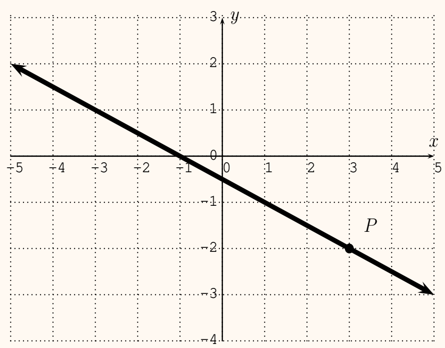
Find the equation of the line in point-slope form (3.3) using the indicated point P.



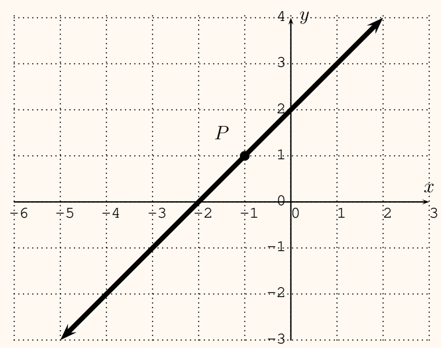
a)



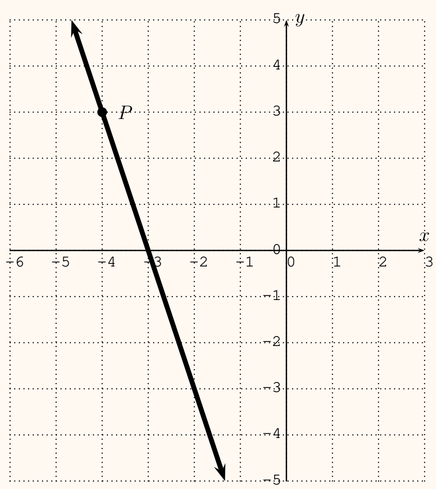
b)



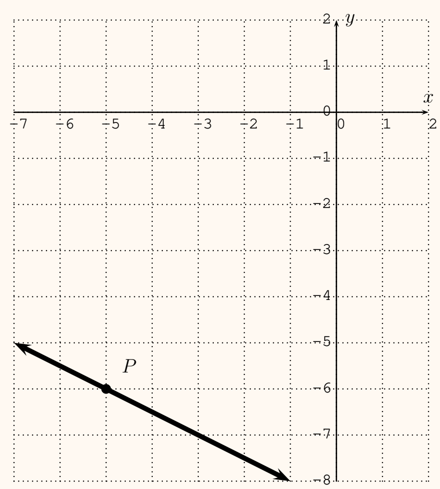
c)



d)



e)

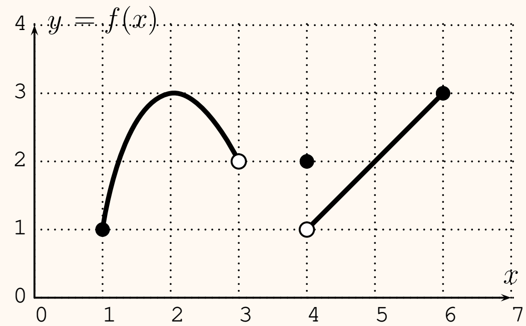


f)

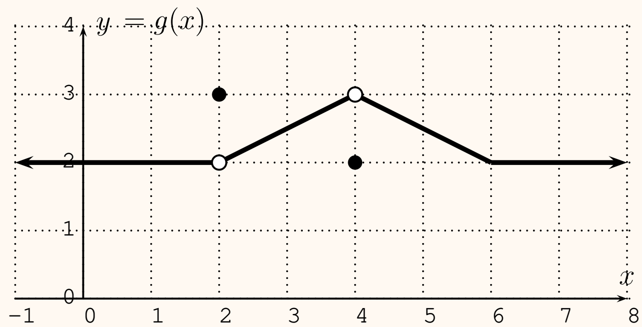
Exercise 3.3

Below are three graphs for the functions f, g, and h.

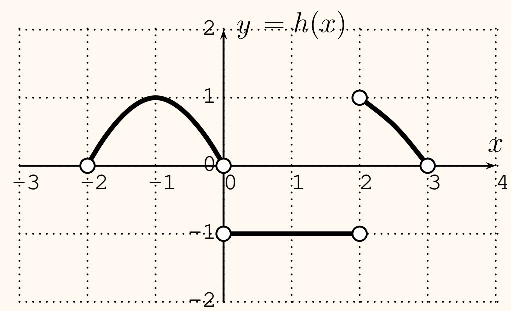
function f :



function g :



function h :



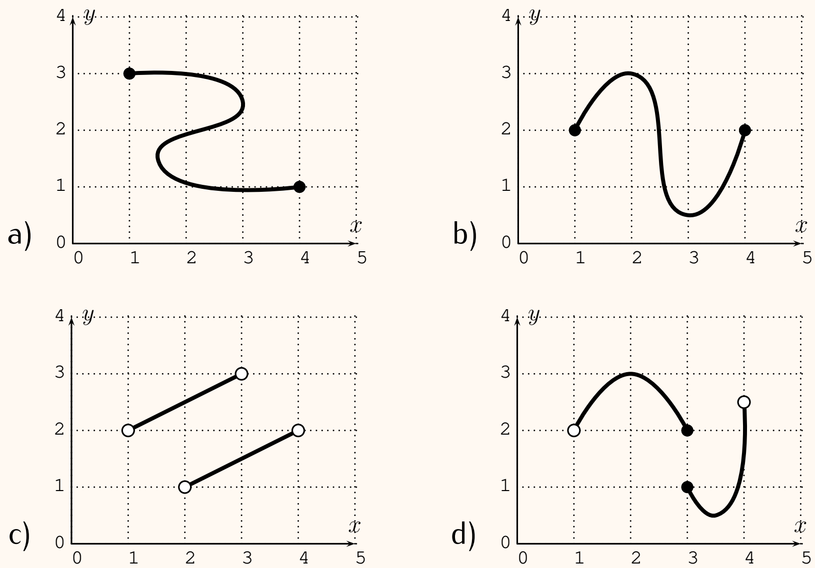
a) Find the domain and range of f.
b) Find the domain and range of g.
c) Find the domain and range of h.

Find the following function values:

d) $f(1)$	e) $f(2)$	f) $f(3)$	g) $f(4)$	h) $f(5)$	i) $f(6)$	j) $f(7)$
k) $g(0)$	l) $g(1)$	m) $g(2)$	n) $g(3)$	o) $g(4)$	p) $g(6)$	q) $g(13.2)$
r) $h(-2)$	s) $h(-1)$	t) $h(0)$	u) $h(1)$	v) $h(2)$	w) $h(3)$	x) $h(\sqrt{2})$

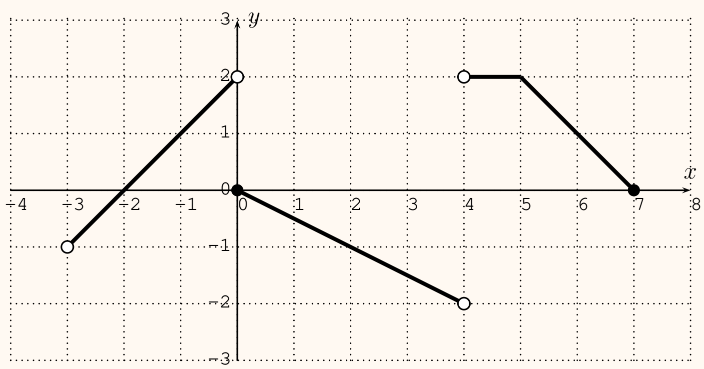
Exercise 3.4

Use the vertical line test to determine which of the following graphs are the graphs of functions.



Exercise 3.5

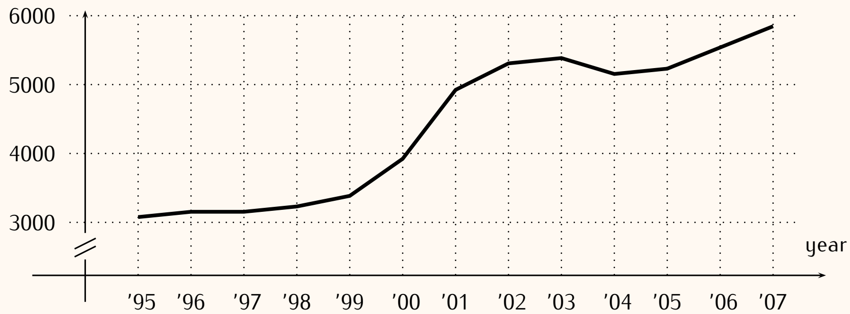
Let f be the function given by the following graph.



a) What is the domain of f?
b) What is the range of f?
c) For which x is $f(x) = 0$?
d) For which x is $f(x) = 2$?
e) For which x is $f(x) \leq 1$?
f) For which x is $f(x) > 0$?
g) Find $f(2)$ and $f(5)$.
h) Find $f(2) + f(5)$.
i) Find $f(2) + 5$.
j) Find $f(2 + 5)$.

Exercise 3.6

The graph below displays the number of students admitted to a college during the years 1995 to 2007.



a) How many students were admitted in the year 2000?
b) In what years were the most students admitted?
c) In what years did the number of admitted students rise fastest?
d) In what year(s) did the number of admitted students decline?

Exercise 3.7

Consider the function described by the following formula:

$$f(x) = \begin{cases} x^2 + 1 & \text{, for } -2 < x \leq 0 \\ x - 1 & \text{, for } 0 < x \leq 2 \\ -x + 4 & \text{, for } 2 < x \leq 5 \end{cases}$$

What is the domain of the function f? Graph the function f.