Precalculus

Third Edition (3.0)

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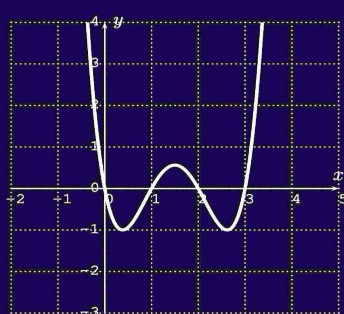

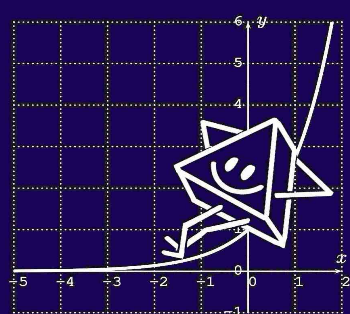

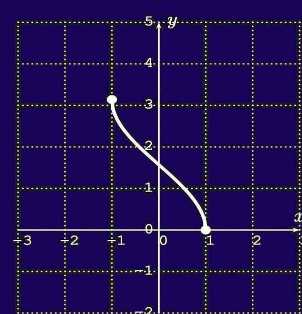

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Chapter 25

The geometric series

We now study another sequence—the geometric sequence. Our analysis follows steps similar to the one of the arithmetic sequence in Section 24.2.

25.1 Finite geometric series

We have already encountered examples of geometric sequences in Example 24.2(b) and (c). A geometric sequence is a sequence for which we *multiply* a constant number to get from one term to the next, for example:

$$5, \underbrace{\quad}_{\times 4} 20, \underbrace{\quad}_{\times 4} 80, \underbrace{\quad}_{\times 4} 320, \underbrace{\quad}_{\times 4} 1280, \ldots$$

Definition 25.1: Geometric sequence

A sequence $\{a_n\}$ is called a **geometric sequence** if any two consecutive terms have a *common ratio* r. The geometric sequence is determined by r and the first value a_1. This can be written recursively as:

$$a_n = a_{n-1} \cdot r \qquad \text{for } n \geq 2$$

Alternatively, we have the general formula for the nth term of the geometric sequence:

$$\boxed{a_n = a_1 \cdot r^{n-1}} \tag{25.1}$$

Example 25.2

Determine whether the terms below are the first terms of an arithmetic sequence, a geometric sequence, neither, or both. If they are the terms of an arithmetic or geometric sequence, then find the general formula a_n of the sequence in the form (24.2) or (25.1).

a)	$3, 6, 12, 24, 48, \ldots$	b)	$100, 50, 25, 12.5, \ldots$
c)	$2, 4, 16, 256, \ldots$	d)	$700, -70, 7, -0.7, 0.07, \ldots$
e)	$3, 10, 17, 24, \ldots$	f)	$-3, -3, -3, -3, -3, \ldots$
g)	$a_n = n^2$	h)	$a_n = \left(\frac{3}{7}\right)^n$

Solution.

a) First, the differences of two consecutive terms $6-3=3$ and $12-6=6$ are different. So, these are not the terms of an arithmetic sequence. On the other hand, the quotient of two consecutive terms always gives the same number $6 \div 3 = 2$, $12 \div 6 = 2$, $24 \div 12 = 2$, etc. Therefore, the common ratio is $r = 2$, which shows that these are the terms in a geometric sequence. Furthermore, the first term is $a_1 = 3$, so the general formula for the nth term is $a_n = 3 \cdot 2^{n-1}$.

b) Since the differences $50 - 100 = -50$ and $25 - 50 = -25$ are not the same, this is not an arithmetic sequence. We see that the common ratio between two terms is $r = \frac{50}{100} = \frac{1}{2}$, so that this is a geometric sequence. Since the first term is $a_1 = 100$, we have the general term $a_n = 100 \cdot (\frac{1}{2})^{n-1}$.

c) The difference between the first two terms is $4 - 2 = 2$, while the next two terms have a difference $16 - 4 = 12$. Therefore, this is also not an arithmetic sequence. Furthermore, the quotient of the first two terms is $4 \div 2 = 2$, whereas the quotient of the next two terms is $16 \div 4 = 4$. Since these quotients are not equal, this is not a geometric sequence.

d) This is not an arithmetic sequence, but these are terms of a geometric sequence. Two consecutive terms have a ratio of $r = -\frac{1}{10}$, and the first term is $a_1 = 700$. The general term of this geometric sequence is $a_n = 700 \cdot (-\frac{1}{10})^{n-1}$.

e) The quotient of the first couple of terms is not equal: $\frac{10}{3} \neq \frac{17}{10}$, so this is not a geometric sequence. The difference between any two terms is $7 = 10 - 3 = 17 - 10 = 24 - 17$, so this is part of an arithmetic sequence with common difference $d = 7$. The general formula is $a_n = a_1 + d \cdot (n-1) = 3 + (n-1) \cdot 7$.

f) The common ratio is $r = (-3) \div (-3) = 1$, so this is a geometric sequence with $a_n = (-3) \cdot 1^{n-1}$. On the other hand, the common difference is $(-3) - (-3) = 0$, so this is *also* an arithmetic sequence with $a_n = (-3) + (n-1) \cdot 0$. Of course, both formulas reduce to the simpler expression $a_n = -3$.

g) We write the first terms in the sequence $\{n^2\}_{n \geq 1}$:

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

Calculating the quotients of consecutive terms, we get $4 \div 1 = 4$ and $9 \div 4 = 2.25$, so this is not a geometric sequence. Also the difference of consecutive terms is $4 - 1 = 3$ and $9 - 4 = 5$, so this is also not an arithmetic sequence.

h) Writing the first couple of terms in the sequence $\{(\frac{3}{7})^n\}$, we obtain:

$$\left(\frac{3}{7}\right)^1, \left(\frac{3}{7}\right)^2, \left(\frac{3}{7}\right)^3, \left(\frac{3}{7}\right)^4, \left(\frac{3}{7}\right)^5, \ldots$$

Thus, we get from one term to the next by multiplying $r = \frac{3}{7}$, so this is a geometric sequence. The first term is $a_1 = \frac{3}{7}$, so $a_n = \frac{3}{7} \cdot \left(\frac{3}{7}\right)^{n-1}$. This is clearly the given sequence, since we may simplify this as

$$a_n = \frac{3}{7} \cdot \left(\frac{3}{7}\right)^{n-1} = \left(\frac{3}{7}\right)^{1+n-1} = \left(\frac{3}{7}\right)^n$$

We can also confirm that this is not an arithmetic sequence.

□

Example 25.3

Find the general formula $a_n = a_1 \cdot r^{n-1}$ of a geometric sequence with the given properties.

a) $r = 4$, and $a_5 = 6400$
b) $a_1 = \frac{2}{5}$, and $a_4 = -\frac{27}{20}$
c) $a_5 = 216$, $a_7 = 24$, and r is positive

Solution.

a) We know that $r = 4$, and we still need to find a_1. Using $a_5 = 64000$, we obtain:

$$6400 = a_5 = a_1 \cdot 4^{5-1} = a_1 \cdot 4^4 = 256 \cdot a_1 \overset{(\div 256)}{\Longrightarrow} a_1 = \frac{6400}{256} = 25$$

The sequence is therefore given by the formula, $a_n = 25 \cdot 4^{n-1}$.

b) The geometric sequence $a_n = a_1 \cdot r^{n-1}$ has $a_1 = \frac{2}{5}$. We calculate r using the second condition.

$$-\frac{27}{20} = a_4 = a_1 \cdot r^{4-1} = \frac{2}{5} \cdot r^3 \overset{(\times \frac{5}{2})}{\Longrightarrow} r^3 = -\frac{27}{20} \cdot \frac{5}{2} = -\frac{27}{4} \cdot \frac{1}{2} = \frac{-27}{8}$$
$$\overset{(\text{take } \sqrt[3]{\ })}{\Longrightarrow} r = \sqrt[3]{\frac{-27}{8}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{8}} = \frac{-3}{2}$$

Therefore, $a_n = \frac{2}{5} \cdot \left(\frac{-3}{2}\right)^{n-1}$.

c) The question provides neither a_1 nor r for our formula $a_n = a_1 \cdot r^{n-1}$. However, we obtain two equations in the two variables a_1 and r:

$$\begin{cases} 216 = a_5 = a_1 \cdot r^{5-1} \\ 24 = a_7 = a_1 \cdot r^{7-1} \end{cases} \Longrightarrow \begin{cases} 216 = a_1 \cdot r^4 \\ 24 = a_1 \cdot r^6 \end{cases}$$

In order to solve this, we need to eliminate one of the variables. Looking at the equations on the right, we see that dividing the top equation by the bottom equation cancels a_1.

$$\frac{216}{24} = \frac{a_1 \cdot r^4}{a_1 \cdot r^6} \Longrightarrow \frac{9}{1} = \frac{1}{r^2} \overset{(\text{take reciprocal})}{\Longrightarrow} \frac{1}{9} = \frac{r^2}{1} \Longrightarrow r^2 = \frac{1}{9}$$

To obtain r, we have to solve this quadratic equation. In general, there are, in fact, two solutions:

$$r = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$$

Since the problem states that r is positive, we see that we need to take the positive solution $r = \frac{1}{3}$. Plugging $r = \frac{1}{3}$ back into either of the two equations, we may solve for a_1. For example, using the first equation $a_5 = 216$, we obtain:

$$216 = a_5 = a_1 \cdot \left(\frac{1}{3}\right)^{5-1} = a_1 \cdot \left(\frac{1}{3}\right)^{4} = a_1 \cdot \frac{1}{3^4} = a_1 \cdot \frac{1}{81}$$
$$\overset{(\times 81)}{\Longrightarrow} \quad a_1 = 81 \cdot 216 = 17,496$$

So, we finally arrive at the general formula for the nth term of the geometric sequence, $a_n = 17,496 \cdot (\frac{1}{3})^{n-1}$.

□

We can find the sum of the first k terms of a geometric sequence using another trick, which is very different from the one we used for the arithmetic sequence.

Note 25.4: Summing over terms in a geometric sequence

Consider the geometric sequence $a_n = 7 \cdot 10^{n-1}$, that is the sequence:

$$7, \quad 70, \quad 700, \quad 7000, \quad 70,000, \quad 700,000, \quad \dots$$

We want to add the first 5 terms of this sequence.

$$7 + 70 + 700 + 7000 + 70,000 = 77,777$$

The above example can, of course, easily be computed by hand. In general, however, much more work is necessary to find a sum of a geometric sequence, especially if the sequence is more complicated and we want to add a lot more terms. To get to a general formula, we will add the terms in the above sum in a different way, which may appear to be more complicated than necessary. However, the advantage of the following calculation is that it is an illustration for a general method, which allows us to find the sum of terms in any geometric sequence. To this end, we multiply $(1-10)$ to the sum $(7+70+700+7000+70,000)$, and simplify this using the distributive law:

$$\begin{aligned} & (1-10)\cdot(7+70+700+7000+70,000) \\ = \;& 7-70+70-700+700-7000 \\ & \qquad\qquad\qquad +7000-70,000+70,000-700,000 \\ = \;& 7-700,000 \end{aligned}$$

The sum in the second line above is called a *telescopic sum*, which is a sum where consecutive terms cancel each other. In the above sum the only remaining terms are the very first and last terms. Dividing by $(1-10)$, we obtain:

$$7+70+700+7000+70,000=\frac{7-700,000}{1-10}=\frac{-699,993}{-9}=77,777$$

An appropriate generalization of the previous note yields a computation that applies to any geometric sequence.

Observation 25.5: Geometric series

Let $\{a_n\}$ be a geometric sequence whose nth term is given by the formula $a_n = a_1 \cdot r^{n-1}$. We furthermore assume that $r \neq 1$. Then the sum $a_1 + a_2 + \cdots + a_{k-1} + a_k$ is given by:

$$\boxed{\sum_{i=1}^{k} a_i = a_1 \cdot \frac{1-r^k}{1-r}} \tag{25.2}$$

Proof. We multiply $(1-r)$ to the sum $(a_1 + a_2 + \cdots + a_{k-1} + a_k)$:

$$\begin{aligned} & (1-r)\cdot(a_1+a_2+\cdots+a_k) \\ = \;& (1-r)\cdot(a_1\cdot r^0 + a_1\cdot r^1 + \cdots + a_1\cdot r^{k-1}) \\ = \;& a_1\cdot r^0 - a_1\cdot r^1 + a_1\cdot r^1 - a_1\cdot r^2 + \cdots + a_1\cdot r^{k-1} - a_1\cdot r^k \\ = \;& a_1\cdot r^0 - a_1\cdot r^k = a_1\cdot(1-r^k) \end{aligned}$$

Dividing by $(1-r)$, we obtain

$$a_1 + a_2 + \cdots + a_k = \frac{a_1 \cdot (1-r^k)}{(1-r)} = a_1 \cdot \frac{1-r^k}{1-r}$$

This is the formula we wanted to prove. □

Example 25.6

Find the value of the geometric series.

a) Find the sum $\sum_{n=1}^{6} a_n$ for the geometric sequence $a_n = 10 \cdot 3^{n-1}$.

b) Determine the value of the geometric series: $\sum_{k=1}^{5} \left(-\frac{1}{2}\right)^k$

c) Find the sum of the first 12 terms of the geometric sequence

$$-3, -6, -12, -24, \ldots$$

Solution.

a) We need to find the sum $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$, and we do so by using the formula provided in Equation (25.2). Since $a_n = 10 \cdot 3^{n-1}$, we have $a_1 = 10$ and $r = 3$, so

$$\sum_{n=1}^{6} a_n = 10 \cdot \frac{1-3^6}{1-3} = 10 \cdot \frac{1-729}{1-3} = 10 \cdot \frac{-728}{-2} = 10 \cdot 364 = 3640$$

b) Again, we use the formula for the geometric series $\sum_{k=1}^{n} a_k = a_1 \cdot \frac{1-r^n}{1-r}$, since $a_k = (-\frac{1}{2})^k$ is a geometric series. We may calculate the first term $a_1 = -\frac{1}{2}$, and the common ratio is also $r = -\frac{1}{2}$. With this, we obtain:

$$\begin{aligned}
\sum_{k=1}^{5} \left(-\frac{1}{2}\right)^k &= \left(-\frac{1}{2}\right) \cdot \frac{1-(-\frac{1}{2})^5}{1-(-\frac{1}{2})} = \left(-\frac{1}{2}\right) \cdot \frac{1-((-1)^5 \frac{1^5}{2^5})}{1-(-\frac{1}{2})} \\
&= \left(-\frac{1}{2}\right) \cdot \frac{1-(-\frac{1}{32})}{1-(-\frac{1}{2})} = \left(-\frac{1}{2}\right) \cdot \frac{1+\frac{1}{32}}{1+\frac{1}{2}} = \left(-\frac{1}{2}\right) \cdot \frac{\frac{32+1}{32}}{\frac{2+1}{2}} \\
&= \left(-\frac{1}{2}\right) \cdot \frac{\frac{33}{32}}{\frac{3}{2}} = \left(-\frac{1}{2}\right) \cdot \frac{33}{32} \cdot \frac{2}{3} = -\frac{1}{2} \cdot \frac{11}{16} = -\frac{11}{32}
\end{aligned}$$

c) Our first task is to find the formula for the provided geometric series $-3, -6, -12, -24, \ldots$. The first term is $a_1 = -3$ and the common ratio is $r = 2$, so that $a_n = (-3) \cdot 2^{n-1}$. The sum of the first 12 terms of this sequence is again given by Equation (25.2):

$$\begin{aligned}\sum_{i=1}^{12} (-3) \cdot 2^{i-1} &= (-3) \cdot \frac{1-2^{12}}{1-2} = (-3) \cdot \frac{1-4096}{1-2} = (-3) \cdot \frac{-4095}{-1} \\ &= (-3) \cdot 4095 = -12,285\end{aligned}$$

□

25.2 Infinite geometric series

In some cases, it makes sense to add not only finitely many terms of a geometric sequence, but all infinitely many terms of the sequence! An informal and intuitive infinite geometric series is exhibited in the next note.

Note 25.7: Summing over all terms in a geometric sequence

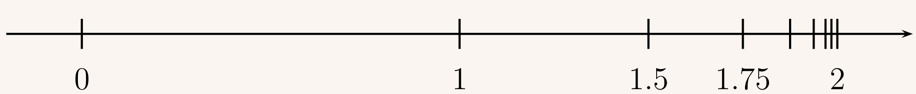
Consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Here, the common ratio is $r = \frac{1}{2}$, and the first term is $a_1 = 1$, so that the formula for a_n is $a_n = \left(\frac{1}{2}\right)^{n-1}$. We are interested in summing *all infinitely many* terms of this sequence:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

We add these terms one by one, and picture these sums on the number line:



$$\begin{aligned} 1 &= 1 \\ 1 + \frac{1}{2} &= 1.5 \end{aligned}$$

$$
\begin{aligned}
1+\frac{1}{2}+\frac{1}{4} &= 1.75 \\
1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= 1.875 \\
1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= 1.9375
\end{aligned}
$$

We see that adding each term takes the sum closer and closer to the number 2. More precisely, adding a term a_n to the partial sum $a_1 + \cdots + a_{n-1}$ decreases the distance between 2 and $a_1 + \cdots + a_{n-1}$ by half. For this reason, we can, in fact, get arbitrarily close to 2, so it is reasonable to expect that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2$$

In the next definition and observation, this equation will be justified and made more precise.

First, we give a definition of an infinite series.

Definition 25.8: Infinite series

An infinite series is given by adding infinitely many terms of a sequence. We write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \tag{25.3}$$

To be more precise, the infinite sum is defined as the limit $\sum\limits_{n=1}^{\infty} a_n := \lim\limits_{k\to\infty} \left(\sum\limits_{n=1}^{k} a_n \right)$. Therefore, an infinite sum is defined precisely when this limit exists.

Observation 25.9: Infinite geometric series

Let $\{a_n\}$ be a geometric sequence with $a_n = a_1 \cdot r^{n-1}$. Then the infinite geometric series is defined whenever $-1 < r < 1$. In this case, we have:

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r}} \tag{25.4}$$

Proof. Informally, this follows from the formula $\sum_{n=1}^{k} a_n = a_1 \cdot \frac{1-r^k}{1-r}$ and the fact that, for $-1 < r < r$, the term r^k approaches zero when k increases without bound.

More formally, the proof uses the notion of limits, and proceeds as follows:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right) = \lim_{k\to\infty}\left(a_1 \cdot \frac{1-r^k}{1-r}\right) = a_1 \cdot \frac{1-\lim\limits_{k\to\infty}(r^k)}{1-r} = a_1 \cdot \frac{1}{1-r}$$

□

Example 25.10

Find the value of the infinite geometric series.

a) $\sum_{j=1}^{\infty} a_j$, for $a_j = 5 \cdot \left(\frac{1}{3}\right)^{j-1}$ b) $\sum_{n=1}^{\infty} 3 \cdot (0.71)^n$

c) $500 - 100 + 20 - 4 + \dots$ d) $3 + 6 + 12 + 24 + 48 + \dots$

Solution.

a) We use formula (25.4) for the geometric series $a_n = 5 \cdot (\frac{1}{3})^{n-1}$, that is $a_1 = 5 \cdot (\frac{1}{3})^{1-1} = 5 \cdot (\frac{1}{3})^0 = 5 \cdot 1 = 5$ and $r = \frac{1}{3}$. Therefore,

$$\sum_{j=1}^{\infty} a_j = a_1 \cdot \frac{1}{1-r} = 5 \cdot \frac{1}{1-\frac{1}{3}} = 5 \cdot \frac{1}{\frac{3-1}{3}} = 5 \cdot \frac{1}{\frac{2}{3}} = 5 \cdot \frac{3}{2} = \frac{15}{2}$$

b) In this case, $a_n = 3 \cdot (0.71)^n$, so that $a_1 = 3 \cdot 0.71^1 = 3 \cdot 0.71 = 2.13$ and $r = 0.71$. Again using formula (25.4), we can find the infinite geometric series as

$$\sum_{n=1}^{\infty} 3 \cdot (0.71)^n = a_1 \cdot \frac{1}{1-r} = 2.13 \cdot \frac{1}{1-0.71} = 2.13 \cdot \frac{1}{0.29} = \frac{2.13}{0.29} = \frac{213}{29}$$

In the last step, we simplified the fraction by multiplying 100 to both numerator and denominator, which had the effect of eliminating the decimals.

c) Our first task is to identify the given sequence as an infinite geometric sequence:

$$\{a_n\} \text{ is given by } 500, -100, 20, -4, \dots$$

Notice that the first term is 500, and each consecutive term is given by dividing by -5, or in other words, by multiplying by the common ratio $r = -\frac{1}{5}$. Therefore, this is an infinite geometric series, which can be evaluated as

$$500 - 100 + 20 - 4 + \cdots = \sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r} = 500 \cdot \frac{1}{1-(-\frac{1}{5})}$$
$$= 500 \cdot \frac{1}{1+\frac{1}{5}} = \frac{500}{\frac{5+1}{5}} = \frac{500}{\frac{6}{5}} = 500 \cdot \frac{5}{6} = \frac{2500}{6} = \frac{1250}{3}$$

d) We want to evaluate the infinite series $3+6+12+24+48+\dots$. The sequence $3, 6, 12, 24, 48, \dots$ is a geometric sequence with $a_1 = 3$ and common ratio $r = 2$. Since $r \geq 1$, we see that formula (25.4) *cannot* be applied, as (25.4) only applies to $-1 < r < 1$. However, since we add larger and larger terms, the series gets larger than any possible bound, so that the whole sum becomes infinite.

$$3 + 6 + 12 + 24 + 48 + \cdots = \infty$$

□

Example 25.11

The fraction $0.55555\dots$ may be written as:

$$0.55555\cdots = 0.5 + 0.05 + 0.005 + 0.0005 + 0.00005 + \dots$$

Noting that the sequence

$$0.5, \underbrace{\quad}_{\times 0.1} 0.05, \underbrace{\quad}_{\times 0.1} 0.005, \underbrace{\quad}_{\times 0.1} 0.0005, \underbrace{\quad}_{\times 0.1} 0.00005, \dots$$

is a geometric sequence with $a_1 = 0.5$ and $r = 0.1$, we can calculate the infinite sum as:

$$0.55555\cdots = \sum_{n=1}^{\infty} 0.5 \cdot (0.1)^{n-1} = 0.5 \cdot \frac{1}{1-0.1} = 0.5 \cdot \frac{1}{0.9} = \frac{0.5}{0.9} = \frac{5}{9},$$

Here we multiplied numerator and denominator by 10 in the last step in order to eliminate the decimals.

25.3 Exercises

Exercise 25.1

Which of these sequences is geometric, arithmetic, neither, or both. Write the sequence in the usual form $a_n = a_1 + (n-1) \cdot d$ if it is an arithmetic sequence, and $a_n = a_1 \cdot r^{n-1}$ if it is a geometric sequence.

a) $7, 14, 28, 56, \dots$

b) $3, -30, 300, -3000, \dots$

c) $81, 27, 9, 3, 1, \frac{1}{3}, \dots$

d) $-7, -5, -3, -1, 1, 3, 5, 7, \dots$

e) $-6, 2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \dots$

f) $-2, -2 \cdot \frac{2}{3}, -2 \cdot \left(\frac{2}{3}\right)^2, -2 \cdot \left(\frac{2}{3}\right)^3, \dots$

g) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

h) $2, 2, 2, 2, 2, \dots$

i) $5, 1, 5, 1, 5, 1, 5, 1, \dots$

j) $-2, 2, -2, 2, -2, 2, \dots$

k) $0, 5, 10, 15, 20, \dots$

l) $5, \frac{5}{3}, \frac{5}{3^2}, \frac{5}{3^3}, \frac{5}{3^4}, \dots$

m) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

n) $\log(2), \log(4), \log(8), \log(16), \dots$

o) $a_n = -4^n$

p) $a_n = -4n$

q) $a_n = 2 \cdot (-9)^n$

r) $a_n = \left(\frac{1}{3}\right)^n$

s) $a_n = -\left(\frac{5}{7}\right)^n$

t) $a_n = \left(-\frac{5}{7}\right)^n$

u) $a_n = \frac{2}{n}$

v) $a_n = 3n + 1$

Exercise 25.2

A geometric sequence, $a_n = a_1 \cdot r^{n-1}$, has the given properties. Find the term a_n of the sequence.

a) $a_1 = 3$, and $r = 5$, find a_4

b) $a_1 = 200$, and $r = -\frac{1}{2}$, find a_6

c) $a_1 = -7$, and $r = 2$, find a_n (for all n)

d) $r = 2$, and $a_4 = 48$, find a_1

e) $r = 100$, and $a_4 = 900,000$, find a_n (for all n)

f) $a_1 = 20$, $a_4 = 2500$, find a_n (for all n)

g) $a_1 = \frac{1}{8}$, and $a_6 = \frac{3^5}{8^6}$, find a_n (for all n)

h) $a_3 = 36$, and $a_6 = 972$, find a_n (for all n)

i) $a_8 = 4000$, $a_{10} = 40$, and r is negative, find a_n (for all n)

Exercise 25.3

Find the value of the finite geometric series using formula (25.2). Confirm the formula either by adding the the summands directly, or alternatively by using the calculator.

a) Find the sum $\sum_{j=1}^{4} a_j$ for the geometric sequence $a_j = 5 \cdot 4^{j-1}$.

b) Find the sum $\sum_{i=1}^{7} a_i$ for the geometric sequence $a_n = \left(\frac{1}{2}\right)^n$.

c) Find: $\sum_{m=1}^{5} \left(-\frac{1}{5}\right)^m$

d) Find: $\sum_{k=1}^{6} 2.7 \cdot 10^k$

e) Find the sum of the first 5 terms of the geometric sequence:

$$2, 6, 18, 54, \dots$$

f) Find the sum of the first 6 terms of the geometric sequence:

$$-5, 15, -45, 135, \dots$$

g) Find the sum of the first 8 terms of the geometric sequence:

$$-1, -7, -49, -343, \dots$$

h) Find the sum of the first 10 terms of the geometric sequence:

$$600, -300, 150, -75, 37.5, \dots$$

i) Find the sum of the first 40 terms of the geometric sequence:

$$5, 5, 5, 5, 5, \dots$$

Exercise 25.4

Find the value of the infinite geometric series.

a) $\sum_{j=1}^{\infty} a_j$, for $a_j = 3 \cdot \left(\frac{2}{3}\right)^{j-1}$

b) $\sum_{j=1}^{\infty} 7 \cdot \left(-\frac{1}{5}\right)^j$

c) $\sum_{j=1}^{\infty} 6 \cdot \frac{1}{3^j}$

d) $\sum_{n=1}^{\infty} -2 \cdot (0.8)^n$

e) $\sum_{n=1}^{\infty} (0.99)^n$

f) $27 + 9 + 3 + 1 + \frac{1}{3} + \dots$

g) $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \dots$

h) $-6 - 2 - \frac{2}{3} - \frac{2}{9} - \dots$

i) $100 + 40 + 16 + 6.4 + \dots$

j) $-54 + 18 - 6 + 2 - \dots$

Exercise 25.5

Rewrite the decimal using an infinite geometric sequence, and then use the formula for the infinite geometric series to rewrite the decimal as a fraction (see Example 25.11).

a) $0.44444\dots$

b) $0.77777\dots$

c) $5.55555\dots$

d) $0.2323232323\dots$

e) $39.393939\dots$

f) $0.248248248\dots$

g) $20.02002\dots$

h) $0.5040504\dots$