Precalculus

Third Edition (3.0)
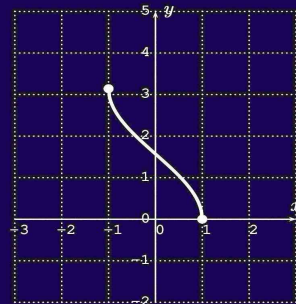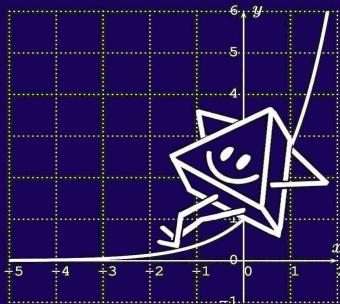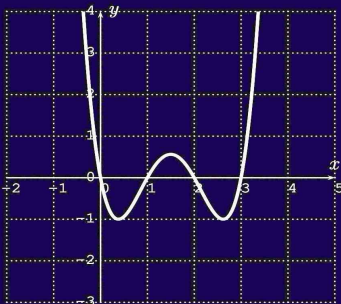
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Chapter 22

Vectors in the plane

So far, we have discussed functions in general, as well as specific examples, such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. In the next chapters we study vectors, complex numbers, sequences, and series. We start in this chapter with vectors in the plane.

22.1   Introduction to vectors

Vectors are used in many applications, as they are useful to describe concepts that have a direction and a magnitude. Examples of these include:

- the velocity of a (moving) object,

- the acceleration of an object,

- the force applied to an object.

> **Definition 22.1: Geometric vector**
>
> A **geometric vector** is a geometric object that is given by a **magnitude** and a **direction**. We denote a vector by $\vec{v}$, that is, by placing an arrow on top of the symbol for the variable.

Note 22.2

Vectors are often represented by directed line segments $\vec{v} = \overrightarrow{PQ}$.
Two directed line segments represent the same vector if one can be moved to the other by parallel translation (without changing its direction or magnitude).
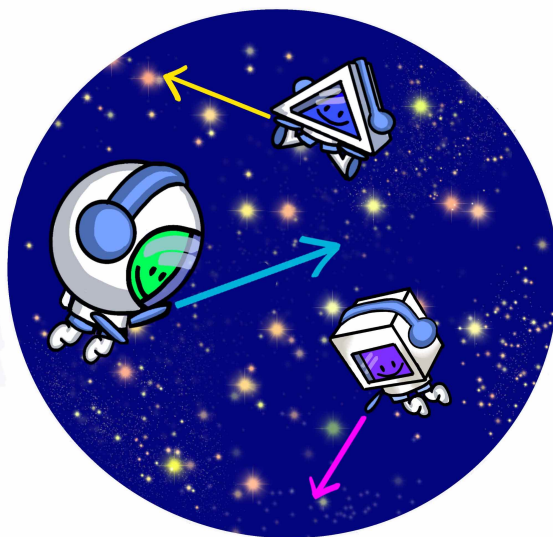
We will now study vectors in the plane $\mathbb{R}^2$ in more detail.

Observation 22.3: Vectors at the origin

A vector $\vec{v} = \overrightarrow{PQ}$ in the plane $\mathbb{R}^2$ can be represented by arranging the starting point of $\vec{v}$ to the origin $O(0,0)$.
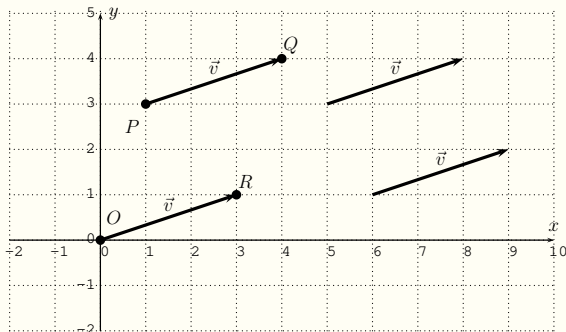
If R is given in coordinates by $R(a,b)$, then we also write for $\vec{v} = \overrightarrow{OR}$,

$$\boxed{\vec{v} = \langle a, b \rangle} \quad \text{or, alternatively,} \quad \boxed{\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}}. \qquad (22.1)$$

Example 22.4

Graph the vectors $\vec{v}, \vec{w}, \vec{r}, \vec{s}, \vec{t}$ in the plane, where $\vec{v} = \overrightarrow{PQ}$ with $P(6,3)$ and $Q(4,-2)$, and

$$\vec{w} = \langle 3, -1 \rangle, \quad \vec{r} = \langle -4, -2 \rangle, \quad \vec{s} = \langle 0, 2 \rangle, \quad \vec{t} = \langle -5, 3 \rangle.$$

Solution.
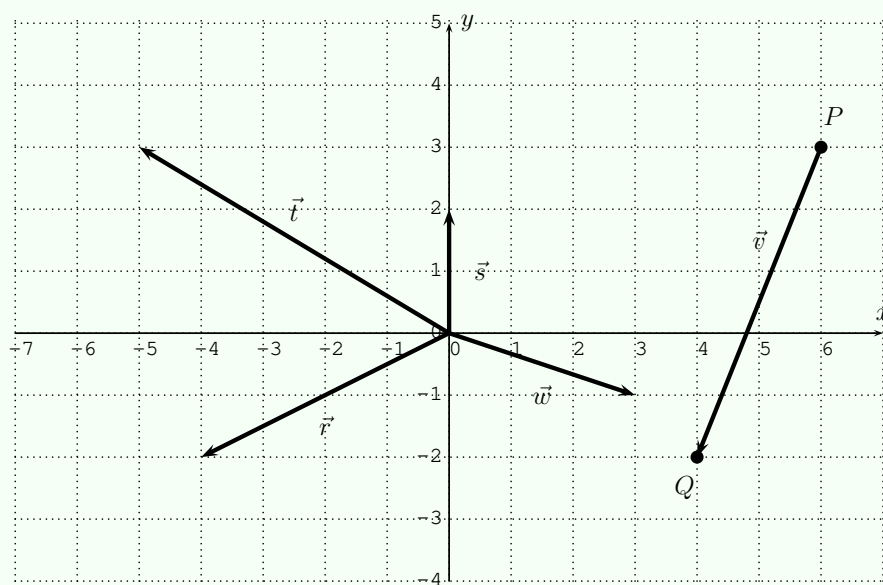
We now compute the magnitude and the direction angle of a vector given in coordinates, $\vec{v} = \langle a, b \rangle$.

Observation 22.5: Magnitude

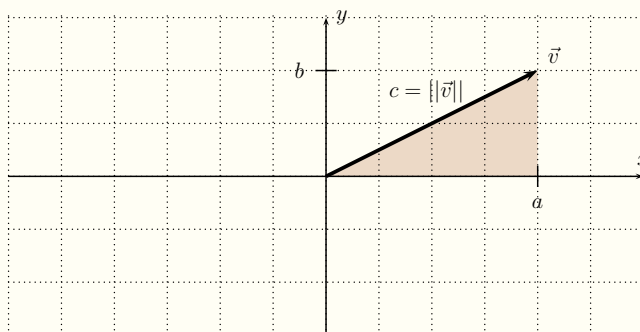Let $\vec{v} = \langle a, b \rangle = \overrightarrow{OR}$ be a vector in the plane, where $R(a, b)$ is the point with coordinates (a, b).

The **magnitude** of $\vec{v}$ is the length c of the line segment $\overrightarrow{OR}$. The Pythagorean theorem for the shaded right triangle above gives

$$a^2 + b^2 = c^2 \qquad \Longrightarrow \qquad c = \sqrt{a^2 + b^2}$$

The magnitude of $\vec{v}$ is usually denoted by $||\vec{v}||$. Therefore, we have:

$$\boxed{||\vec{v}|| = \sqrt{a^2 + b^2}} \tag{22.2}$$

Example 22.6

Find the magnitude of the given vectors.

$$\text{a) } \vec{v} = \langle 8, -6 \rangle \quad \text{b) } \vec{v} = \langle -5, -5 \rangle \quad \text{c) } \vec{v} = \langle 4, 4\sqrt{3} \rangle$$

Solution.

a) $||\vec{v}|| = ||\langle 8, -6 \rangle|| = \sqrt{8^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$

b) $||\vec{v}|| = ||\langle -5, -5 \rangle|| = \sqrt{(-5)^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50} = \sqrt{25 \cdot 2} = 4\sqrt{2}$
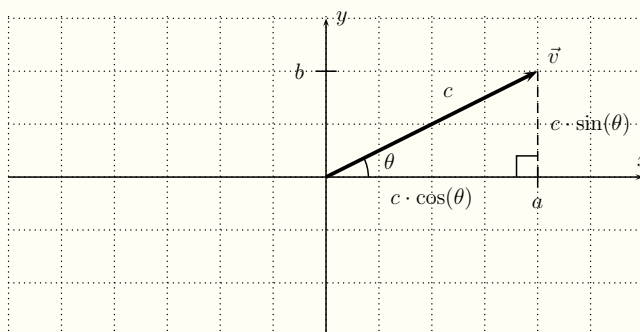
c) $||\vec{v}|| = ||\langle 4, 4\sqrt{3} \rangle|| = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{4^2 + 4^2\sqrt{3}^2} = \sqrt{16 + 16 \cdot 3} = \sqrt{16 + 48} = \sqrt{64} = 8$

$\square$

Next, we identify the direction angle of a vector $\vec{v} = \langle a, b \rangle$.

Observation 22.7: Direction angle

Let $\vec{v} = \langle a, b \rangle = \overrightarrow{OR}$ be a vector in the plane, where $R(a, b)$ is the point with coordinates (a, b).

The **direction angle** of $\vec{v}$ is the angle θ (read as "theta") determined by the line segment $\overrightarrow{OR}$. Denoting by $c = ||\vec{v}||$, the length of the vector $\vec{c}$, then, by (21.2), we have $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, and so, $\tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \cdot \frac{c}{a} = \frac{b}{a}$:

$$\boxed{\tan\theta = \frac{b}{a}} \tag{22.3}$$

Note that we can recover the vector $\vec{v} = \langle a, b \rangle$ in coordinate form from the magnitude $||\vec{v}|| = c$ and the angle θ, since $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$ gives

$$a = c \cdot \cos(\theta) \qquad \text{and} \qquad b = c \cdot \sin(\theta) \tag{22.4}$$

and therefore:

$$\vec{v} = \langle a, b \rangle = \langle\ ||\vec{v}|| \cdot \cos(\theta)\ ,\ ||\vec{v}|| \cdot \sin(\theta)\ \rangle \tag{22.5}$$

To find the angle θ from identity $\tan\theta = \frac{b}{a}$, we need to be a bit careful, because there is more than one angle θ whose tangent is $\frac{b}{a}$. This is illustrated in the next example.
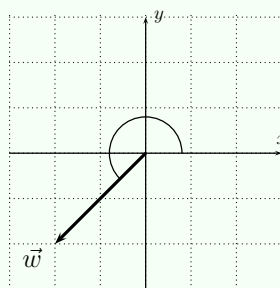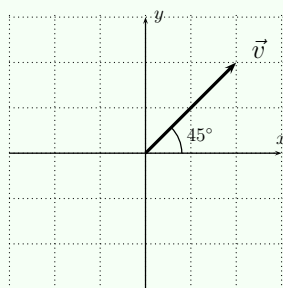
Example 22.8

Find the direction angle for the vectors

$$\vec{v} = \langle 2, 2 \rangle \qquad \text{and} \qquad \vec{w} = \langle -2, -2 \rangle$$

Solution.

For $\vec{v} = \langle 2, 2 \rangle$, the direction angle satisfies $\tan(\theta) = \frac{2}{2} = 1$, and we can compute $\theta = \tan^{-1}(1) = 45°$. This fits well with the depiction of $\vec{v}$ in the plane:

Now, the direction angle for $\vec{w} = \langle -2, -2 \rangle$ satisfies $\tan(\theta) = \frac{-2}{-2} = 1$, but $\tan^{-1}(1) = 45°$ is *not* the angle for $\vec{w}$, since $\vec{w}$ is in the third quadrant (while $45°$ is in the first quadrant). The issue is that the tangent function has the same output values when adding $180°$, that is, $\tan(x + 180°) = \tan(x)$ for all angle x. On the other hand, the outputs of the $\tan^{-1}$ function are in the interval $(-\frac{\pi}{2}, \frac{\pi}{2}) = (-90°, 90°)$ (see Definition 19.2) which are in the first and fourth quadrant. We therefore need to add $180°$ to the $\tan^{-1}(\frac{b}{a})$ whenever the vector lies in the second or third quadrant. We thus get:

$$(\text{angle of } \vec{v}) = 45°$$
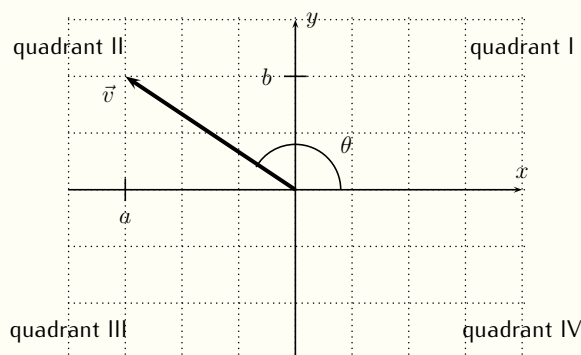$$(\text{angle of } \vec{w}) = 45° + 180° = 225°$$

☐

We can thus summarize the formulas for the magnitude and direction angle as follows.

Observation 22.9: Magnitude and direction angle

Let $\vec{v} = \langle a, b \rangle = \overrightarrow{OR}$ be a vector in the plane $\mathbb{R}^2$ pointing from the origin to $R(a, b)$.

Then the magnitude and direction angle of $\vec{v}$ are given by:

$$\boxed{||\vec{v}|| = \sqrt{a^2 + b^2}} \quad \text{and} \tag{22.6}$$

$$\boxed{\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & \text{if } R \text{ is in quadrant I or IV} \\ \\ \tan^{-1}\left(\frac{b}{a}\right) + 180° & \text{if } R \text{ is in quadrant II or III} \end{cases}} \tag{22.7}$$

Here R is the endpoint of the vector $\vec{v} = \overrightarrow{OR}$ when placed at the origin $(0, 0)$.

Example 22.10

Find the magnitude and direction angle of the given vectors.

a) $\langle -6, 6 \rangle$ b) $\langle 4, -3 \rangle$ c) $\langle -2\sqrt{3}, -2 \rangle$
d) $\langle 8, 4\sqrt{5} \rangle$ e) $\overrightarrow{PQ}$,   where $P(9, 2)$ and $Q(3, 10)$
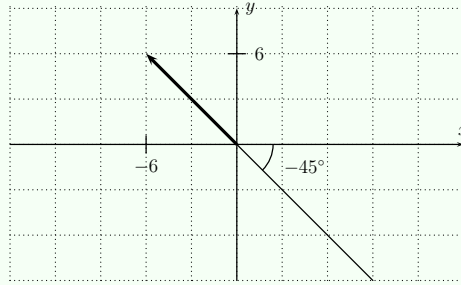
Solution.

a) We use formulas (22.6) and (22.7). The magnitude of $\vec{v} = \langle -6, 6 \rangle$ is

$$||\vec{v}|| = \sqrt{(-6)^2 + 6^2} = \sqrt{36 + 36} = \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

The direction angle θ is given by $\tan(\theta) = \frac{6}{-6} = -1$. Since $\tan^{-1}(-1) = -\tan^{-1}(1) = -45°$ is in the fourth quadrant, but $\vec{v} = \langle -6, 6 \rangle$ drawn at the origin $O(0,0)$ has its endpoint in the second quadrant, we see that the angle $\theta = -45° + 180° = 135°$.

b) The magnitude of $\vec{v} = \langle 4, -3 \rangle$ is

$$||\vec{v}|| = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

The direction angle is given by $\tan(\theta) = \frac{-3}{4}$. Since $\tan^{-1}(\frac{-3}{4}) \approx -36.9°$ is in the fourth quadrant, and $\vec{v} = \langle 4, -3 \rangle$ is indeed in the fourth quadrant, we see that

$$\theta = \tan^{-1}\left(\frac{-3}{4}\right) \approx -36.9°.$$

Sometimes it may be preferable to describe the angle between $0°$ and $360°$. To obtain such and angle for $\vec{v}$, we can add $360°$, which gives and angle of $\approx -36.9° + 360° = 323.1°$ for $\vec{v}$.

c) The magnitude of $\vec{v} = \langle -2\sqrt{3}, -2 \rangle$ is

$$||\vec{v}|| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{4 \cdot 3 + 4} = \sqrt{12 + 4} = \sqrt{16} = 4.$$

The direction angle is given by $\tan(\theta) = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Note that $\tan^{-1}(\frac{\sqrt{3}}{3}) = 30°$ is in the first quadrant, whereas $\vec{v} = \langle -2\sqrt{3}, -2 \rangle$ is in the third quadrant. Therefore, the angle is given by adding an additional $180°$ to the angle:

$$\theta = 30° + 180° = 210°$$

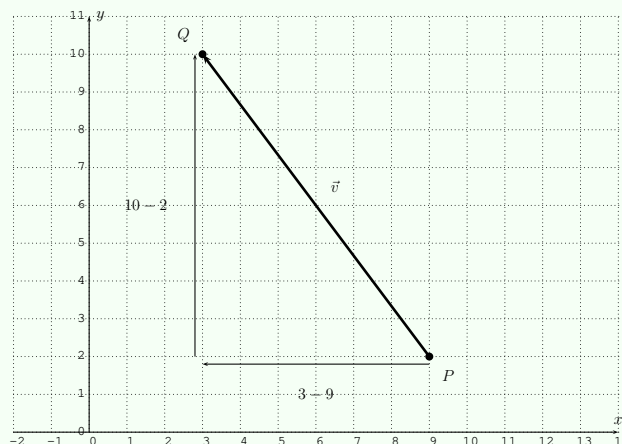d) The magnitude of $\vec{v} = \langle 8, 4\sqrt{5} \rangle$ is

$$||\vec{v}|| = \sqrt{8^2 + (4\sqrt{5})^2} = \sqrt{64 + 16 \cdot 5} = \sqrt{64 + 80} = \sqrt{144} = 12.$$

The direction angle is given by the equation $\tan(\theta) = \frac{4\sqrt{5}}{8} = \frac{\sqrt{5}}{2}$. Since both $\tan^{-1}(\frac{\sqrt{5}}{2}) \approx 48.2°$ and the endpoint of $\vec{v}$ (represented with beginning point at the origin) are in the first quadrant, we have:

$$\theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) \approx 48.2°.$$

e) We first need to find the vector $\vec{v} = \overrightarrow{PQ}$ in the form $\vec{v} = \langle a, b \rangle$. The vector in the plane below shows that $\vec{v}$ is given by

$$\vec{v} = \langle 3 - 9, 10 - 2 \rangle = \langle -6, 8 \rangle$$

From this we calculate the magnitude to be

$$||\vec{v}|| = \sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

The direction angle is given by $\tan(\theta) = \frac{8}{-6} = -\frac{4}{3}$. Note that $\tan^{-1}(-\frac{4}{3}) \approx -53.1°$ is in quadrant IV, whereas $\vec{v} = \langle -6, 8 \rangle$ has its endpoint in quadrant II (when representing $\vec{v}$ with starting point at the origin $O(0,0)$). Therefore, the direction angle is

$$\theta = \tan^{-1}\left(-\frac{4}{3}\right) + 180° \approx 126.9°$$

□

22.2   Operations on vectors

There are two basic operations on vectors, which are the *scalar multiplication* and the *vector addition.* First, let's look at the scalar multiplication.

Definition 22.11: Scalar multiplication

The **scalar multiplication** of a real number r with a vector $\vec{v} = \langle a, b \rangle$ is defined to be the vector given by multiplying r to each coordinate.

$$\boxed{r \cdot \langle a, b \rangle := \langle r \cdot a, r \cdot b \rangle}$$
(22.8)

We study the effect of scalar multiplication with an example.

Example 22.12
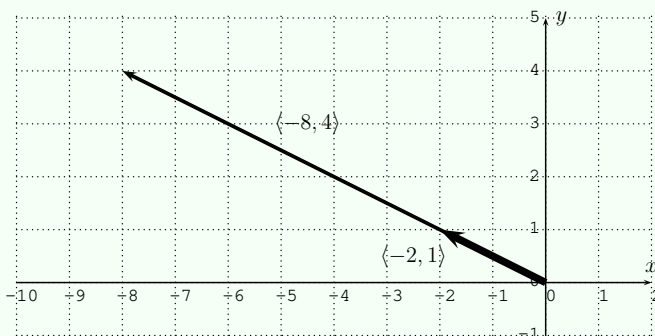
Multiply and graph the vectors.

$$\text{a) } 4 \cdot \langle -2, 1 \rangle \qquad \text{b) } (-3) \cdot \langle -6, -2 \rangle$$

Solution.

a) The calculation is straightforward.

$$4 \cdot \langle -2, 1 \rangle = \langle 4 \cdot (-2), 4 \cdot 1 \rangle = \langle -8, 4 \rangle$$
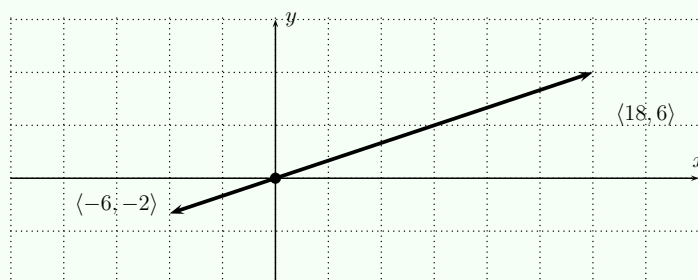
The vectors are displayed below. We see that $\langle -2, 1 \rangle$ and $\langle -8, 4 \rangle$ both have the same direction angle, and the latter stretches the former by a factor 4.

b) Algebraically, we calculate the scalar multiplication as follows:

$$(-3) \cdot \langle -6, -2 \rangle = \langle (-3) \cdot (-6), (-3) \cdot (-2) \rangle = \langle 18, 6 \rangle$$

Furthermore, graphing the vectors gives:

We see that the direction angle of the two vectors differs by $180°$. Indeed, $\langle 18, 6 \rangle$ is obtained from $\langle -6, -2 \rangle$ by reflecting it at the origin $O(0,0)$ and then stretching the result by a factor 3.

☐

We see from the above example, that scalar multiplication by a positive number c does not change the angle of the vector, but it multiplies the magnitude by c.

Observation 22.13: Geometric interpretation of scalar multiplication

Let $\vec{v}$ be a vector with magnitude $||\vec{v}||$ and angle $\theta_{\vec{v}}$. Then, for a positive scalar, $r > 0$, the scalar multiple $r \cdot \vec{v}$ has the same angle as $\vec{v}$, and a magnitude that is r times the magnitude of $\vec{v}$:

$$\text{for } r > 0: \quad ||r \cdot \vec{v}|| = r \cdot ||\vec{v}|| \quad \text{and} \quad \theta_{r \cdot \vec{v}} = \theta_{\vec{v}}$$
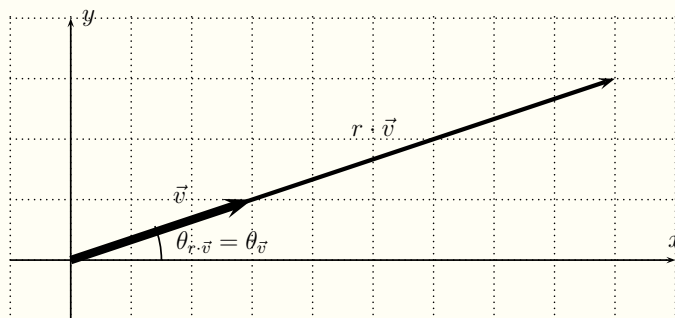
Definition 22.14: Unit vector

A vector $\vec{u}$ is called a **unit vector** if it has a magnitude of 1.

$$\vec{u} \text{ is a unit vector} \quad \Longleftrightarrow \quad ||\vec{u}|| = 1$$

There are two special unit vectors, which are the vectors pointing in the x- and the y-direction.

$$\boxed{\vec{i} := \langle 1, 0 \rangle} \quad \text{and} \quad \boxed{\vec{j} := \langle 0, 1 \rangle} \qquad (22.9)$$

Example 22.15

Find a unit vector in the direction of $\vec{v}$.

$$\text{a) } \langle 8, 6 \rangle \qquad \text{b) } \langle -2, 3\sqrt{7} \rangle$$

Solution.

a) Note that the magnitude of $\vec{v} = \langle 8, 6 \rangle$ is

$$||\langle 8, 6 \rangle|| = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$$

Therefore, if we multiply $\langle 8, 6 \rangle$ by $\frac{1}{10}$ we obtain $\frac{1}{10} \cdot \langle 8, 6 \rangle = \langle \frac{8}{10}, \frac{6}{10} \rangle = \langle \frac{4}{5}, \frac{3}{5} \rangle$, which (according to Observation 22.13 above) has the same direction angle as $\langle 8, 6 \rangle$, and has a magnitude of 1:

$$\left|\left| \frac{1}{10} \cdot \langle 8, 6 \rangle \right|\right| = \frac{1}{10} \cdot ||\langle 8, 6 \rangle|| = \frac{1}{10} \cdot 10 = 1$$

b) The magnitude of $\langle -2, 3\sqrt{7} \rangle$ is

$$||\langle -2, 3\sqrt{7} \rangle|| = \sqrt{(-2)^2 + (3\sqrt{7})^2} = \sqrt{4 + 9 \cdot 7} = \sqrt{4 + 63} = \sqrt{67}.$$

Therefore, we have the unit vector

$$\frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7} \rangle = \langle \frac{-2}{\sqrt{67}}, \frac{3\sqrt{7}}{\sqrt{67}} \rangle$$

$$= \langle \frac{-2\sqrt{67}}{67}, \frac{3\sqrt{7}\sqrt{67}}{67} \rangle = \langle \frac{-2\sqrt{67}}{67}, \frac{3\sqrt{469}}{67} \rangle$$

which again has the same direction angle as $\langle -2, 3\sqrt{7} \rangle$. We can also check that $\frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7} \rangle$ is a *unit vector*, because $||\frac{1}{\sqrt{67}} \cdot \langle -2, 3\sqrt{7} \rangle|| = \frac{1}{\sqrt{67}} \cdot ||\langle -2, 3\sqrt{7} \rangle|| = \frac{1}{\sqrt{67}} \cdot \sqrt{67} = 1.$

$\square$

The second operation on vectors is *vector addition*, which we discuss now.

Definition 22.16: Vector addition

Let $\vec{v} = \langle a, b \rangle$ and $\vec{w} = \langle c, d \rangle$ be two vectors. Then the **vector addition** $\vec{v} + \vec{w}$ is defined by component-wise addition:

$$\boxed{\langle a, b \rangle + \langle c, d \rangle := \langle a + c, b + d \rangle} \tag{22.10}$$

In terms of the plane, the vector addition corresponds to placing the vectors $\vec{v}$ and $\vec{w}$ as the edges of a parallelogram, so that $\vec{v} + \vec{w}$ becomes its diagonal. This is displayed below.
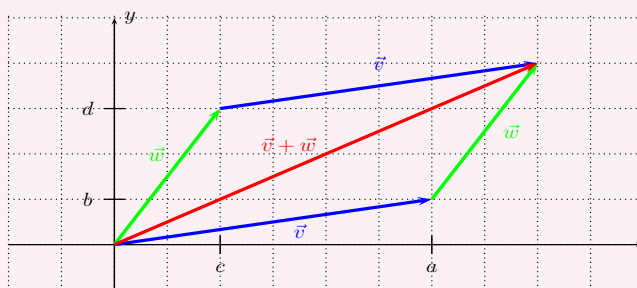
Example 22.17

Perform the vector addition and simplify as much as possible.

a) $\langle 3, -5 \rangle + \langle 6, 4 \rangle$ b) $5 \cdot \langle -6, 2 \rangle - 7 \cdot \langle 1, -3 \rangle$ c) $4\vec{i} + 9\vec{j}$

d) find $2\vec{v} + 3\vec{w}$ for $\vec{v} = -6\vec{i} - 4\vec{j}$ and $\vec{w} = 10\vec{i} - 7\vec{j}$
e) find $-3\vec{v} + 5\vec{w}$ for $\vec{v} = \langle 8, \sqrt{3} \rangle$ and $\vec{w} = \langle 0, 4\sqrt{3} \rangle$

Solution.
We can find the answer by direct algebraic computation.

a) $\langle 3, -5 \rangle + \langle 6, 4 \rangle = \langle 3 + 6, -5 + 4 \rangle = \langle 9, -1 \rangle$

b) $5 \cdot \langle -6, 2 \rangle - 7 \cdot \langle 1, -3 \rangle = \langle -30, 10 \rangle + \langle -7, 21 \rangle = \langle -37, 31 \rangle$

c) $4\vec{i} + 9\vec{j} = 4 \cdot \langle 1, 0 \rangle + 9 \cdot \langle 0, 1 \rangle = \langle 4, 0 \rangle + \langle 0, 9 \rangle = \langle 4, 9 \rangle$

From the last calculation, we see that every vector can be written as a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

$$\boxed{\langle a, b \rangle = a \cdot \vec{i} + b \cdot \vec{j}} \qquad (22.11)$$

We will use this equation for the next example (d). Here, $\vec{v} = -6\vec{i} - 4\vec{j} = \langle -6, -4 \rangle$ and $\vec{w} = 10\vec{i} - 7\vec{j} = \langle 10, -7 \rangle$. Therefore, we obtain:

d) $2\vec{v} + 3\vec{w} = 2 \cdot \langle -6, -4 \rangle + 3 \cdot \langle 10, -7 \rangle$

$= \langle -12, -8 \rangle + \langle 30, -21 \rangle = \langle 18, -29 \rangle$

e) $-3\vec{v} + 5\vec{w} = -3 \cdot \langle 8, \sqrt{3} \rangle + 5 \cdot \langle 0, 4\sqrt{3} \rangle$

$= \langle -24, -3\sqrt{3} \rangle + \langle 0, 20\sqrt{3} \rangle = \langle -24, 17\sqrt{3} \rangle$

Note that the answer could also be written as $-3\vec{v} + 5\vec{w} = -24\vec{i} + 17\sqrt{3}\vec{j}$. $\square$

In many applications in the sciences, vectors play an important role, since many quantities are naturally described by vectors. Examples for this in physics include the velocity $\vec{v}$, acceleration $\vec{a}$, and the force $\vec{F}$ applied to an object.

Example 22.18

The forces $\vec{F}_1$ and $\vec{F}_2$ are applied to an object. Find the resulting total force $\vec{F} = \vec{F}_1 + \vec{F}_2$. Determine the magnitude and direction angle of the total force $\vec{F}$. Approximate these values as necessary. Recall that the international system of units for force is the *newton* $\left[1N = 1\frac{kg \cdot m}{s^2}\right]$.

a) $\vec{F}_1$ has magnitude 3 newtons, and angle $\theta_1 = 45°$,
$\vec{F}_2$ has magnitude 5 newtons, and angle $\theta_2 = 135°$

b) $||\vec{F}_1|| = 7$ newtons, and $\theta_1 = \frac{\pi}{6}$, and
$||\vec{F}_2|| = 4$ newtons, and $\theta_2 = \frac{5\pi}{3}$

Solution.

a) The vectors $\vec{F}_1$ and $\vec{F}_2$ are given by their magnitudes and direction angles. However, the addition of vectors (in Definition 22.16) is defined in terms of their components. Therefore, our first task is to find the vectors in component form. As was stated in Equation (22.5) on page 383, the vectors are calculated by $\vec{v} = \langle\ ||\vec{v}||\cdot\cos(\theta)\ ,\ ||\vec{v}||\cdot\sin(\theta)\ \rangle$. Therefore,

$$
\begin{aligned}
\vec{F}_1 &= \langle 3\cdot\cos(45°), 3\cdot\sin(45°)\rangle \\
&= \langle 3\cdot\frac{\sqrt{2}}{2}, 3\cdot\frac{\sqrt{2}}{2}\rangle = \langle\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\rangle \\
\vec{F}_2 &= \langle 5\cdot\cos(135°), 5\cdot\sin(135°)\rangle \\
&= \langle 5\cdot\frac{-\sqrt{2}}{2}, 5\cdot\frac{\sqrt{2}}{2}\rangle = \langle\frac{-5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\rangle
\end{aligned}
$$

The total force is the sum of the forces.
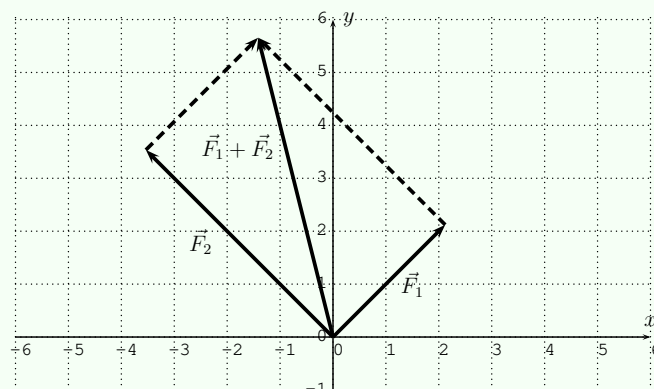
$$
\begin{aligned}
\vec{F} &= \vec{F}_1 + \vec{F}_2 = \langle\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\rangle + \langle\frac{-5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\rangle \\
&= \langle\frac{3\sqrt{2}}{2} + \frac{-5\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}\rangle = \langle\frac{3\sqrt{2} - 5\sqrt{2}}{2}, \frac{3\sqrt{2} + 5\sqrt{2}}{2}\rangle \\
&= \langle\frac{-2\sqrt{2}}{2}, \frac{8\sqrt{2}}{2}\rangle = \langle-\sqrt{2}, 4\sqrt{2}\rangle
\end{aligned}
$$

The total force applied in components is $\vec{F} = \langle-\sqrt{2}, 4\sqrt{2}\rangle$. It has a magnitude of $||\vec{F}|| = \sqrt{(-\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{2 + 16\cdot 2} = \sqrt{34} \approx$

5.83 newton. The direction angle is given by $\tan(\theta) = \frac{4\sqrt{2}}{-\sqrt{2}} = -4$.
Since $\tan^{-1}(-4) \approx -76.0°$ is in quadrant IV, and $\vec{F} = \langle -\sqrt{2}, 4\sqrt{2} \rangle$
is in quadrant II, we see that the direction angle of $\vec{F}$ is

$$\theta = 180° + \tan^{-1}(-4) \approx 180° - 76.0° \approx 104°.$$

b) We solve this example in much the same way we solved part (a).
First, $\vec{F_1}$ and $\vec{F_2}$ in components is given by

$$\vec{F_1} = \langle 7 \cdot \cos\left(\frac{\pi}{6}\right), 7 \cdot \sin\left(\frac{\pi}{6}\right) \rangle = \langle 7 \cdot \frac{\sqrt{3}}{2}, 7 \cdot \frac{1}{2} \rangle = \langle \frac{7\sqrt{3}}{2}, \frac{7}{2} \rangle$$

$$\vec{F_2} = \langle 4 \cdot \cos\left(\frac{5\pi}{3}\right), 4 \cdot \sin\left(\frac{5\pi}{3}\right) \rangle = \langle 4 \cdot \frac{1}{2}, 4 \cdot \frac{-\sqrt{3}}{2} \rangle = \langle 2, -2\sqrt{3} \rangle$$

The total force is therefore:

$$\vec{F} = \vec{F_1} + \vec{F_2} = \langle \frac{7\sqrt{3}}{2}, \frac{7}{2} \rangle + \langle 2, -2\sqrt{3} \rangle = \langle \frac{7\sqrt{3}}{2} + 2, \frac{7}{2} - 2\sqrt{3} \rangle$$
$$\approx \langle 8.06, 0.04 \rangle$$

The magnitude is approximately

$$||\vec{F}|| \approx \sqrt{(8.06)^2 + (0.04)^2} \approx 8.06 \text{ newton}.$$

The direction angle is given by $\tan(\theta) \approx \frac{0.04}{8.06}$. Since $\vec{F}$ is in quadrant
I, we see that $\theta \approx \tan^{-1}\left(\frac{0.04}{8.06}\right) \approx 0.3°$.

□

Note 22.19: Vector space

In general, a *vector space* is an abstract algebraic notion that is fundamental to many areas of mathematics. Although we do not explicitly use this structure in this text, we will state its definition. A **vector space** is a set V, with the following structures and properties. The elements of V are called vectors, denoted by the usual symbol $\vec{v}$. For any vectors $\vec{v}$ and $\vec{w}$ there is a vector $\vec{v} + \vec{w}$, called the vector addition. For any real number r and vector $\vec{v}$, there is a vector $r \cdot \vec{v}$ called

the scalar product. These operations have to satisfy the following properties.

Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Zero element: there is a vector $\vec{o}$ such that $\vec{o} + \vec{v} = \vec{v}$ and $\vec{v} + \vec{o} = \vec{v}$ for every vector $\vec{v}$

Negative element: for every $\vec{v}$ there is a vector $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \vec{o}$ and $(-\vec{v}) + \vec{v} = \vec{o}$

Distributivity (1): $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$

Distributivity (2): $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

Scalar compatibility: $(r \cdot s) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

Identity: $1 \cdot \vec{v} = \vec{v}$

An important example of a vector space is the 2-dimensional plane $V = \mathbb{R}^2$ as it was discussed in this chapter. A thorough introduction to this topic will be provided in a course in linear algebra.

22.3   Exercises

Exercise 22.1

Graph the vectors in the plane.

a) $\overrightarrow{PQ}$ with $P(2, 1)$ and $Q(4, 7)$ b) $\overrightarrow{PQ}$ with $P(-3, 3)$ and $Q(-5, -4)$

c) $\overrightarrow{PQ}$ with $P(0, -4)$ and $Q(6, 0)$ d) $\langle -2, 4 \rangle$

e) $\langle -3, -3 \rangle$ f) $\langle 5, 5\sqrt{2} \rangle$

Exercise 22.2

Find the magnitude and direction angle of the vector.

a) $\langle 6, 8 \rangle$ b) $\langle -2, 5 \rangle$ c) $\langle -4, -4 \rangle$

d) $\langle 3, -3 \rangle$ e) $\langle 2, -2 \rangle$ f) $\langle 4\sqrt{3}, 4 \rangle$

g) $\langle -\sqrt{3}, -1 \rangle$ h) $\langle -4, 4\sqrt{3} \rangle$ i) $\langle -2\sqrt{3}, -2 \rangle$

j) $\overrightarrow{PQ}$,   where $P(3, 1)$ and $Q(7, 4)$

k) $\overrightarrow{PQ}$,   where $P(4, -2)$ and $Q(-5, 7)$

Exercise 22.3

Perform the operation on the vectors.

a) $5 \cdot \langle 3, 2 \rangle$

b) $2 \cdot \langle -1, 4 \rangle$

c) $(-10) \cdot \langle -\frac{3}{2}, -\frac{7}{5} \rangle$

d) $\langle 2, 3 \rangle + \langle 6, 1 \rangle$

e) $\langle 5, -4 \rangle - \langle -8, -9 \rangle$

f) $3 \cdot \langle 5, 3 \rangle + 4 \cdot \langle 2, 8 \rangle$

g) $(-2)\langle -5, -4 \rangle - 6\langle -1, -2 \rangle$

h) $\frac{2}{3}\langle -3, 6 \rangle - \frac{7}{5}\langle 10, -15 \rangle$

i) $\sqrt{2} \cdot \langle \frac{\sqrt{8}}{6}, \frac{-5\sqrt{2}}{12} \rangle - 2\langle \frac{2}{3}, \frac{5}{3} \rangle$

j) $6\vec{i} - 4\vec{j}$

k) $-5\vec{i} + \vec{j} + 3\vec{i}$

l) $3 \cdot \langle -4, 2 \rangle - 8\vec{j} + 12\vec{i}$

m) find $4\vec{v} + 7\vec{w}$ for $\vec{v} = \langle 2, 3 \rangle$ and $\vec{w} = \langle 5, 1\sqrt{3} \rangle$

n) find $\vec{v} - 2\vec{w}$ for $\vec{v} = \langle -11, -6 \rangle$ and $\vec{w} = \langle -3, 2 \rangle$

o) find $3\vec{v} - \vec{w}$ for $\vec{v} = -4\vec{i} + 7\vec{j}$ and $\vec{w} = 6\vec{i} + \vec{j}$

p) find $-\vec{v} - \sqrt{5}\vec{w}$ for $\vec{v} = 5\vec{j}$ and $\vec{w} = -8\vec{i} + \sqrt{5}\vec{j}$

Exercise 22.4

Find a unit vector in the direction of the given vector.

a) $\langle 8, -6 \rangle$

b) $\langle -3, -\sqrt{7} \rangle$

c) $\langle 9, 2 \rangle$

d) $\langle -\sqrt{5}, \sqrt{31} \rangle$

e) $\langle 5\sqrt{2}, 3\sqrt{10} \rangle$

f) $\langle 0, -\frac{3}{5} \rangle$

Exercise 22.5

Find the approximate magnitude and direction angle of sum $\vec{v} = \vec{v_1} + \vec{v_2}$ of the given vectors $\vec{v_1}$ and $\vec{v_2}$ (see Example 22.18).

a) $\|\vec{v_1}\| = 6$, and $\theta_1 = 60°$, and
$\|\vec{v_2}\| = 2$, and $\theta_2 = 180°$

b) $\|\vec{v_1}\| = 3.7$, and $\theta_1 = 92°$, and
$\|\vec{v_2}\| = 2.2$, and $\theta_2 = 253°$

c) $\|\vec{v_1}\| = 8$, and $\theta_1 = \frac{3\pi}{4}$, and
$\|\vec{v_2}\| = 8\sqrt{2}$, and $\theta_2 = \frac{3\pi}{2}$