Precalculus

Third Edition (3.0)

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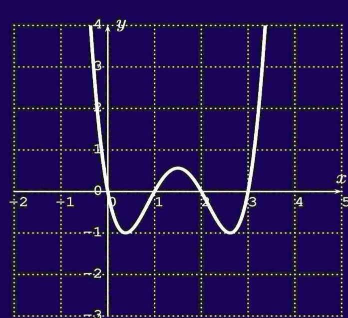

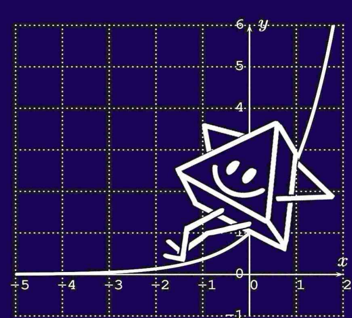

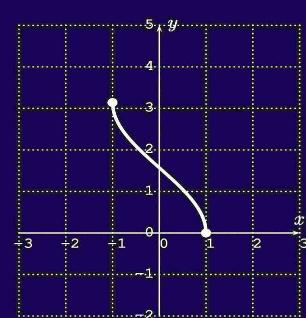

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Chapter 21

Trigonometric identities

In this section, we state and summarize various important identities of trigonometric functions, some of which we have already used in previous sections. We will look at four kinds of identities:

1. Reciprocal identities and quotient identities
2. Pythagorean identities
3. Identities involving signs
4. Identities from adding $\frac{\pi}{2}$ or π to an angle
5. Addition, subtraction of angles formulas, half- and double-angle formulas

21.1 Reciprocal, Pythagorean, and sign identities

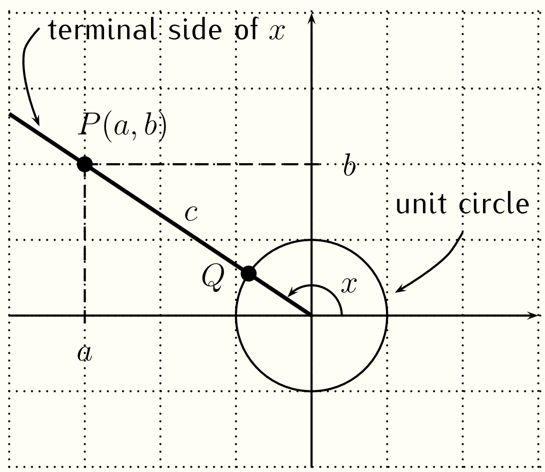
We start by recalling the definition of the trigonometric functions. In fact, going beyond the unit circle, we will restate the definition in a slightly more general setting, that is, stating the trigonometric functions for *any* point on the terminal side of the angle.

Observation 21.1: $\sin$, $\cos$, $\tan$ via point on the terminal side

Let x be an angle. Consider the terminal side of the angle x, and assume that the point $P(a, b)$ is a point on the terminal side of x (not necessarily on the unit circle). Let c be the distance from P to the origin $(0,0)$. Note that the the Pythagorean theorem states that

$$a^2+b^2=c^2 \quad \Longrightarrow \quad \boxed{c=\sqrt{a^2+b^2}} \tag{21.1}$$

Now, dividing the coordinates of P by c gives another point Q also on the terminal side of x with coordinates $(\frac{a}{c},\frac{b}{c})$, and moreover, Q is on the unit circle, since its distance to the origin is $\sqrt{(\frac{a}{c})^2+(\frac{b}{c})^2} = \sqrt{\frac{a^2}{c^2}+\frac{b^2}{c^2}} = \sqrt{\frac{a^2+b^2}{c^2}} = \sqrt{\frac{c^2}{c^2}} = \sqrt{1} = 1$.



Therefore, the trigonometric function values of x are given by the coordinates of $Q(\frac{a}{c},\frac{b}{c})$, that is:

$$\boxed{\sin(x)=\frac{b}{c}} \qquad \boxed{\cos(x)=\frac{a}{c}} \qquad \boxed{\tan(x)=\frac{b}{a}} \tag{21.2}$$

where we used that $\tan(x)=\frac{\sin(x)}{\cos(x)}=\frac{\frac{b}{c}}{\frac{a}{c}}=\frac{b}{c}\cdot\frac{c}{a}=\frac{b}{a}$. Moreover, the cosecant, the secant, and the cotangent are given by:

$$\csc(x)=\frac{1}{\sin(x)}=\frac{c}{b} \qquad \sec(x)=\frac{1}{\cos(x)}=\frac{c}{a} \qquad \cot(x)=\frac{1}{\tan(x)}=\frac{a}{b}$$

1. Reciprocal identities and quotient identities

From the above observation, we have the following immediate identities between the trigonometric functions:

Observation 21.2: Reciprocal identities and quotient identities

The following reciprocal identities hold true:

$$\boxed{\sin(x) = \frac{1}{\csc(x)}} \quad \boxed{\cos(x) = \frac{1}{\sec(x)}} \quad \boxed{\tan(x) = \frac{1}{\cot(x)}} \tag{21.3}$$

$$\boxed{\csc(x) = \frac{1}{\sin(x)}} \quad \boxed{\sec(x) = \frac{1}{\cos(x)}} \quad \boxed{\cot(x) = \frac{1}{\tan(x)}} \tag{21.4}$$

The quotient identities hold true:

$$\boxed{\tan(x) = \frac{\sin(x)}{\cos(x)}} \qquad \boxed{\cot(x) = \frac{\cos(x)}{\sin(x)}} \tag{21.5}$$

Example 21.3

Write the expression as one of the six trigonometric functions.

a) $\sin(x) \cdot \cot(x)$ b) $\frac{\cot(x)}{\csc(x)\cos(x)} \cdot \frac{\tan(x)}{\sin(x)}$

Solution.

a) $\sin(x) \cdot \cot(x) = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x)$

b) We rewrite in terms of $\sin(x)$ and $\cos(x)$ and cancel:

$$\frac{\cot(x)}{\csc(x)\cos(x)} \cdot \frac{\tan(x)}{\sin(x)} = \frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}\cos(x)} \cdot \frac{\frac{\sin(x)}{\cos(x)}}{\sin(x)} = \frac{1}{\cos(x)} = \sec(x)$$

□

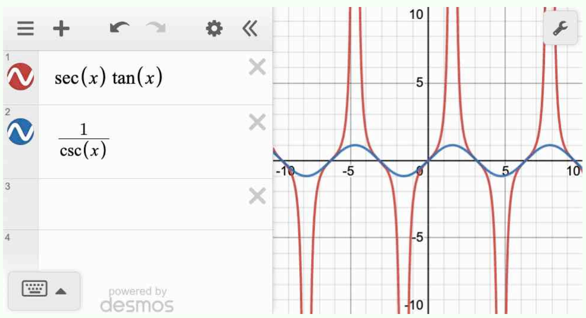
Example 21.4

Determine whether the identity is true or false.

a) $\sec(x) \cdot \tan(x) = \frac{1}{\csc(x)}$ b) $\frac{\cos(x)}{\csc(x)} = \frac{\sin^2(x)}{\tan(x)}$

Solution.

a) Since $\sec(x) \cdot \tan(x) = \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)}$ and $\frac{1}{\csc(x)} = \frac{1}{\frac{1}{\sin(x)}} = \sin(x)$, we see that the expression on the right-hand side is different from the expression on the left-hand side, and so, the identity is false. We can also check this by graphing both sides with the calculator, which, indeed, shows that the two expressions are different.



b) Note that the calculator appears to show the same graph for $\frac{\cos(x)}{\csc(x)}$ and $\frac{\sin^2(x)}{\tan(x)}$.

To confirm this, calculate $\frac{\cos(x)}{\csc(x)} = \frac{\cos(x)}{\frac{1}{\sin(x)}} = \cos(x) \cdot \sin(x)$ and $\frac{\sin^2(x)}{\tan(x)} = \frac{\sin^2(x)}{\frac{\sin(x)}{\cos(x)}} = \sin^2(x)\frac{\cos(x)}{\sin(x)} = \sin(x) \cdot \cos(x)$, which are, indeed, equal. Therefore, the identity is true.

□

2. Pythagorean identities

The next identities come from the Pythagorean theorem.

Observation 21.5: Pythagorean identities

The Pythagorean identities hold true:

$$\boxed{\sin^2(x) + \cos^2(x) = 1} \tag{21.6}$$

$$\boxed{\sec^2(x) = 1 + \tan^2(x)} \tag{21.7}$$

$$\boxed{\csc^2(x) = 1 + \cot^2(x)} \tag{21.8}$$

Proof. In the notation of Observation 21.1, where $P(a, b)$ is a point on the terminal side of x with distance c to the origin, the Pythagorean theorem states $a^2 + b^2 = c^2$ (see (21.1)). Therefore, from (21.2), we have

$$\sin^2(x) + \cos^2(x) = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1$$

Similarly, $1 + \tan^2(x) = 1 + \left(\frac{b}{a}\right)^2 = \frac{a^2+b^2}{a^2} = \frac{c^2}{a^2} = \left(\frac{c}{a}\right)^2 = \sec^2(x)$, and $1 + \cot^2(x) = 1 + \left(\frac{a}{b}\right)^2 = \frac{b^2+a^2}{b^2} = \frac{c^2}{b^2} = \left(\frac{c}{b}\right)^2 = \csc^2(x)$. □

Example 21.6

Simplify the expression as much as possible.

a) $(\cos(x) - 1) \cdot (\cos(x) + 1)$ b) $\frac{1-\sec^2(x)}{\cot(x)}$ c) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}$

Solution.

a) $(\cos(x) - 1) \cdot (\cos(x) + 1) = \cos^2(x) - 1 \overset{(21.6)}{=} -\sin^2(x)$

b) $\frac{1-\sec^2(x)}{\cot(x)} \overset{(21.7)}{=} \frac{-\tan^2(x)}{\frac{1}{\tan(x)}} = -\tan^2(x) \cdot \frac{\tan(x)}{1} = -\tan^3(x)$

c) $\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} = \frac{\sin^2(x)}{\sin(x)\cos(x)} + \frac{\cos^2(x)}{\sin(x)\cos(x)} = \frac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)} = \frac{1}{\sin(x)\cos(x)}$

□

Example 21.7

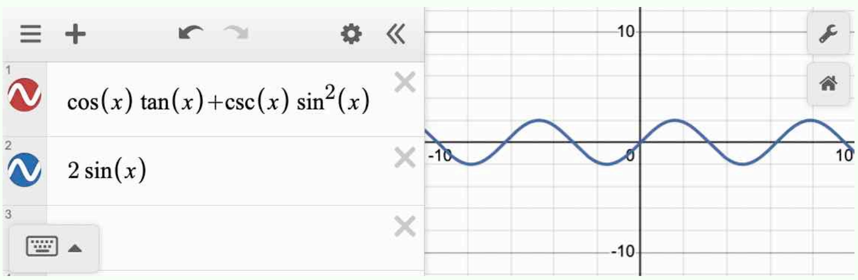
Determine whether the identity is true or false.

a) $\cos(x)\tan(x) + \csc(x)\sin^2(x) = 2\sin(x)$

b) $\frac{\cos^2(x)-1}{1-\sec^2(x)} = \frac{\cos(x)+1}{2}$

Solution.

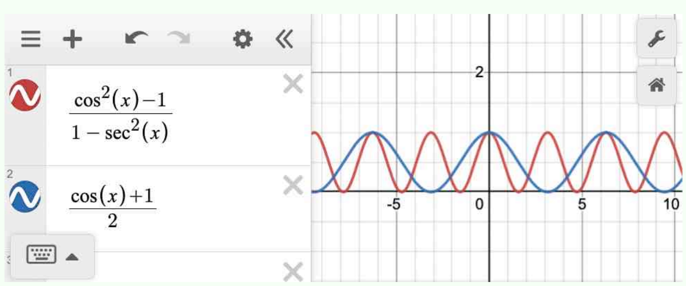
a) We use the calculator to check for differences between the right- and left-hand sides of the equation. The two sides appear to be equal.



To verify the identity, we compute:

$$\begin{aligned}\cos(x)\tan(x) + \csc(x)\sin^2(x) &= \cos(x)\frac{\sin(x)}{\cos(x)} + \frac{1}{\sin(x)}\sin^2(x)\\ &= \sin(x) + \sin(x) = 2\sin(x)\end{aligned}$$

b) The calculator shows that the two sides differ. The identity is false.



□

3. Identities involving signs

When changing an angle to its negative angle, the trigonometric functions also transform in a well-behaved manner. We now state and prove these transformation identities.

Observation 21.8: Identities involving signs

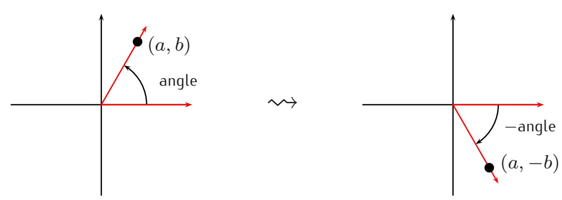
The definition of $\sin(x)$ and $\cos(x)$ gives the following behavior under sign change.

$$\boxed{\sin(-x) = -\sin(x)} \qquad \boxed{\cos(-x) = \cos(x)} \tag{21.9}$$

$$\boxed{\csc(-x) = -\csc(x)} \qquad \boxed{\sec(-x) = \sec(x)} \tag{21.10}$$

$$\boxed{\tan(-x) = -\tan(x)} \qquad \boxed{\cot(-x) = -\cot(x)} \tag{21.11}$$

Proof. The negative of an angle has a terminal side that is reflected about the x-axis, so that the cosine (which is the x-coordinate of a point on the terminal side) stays the same, and the sine (which is the y-coordinate of a point on the terminal side) becomes the negative of the original angle.



While the above picture is for an angle in the first quadrant, the argument holds in general. Convince yourself that the same holds in other quadrants as well!

The other identities (21.10) and (21.11) then follow from the reciprocal and quotient identities (21.4) and (21.5). □

21.2 *Optional section:* Further identities revisited

To give a more complete picture, we now state and provide a proof for some further identities. Several of these identities have already been encountered in previous sections.

4. Identities from adding $\frac{\pi}{2}$ or π to an angle

Observation 21.9: Identities from adding $\frac{\pi}{2}$ or π to an angle

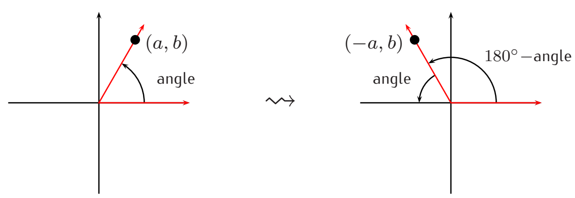
$$\boxed{\sin(\pi - x) = \sin(x)} \qquad \boxed{\cos(\pi - x) = -\cos(x)} \tag{21.12}$$

$$\boxed{\sin(x + \frac{\pi}{2}) = \cos x} \qquad \boxed{\cos(x + \frac{\pi}{2}) = -\sin(x)} \tag{21.13}$$

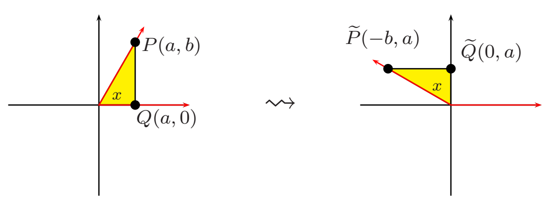
$$\boxed{\sin(x - \frac{\pi}{2}) = -\cos x} \qquad \boxed{\cos(x - \frac{\pi}{2}) = \sin(x)} \tag{21.14}$$

Proof. Just as in the proof to Observation 21.8, we will give the argument in general, but only provide pictures for an angle in the first quadrant. Convince yourself that the same argument holds in other quadrants as well!

Taking $(\pi - \text{angle}) = (180° - \text{angle})$ reflects the terminal side of the angle about the y-axis. Therefore, the cosine becomes negative, and the sine stays the same.



For the identity $\sin(x + \frac{\pi}{2}) = \cos x$, note that adding $\frac{\pi}{2} = 90°$ rotates the terminal side by $90°$. If $P(a, b)$ are coordinates on the terminal side of the angle, then consider the triangle $\triangle OQP$, given by the points $O(0, 0)$ and $Q(a, 0)$. Note that triangle $\triangle OQP$ is congruent to triangle $\triangle O\widetilde{Q}\widetilde{P}$ where $\widetilde{Q}(0, a)$ and $\widetilde{P}(-b, a)$, and they have the same angle x at the origin.



Note that the point $\widetilde{P}(-b, a)$ lies on the terminal side of $x + 90°$. Thus, $\sin(x + 90°) = a$ is the vertical coordinate of $\widetilde{P}(-b, a)$, which equals $\cos(x) = a$, the horizontal coordinate of $P(a, b)$. We have shown that $\sin(x + \frac{\pi}{2}) = \cos x$ for all x.

We check the remaining identities with the identities we have already proved. Applying $\cos(u) = \sin(u + \frac{\pi}{2})$ to $u = x + \frac{\pi}{2}$ gives:

$$\cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + \pi) \overset{(21.12)}{=} \sin(\pi - (x + \pi)) \overset{(21.9)}{=} \sin(-x) = -\sin(x)$$

This proves (21.13). For (21.14), note that:

$$-\cos(x) \overset{(21.12)}{=} \cos(\pi - x) \overset{(21.9)}{=} \cos(x-\pi) \overset{(21.13)}{=} \sin(x-\pi+\frac{\pi}{2}) = \sin(x-\frac{\pi}{2})$$

$$\sin(x) \overset{(21.12)}{=} \sin(\pi - x) \overset{(21.9)}{=} -\sin(x-\pi) \overset{(21.13)}{=} \cos(x-\pi+\frac{\pi}{2}) = \cos(x-\frac{\pi}{2})$$

□

Example 21.10

Simplify the expression as much as possible.

a) $\cos(x+\pi)$ b) $\tan(x+\frac{\pi}{2})$

Solution.

a) $\cos(x+\pi) = \cos(\pi-(-x)) \overset{(21.12)}{=} -\cos(-x) \overset{(21.9)}{=} -\cos(x)$

b) $\tan(x+\frac{\pi}{2}) = \frac{\sin(x+\frac{\pi}{2})}{\cos(x+\frac{\pi}{2})} \overset{(21.13)}{=} \frac{\cos(x)}{-\sin(x)} = -\cot(x)$

□

5. Addition, subtraction of angles formulas, half- and double-angle formulas

We end this section by revisiting the addition and subtraction of angles formulas, and the half- and double-angle formulas. In fact, we will give a proof of these identities. We first recall the identities and give an example.

Proposition 21.11: Addition and subtraction of angles formulas

For any angles α and β, we have the following **addition and subtraction of angles formulas**:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

Proposition 21.12: Half- and double-angle formulas

Let α be an angle. Then we have the **half-angle formulas**:

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

Here, the signs "$\pm$" are determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies. (For more on the signs, see also page 310.)
Furthermore, we have the **double-angle formulas**:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1 \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Example 21.13

Find the trigonometric functions of 2α when α has the properties below.

a) $\sin(\alpha) = \frac{3}{5}$, and α is in quadrant II
b) $\tan(\alpha) = \frac{12}{5}$, and α is in quadrant III

Solution.

a) From $\sin^2(\alpha)+\cos^2(\alpha)=1$, we find that $\cos^2(\alpha) = 1-\sin^2(\alpha)$, and since α is in the second quadrant, $\cos(\alpha)$ is negative, so that

$$\begin{aligned}
\cos(\alpha) &= -\sqrt{1-\sin^2(\alpha)} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\sqrt{1-\frac{9}{25}} \\
&= -\sqrt{\frac{25-9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5},
\end{aligned}$$

and

$$\tan(\alpha) = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{-4}{5}} = \frac{3}{5}\cdot\frac{5}{-4} = -\frac{3}{4}$$

From this we can calculate the solution by plugging these values into the double-angle formulas.

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{(-4)}{5} = \frac{-24}{25}\\
\cos(2\alpha) &= \cos^2(\alpha)-\sin^2(\alpha) = \left(\frac{-4}{5}\right)^2-\left(\frac{3}{5}\right)^2 = \frac{16}{25}-\frac{9}{25} = \frac{7}{25}\\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\left(\frac{-3}{4}\right)}{1-\left(\frac{-3}{4}\right)^2} = \frac{\frac{-3}{2}}{1-\frac{9}{16}} = \frac{\frac{-3}{2}}{\frac{16-9}{16}}\\
&= \frac{-3}{2}\cdot\frac{16}{7} = \frac{-24}{7}
\end{aligned}$$

b) Similar to the calculation in part (a), we first calculate $\sin(\alpha)$ and $\cos(\alpha)$, which are both negative in the third quadrant. Recall from Equation (21.6) on page 364 that $\sec^2\alpha = 1+\tan^2\alpha$, where $\sec\alpha = \frac{1}{\cos\alpha}$. Therefore,

$$\sec^2\alpha = 1+\left(\frac{12}{5}\right)^2 = 1+\frac{144}{25} = \frac{25+144}{25} = \frac{169}{25} \implies \sec\alpha = \pm\frac{13}{5}$$

Since $\cos(\alpha)$ is negative (in quadrant III), so is $\sec(\alpha)$, so that we get,

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{-\frac{13}{5}} = -\frac{5}{13}$$

Furthermore, $\sin^2\alpha = 1-\cos^2\alpha$, and $\sin\alpha$ is negative (in quadrant III), we have

$$\begin{aligned}
\sin\alpha &= -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\left(-\frac{5}{13}\right)^2} = -\sqrt{1-\frac{25}{169}}\\
&= -\sqrt{\frac{169-25}{169}} = -\sqrt{\frac{144}{169}} = -\frac{12}{13}
\end{aligned}$$

Thus, we obtain the solution as follows:

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha = 2\cdot\frac{(-12)}{13}\cdot\frac{(-5)}{13} = \frac{120}{169}\\
\cos(2\alpha) &= \cos^2(\alpha)-\sin^2(\alpha) = \left(\frac{-5}{13}\right)^2-\left(\frac{-12}{13}\right)^2
\end{aligned}$$

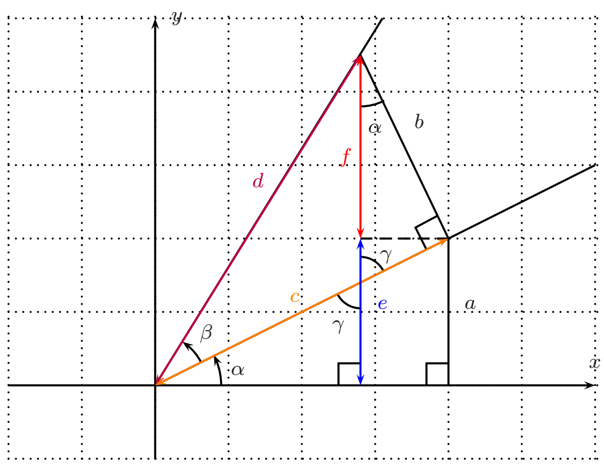
$$
\begin{aligned}
&= \frac{25}{169} - \frac{144}{169} = \frac{-119}{169}\\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\cdot\frac{12}{5}}{1-\left(\frac{12}{5}\right)^2} = \frac{\frac{24}{5}}{1-\frac{144}{25}} = \frac{\frac{24}{5}}{\frac{25-144}{25}}\\
&= \frac{24}{5}\cdot\frac{25}{-119} = \frac{120}{-119}
\end{aligned}
$$

□

We now give a proof of Proposition 21.11.

Proof of Proposition 21.11. We start with the proof of the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are angles between 0 and $\frac{\pi}{2} = 90°$. We prove the addition formulas (for $\alpha, \beta \in (0, \frac{\pi}{2})$) in a quite elementary way, and then show that the addition formulas also hold for arbitrary angles α and β.

To find $\sin(\alpha+\beta)$, consider the following setup.

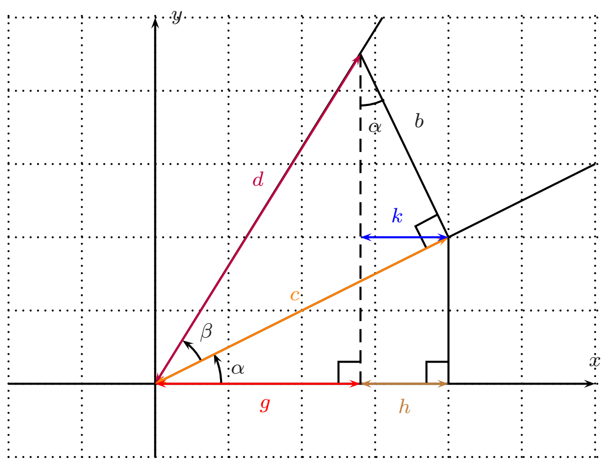


Note that there are vertically opposite angles, labelled by γ, which are therefore equal. These angles are angles in two right triangles, with the third angle being α. We therefore see that the angle α appears again as the angle among the sides b and f. With this, we can now calculate $\sin(\alpha+\beta)$.

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{e+f}{d} = \frac{e}{d} + \frac{f}{d} = \frac{a}{d} + \frac{f}{d} = \frac{a}{c}\cdot\frac{c}{d} + \frac{f}{b}\cdot\frac{b}{d}\\
&= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)
\end{aligned}
$$

The above figure displays the situation when $\alpha+\beta \leq \frac{\pi}{2}$. There is a similar figure for $\frac{\pi}{2} < \alpha+\beta < \pi$. (We recommend as an exercise to draw the corresponding figure for the case of $\frac{\pi}{2} < \alpha+\beta < \pi$.)

Next, we prove the addition formula for $\cos(\alpha+\beta)$. The following figure depicts the relevant objects.



We calculate $\cos(\alpha+\beta)$ as follows.

$$\begin{aligned}\cos(\alpha+\beta) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{g}{d} = \frac{g+h}{d} - \frac{h}{d} = \frac{g+h}{d} - \frac{k}{d} = \frac{g+h}{c}\cdot\frac{c}{d} - \frac{k}{b}\cdot\frac{b}{d} \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{aligned}$$

Again, there is a corresponding figure when the angle $\alpha+\beta$ is greater than $\frac{\pi}{2}$. (We encourage the student to check the addition formula for this situation as well.)

We therefore have proved the addition formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are angles between 0 and $\frac{\pi}{2}$, which we will now extend to all angles α and β. First, note that the addition formulas are trivially true when α or β are 0. (Check this!) Next, by observing that $\sin(x)$ and $\cos(x)$ can be converted to each other via shifts of $\frac{\pi}{2}$, (that is, by using the identities (21.13) and (21.14)), we obtain that

$$\begin{aligned}\sin(x+\frac{\pi}{2}) &= \cos x, &\qquad \cos(x+\frac{\pi}{2}) &= -\sin(x),\\ \sin(x-\frac{\pi}{2}) &= -\cos x, &\qquad \cos(x-\frac{\pi}{2}) &= \sin(x).\end{aligned}$$

With this, we extend the addition identities for α by $\pm\frac{\pi}{2}$ as follows:

$$\begin{aligned}\sin\left((\alpha+\frac{\pi}{2})+\beta\right) &= \sin(\alpha+\beta+\frac{\pi}{2}) = \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\\ &= \sin(\alpha+\frac{\pi}{2})\cos(\beta)+\cos(\alpha+\frac{\pi}{2})\sin(\beta),\\ \sin\left((\alpha-\frac{\pi}{2})+\beta\right) &= \sin(\alpha+\beta-\frac{\pi}{2}) = -\cos(\alpha+\beta) = -\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\\ &= \sin(\alpha-\frac{\pi}{2})\cos(\beta)+\cos(\alpha-\frac{\pi}{2})\sin(\beta),\\ \cos\left((\alpha+\frac{\pi}{2})+\beta\right) &= \cos(\alpha+\beta+\frac{\pi}{2}) = -\sin(\alpha+\beta) = -\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \cos(\alpha+\frac{\pi}{2})\cos(\beta)-\sin(\alpha+\frac{\pi}{2})\sin(\beta),\\ \cos\left((\alpha-\frac{\pi}{2})+\beta\right) &= \cos(\alpha+\beta-\frac{\pi}{2}) = \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ &= \cos(\alpha-\frac{\pi}{2})\cos(\beta)-\sin(\alpha-\frac{\pi}{2})\sin(\beta).\end{aligned}$$

There are similar proofs to extend the identities for β. An induction argument shows the validity of the addition formulas for arbitrary angles α and β.

The remaining formulas now follow via the use of trigonometric identities.

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}}$$

This shows that $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$. For the relations with $\alpha-\beta$, we use the fact that sin and tan are odd functions, whereas cos is an even function. See identities (18.2) and (18.4).

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))=\sin(\alpha)\cos(-\beta)+\cos(\alpha)\sin(-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,\\
\cos(\alpha-\beta) &= \cos(\alpha+(-\beta))=\cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta,\\
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta))=\frac{\tan(\alpha)+\tan(-\beta)}{1-\tan(\alpha)\tan(-\beta)}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
\end{aligned}$$

This completes the proof of the proposition. □

Finally, using Proposition 21.11, we also prove Proposition 21.12.

Proof of Proposition 21.12. We start with the double angle formulas. Using Proposition 21.11, we have:

$$\begin{aligned}
\sin(2\alpha) &= \sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha=2\sin\alpha\cos\alpha\\
\cos(2\alpha) &= \cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha=\cos^2\alpha-\sin^2\alpha\\
\tan(2\alpha) &= \tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}=\frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Notice that $\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$ can be rewritten using $\sin^2\alpha+\cos^2\alpha=1$ as follows:

$$\begin{aligned}
& \cos^2\alpha-\sin^2\alpha = (1-\sin^2\alpha)-\sin^2\alpha=1-2\sin^2\alpha\\
\text{and}\quad & \cos^2\alpha-\sin^2\alpha = \cos^2\alpha-(1-\cos^2\alpha)=2\cos^2\alpha-1
\end{aligned}$$

This shows the double-angle formulas. These formulas can now be used to prove the half-angle formulas.

$$\begin{aligned}
\cos(2\alpha) &= 1-2\sin^2\alpha \implies 2\sin^2\alpha=1-\cos(2\alpha)\implies \sin^2\alpha=\frac{1-\cos(2\alpha)}{2}\\
&\implies \sin\alpha=\pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \overset{\text{replace } \alpha \text{ by } \frac{\alpha}{2}}{\implies} \sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}\\
\cos(2\alpha) &= 2\cos^2\alpha-1 \implies 2\cos^2\alpha=1+\cos(2\alpha)\implies \cos^2\alpha=\frac{1+\cos(2\alpha)}{2}\\
&\implies \cos\alpha=\pm\sqrt{\frac{1+\cos(2\alpha)}{2}} \overset{\text{replace } \alpha \text{ by } \frac{\alpha}{2}}{\implies} \cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}\\
\text{in particular:}\quad & \tan\frac{\alpha}{2}=\frac{\sin(\frac{\alpha}{2})}{\cos(\frac{\alpha}{2})}=\frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

For the first two formulas for $\tan\frac{\alpha}{2}$ we simplify $\sin(2\alpha)\cdot\tan(\alpha)$ and $(1+\cos(2\alpha))\cdot\tan(\alpha)$ as follows.

$$\begin{aligned}
\sin(2\alpha)\cdot\tan(\alpha) &= 2\sin\alpha\cos\alpha\cdot\frac{\sin\alpha}{\cos\alpha}=2\sin^2\alpha=1-\cos(2\alpha)\\
&\implies \tan(\alpha)=\frac{1-\cos(2\alpha)}{\sin(2\alpha)} \overset{\text{replace } \alpha \text{ by } \frac{\alpha}{2}}{\implies} \tan(\frac{\alpha}{2})=\frac{1-\cos(\alpha)}{\sin(\alpha)}\\
(1+\cos(2\alpha))\cdot\tan(\alpha) &= 2\cos^2\alpha\cdot\frac{\sin\alpha}{\cos\alpha}=2\sin\alpha\cos\alpha=\sin(2\alpha)\\
&\implies \tan(\alpha)=\frac{\sin(2\alpha)}{1+\cos(2\alpha)} \overset{\text{replace } \alpha \text{ by } \frac{\alpha}{2}}{\implies} \tan(\frac{\alpha}{2})=\frac{\sin(\alpha)}{1+\cos(\alpha)}
\end{aligned}$$

This completes the proof of the proposition. □

21.3 Exercises

Exercise 21.1

Write the expression as one of the six trigonometric functions.

a) $\cos(x) \cdot \tan(x)$ b) $\sec(x) \cdot \cot(x)$ c) $\frac{\csc(x)}{\sec(x)}$

d) $\tan(x) \cdot \frac{\cot(x)}{\sin(x)}$ e) $\frac{\cot(x)}{\csc(x)}$ f) $\frac{\sin(x)}{\cot(x)} \cdot \csc^2(x)$

Exercise 21.2

Determine if the identity is true or false. If the identity is true, then give an argument for why it is true.

a) $\cos(x) \cdot \csc(x) = \sin(x) \cdot \sec(x)$

b) $\frac{\sin(x)}{\cot(x)} = \frac{\tan(x)}{\csc(x)}$

c) $\frac{\csc(x)}{\sin(x)} = \frac{\cot(x)}{\tan(x)}$

d) $\sin(x) \cdot \cos(x) \cdot \csc^2(x) = \frac{\csc(x)}{\sec(x)}$

Exercise 21.3

Simplify the expression as much as possible.

a) $\frac{\cos^2(x)-1}{\sin(x)}$ b) $\frac{1-\sin^2(x)}{\cot(x)}$

c) $1 + \frac{\cos^2(x)}{\sin^2(x)}$ d) $\frac{\tan^2(x)}{\sec^2(x)} - 1$

e) $\cos(x) + \frac{\sin^2(x)}{\cos(x)}$ f) $\sec(x) - \frac{\tan^2(x)}{\sec(x)}$

g) $(1+\sin(x)) \cdot (1-\sin(x))$ h) $(1-\sec(x)) \cdot (1+\sec(x))$

i) $(\csc(x)-1) \cdot (\csc(x)+1)$ j) $\frac{\sec(x)}{\tan(x)} - \frac{\tan(x)}{\sec(x)}$

k) $\cos^4(x) - \sin^4(x)$ l) $\tan^4(x) - \sec^4(x)$

Exercise 21.4

Determine whether the identity is true or false. If the identity is true, then give an argument for why it is true.

a) $\sin(x) - \sin(x)\cos^2(x) = \sin^3(x)$
b) $\cot^2(x) - \csc^2(x) = \tan^2(x) - \sec^2(x)$
c) $\tan^2(x) + \sec^2(x) = 1$
d) $\sin^3(x) - \sin(x) = -\sin(x) \cdot \cos^2(x)$
e) $\sin(x) \cdot (\cos(x) - \sin(x)) = \cos^2(x)$
f) $(\sin(x) - \cos(x))^2 = 1 - 2\sin(x)\cos(x)$

Exercise 21.5

Simplify the expression as much as possible.

a) $\sin(x + \pi)$ b) $\tan(\pi - x)$ c) $\cot(x + \frac{\pi}{2})$ d) $\cos(x + \frac{3\pi}{2})$

Exercise 21.6

Find the exact values of the trigonometric functions of $\frac{\alpha}{2}$ and of 2α by using the half-angle and double-angle formulas.

a) $\sin(\alpha) = \frac{4}{5}$, and α in quadrant I
b) $\cos(\alpha) = \frac{7}{13}$, and α in quadrant IV
c) $\sin(\alpha) = \frac{-3}{5}$, and α in quadrant III
d) $\tan(\alpha) = \frac{4}{3}$, and α in quadrant III
e) $\tan(\alpha) = \frac{-5}{12}$, and α in quadrant II
f) $\cos(\alpha) = \frac{-2}{3}$, and α in quadrant II