Precalculus

Third Edition (3.0)

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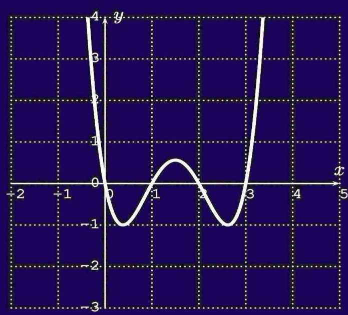

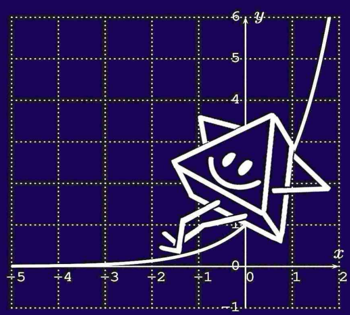

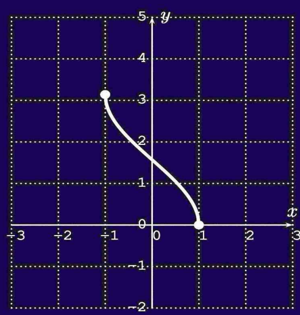

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Chapter 20

Solving trigonometric equations

Using the inverse trigonometric functions, we now solve equations that contain $\sin$, $\cos$, or $\tan$.

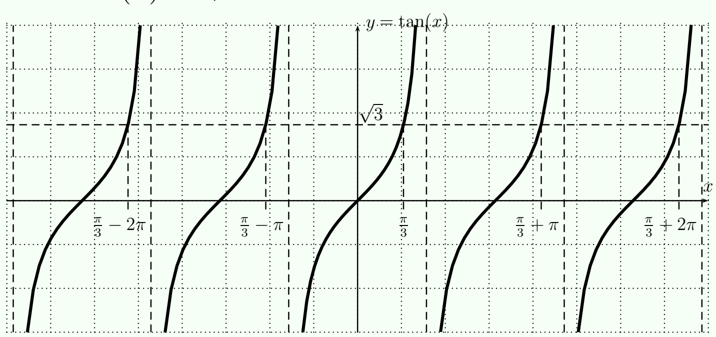
20.1 Basic trigonometric equations

In this section, we solve equations such as $\tan(x) = \sqrt{3}$. We can easily check that $x = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ solves this equation. However, there are other solutions, such as $x = \frac{4\pi}{3}$ or $x = \frac{7\pi}{3}$. Below, we will find all solutions of equations of the form $\sin(x) = c$, $\cos(x) = c$, and $\tan(x) = c$. We start with equations involving the tangent.

The equation $\tan(x) = c$

Example 20.1

Solve for x: $\quad \tan(x) = \sqrt{3}$



Solution.
There is an obvious solution given by $x = \tan^{-1}(\sqrt{3}) = 60° = \frac{\pi}{3}$, as we studied in the last section. However, there are more solutions, as the graph of $\tan(x) = \sqrt{3}$ above shows. To find the other solutions, we look for all points where the graph of the $y = \tan(x)$ intersects with the horizontal line $y = \sqrt{3}$. Since the function $y = \tan(x)$ is periodic with period π, we see that the other solutions of $\tan(x) = \sqrt{3}$ besides $x = \frac{\pi}{3}$ are

$$\frac{\pi}{3}+\pi, \quad \frac{\pi}{3}+2\pi, \quad \frac{\pi}{3}+3\pi, \dots, \qquad \text{and} \qquad \frac{\pi}{3}-\pi, \quad \frac{\pi}{3}-2\pi, \quad \frac{\pi}{3}-3\pi, \dots$$

In general, we write the solution as

$$x = \frac{\pi}{3} + n \cdot \pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

The graph also shows that these are *all* solutions of $\tan(x) = \sqrt{3}$. □

In a similar fashion, we can solve the equation $\tan(x) = c$ by replacing $\sqrt{3}$ in the above example with c. We get the following general solution of $\tan(x) = c$.

Observation 20.2: Solving $\tan(x) = c$

To solve $\tan(x) = c$, we first determine one solution $x = \tan^{-1}(c)$. Then the general solution is given by

$$\boxed{x = \tan^{-1}(c) + n \cdot \pi} \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots \tag{20.1}$$

Example 20.3

Solve for x:

a) $\tan(x) = 1$ b) $\tan(x) = -1$ c) $\tan(x) = 5.1$ d) $\tan(x) = -3.7$

Solution.

a) First, we find $\tan^{-1}(1) = 45° = \frac{\pi}{4}$. The general solution is thus:

$$x = \frac{\pi}{4} + n \cdot \pi \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

b) We first compute one solution, $\tan^{-1}(-1) = -45° = -\frac{\pi}{4}$. The general solution of $\tan(x) = -1$ is therefore,

$$x = -\frac{\pi}{4} + n \cdot \pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

For parts (c) and (d), we do not have an exact solution, so that the solution can only be approximated with the calculator.

c)

$$x = \tan^{-1}(5.1) + n\pi \quad \approx 1.377 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2$$

d)

$$x = \tan^{-1}(-3.7) + n\pi \quad \approx -1.307 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2$$

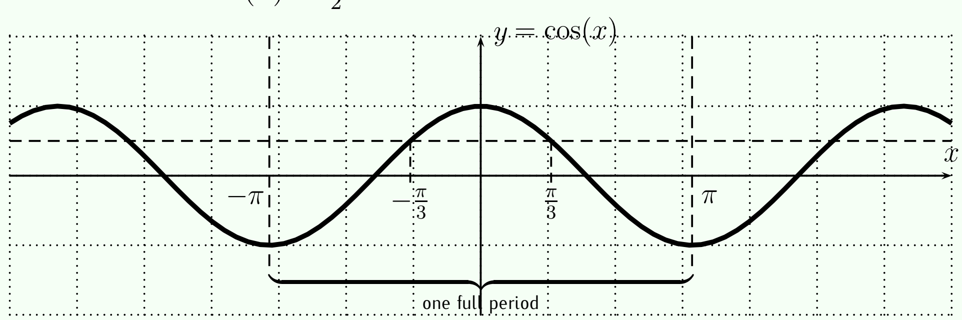
□

The equation $\cos(x) = c$

Next, we consider equations that contain a cosine. We start again by solving a specific example from which we infer the general solution.

Example 20.4

Solve for x: $\quad \cos(x) = \frac{1}{2}$



Solution.

We have the obvious solution to the equation $x = \cos^{-1}(\frac{1}{2}) = 60° = \frac{\pi}{3}$. However, since $\cos(-x) = \cos(x)$, there is another solution given by taking $x = -\frac{\pi}{3}$:

$$\cos\Big(-\frac{\pi}{3}\Big) = \cos\Big(\frac{\pi}{3}\Big) = \frac{1}{2}$$

Moreover, the $y = \cos(x)$ function is periodic with period 2π, that is, we have $\cos(x + 2\pi) = \cos(x)$. Thus, all of the following numbers are solutions of the equation $\cos(x) = \frac{1}{2}$:

$$\begin{array}{lccccccc} & \dots, & \frac{\pi}{3} - 4\pi, & \frac{\pi}{3} - 2\pi, & \frac{\pi}{3}, & \frac{\pi}{3} + 2\pi, & \frac{\pi}{3} + 4\pi, & \dots, \\ \text{and:} & \dots, & -\frac{\pi}{3} - 4\pi, & -\frac{\pi}{3} - 2\pi, & -\frac{\pi}{3}, & -\frac{\pi}{3} + 2\pi, & -\frac{\pi}{3} + 4\pi, & \dots. \end{array}$$

From the graph we see that there are only two solutions of $\cos(x) = \frac{1}{2}$ within one period. Thus, the above list constitutes *all* solutions of the equation. With this observation, we may write the general solution as:

$$\begin{array}{rl} x = & \frac{\pi}{3} + 2n \cdot \pi \\ \text{or} \quad x = & -\frac{\pi}{3} + 2n \cdot \pi \end{array} \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

In short, we write this as: $x = \pm\frac{\pi}{3} + 2n \cdot \pi$ with $n = 0, \pm 1, \pm 2, \pm 3, \dots$. □

We generalize this example as follows.

Observation 20.5: Solving $\cos(x) = c$

To solve $\cos(x) = c$, we first determine one solution $x = \cos^{-1}(c)$. Then the general solution is given by

$$\boxed{\begin{array}{rl} x = & \cos^{-1}(c) + 2n \cdot \pi \\ \text{or} \quad x = & -\cos^{-1}(c) + 2n \cdot \pi \end{array}} \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots \tag{20.2}$$

In short, we can also write this as

$$x = \pm\cos^{-1}(c) + 2n \cdot \pi \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Example 20.6

Solve for x.

a) $\cos(x) = -\frac{\sqrt{2}}{2}$ b) $\cos(x) = 0.6$ c) $\cos(x) = -3$ d) $\cos(x) = -1$

Solution.

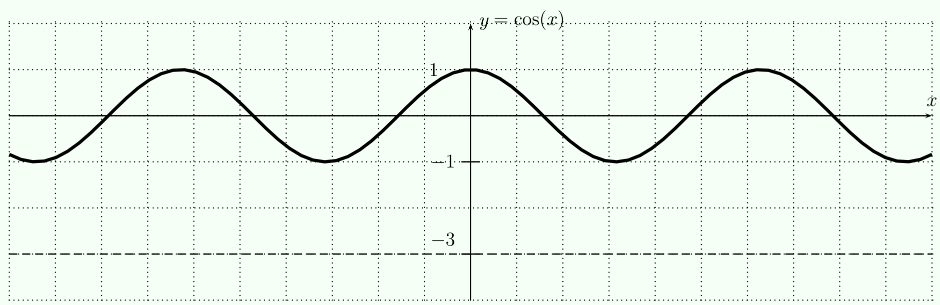
a) First, we need to find $\cos^{-1}(-\frac{\sqrt{2}}{2}) = 135° = \frac{3\pi}{4}$. The general solution is therefore,

$$x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) We calculate $\cos^{-1}(0.6) \approx 0.927$ with the calculator. The general solution is therefore,

$$x = \pm\cos^{-1}(0.6) + 2n\pi \quad \approx \pm 0.927 + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

c) Since the cosine is always $-1 \leq \cos(x) \leq 1$, the cosine can never be -3. Therefore, there is *no solution* to the equation $\cos(x) = -3$. This can also be seen from the graph, which does not intersect with the horizontal line $y = -3$.



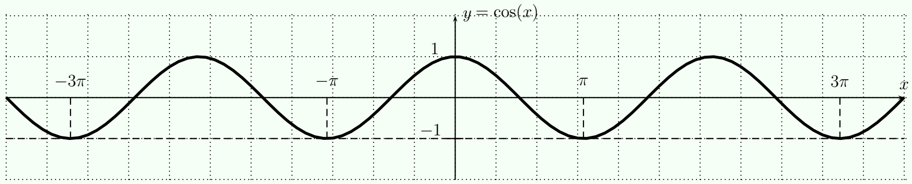
d) A special solution of $\cos(x) = -1$ is $\cos^{-1}(-1) = 180° = \pi$, so that the general solution is

$$x = \pm\pi + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

However, since $-\pi + 2\pi = +\pi$, the solutions $\pi + 2n\pi$ and $-\pi + 2n\pi$ (for $n = 0, \pm 1, \pm 2, \ldots$) can be identified with each other, and there is only *one* solution in each period. Thus, the general solution can be written as

$$x = \pi + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The graphs of $y = \cos(x)$ and $y = -1$ confirm this solution.



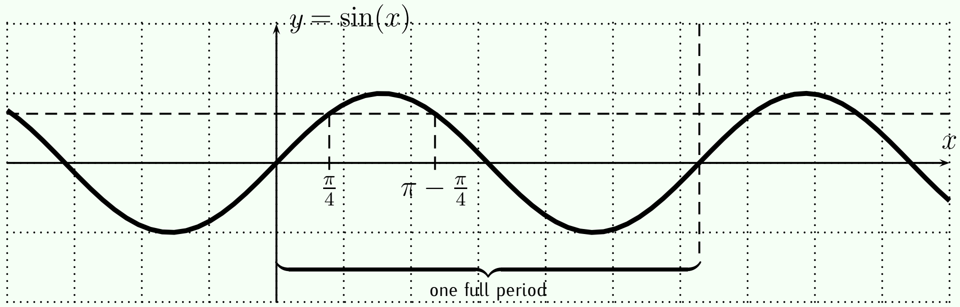
□

The equation $\sin(x) = c$

Finally, we consider equations with a sine.

Example 20.7

Solve for x: $\sin(x) = \frac{\sqrt{2}}{2}$



Solution.

First, we can find one obvious solution $x = \sin^{-1}(\frac{\sqrt{2}}{2}) = 45° = \frac{\pi}{4}$. Furthermore, another solution appears to be given at an input with the same distance $\frac{\pi}{4}$ from π, that is at $\pi - \frac{\pi}{4}$:

$$\sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

In fact, we have the general identity $\sin(\pi - x) = \sin(x)$ for any x, which will be shown in (21.12). These are all solutions within one period, as can be confirmed from the graph above. The function $y = \sin(x)$ is periodic with period 2π, so that adding $2n \cdot \pi$ for any $n = 0, \pm1, \pm2, \ldots$

gives all solutions of $\sin(x) = \frac{\sqrt{2}}{2}$. This means that the general solution is given by:

$$\begin{array}{rl} & x = \frac{\pi}{4} + 2n \cdot \pi \\ \text{or} & x = (\pi - \frac{\pi}{4}) + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We can evaluate the second solution a bit further, since $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$, so that the final answer is:

$$\begin{array}{rl} & x = \frac{\pi}{4} + 2n \cdot \pi \\ \text{or} & x = \frac{3\pi}{4} + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

□

We have the following general statement.

Observation 20.8: Solving $\sin(x) = c$

To solve $\sin(x) = c$, we first determine one solution $x = \sin^{-1}(c)$. Then the general solution is given by

$$\boxed{\begin{array}{rl} & x = \sin^{-1}(c) + 2n \cdot \pi \\ \text{or} & x = (\pi - \sin^{-1}(c)) + 2n \cdot \pi \end{array}} \quad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{20.3}$$

Example 20.9

Solve for x.

a) $\sin(x) = \frac{1}{2}$ b) $\sin(x) = -\frac{1}{2}$ c) $\sin(x) = -\frac{5}{7}$ d) $\sin(x) = -1$

Solution.

a) First, we calculate $\sin^{-1}(\frac{1}{2}) = 30° = \frac{\pi}{6}$. A second solution is then given by $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The general solution is therefore,

$$\begin{array}{rl} & x = \frac{\pi}{6} + 2n \cdot \pi \\ \text{or} & x = \frac{5\pi}{6} + 2n \cdot \pi \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) First, we calculated $\sin^{-1}(-\frac{1}{2}) = -30° = -\frac{\pi}{6}$. We find a second solution by taking $\pi - (-\frac{\pi}{6}) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$. We thus state the general solution as

$$\begin{aligned} & x = -\tfrac{\pi}{6} + 2n\cdot\pi \\ \text{or} \quad & x = \tfrac{7\pi}{6} + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

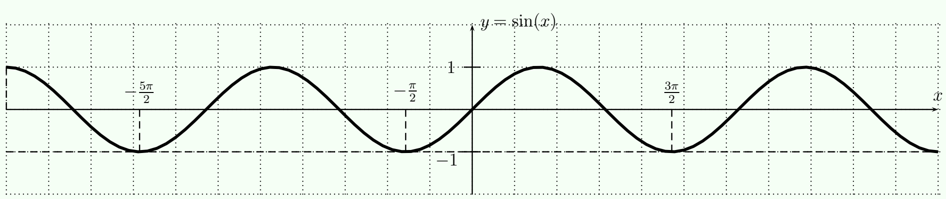
c) We do not have an exact value of $\sin^{-1}(-\frac{5}{7})$, so that we either need to leave it as is, or approximate it with the calculator to be $\sin^{-1}(-\frac{5}{7}) \approx -0.796$. A second solution is given by $\pi - \sin^{-1}(-\frac{5}{7}) \approx 3.937$. We get the solution:

$$\begin{aligned} & x \approx -0.796 + 2n\cdot\pi \\ \text{or} \quad & x \approx 3.937 + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

d) We calculate $\sin^{-1}(-1) = -90^\circ = -\frac{\pi}{2}$, and $\pi - (-\frac{\pi}{2}) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$. The solution is therefore,

$$\begin{aligned} & x = -\tfrac{\pi}{2} + 2n\cdot\pi \\ \text{or} \quad & x = \tfrac{3\pi}{2} + 2n\cdot\pi \end{aligned} \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Note however, that $\frac{3\pi}{2} = -\frac{\pi}{2} + 2\pi$, so that the various solutions can be identified with each other. This can also be seen on the graph of $y = \sin(x)$:



A complete solution is therefore given by

$$x = -\frac{\pi}{2} + 2n\cdot\pi, \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Writing the solution in this way has the advantage that it does not repeat any of the solutions, and is therefore preferred.

□

Summary

We summarize the different formulas we used to solve the basic trigonometric equations in the following table.

Solve: $\sin(x) = c$	Solve: $\cos(x) = c$	Solve: $\tan(x) = c$
First, find one solution: $\sin^{-1}(c)$	First, find one solution: $\cos^{-1}(c)$	First, find one solution: $\tan^{-1}(c)$
The general solution is: $\boxed{\begin{array}{ll} x = & \sin^{-1}(c) + 2n\pi \\ x = & (\pi - \sin^{-1}(c)) + 2n\pi \end{array}}$ where $n = 0, \pm 1, \pm 2, \dots$	The general solution is: $\boxed{\begin{array}{ll} x = & \cos^{-1}(c) + 2n\pi \\ x = & -\cos^{-1}(c) + 2n\pi \end{array}}$ where $n = 0, \pm 1, \pm 2, \dots$	The general solution is: $\boxed{x = \tan^{-1}(c) + n\pi}$ where $n = 0, \pm 1, \pm 2, \dots$

Example 20.10

Find the general solution of the equation, and state at least 6 distinct solutions.

a) $\sin(x) = -\frac{1}{2}$ b) $\cos(x) = -\frac{\sqrt{3}}{2}$

Solution.

a) We already calculated the general solution in Example 20.9(b). The solution is

$$\begin{array}{ll} x = -\frac{\pi}{6} + n \cdot 2\pi & \\ \text{or} \quad x = \frac{7\pi}{6} + n \cdot 2\pi & \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots \end{array}$$

We simplify the solutions for $n = 0, 1, -1$:

$$\begin{aligned} n = 0: \quad x &= -\frac{\pi}{6} + 0 \cdot 2\pi = -\frac{\pi}{6} \\ n = 1: \quad x &= -\frac{\pi}{6} + 1 \cdot 2\pi = -\frac{\pi}{6} + \frac{12\pi}{6} = \frac{11\pi}{6} \\ n = -1: \quad x &= -\frac{\pi}{6} + (-1) \cdot 2\pi = -\frac{\pi}{6} - \frac{12\pi}{6} = -\frac{13\pi}{6} \end{aligned}$$

and

$$n = 0: \quad x = \frac{7\pi}{6} + 0 \cdot 2\pi = \frac{7\pi}{6}$$

$$\begin{aligned} n = 1: \quad x = &\ \frac{7\pi}{6} + 1 \cdot 2\pi = \frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6} \\ n = -1: \quad x = &\ \frac{7\pi}{6} + (-1) \cdot 2\pi = \frac{7\pi}{6} - \frac{12\pi}{6} = - - \frac{5\pi}{6} \end{aligned}$$

b) Since $\cos^{-1}(-\frac{\sqrt{3}}{2}) = 150^\circ = \frac{5\pi}{6}$, the solutions of $\cos(x) = -\frac{\sqrt{3}}{2}$ are:

$$x = \pm\frac{5\pi}{6} + n \cdot 2\pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We write the 6 solutions with $n = 0, +1, -1$, and for each use the two distinct first terms $+\frac{5\pi}{6}$ and $-\frac{5\pi}{6}$.

$$\begin{aligned} n = 0: \quad x = &\ +\frac{5\pi}{6} + 0 \cdot 2\pi = \frac{5\pi}{6} \\ n = 1: \quad x = &\ +\frac{5\pi}{6} + 1 \cdot 2\pi = \frac{5\pi}{6} + 2\pi = \frac{5\pi + 12\pi}{6} = \frac{17\pi}{6} \\ n = -1: \quad x = &\ +\frac{5\pi}{6} + (-1) \cdot 2\pi = \frac{5\pi}{6} - 2\pi = \frac{5\pi - 12\pi}{6} = \frac{-7\pi}{6} \end{aligned}$$

and

$$\begin{aligned} n = 0: \quad x = &\ -\frac{5\pi}{6} + 0 \cdot 2\pi = -\frac{5\pi}{6} \\ n = 1: \quad x = &\ -\frac{5\pi}{6} + 1 \cdot 2\pi = -\frac{5\pi}{6} + 2\pi = \frac{-5\pi + 12\pi}{6} = \frac{7\pi}{6} \\ n = -1: \quad x = &\ -\frac{5\pi}{6} + (-1) \cdot 2\pi = -\frac{5\pi}{6} - 2\pi = \frac{-5\pi - 12\pi}{6} \\ = &\ \frac{-17\pi}{6} \end{aligned}$$

Further solutions can be found by taking values $n = +2, -2, +3, -3, \ldots$. $\square$

20.2 Equations involving trigonometric functions

The previous section explained how to solve the basic trigonometric equations

$$\sin(x) = c, \quad \cos(x) = c, \qquad \text{and} \qquad \tan(x) = c.$$

The next examples can be reduced to these basic equations.

Example 20.11

Solve for x.

a) $2\sin(x) - \sqrt{3} = 0$ b) $\sec(x) = -\sqrt{2}$ c) $7\cot(x) + 3 = 0$

Solution.

a) Solving for $\sin(x)$, we get

$$2\sin(x) - \sqrt{3} = 0 \overset{(+\sqrt{3})}{\Longrightarrow} 2\sin(x) = \sqrt{3} \overset{(\div 2)}{\Longrightarrow} \sin(x) = \frac{\sqrt{3}}{2}$$

One solution of $\sin(x) = \frac{\sqrt{3}}{2}$ is $\sin^{-1}(\frac{\sqrt{3}}{2}) = 60° = \frac{\pi}{3}$. Another solution is given by $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. The general solution is

$$\begin{array}{lll} & x = \frac{\pi}{3} + 2n\pi & \\ \text{or} & x = \frac{2\pi}{3} + 2n\pi & \end{array} \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) Recall that $\sec(x) = \frac{1}{\cos(x)}$. Therefore,

$$\sec(x) = -\sqrt{2} \Longrightarrow \frac{1}{\cos(x)} = -\sqrt{2} \overset{\text{(reciprocal)}}{\Longrightarrow} \cos(x) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

A special solution of $\cos(x) = -\frac{\sqrt{2}}{2}$ is $\cos^{-1}(-\frac{\sqrt{2}}{2}) = 135° = \frac{3\pi}{4}$. The general solution is

$$x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

c) Recall that $\cot(x) = \frac{1}{\tan(x)}$. So

$$7\cot(x) + 3 = 0 \overset{(-3)}{\Longrightarrow} 7\cot(x) = -3 \overset{(\div 7)}{\Longrightarrow} \cot(x) = -\frac{3}{7}$$
$$\Longrightarrow \frac{1}{\tan(x)} = -\frac{3}{7} \overset{\text{(reciprocal)}}{\Longrightarrow} \tan(x) = -\frac{7}{3}$$

The solution is

$$x = \tan^{-1}\left(-\frac{7}{3}\right) + n\pi \approx -1.166 + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

□

For the next problems we combine quadratic functions with trigonometric functions. It is customary to use the following notation.

Convention 20.12: Square of a trigonometric function

We denote the square of a trigonometric function as follows:

$$\sin^2\alpha := (\sin\alpha)^2 \qquad \cos^2\alpha := (\cos\alpha)^2 \qquad \tan^2\alpha := (\tan\alpha)^2$$

In order to solve quadratic trigonometric equations, it can be helpful to substitute u for a trigonometric expression first, then solve for u, and finally apply the rules from the previous section to solve for the wanted variable. This method is shown in the next example.

Example 20.13

Solve for x.

a) $2\sin^2(x) + \sqrt{3}\sin(x) = 0$ b) $2\cos^2(x) - 1 = 0$
c) $\tan^2(x) + 2\tan(x) + 1 = 0$

Solution.

a) We first need to solve $2\sin^2(x)+\sqrt{3}\sin(x) = 0$ for $\sin(x)$. In this case, this can be done either by factoring $\sin(x)$ directly, that is, by writing $\sin(x)\cdot(2\sin(x)+\sqrt{3}) = 0$, or, more thoroughly, by substituting $u = \sin(x)$, and then solving for u, for which we get:

$$2u^2+\sqrt{3}u = 0 \overset{\text{(factor } u)}{\Longrightarrow} u\cdot(2u+\sqrt{3}) = 0 \implies u = 0 \text{ or } 2u+\sqrt{3} = 0$$

We get two trigonometric equations that we need to solve:

$\sin(x) = 0$	$2\sin(x)+\sqrt{3} = 0$ $\implies 2\sin(x) = -\sqrt{3}$ $\implies \sin(x) = -\frac{\sqrt{3}}{2}$
then: $\sin^{-1}(0) = 0^\circ = 0$	then: $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60^\circ = -\frac{\pi}{3}$
and: $\pi - 0 = \pi$	and: $\pi - (-\frac{\pi}{3}) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
$\implies x = 0 + 2n\pi$ or $x = \pi + 2n\pi$ where $n = 0, \pm1, \pm2, \ldots$	$\implies x = -\frac{\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi$ where $n = 0, \pm1, \pm2, \ldots$

The general solution is therefore,

$$\begin{aligned} x &= 0 + 2n\pi, \quad \text{or} \quad x = -\frac{\pi}{3} + 2n\pi, \\ \text{or } x &= \pi + 2n\pi, \quad \text{or} \quad x = \frac{4\pi}{3} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

b) Substituting $u = \cos(x)$, we get

$$\begin{aligned} 2u^2 - 1 = 0 \quad &\overset{(+1)}{\Longrightarrow} \quad 2u^2 = 1 \quad \overset{(\div 2)}{\Longrightarrow} \quad u^2 = \frac{1}{2} \\ &\Longrightarrow \quad u = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \\ &\Longrightarrow \quad u = +\frac{\sqrt{2}}{2} \quad \text{or} \quad u = -\frac{\sqrt{2}}{2} \end{aligned}$$

For each of the two cases, we need to solve the corresponding trigonometric equation after replacing $u = \cos(x)$.

$\cos(x) = \frac{\sqrt{2}}{2}$	$\cos(x) = -\frac{\sqrt{2}}{2}$
then: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45° = \frac{\pi}{4}$	then: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135° = \frac{3\pi}{4}$
$\Longrightarrow x = \pm\frac{\pi}{4} + 2n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$	$\Longrightarrow x = \pm\frac{3\pi}{4} + 2n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$

Thus, the general solution is,

$$x = \pm\frac{\pi}{4} + 2n\pi, \quad \text{or} \quad x = \pm\frac{3\pi}{4} + 2n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

c) Substituting $u = \tan(x)$, we have to solve the equation

$$u^2 + 2u + 1 = 0 \overset{\text{(factor)}}{\Longrightarrow} (u+1)(u+1) = 0 \Longrightarrow u + 1 = 0 \overset{(-1)}{\Longrightarrow} u = -1$$

Resubstituting $u = \tan(x)$, we have to solve $\tan(x) = -1$. Using the fact that $\tan^{-1}(-1) = -45° = -\frac{\pi}{4}$, we have the general solution

$$x = -\frac{\pi}{4} + n\pi, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

□

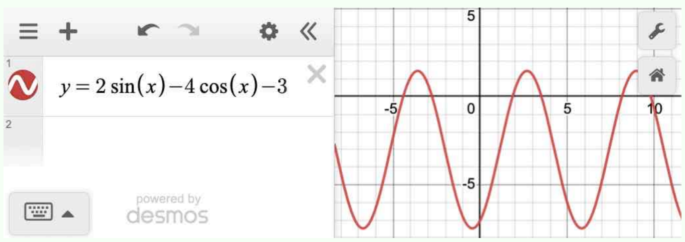
Example 20.14

Solve the equation with the calculator. Approximate the solution to the nearest thousandth.

a) $2\sin(x) = 4\cos(x) + 3$ b) $5\cos(2x) = \tan(x)$

Solution.

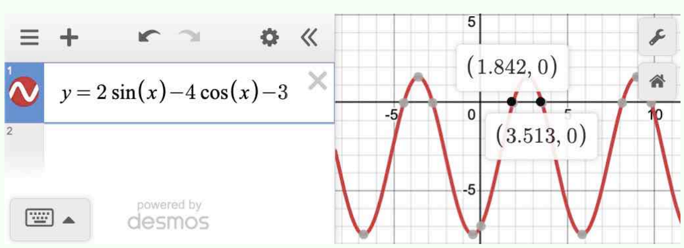
a) We rewrite the equation as $2\sin(x) - 4\cos(x) - 3 = 0$, and use the calculator to find the graph of the function $f(x) = 2\sin(x) - 4\cos(x) - 3$. The zeros of the function f are the solutions of the initial equation. The graph that we obtain is displayed below.



The graph indicates that the function $f(x) = 2\sin(x) - 4\cos(x) - 3$ is periodic. This can be confirmed by observing that both $\sin(x)$ and $\cos(x)$ are periodic with period 2π, and thus also $f(x)$.

$$\begin{aligned} f(x+2\pi) &= 2\sin(x+2\pi) - 4\cos(x+2\pi) - 3 \\ &= 2\sin(x) - 4\cos(x) - 3 = f(x) \end{aligned}$$

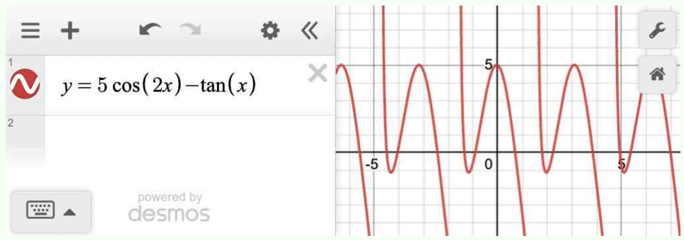
The solution of $f(x) = 0$ can be approximated by clicking on the roots.



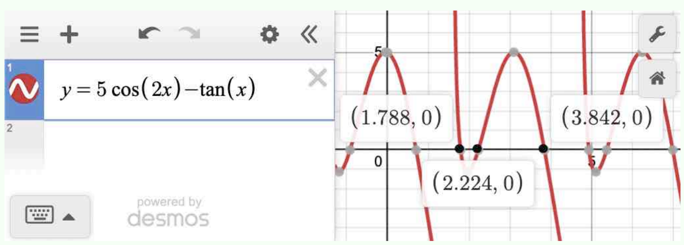
The general solution is thus

$x \approx 1.842 + 2n\pi$ or $x \approx 3.513 + 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$

b) We rewrite the equation as $5\cos(2x) - \tan(x) = 0$ and graph the function $f(x) = 5\cos(2x) - \tan(x)$ in the standard window.



Note again that the function f is periodic. The period of $\cos(2x)$ is $\frac{2\pi}{2} = \pi$ (see Definition 18.7 on page 321), and the period of $\tan(x)$ is also π (see Equation (18.3) on page 317). Thus, f is also periodic with period π. The solutions in one period are approximated by finding the zeros with the calculator.



The general solution is given by any of these numbers, with possibly an additional shift by any multiple of π.

$$x \approx 1.788 + n\pi \quad \text{or} \quad x \approx 2.224 + n\pi \quad \text{or} \quad x \approx 3.842 + n\pi,$$
$$\text{where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$\square$

20.3 Exercises

Exercise 20.1

Find all solutions of the equation, and simplify as much as possible. Do not approximate the solution.

a) $\tan(x) = \frac{\sqrt{3}}{3}$ b) $\sin(x) = \frac{\sqrt{3}}{2}$ c) $\sin(x) = -\frac{\sqrt{2}}{2}$ d) $\cos(x) = \frac{\sqrt{3}}{2}$
e) $\cos(x) = 0$ f) $\cos(x) = -0.5$ g) $\cos(x) = 1$ h) $\sin(x) = 5$
i) $\sin(x) = 0$ j) $\sin(x) = -1$ k) $\tan(x) = -\sqrt{3}$ l) $\cos(x) = 0.2$

Exercise 20.2

Find all solutions of the equation. Approximate your solution with the calculator.

a) $\tan(x) = 6.2$ b) $\cos(x) = 0.45$ c) $\sin(x) = 0.91$
d) $\cos(x) = -.772$ e) $\tan(x) = -0.2$ f) $\sin(x) = -0.06$

Exercise 20.3

Find at least 5 distinct solutions of the equation.

a) $\tan(x) = -1$ b) $\cos(x) = \frac{\sqrt{2}}{2}$ c) $\sin(x) = -\frac{\sqrt{3}}{2}$ d) $\tan(x) = 0$
e) $\cos(x) = 0$ f) $\cos(x) = 0.3$ g) $\sin(x) = 0.4$ h) $\sin(x) = -1$

Exercise 20.4

Solve for x. State the general solution without approximation.

a) $\tan(x) - 1 = 0$ b) $2\sin(x) = 1$ c) $2\cos(x) + \sqrt{3} = 0$
d) $\sqrt{2}\cos(x) - 1 = 0$ e) $\sec(x) = -2$ f) $\cot(x) = \sqrt{3}$

Exercise 20.5

Solve for x. State the general solution without approximation.

a) $2\sin^2(x) - \sqrt{2}\sin(x) = 0$
b) $\tan^2(x) + \tan(x) = 0$
c) $2\cos^2(x) + \sqrt{3}\cos(x) = 0$
d) $\sin^2(x) + \sin(x) = 0$
e) $\tan^2(x) - 3 = 0$
f) $4\cos^2(x) - 1 = 0$
g) $4\sin^2(x) - 3 = 0$
h) $\cos(x)\sin(x) + \sin(x) = 0$
i) $\tan(x)\cos(x) + \sqrt{3}\cos(x) = 0$
j) $\cos^2(x) + 7\cos(x) + 6 = 0$
k) $4\cos^2(x) - 4\cos(x) + 1 = 0$
l) $2\sin^2(x) + 11\sin(x) = -5$
m) $2\sin^2(x) + \sin(x) - 1 = 0$
n) $2\cos^2(x) - 3\cos(x) + 1 = 0$
o) $2\cos^2(x) + 9\cos(x) = 5$
p) $\tan^3(x) - \tan(x) = 0$

Exercise 20.6

Use the calculator to find all solutions of the given equation. Approximate the answer to the nearest thousandth.

a) $2\cos(x) = 2\sin(x) + 1$
b) $7\tan(x) \cdot \cos(2x) = 1$
c) $4\cos^2(3x) + \cos(3x) = \sin(3x) + 2$
d) $\sin(x) + \tan(x) = \cos(x)$