Precalculus

Third Edition (3.0)

Thomas Tradler

Holly Carley

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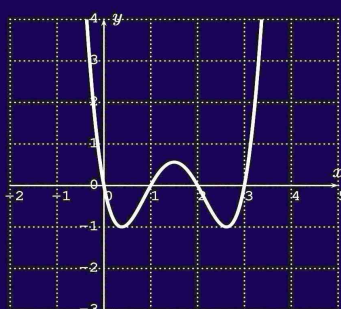

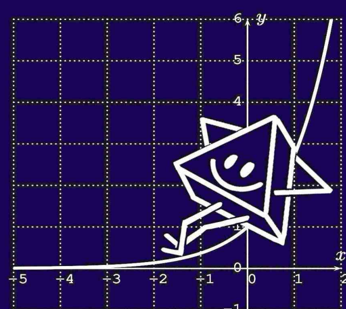

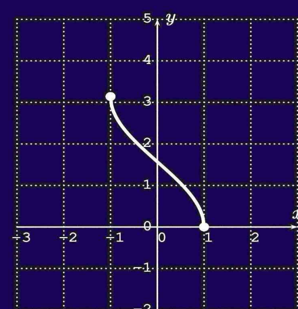

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Chapter 2

Functions via formulas

Most of the time we will discuss functions that take some real numbers as inputs, and give real numbers as outputs. Such functions are often described with a formula.

2.1 Functions given by formulas

Here are some examples of functions given by a formula.

Example 2.1

For the given function f, calculate the outputs $f(2)$, $f(-3)$, and $f(-1)$.

a) $f(x) = 3x + 4$

b) $f(x) = \sqrt{x^2 - 3}$

c) $f(x) = \begin{cases} 5x - 6 & \text{, for } -1 \leq x \leq 1 \\ x^3 + 2x & \text{, for } 1 < x \leq 5 \end{cases}$

d) $f(x) = \frac{x+2}{x+3}$

Solution.

a) We substitute the input values into the function and simplify.

$$\begin{aligned} f(2) &= 3 \cdot 2 + 4 = 6 + 4 = 10, \\ f(-3) &= 3 \cdot (-3) + 4 = -9 + 4 = -5, \\ f(-1) &= 3 \cdot (-1) + 4 = -3 + 4 = 1. \end{aligned}$$

b) Similarly, we calculate

$$f(2) = \sqrt{2^2 - 3} = \sqrt{4 - 3} = \sqrt{1} = 1,$$

$$\begin{aligned} f(-3) &= \sqrt{(-3)^2-3} = \sqrt{9-3} = \sqrt{6}, \\ f(-1) &= \sqrt{(-1)^2-3} = \sqrt{1-3} = \sqrt{-2} \text{ is undefined.} \end{aligned}$$

Note that in the last evaluation, we obtained an output of $\sqrt{-2}$. As you are probably aware, $\sqrt{-2}$ is a complex number. However, at this point, we will only allow outputs that are real numbers! Since $\sqrt{-2}$ is *not* a real number (but only a complex number), there is no real output for $f(-1)$, and we say that $f(-1)$ is undefined.

c) The function $f(x) = \begin{cases} 5x-6 & \text{, for } -1 \le x \le 1 \\ x^3+2x & \text{, for } 1 < x \le 5 \end{cases}$ is given as a piecewise defined function. We have to substitute the values into the correct branch:

$$\begin{aligned} f(2) &= 2^3 + 2\cdot 2 = 8+4 = 12, \text{ since } 1 < 2 \le 5, \\ f(-3) &= \text{undefined, since } -3 \text{ is not in any of the two branches,} \\ f(-1) &= 5\cdot(-1) - 7 = -5-6 = -11, \text{ since } -1 \le -1 \le 1. \end{aligned}$$

d) Finally for $f(x) = \frac{x+2}{x+3}$, we have:

$$\begin{aligned} f(2) &= \frac{2+2}{2+3} = \frac{4}{5}, \qquad f(-3) = \frac{-3+2}{-3+3} = \frac{-1}{0} \text{ is undefined,} \\ f(-1) &= \frac{-1+2}{-1+3} = \frac{1}{2}. \end{aligned}$$

□

Example 2.2

Let f be the function given by $f(x) = x^2 + 2x - 3$. Find the following function values.

a) $f(5)$	b) $f(2)$	c) $f(-2)$	d) $f(0)$
e) $f(\sqrt{5})$	f) $f(\sqrt{3}+1)$	g) $f(a)$	h) $f(a)+5$
i) $f(x+h)$	j) $f(x+h)-f(x)$	k) $\frac{f(x+h)-f(x)}{h}$	l) $\frac{f(x)-f(a)}{x-a}$

Solution.

The first four function values ((a)-(d)) can be calculated directly:

$$f(5) = 5^2 + 2\cdot 5 - 3 = 25 + 10 - 3 = 32,$$

$$\begin{aligned} f(2) &= 2^2 + 2 \cdot 2 - 3 = 4 + 4 - 3 = 5, \\ f(-2) &= (-2)^2 + 2 \cdot (-2) - 3 = 4 + -4 - 3 = -3, \\ f(0) &= 0^2 + 2 \cdot 0 - 3 = 0 + 0 - 3 = -3. \end{aligned}$$

The next two values ((e) and (f)) are similar, but the arithmetic is a bit more involved.

$$\begin{aligned} f(\sqrt{5}) &= \sqrt{5}^2 + 2 \cdot \sqrt{5} - 3 = 5 + 2 \cdot \sqrt{5} - 3 = 2 + 2 \cdot \sqrt{5}, \\ f(\sqrt{3}+1) &= (\sqrt{3}+1)^2 + 2 \cdot (\sqrt{3}+1) - 3 \\ &= (\sqrt{3}+1) \cdot (\sqrt{3}+1) + 2 \cdot (\sqrt{3}+1) - 3 \\ &= \sqrt{3} \cdot \sqrt{3} + 2 \cdot \sqrt{3} + 1 \cdot 1 + 2 \cdot \sqrt{3} + 2 - 3 \\ &= 3 + 2 \cdot \sqrt{3} + 1 + 2 \cdot \sqrt{3} + 2 - 3 \\ &= 3 + 4 \cdot \sqrt{3}. \end{aligned}$$

The last five values ((g)-(l)) are purely algebraic:

$$\begin{aligned} f(a) &= a^2 + 2 \cdot a - 3, \\ f(a) + 5 &= a^2 + 2 \cdot a - 3 + 5 = a^2 + 2 \cdot a + 2, \\ f(x+h) &= (x+h)^2 + 2 \cdot (x+h) - 3 \\ &= x^2 + 2xh + h^2 + 2x + 2h - 3, \\ f(x+h) - f(x) &= (x^2 + 2xh + h^2 + 2x + 2h - 3) - (x^2 + 2x - 3) \\ &= x^2 + 2xh + h^2 + 2x + 2h - 3 - x^2 - 2x + 3 \\ &= 2xh + h^2 + 2h, \\ \frac{f(x+h) - f(x)}{h} &= \frac{2xh + h^2 + 2h}{h} \\ &= \frac{h \cdot (2x + h + 2)}{h} = 2x + h + 2, \end{aligned}$$

and

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{(x^2 + 2x - 3) - (a^2 + 2a - 3)}{x - a} \\ &= \frac{x^2 + 2x - 3 - a^2 - 2a + 3}{x - a} = \frac{x^2 - a^2 + 2x - 2a}{x - a} \\ &= \frac{(x+a)(x-a) + 2(x-a)}{x - a} = \frac{(x-a)(x+a+2)}{(x-a)} = x + a + 2. \end{aligned}$$

□

The quotients in the last two examples 2.2(k) and (l) will become particularly important in calculus. They are called difference quotients.

Definition 2.3: Difference quotient

Let $y = f(x)$ be a function. We call the expressions

$$\boxed{\frac{f(x+h)-f(x)}{h}} \quad \text{or} \quad \boxed{\frac{f(x)-f(a)}{x-a}} \tag{2.1}$$

difference quotients for the function f.

We next calculate some more examples of difference quotients.

Example 2.4

Calculate the difference quotient $\frac{f(x+h)-f(x)}{h}$ for

a) $f(x) = x^2 - 4x$ b) $f(x) = 3x^2 + 8x - 5$

Solution.

a) For $f(x) = x^2 - 4x$, we get:

$$\begin{aligned} f(x+h) &= (x+h)^2 - 4\cdot(x+h) \\ &= x^2 + 2xh + h^2 - 4x - 4h, \\ f(x+h) - f(x) &= (x^2 + 2xh + h^2 - 4x - 4h) - (x^2 - 4x) \\ &= x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x \\ &= 2xh + h^2 - 4h, \\ \frac{f(x+h)-f(x)}{h} &= \frac{2xh + h^2 - 4h}{h} \\ &= \frac{h\cdot(2x+h-4)}{h} = 2x + h - 4 \end{aligned}$$

b) For $f(x) = 3x^2 + 8x - 5$, we get:

$$\begin{aligned} f(x+h) &= 3(x+h)^2 + 8\cdot(x+h) - 5 \\ &= 3(x^2 + 2xh + h^2) + 8x + 8h - 5, \\ &= 3x^2 + 6xh + 3h^2 + 8x + 8h - 5, \end{aligned}$$

$$\begin{aligned} f(x+h)-f(x) &= (3x^2+6xh+3h^2+8x+8h-5) \\ &\qquad -(3x^2+8x-5) \\ &= 3x^2+6xh+3h^2+8x+8h-5-3x^2-8x+5 \\ &= 6xh+3h^2+8h, \\ \frac{f(x+h)-f(x)}{h} &= \frac{6xh+3h^2+8h}{h} \\ &= \frac{h\cdot(6x+3h+8)}{h} = 6x+3h+8 \end{aligned}$$

□

Example 2.5

Calculate the difference quotient $\frac{f(x)-f(a)}{x-a}$ for

a) $f(x)=x^2-7x-2$ b) $f(x)=-2x^2+3x$

Solution.

a) For $f(x)=x^2-7x-2$, we get:

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} &= \frac{(x^2-7x-2)-(a^2-7a-2)}{x-a} \\ &= \frac{x^2-7x-2-a^2+7a+2}{x-a} = \frac{x^2-a^2-7x+7a}{x-a} \\ &= \frac{(x+a)(x-a)-7(x-a)}{x-a} = \frac{(x-a)(x+a-7)}{(x-a)} = x+a-7. \end{aligned}$$

b) For $f(x)=-2x^2+3x$, we get:

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} &= \frac{(-2x^2+3x)-(-2a^2+3a)}{x-a} \\ &= \frac{-2x^2+3x+2a^2-3a}{x-a} = \frac{-2x^2+2a^2+3x-3a}{x-a} \\ &= \frac{-2(x^2-a^2)+3x-3a}{x-a} = \frac{-2(x+a)(x-a)+3(x-a)}{x-a} \\ &= \frac{(x-a)(-2(x+a)+3)}{(x-a)} = -2(x+a)+3 = -2x-2a+3. \end{aligned}$$

□

Here are some difference quotients of a degree 3 polynomial, a rational function, and a square root function.

Example 2.6

Calculate the difference quotient $\frac{f(x+h)-f(x)}{h}$ for

a) $f(x) = x^3 + 2$ b) $f(x) = \dfrac{1}{x}$ c) $f(x) = \sqrt{2x+3}$

Solution.

a) We calculate first the difference quotient step by step.

$$\begin{aligned} f(x+h) &= (x+h)^3 + 2 = (x+h)\cdot(x+h)\cdot(x+h) + 2 \\ &= (x^2 + 2xh + h^2)\cdot(x+h) + 2 \\ &= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 + 2, \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 2. \end{aligned}$$

Subtracting $f(x)$ from $f(x+h)$ gives

$$\begin{aligned} f(x+h) - f(x) &= (x^3 + 3x^2h + 3xh^2 + h^3 + 2) - (x^3 + 2) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 + 2 - x^3 - 2 \\ &= 3x^2h + 3xh^2 + h^3. \end{aligned}$$

With this we obtain:

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \frac{h\cdot(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2. \end{aligned}$$

b) The computation for (b) is similar.

$$f(x+h) = \frac{1}{x+h},$$

so that

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}.$$

We obtain the solution after simplifying the double fraction:

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{(x+h)\cdot x}}{h} = \frac{\frac{x-x-h}{(x+h)\cdot x}}{h} = \frac{\frac{-h}{(x+h)\cdot x}}{h} \\ &= \frac{-h}{(x+h)\cdot x}\cdot\frac{1}{h} = \frac{-1}{(x+h)\cdot x}. \end{aligned}$$

c)

$$\begin{aligned} f(x+h) &= \sqrt{2(x+h)+3} = \sqrt{2x+2h+3} \\ \Longrightarrow\ f(x+h)-f(x) &= \sqrt{2x+2h+3}-\sqrt{2x+3} \\ \Longrightarrow\ \frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2x+2h+3}-\sqrt{2x+3}}{h} \end{aligned}$$

We can simplify this expression by multiplying both numerator and denominator with $\sqrt{2x+2h+3}+\sqrt{2x+3}$:

$$\begin{aligned} \Longrightarrow\ \frac{f(x+h)-f(x)}{h} &= \frac{(\sqrt{2x+2h+3}-\sqrt{2x+3})\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{(\sqrt{2x+2h+3})^2-(\sqrt{2x+3})^2}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{(2x+2h+3)-(2x+3)}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2x+2h+3-2x-3}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2h}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})} \\ &= \frac{2}{\sqrt{2x+2h+3}+\sqrt{2x+3}} \end{aligned}$$

□

So far, we have not mentioned the domain and range of the functions defined above. Indeed, we will often not describe the domain explicitly but use the following convention:

Convention 2.7: Standard convention of the domain

Unless stated otherwise, a function f is assumed to allow any real numbers x as an input for which the output $f(x)$ is a well-defined real number. We refer to this as the **standard convention of the domain**.
In this case, both domain and range are then subsets of the set $\mathbb{R}$ of real numbers. The range is, of course, the set of all outputs obtained by f from the inputs (see also Note 1.10 on page 7).
In particular, under this convention, any polynomial has the domain $\mathbb{R}$ of all real numbers.

Example 2.8

Find the domain of each of the following functions according to the standard convention of the domain.

a) $f(x) = 4x^3 - 2x + 5$

b) $f(x) = |x|$

c) $f(x) = \sqrt{x}$

d) $f(x) = \sqrt{x-3}$

e) $f(x) = \frac{1}{x-5}$

f) $f(x) = \frac{x-2}{x^2+8x+15}$

g) $f(x) = \begin{cases} x+1 & \text{, for } 2 < x \leq 4 \\ 2x-1 & \text{, for } 5 \leq x \end{cases}$

Solution.

a) There is no problem taking a real number x to the power of any positive integer. Therefore, f is defined for all real numbers x, and the domain is written as $D = \mathbb{R}$.

b) Again, we can take the absolute value for any real number x. The domain is all real numbers, $D = \mathbb{R}$.

c) The square root $\sqrt{x}$ is only defined for $x \geq 0$ (remember we are not using complex numbers yet!). Thus, the domain is $D = [0, \infty)$.

d) Again, the square root is only defined for non-negative numbers. Thus, the argument in the square root has to be greater than or equal to zero: $x - 3 \geq 0$. Solving this for x gives

$$x - 3 \geq 0 \quad \overset{\text{(add 3)}}{\Longrightarrow} \quad x \geq 3.$$

The domain is therefore, $D = [3, \infty)$.

e) A fraction is defined whenever the denominator is not zero, so in this case, $\frac{1}{x-5}$ is defined whenever $x \neq 5$. Therefore, the domain is all real numbers except five, $D = \mathbb{R} - \{5\}$.

f) Again, we need to make sure that the denominator does not become zero, and we disregard the numerator. The denominator is zero exactly when $x^2 + 8x + 15 = 0$. Solving this for x gives:

$$\begin{aligned} x^2 + 8x + 15 = 0 \quad &\Longrightarrow \quad (x+3) \cdot (x+5) = 0 \\ &\Longrightarrow \quad x + 3 = 0 \text{ or } x + 5 = 0 \\ &\Longrightarrow \quad x = -3 \text{ or } x = -5. \end{aligned}$$

The domain is all real numbers except for -3 and -5, that is $D = \mathbb{R} - \{-5, -3\}$.

g) The function is explicitly defined for all $2 < x \leq 4$ and $5 \leq x$. Therefore, the domain is $D = (2, 4] \cup [5, \infty)$.

□

2.2 Exercises

Exercise 2.1

For each of the following functions,

a) $f(x) = 3x + 1$ b) $f(x) = x^2 - x$ c) $f(x) = \sqrt{x^2 - 9}$
d) $f(x) = \frac{1}{x}$ e) $f(x) = \frac{x-5}{x+2}$ f) $f(x) = -x^3$

calculate the function values

i) $f(3)$ ii) $f(5)$ iii) $f(-2)$ iv) $f(0)$ v) $f(\sqrt{13})$
vi) $f(\sqrt{2} + 3)$ vii) $f(-x)$ viii) $f(x + 2)$ ix) $f(x) + h$ x) $f(x + h)$

Exercise 2.2

Let f be the piecewise defined function

$$f(x) = \begin{cases} x - 5 & \text{, for } -4 < x < 3 \\ x^2 & \text{, for } 3 \leq x \leq 6 \end{cases}$$

a) State the domain of the function.
Find the function values

b) $f(2)$ c) $f(5)$ d) $f(-3)$ e) $f(3)$

Exercise 2.3

Let f be the piecewise defined function

$$f(x) = \begin{cases} |x| - x^2 & \text{, for } x < 2 \\ 7 & \text{, for } 2 \leq x < 5 \\ x^2 - 4x + 1 & \text{, for } 5 < x \end{cases}$$

a) State the domain of the function.
Find the function values

b) $f(1)$ c) $f(-2)$ d) $f(3)$
e) $f(2)$ f) $f(5)$ g) $f(7)$

Exercise 2.4

Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the following functions:

a) $f(x) = 5x$	b) $f(x) = 2x - 6$	c) $f(x) = x^2$
d) $f(x) = x^2 + 5x$	e) $f(x) = x^2 - 7$	f) $f(x) = x^2 + 3x + 4$
g) $f(x) = x^2 + 4x - 9$	h) $f(x) = 3x^2 - 2x$	i) $f(x) = 4x^2 + 6x$
j) $f(x) = 2x^2 - 8x - 3$	k) $f(x) = -5x^2 + 3$	l) $f(x) = x^3$

Exercise 2.5

Find the difference quotient $\frac{f(x)-f(a)}{x-a}$ for the following functions:

a) $f(x) = 3x$	b) $f(x) = 4x - 7$	c) $f(x) = x^2 - 3x$
d) $f(x) = x^2 + 4x - 5$	e) $f(x) = 7x^2 + 2x$	f) $f(x) = \frac{1}{x}$

Exercise 2.6

Find the domains of the following functions.

a) $f(x) = x^2 + 3x + 5$	b) $f(x) = \lvert x - 2\rvert$	c) $f(x) = \sqrt{x - 2}$
d) $f(x) = \sqrt{8 - 2x}$	e) $f(x) = \sqrt{\lvert x + 3\rvert}$	f) $f(x) = \frac{1}{x+6}$
g) $f(x) = \frac{x-5}{x-7}$	h) $f(x) = \frac{x+1}{x^2-7x+10}$	i) $f(x) = \frac{x}{\lvert x-2\rvert}$
j) $f(x) = \begin{cases} \lvert x\rvert & \text{for } 1 < x < 2 \\ 2x & \text{for } 3 \leq x \end{cases}$	k) $f(x) = \frac{\sqrt{x}}{x-9}$	l) $f(x) = \frac{5}{\sqrt{x+4}}$