

Precalculus

Third Edition (3.0)

Thomas Tradler

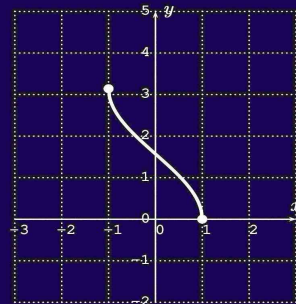
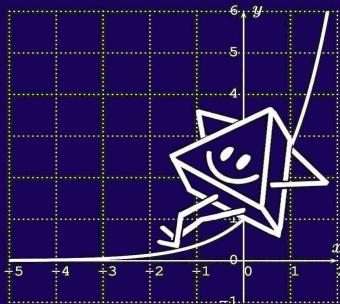
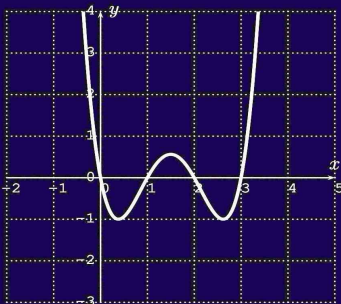
Holly Carley

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Chapter 19

Inverse trigonometric functions

The inverse trigonometric functions are the inverse functions of the $y = \sin x$, $y = \cos x$, and $y = \tan x$ functions restricted to appropriate domains. In this chapter we give a precise definition of these functions.

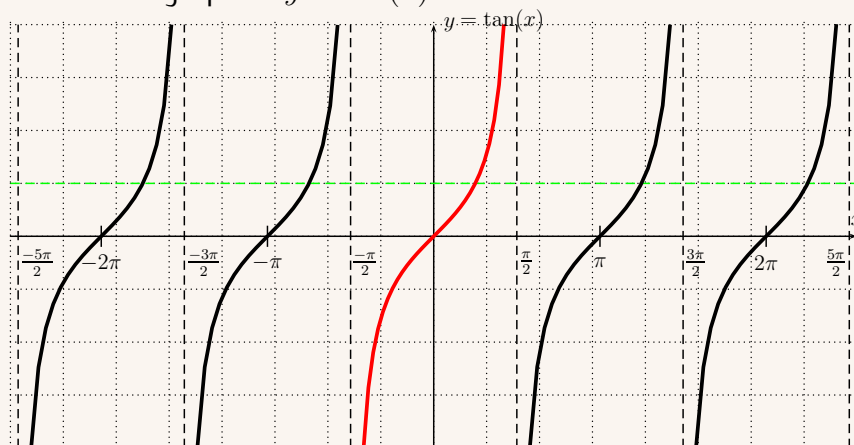
19.1 The functions \sin^{-1} , \cos^{-1} , and \tan^{-1}

We start with the inverse to the tangent function $y = \tan(x)$.

The function $y = \tan^{-1}(x)$

Note 19.1

Recall that the graph of $y = \tan(x)$:



The graph has vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$. Note that $y = \tan(x)$ is not a one-to-one function in the sense of definition 6.1 on page 92. (For example, the horizontal line $y = 1$ intersects the graph at $x = \frac{\pi}{4}, x = \frac{\pi}{4} \pm \pi, x = \frac{\pi}{4} \pm 2\pi$, etc.) However, when we restrict the function to the domain $D = (-\frac{\pi}{2}, \frac{\pi}{2})$, that is, the red part of the above graph, then the restricted function is one-to-one, and for this restricted function, we may take its inverse function.

Definition 19.2: Inverse tangent function

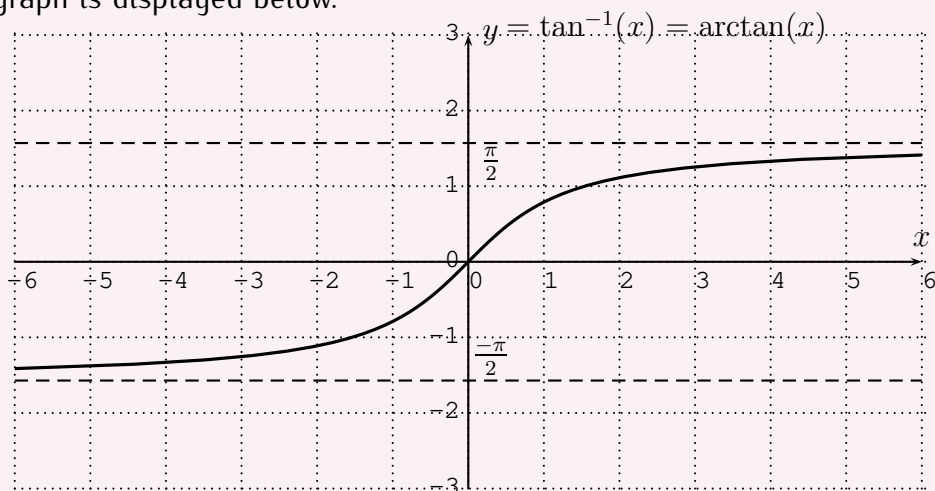
The inverse of the function $y = \tan(x)$ with restricted domain $D = (-\frac{\pi}{2}, \frac{\pi}{2})$ and range $R = \mathbb{R}$ is called the **inverse tangent** function. It is defined by

$$\boxed{x = \tan(y) \iff y = \tan^{-1}(x)} \quad , \text{ for } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Alternatively, the inverse tangent function is also written as the **arc-tangent** function:

$$y = \tan^{-1}(x) = \arctan(x)$$

The arctangent reverses the input and output of the tangent function, so that the arctangent has domain $D = \mathbb{R}$ and range $R = (-\frac{\pi}{2}, \frac{\pi}{2})$. The graph is displayed below.



The inverse tangent function has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

Note 19.3: \tan^{-1} is odd

We note that the inverse tangent function is an odd function:

$$\boxed{\tan^{-1}(-x) = -\tan^{-1}(x)} \quad (19.1)$$

This can be seen by observing that the tangent $y = \tan(x)$ is an odd function (that is $\tan(-x) = -\tan(x)$), and this also is confirmed by the symmetry of the graph of $y = \tan^{-1}(x)$ with respect to the origin $(0, 0)$.

Note 19.4

The exponent notation of $\tan^{-1}(x)$ is unfortunately somewhat inconsistent, since the exponent can refer to two different concepts. Indeed, writing $\tan^{-1}(x) = \arctan(x)$ means that we consider the *inverse function* of the $\tan(x)$ function. However, when we write $\tan^2(x)$, we mean

$$\tan^2(x) = (\tan(x))^2 = \tan(x) \cdot \tan(x)$$

Therefore, $\tan^{-1}(x)$ is the inverse function of $\tan(x)$ with respect to the *composition* operation, whereas $\tan^2(x)$ is the square with respect to the *product* in \mathbb{R} . Note also that the inverse function of the tangent with respect to the product in \mathbb{R} is $y = \frac{1}{\tan(x)} = \cot(x)$, which is the cotangent.

The next example calculates some inverse tangent function values.

Example 19.5

Compute the inverse tangent function values.

$$\text{a) } \tan^{-1}(\sqrt{3}) \quad \text{b) } \tan^{-1}(-1) \quad \text{c) } \tan^{-1}(4.3)$$

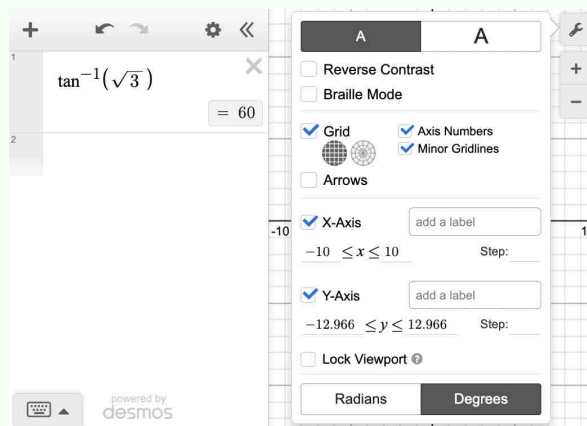
Solution.

- a) We will first show how to compute $\tan^{-1}(\sqrt{3})$ without the use of a calculator, and then more easily with the use of a calculator. Recall the exact values of the tangent function from Section 17.1:

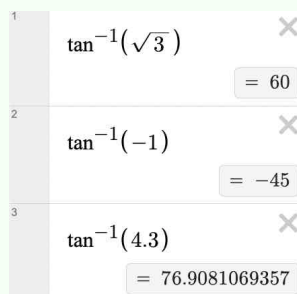
x	$0 = 0^\circ$	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$	$\frac{\pi}{2} = 90^\circ$
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

Since, by Definition 19.2, $y = \tan^{-1}(x)$ is given by $x = \tan(y)$ (for $-\frac{\pi}{2} < y < \frac{\pi}{2}$), we can rewrite $y = \tan^{-1}(\sqrt{3})$ as $\sqrt{3} = \tan(y)$. The above table shows that $\tan(\frac{\pi}{3}) = \sqrt{3}$ and so, $\tan^{-1}(\sqrt{3}) = y = \frac{\pi}{3} = 60^\circ$.

Alternatively, we can use a calculator. Many calculators will not display the exact radian measure for the angle, but only an approximation. Nevertheless, we can use degree measure and then convert to radian if needed.



b) Similarly, we can compute the other values with the calculator.



We see that $\tan^{-1}(-1) = -45^\circ = -\frac{\pi}{4}$.

c) For $\tan^{-1}(4.3)$, we do not have an exact value that appears in our table above. However, we can still find an approximate answer using the calculator, $\tan^{-1}(4.3) \approx 76.91$.

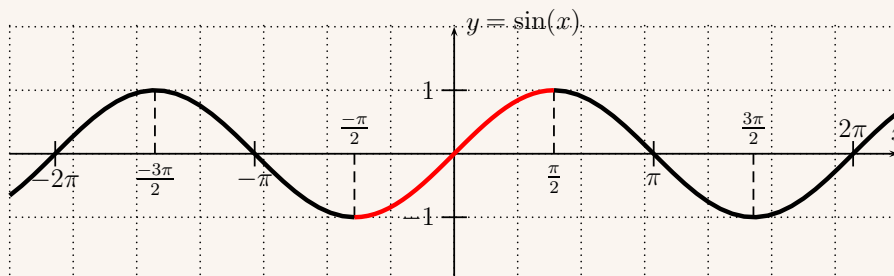
□

The function $y = \sin^{-1}(x)$

Next, we define the inverse sine function.

Note 19.6

We recall the graph of the $y = \sin(x)$ function, and note that it is *not* one-to-one.



However, when restricting the sine to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (drawn in the red part in the above graph), the restricted function *is* one-to-one. Note furthermore, that when restricting the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the range is $[-1, 1]$, and therefore we cannot extend this to a larger domain in a way such that the function remains a one-to-one function. We use the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to define the inverse sine function.

Definition 19.7: Inverse sine function

The inverse of the function $y = \sin(x)$ with restricted domain $D = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and range $R = [-1, 1]$ is called the **inverse sine** function. It is defined by

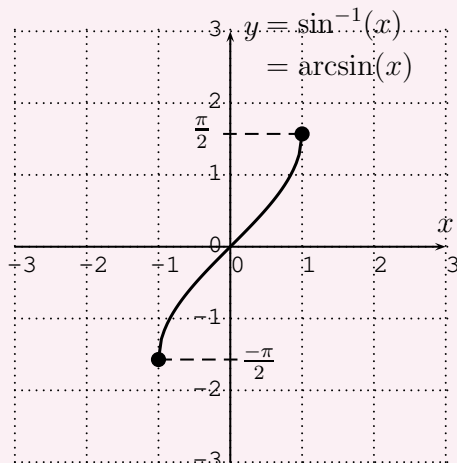
$$\boxed{x = \sin(y) \iff y = \sin^{-1}(x)} \quad , \text{ for } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Alternatively, the inverse sine function is also written as the **arcsine** function:

$$y = \sin^{-1}(x) = \arcsin(x)$$

The arcsine reverses the input and output of the sine function, so that the arcsine has domain $D = [-1, 1]$ and range $R = [-\frac{\pi}{2}, \frac{\pi}{2}]$. The graph

of the arcsine is drawn below.



Note 19.8: \sin^{-1} is odd

The inverse sine function is odd:

$$\boxed{\sin^{-1}(-x) = -\sin^{-1}(x)} \quad (19.2)$$

This can again be seen by observing that the sine $y = \sin(x)$ is an odd function (that is, $\sin(-x) = -\sin(x)$), and is also confirmed by the symmetry of the graph with respect to the origin $(0,0)$.

We calculate some function values of the inverse sine.

Example 19.9

Compute the inverse sine function values.

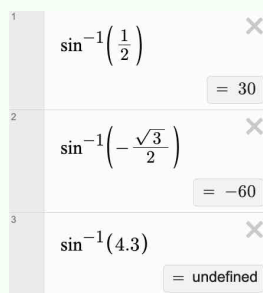
$$\text{a) } \sin^{-1}\left(\frac{1}{2}\right) \quad \text{b) } \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \quad \text{c) } \sin^{-1}(4.3)$$

Solution.

a) We may either use the definition or a calculator to evaluate the expressions. Since $y = \sin^{-1}\left(\frac{1}{2}\right)$ is equivalent to $\frac{1}{2} = \sin(y)$, we need to find such a y with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. For this, recall the known values of the sine.

x	$0 = 0^\circ$	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$	$\frac{\pi}{2} = 90^\circ$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

We see that $y = \frac{\pi}{6}$, so that $\sin^{-1}\left(\frac{1}{2}\right) = y = \frac{\pi}{6} = 30^\circ$. Alternatively, we can obtain the answers using the calculator in degree mode.



b) From the calculator we get that $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60^\circ = -\frac{\pi}{3}$.

c) As for $\sin^{-1}(4.3)$, we note that this is undefined, since the \sin^{-1} has a domain of $[-1, 1]$ and so is only defined for x with $-1 \leq x \leq 1$.

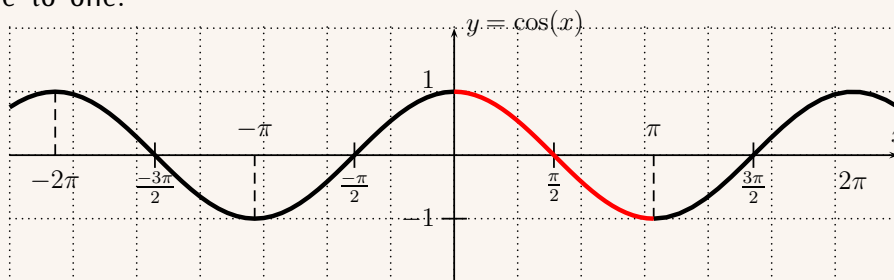
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The function $y = \cos^{-1}(x)$

Lastly, we define the inverse cosine.

Note 19.10

Recall the graph of $y = \cos(x)$, and note again that the function is *not* one-to-one.



Again, we need to restrict the cosine to a smaller domain so that the restricted function becomes one-to-one. By convention, the cosine is restricted to the domain $[0, \pi]$ (see the red part above). This provides a function that is one-to-one, which is used to define the inverse cosine.

Definition 19.11: Inverse cosine function

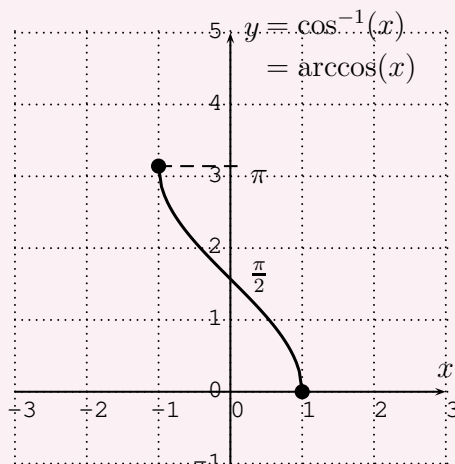
The inverse of the function $y = \cos(x)$ with restricted domain $D = [0, \pi]$ and range $R = [-1, 1]$ is called the **inverse cosine** function. It is defined by

$$\boxed{x = \cos(y) \iff y = \cos^{-1}(x)} \quad , \text{ for } y \in [0, \pi]$$

Alternatively, the inverse cosine function is also written as the **arccosine** function:

$$y = \cos^{-1}(x) = \arccos(x)$$

The arccosine reverses the input and output of the cosine function, so that the arccosine has domain $D = [-1, 1]$ and range $R = [0, \pi]$. The graph of the arccosine is drawn below.

**Note 19.12: \cos^{-1} is neither even nor odd**

The inverse cosine function is *neither even nor odd*. That is, the function $\cos^{-1}(-x)$ *cannot* be computed by simply taking $\pm \cos^{-1}(x)$. But it does have some symmetry given algebraically by the more complicated relation

$$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}(x)} \quad (19.3)$$

Proof of Equation (19.3). We can see that if we shift the graph down by $\frac{\pi}{2}$ the resulting function is odd. That is to say the function with the rule $\cos^{-1}(x) - \frac{\pi}{2}$ is odd:

$$\cos^{-1}(-x) - \frac{\pi}{2} = -(\cos^{-1}(x) - \frac{\pi}{2}),$$

which yields 19.3 upon distributing and adding $\frac{\pi}{2}$.

Another, more formal approach, is as follows. The right relation of (21.12) on page 367 states that we have the relation $\cos(\pi - y) = -\cos(y)$ for all y . Let $-1 \leq x \leq 1$, and denote by $y = \cos^{-1}(x)$. This means that y is the number $0 \leq y \leq \pi$ with $\cos(y) = x$. Then we have

$$-x = -\cos(y) = \cos(\pi - y) \quad (\text{by Equation (21.12)})$$

Applying \cos^{-1} to both sides gives:

$$\cos^{-1}(-x) = \cos^{-1}(\cos(\pi - y)) = \pi - y$$

The last equality follows, since \cos and \cos^{-1} are inverse to each other, and $0 \leq y \leq \pi$, so that $0 \leq \pi - y \leq \pi$ are also in the range of the \cos^{-1} . Rewriting $y = \cos^{-1}(x)$ gives the wanted result:

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

□

Example 19.13

Compute the inverse cosine function values.

$$\text{a) } \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \quad \text{b) } \cos^{-1}\left(-\frac{1}{2}\right) \quad \text{c) } \cos^{-1}(4.3)$$

Solution.

a) Evaluating these expressions by hand requires the use of specific values of the cosine function. We recall the known values of the cosine.

x	$0 = 0^\circ$	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$	$\frac{\pi}{2} = 90^\circ$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Since $y = \cos^{-1}(x)$ is given by $x = \cos(y)$ for $0 \leq y \leq \pi$, we see that for $y = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$, we need a y with $\frac{\sqrt{2}}{2} = \cos(y)$. According to the above table, we get $y = \frac{\pi}{4}$, so that $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = y = \frac{\pi}{4} = 45^\circ$.

Alternatively, we can obtain the answer with the calculator.

1	$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$	\times
	= 45	
2	$\cos^{-1}\left(-\frac{1}{2}\right)$	\times
	= 120	
3	$\cos^{-1}(4.3)$	\times
	= undefined	

- b) Using the calculator, we obtain $\cos^{-1}(-\frac{1}{2}) = 120^\circ = \frac{2\pi}{3}$. Note that this is *not* the same as the negative of $\cos^{-1}(\frac{1}{2}) = 60^\circ$, but the identity (19.3) holds: $\cos^{-1}(-\frac{1}{2}) = 180^\circ - \cos^{-1}(\frac{1}{2})$, that is, $120^\circ = 180^\circ - 60^\circ$.
- c) $\cos^{-1}(4.3)$ is undefined, since the domain of $y = \cos^{-1}(x)$ is $D = [-1, 1]$.

□

19.2 Exercises

Exercise 19.1

Graph the function with the calculator. Use both radian and degree mode to display your graph. Zoom to an appropriate window for each mode to display a graph which includes the main features of the graph.

a) $y = \sin^{-1}(x)$ b) $y = \cos^{-1}(x)$ c) $y = \tan^{-1}(x)$

Exercise 19.2

Find the exact value of the inverse trigonometric function.

a) $\tan^{-1}(\sqrt{3})$ b) $\sin^{-1}(\frac{1}{2})$ c) $\cos^{-1}(\frac{1}{2})$ d) $\tan^{-1}(0)$
 e) $\cos^{-1}(\frac{\sqrt{2}}{2})$ f) $\cos^{-1}(-\frac{\sqrt{2}}{2})$ g) $\sin^{-1}(-1)$ h) $\tan^{-1}(-\sqrt{3})$
 i) $\cos^{-1}(-\frac{\sqrt{3}}{2})$ j) $\sin^{-1}(-\frac{\sqrt{2}}{2})$ k) $\sin^{-1}(-\frac{\sqrt{3}}{2})$ l) $\tan^{-1}(-\frac{1}{\sqrt{3}})$

Exercise 19.3

Find the inverse trigonometric function value using the calculator. Approximate your answer to the nearest hundredth.

- For parts (a)–(f), write your answer in radian mode.

a) $\cos^{-1}(0.2)$ b) $\sin^{-1}(-0.75)$ c) $\cos^{-1}(\frac{1}{3})$
 d) $\tan^{-1}(100,000)$ e) $\tan^{-1}(-2)$ f) $\cos^{-1}(-2)$

- For parts (g)-(l), write your answer in degree mode.

g) $\cos^{-1}(0.68)$ h) $\tan^{-1}(-1)$ i) $\sin^{-1}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$
j) $\tan^{-1}(100,000)$ k) $\cos^{-1}\left(\frac{\sqrt{2}-\sqrt{2}}{2}\right)$ l) $\tan^{-1}(2 + \sqrt{3} - \sqrt{6} - \sqrt{2})$