Precalculus

Third Edition (3.0)

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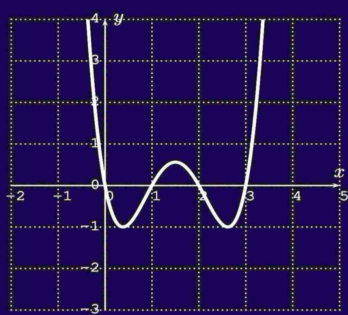

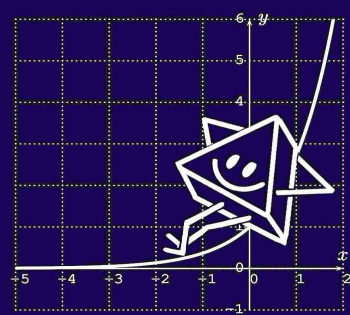

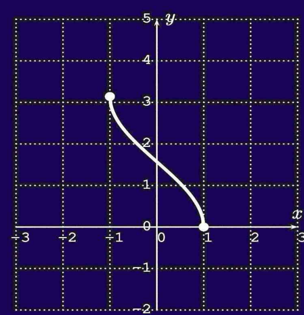

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Chapter 17

Trigonometric functions reviewed

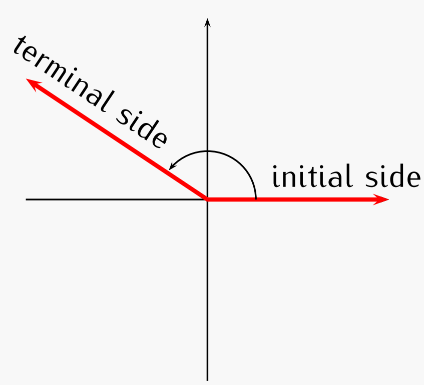
In the next chapters, we will study trigonometric functions, such as $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$ in terms of their function theoretic aspects.

17.1 Review of unit circle trigonometry

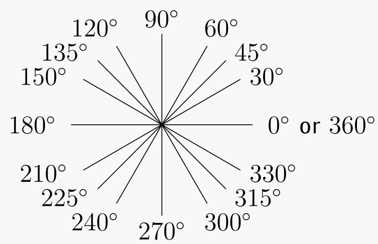
We start by reviewing some basic definitions and facts about trigonometry on the unit circle.

Review 17.1: Angle in standard position

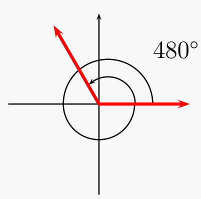
An angle in the plane is in **standard position** if its vertex is at the origin and the initial side is at the positive x-axis.

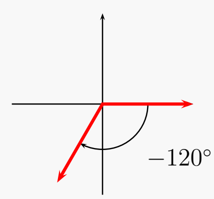


The angle is measured in the counterclockwise direction, where a full rotation measures as $360°$.



Angles greater than $360°$ will span more than a full rotation, angles less than $0°$ will rotate in the clockwise direction. Adding or subtracting $360°$ will give the same terminal side.

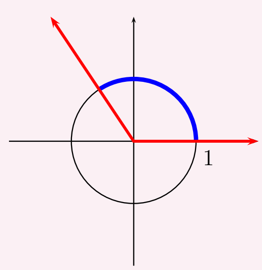




Besides degree measure, we will also need to use radians for the measure of an angle.

Definition 17.2: Radian

The **radian** measure of an angle is the length of the arc (shown below in blue) on the unit circle from the initial side to the terminal side.



Note that a full rotation has a radian measure of 2π, and that degrees and radians are linearly related via the conversion formula:

$$\boxed{\pi = 180°} \tag{17.1}$$

Example 17.3

Convert from radian to degree measure and vice versa.

a) $\frac{5\pi}{4}$ b) $\frac{11\pi}{6}$ c) $150°$ d) $240°$ e) $315°$

Solution.
Replace π with $180°$ and simplify as needed.

a) $\frac{5\pi}{4} = \frac{5\cdot 180°}{4} = 225°$

b) $\frac{11\pi}{6} = \frac{11\cdot 180°}{6} = 330°$

Conversely, using $180° = \pi$, we get that $1° = \frac{\pi}{180}$. With this, we have:

c) $150° = 150 \cdot \frac{\pi}{180} = \frac{150\pi}{180} = \frac{5\pi}{6}$ (cancel with 30)

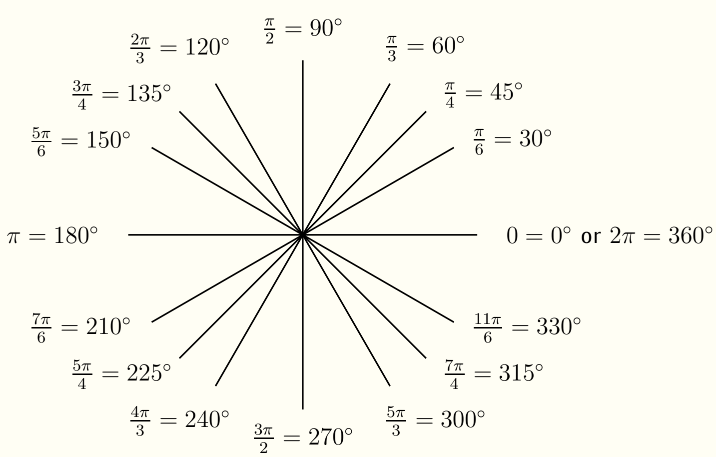
d) $240° = 240 \cdot \frac{\pi}{180} = \frac{4\pi}{3}$ (cancel with 60)

e) $315° = 315 \cdot \frac{\pi}{180} = \frac{7\pi}{4}$ (cancel with 45)

$\square$

Observation 17.4: Radian and degree for multiples of $30°$ and $45°$

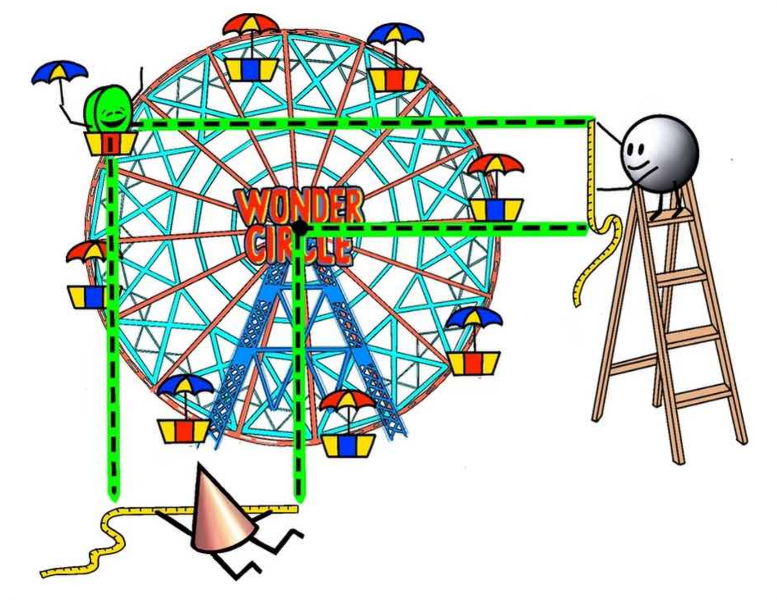
Below are all angles that are multiples of $30°$ or $45°$ between $0°$ and $360°$ in both degree and radian measure.



We next define the "sine", "cosine", and "tangent" of an angle.

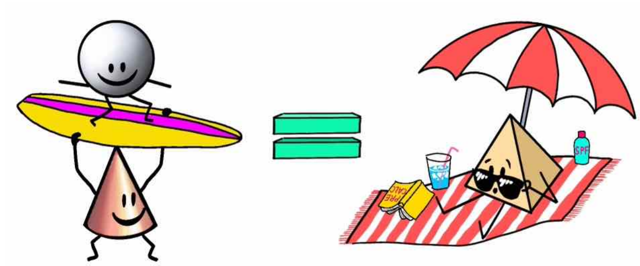
Motivation: Sine and cosine of an angle

Imagine riding in a Ferris wheel.



We may be interested in calculating the distance from the ground to a passenger car when its position around the wheel is rotated at an angle x. Alternatively, we can measure the vertical displacement of the passenger car from the center of the Ferris wheel. Note that, in the above picture, this distance is measured by the "silver sphere" figure atop the ladder. (We think of placing a coordinate system so that the center of the Ferris wheel is the origin $(0, 0)$ of the coordinate system.) If the Ferris wheel has a radius of 1, then the number $\sin(x)$ for a given angle x is precisely what measures this vertical displacement of the passenger car from the center of the wheel.
Similarly, we may ask how far the passenger car is displaced horizontally from the center of the Ferris wheel. Note that this is what the "copper cone" figure at the bottom left is measuring in the above picture. This distance is calculated by $\cos(x)$ for the angle x in the case in which the Ferris wheel has a radius of 1.

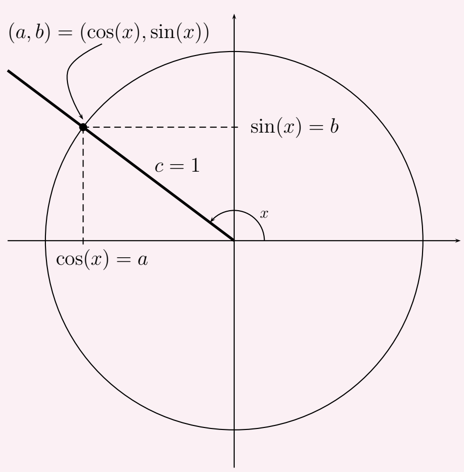
The sine over the cosine is the tangent: $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$.



Using the idea of vertical and horizontal displacement from the center of the Ferris wheel, we now define the trigonometric functions of an angle x.

Definition 17.5: Trigonometric functions

For an angle x, let $P(a, b)$ be the intersection point of the terminal side of x with the unit circle.
Then, we define the **cosine** of x to be the horizontal coordinate a of P, that is, $\cos(x) = a$. We define the **sine** of x to be the vertical coordinate b of P, that is, $\sin(x) = b$. Moreover, we define the **tangent** of x to be the quotient $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{b}{a}$.



$$\boxed{\begin{aligned} \cos(x) &= a \\ \sin(x) &= b \\ \tan(x) &= \tfrac{b}{a} \end{aligned}} \tag{17.2}$$

We also define the multiplicative inverses of these functions, which are called the **secant**, the **cosecant**, and the **cotangent**.

$$\boxed{\begin{aligned} \sec(x) &= \tfrac{1}{\cos(x)} \\ \csc(x) &= \tfrac{1}{\sin(x)} \\ \cot(x) &= \tfrac{1}{\tan(x)} \end{aligned}} \tag{17.3}$$

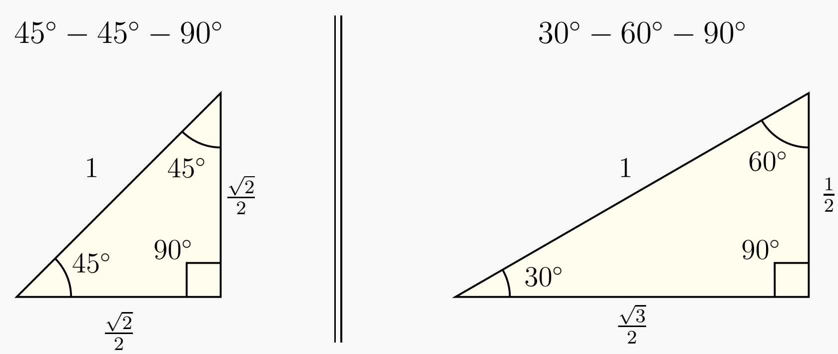
Many elementary facts and identities immediately follow from the above definition, and we will come back to these in Chapter 21. In the following section we will instead show how to compute the trigonometric function values for suitable angles (angles that are multiples of $30°$ or $45°$).

17.2 Computing trigonometric function values

First, we show how to compute $\sin(x)$, $\cos(x)$, and $\tan(x)$ for certain angles x by hand. One way to do so is by using the special $45° - 45° - 90°$ and $30° - 60° - 90°$ triangles. We will review these triangles now.

Review 17.6: Special right triangles

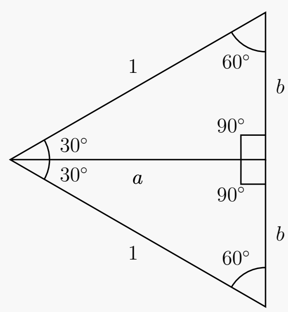
Consider right triangles with angles either $45° - 45° - 90°$ or $30° - 60° - 90°$. If the hypotenuse of the triangles are taken to be 1, then the other side lengths are given as follows:



Proof. In the $45° - 45° - 90°$ triangle, denote the side lengths by a, b, and c. By assumption, the hypotenuse is $c = 1$, and by symmetry, the two side lengths $a = b$ are equal. Using the Pythagorean theorem $a^2 + b^2 = c^2$, we get $a^2 + a^2 = 1^2$, which gives

$$2a^2 = 1 \implies a^2 = \frac{1}{2} \implies a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

For the $30° - 60° - 90°$ triangle, we again denote the side lengths by a, b, and c, with the hypotenuse $c = 1$ by assumption. Now reflect the triangle on its edge opposite to the $60°$ angle.



The outer triangle is an equilateral triangle having all side lengths equal to 1, so that $2b = 1$, or $b = \frac{1}{2}$. Finally, we find a from the Pythagorean theorem $a^2 + b^2 = c^2$, that is, $a^2 + (\frac{1}{2})^2 = 1^2$, so that:

$$a^2 + \frac{1}{4} = 1 \implies a^2 = 1 - \frac{1}{4} = \frac{3}{4} \implies a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

□

We use the above Review 17.6 to compute some trigonometric function values by hand.

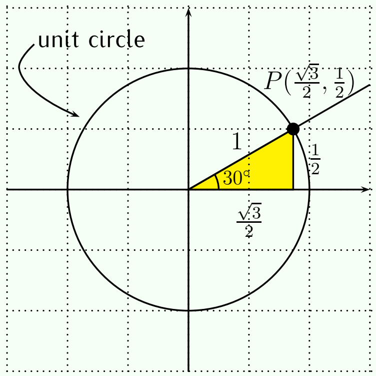
Example 17.7

Find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for the angles

a) $x = 30°$	b) $x = 45°$	c) $x = 60°$	d) $x = 90°$
e) $x = 150°$	f) $x = 225°$	g) $x = 300°$	

Solution.

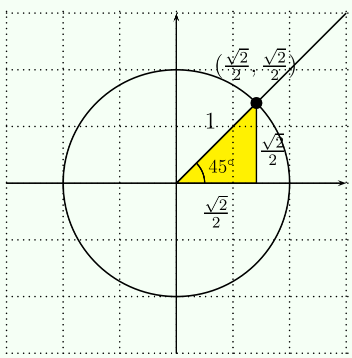
a) We place a $30° - 60° - 90°$ right triangle (drawn below in yellow) in the coordinate plane as shown below.



$$\cos(30°) = a = \frac{\sqrt{3}}{2}$$
$$\sin(30°) = b = \frac{1}{2}$$
$$\tan(30°) = \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

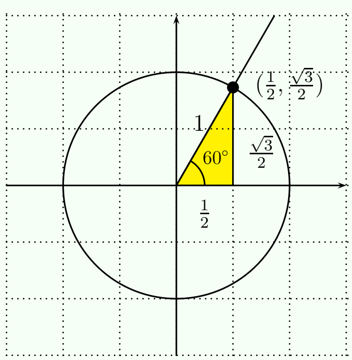
The coordinates of the intersection point P of the terminal side of $30°$ with the unit circle are $(a, b) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, so these give the $\cos(30°) = a = \frac{\sqrt{3}}{2}$ and $\sin(30°) = b = \frac{1}{2}$.

b) For the $45°$ angle, we place a $45° - 45° - 90°$ right triangle (drawn in yellow) in the coordinate plane as shown below.



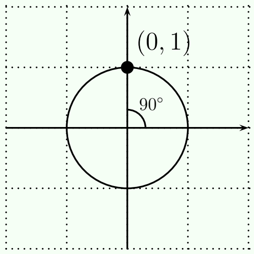
$$\cos(45°) = a = \frac{\sqrt{2}}{2}$$
$$\sin(45°) = b = \frac{\sqrt{2}}{2}$$
$$\tan(45°) = \frac{b}{a} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

c) We place a $30° - 60° - 90°$ triangle with the $60°$ angle at the origin.



$$\cos(60°) = a = \frac{1}{2}$$
$$\sin(60°) = b = \frac{\sqrt{3}}{2}$$
$$\tan(60°) = \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

d) The terminal side of a 90° angle is the positive y-axis, intersecting with the unit circle at $(0, 1)$.

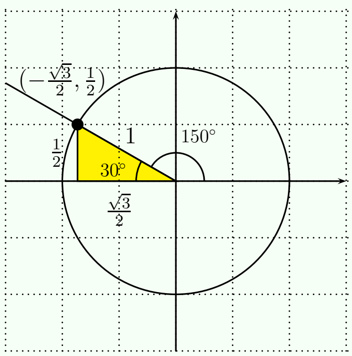


$$\cos(90^\circ) = a = 0$$

$$\sin(90^\circ) = b = 1$$

$\tan(90^\circ) = \frac{b}{a}$ is undefined (since $a = 0$)

e) To obtain the intersection point for terminal side of 150° with the unit circle, we place a $30^\circ - 60^\circ - 90^\circ$ triangle in quadrant II. Thus, the coordinates of the intersection point $P(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ are in quadrant II and therefore have a negative x-coordinate and a positive y-coordinate.



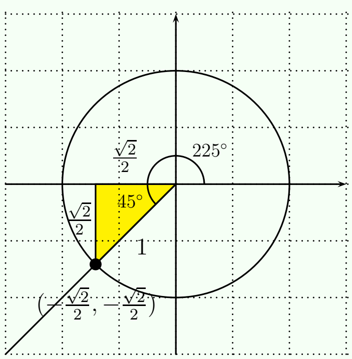
$$\cos(150^\circ) = a = -\frac{\sqrt{3}}{2}$$

$$\sin(150^\circ) = b = \frac{1}{2}$$

$$\tan(150^\circ) = \frac{b}{a} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

f)

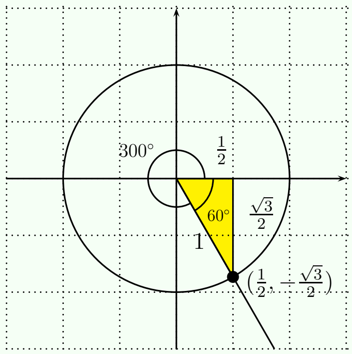


$$\cos(225^\circ) = a = \frac{-\sqrt{2}}{2}$$

$$\sin(225^\circ) = b = \frac{-\sqrt{2}}{2}$$

$$\tan(225^\circ) = \frac{b}{a} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

Note that for an angle in quadrant III, both coordinates are negative, and thus both sine and cosine are negative.

g) 

$$\cos(300°) = a = \frac{1}{2}$$

$$\sin(300°) = b = -\frac{\sqrt{3}}{2}$$

$$\tan(300°) = \frac{b}{a} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1}$$
$$= -\sqrt{3}$$

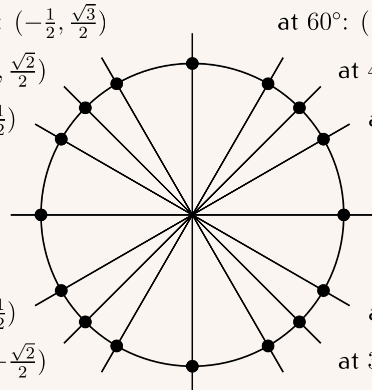
For an angle in quadrant IV, the cosine is positive and the sine is negative.

□

Note 17.8

The method from the previous example can be used to obtain the sine, cosine, and tangent of any angle x that is a multiple of $30°$ or $45°$.

- The sine, cosine, and tangent are given by the coordinates of the intersection of the terminal side of x with the unit circle. These intersection points for angles x from $0°$ to $360°$ (where x are multiples of $30°$ or $45°$) are:



- This gives the trigonometric function values for all angles that are multiples of $30°$ or $45°$. In particular:

x	$0° = 0$	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

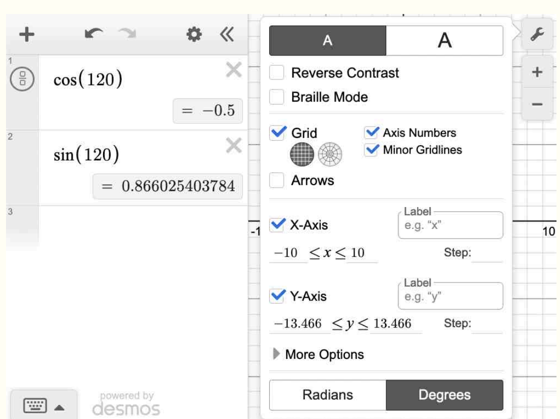
- Angles that differ by $360°$ have the same terminal side and, therefore, the same trigonometric function values.

$$\boxed{\sin(x \pm 360°) = \sin(x)} \quad \boxed{\cos(x \pm 360°) = \cos(x)} \tag{17.4}$$

When using a scientific calculator, the computations of the trigonometric function values becomes significantly easier. However, some of the answers from the calculator may require an appropriate interpretation. We demonstrate this in the following observation.

Observation 17.9: Trigonometric function values with the calculator

We want to use a calculator to find $\cos(120°)$ and $\sin(120°)$. We enter "cos(120)" and "sin(120)" into the calculator and, since our angles are in degree, we check in our settings that we have the degree mode selected. (The mode can be changed with the wrench symbol .)



Note that the calculator correctly shows the cosine $\cos(120°) = -0.5 = -\frac{1}{2}$. However, for sine, we only get an approximation $\sin(120°) \approx 0.866$, which *we have to interpret* as $\sin(120°) = \frac{\sqrt{3}}{2}$. Fortunately, there are only a few possible numbers that appear as the coordinates of the points on the unit circle in Note 17.8. Up to sign, these are the following:

$$\boxed{0 \qquad \frac{1}{2} \qquad \frac{\sqrt{2}}{2} \approx 0.707 \qquad \frac{\sqrt{3}}{2} \approx 0.866 \qquad 1} \tag{17.5}$$

Thus, these are the possible sine and cosine values (up to $\pm$-sign) for the angles shown in Note 17.8.
For the tangent $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we need to look at quotients of the above numbers. These are (compare Example 17.7):

$$\boxed{0 \qquad \frac{\sqrt{3}}{3} \approx 0.577 \qquad 1 \qquad \sqrt{3} \approx 1.732} \tag{17.6}$$

We demonstrate the above observation on the method for finding trigonometric function values with the calculator in the following example.

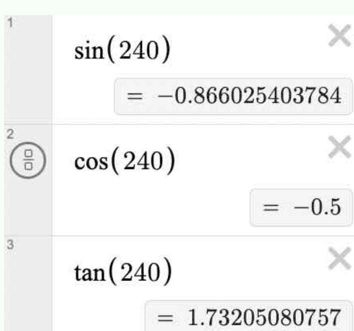
Example 17.10

Use the calculator to find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for:

a) $x = 240°$ b) $x = 495°$ c) $x = \frac{11\pi}{6}$ d) $x = -\frac{9\pi}{4}$

Solution.

We use the calculator and interpret approximations as shown in (17.5) and (17.6).

a) 

$\sin(240°) = -\frac{\sqrt{3}}{2}$

$\cos(240°) = -\frac{1}{2}$

$\tan(240°) = \sqrt{3}$

b) 

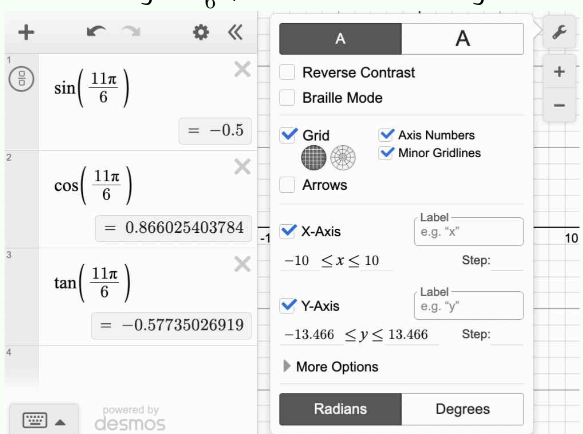
$\sin(495°) = \frac{\sqrt{2}}{2}$

$\cos(495°) = -\frac{\sqrt{2}}{2}$

$\tan(495°) = -1$

Note that $495° - 360° = 135°$, so that $495°$ and $135°$ have the same trigonometric function values.

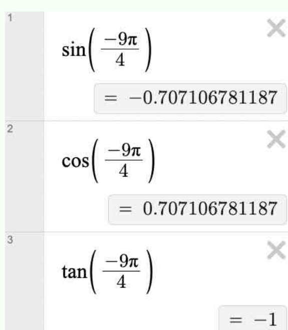
c) For the angle $\frac{11\pi}{6}$, we need to change the calculator to radian mode.



$\sin(\frac{11\pi}{6}) = -\frac{1}{2}$

$\cos(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2}$

$\tan(\frac{11\pi}{6}) = -\frac{\sqrt{3}}{3}$

d) 

$\sin(-\frac{9\pi}{4}) = -\frac{\sqrt{2}}{2}$

$\cos(-\frac{9\pi}{4}) = \frac{\sqrt{2}}{2}$

$\tan(-\frac{9\pi}{4}) = -1$

$\square$

We can also compute exact trigonometric function values for at least some angles other than those in Note 17.8. One possible way to do so is to use the addition and subtraction of angles formulas. We will state these formulas in this section and show they apply in our setting. A proof for these formulas will be given in Section 21.2.

Proposition 17.11: Addition and subtraction of angles formulas

For any angles α and β, we have the following **addition and subtraction of angles formulas**:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

Example 17.12

Find the exact values of the trigonometric functions.

a) $\cos(75^\circ)$ b) $\sin\left(\frac{11\pi}{12}\right)$ c) $\tan(15^\circ)$

Solution.

a) Note that 75° is not one of the angles we computed in Note 17.8, but it is the sum of two such angles, since $75^\circ = 30^\circ + 45^\circ$. To compute $\cos(75^\circ)$, we therefore use the formula for $\cos(\alpha+\beta)$ with $\alpha = 30^\circ$ and $\beta = 45^\circ$.

$$\begin{aligned}
\cos(75^\circ) = \cos(30^\circ+45^\circ) &= \cos(30^\circ)\cos(45^\circ) - \sin(30^\circ)\sin(45^\circ) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

b) Again we want to write the angle $\frac{11\pi}{12}$ as a sum or difference of angles from Note 17.8. It might be a bit easier to first convert the angle from radian into degree:

$$\frac{11\pi}{12} = \frac{11\cdot 180}{12} = 165^\circ$$

Now, there are several ways in which we can write $165°$ as a sum or difference of known angles: $165° = 45° + 120°$ or $165° = 210° - 45°$, etc. We will use the difference $165° = 210° - 45°$ for our computation, but we note that other choices would work just as well. We thus get:

$$\begin{aligned}\sin(165°) &= \sin(210° - 45°) = \sin 210° \cos 45° - \cos 210° \sin 45° \\ &= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{3}}{2}\right) \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

c) For $\tan(15°)$ we note that $15° = 60° - 45°$. We get that:

$$\begin{aligned}\tan(15°) = \tan(60° - 45°) &= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°} \\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}\end{aligned}$$

To fully simplify this expression, we rationalize the denominator by multiplying $1 - \sqrt{3}$ to both numerator and denominator:

$$\begin{aligned}\tan(15°) &= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{\sqrt{3} - \sqrt{3}^2 - 1 + \sqrt{3}}{1^2 - \sqrt{3}^2} \\ &= \frac{2\sqrt{3} - 3 - 1}{1 - 3} = \frac{2\sqrt{3} - 4}{-2} = -\sqrt{3} + 2 = 2 - \sqrt{3}\end{aligned}$$

□

Using Proposition 17.11 we can also obtain certain identities among the trigonometric functions.

Example 17.13

Rewrite $\cos(x + \frac{\pi}{2})$ by using the addition formula.

Solution.

$$\cos\left(x + \frac{\pi}{2}\right) = \cos x \cdot \cos \frac{\pi}{2} - \sin x \cdot \sin \frac{\pi}{2} = \cos x \cdot 0 + \sin x \cdot 1 = \sin(x)$$

□

Another set of useful formulas concerns the trigonometric function values of half- and double-angles.

Proposition 17.14: Half- and double-angle formulas

Let α be an angle. Then we have the **half-angle formulas**:

$$\begin{aligned} \sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ \cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ \tan\frac{\alpha}{2} &= \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \end{aligned}$$

Here, the signs "$\pm$" are determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies. (For more on the signs, see also page 310.)
Furthermore, we have the **double-angle formulas**:

$$\begin{aligned} \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 \\ \tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} \end{aligned}$$

Here is an example involving the half-angle identities.

Example 17.15

Find the trigonometric functions using the half-angle formulas.

a) $\sin(22.5°)$ b) $\cos\left(\frac{7\pi}{8}\right)$ c) $\tan\left(\frac{\pi}{8}\right)$

Solution.

a) Since $22.5° = \frac{45°}{2}$, we use the half-angle formula with $\alpha = 45°$.

$$\begin{aligned} \sin(22.5°) &= \sin\left(\frac{45°}{2}\right) = \pm\sqrt{\frac{1-\cos 22.5°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\ &= \pm\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

Since $22.5°$ is in the first quadrant, the sine is positive, so that $\sin(22.5°) = \frac{\sqrt{2-\sqrt{2}}}{2}$.

b) Note that $\frac{7\pi}{8} = \frac{7\cdot 180°}{2} = 157.5°$ and $157.5° = \frac{315°}{2}$, so that we use the half-angle formulas for $\alpha = 315°$.

$$\cos(157.5°) = \cos\left(\frac{315°}{2}\right) = \pm\sqrt{\frac{1+\cos 315°}{2}}$$

Now, $157.5°$ is in the second quadrant, so that the cosine is negative. Using that $\cos(315°) = \frac{\sqrt{2}}{2}$, we thus get

$$\cos(157.5°) = -\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

c) Note that $\frac{\pi}{8} = \frac{180°}{8} = 22.5°$ and that $22.5° = \frac{45°}{2}$. We therefore get (using the first formula for $\tan\frac{\alpha}{2}$ from Proposition 17.14):

$$\tan(22.5°) = \tan\left(\frac{45°}{2}\right) = \frac{1-\cos 45°}{\sin 45°} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{2}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} - 1 = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2}-1$$

□

We end this section by noting where the trigonometric functions are positive or negative.

Note 17.16: Signs by quadrant

Following the notation from Defintion 17.5, let x be an angle, and let $P(a,b)$ be the intersection point of the terminal side of x with the unit circle. Then, the horizontal coordinate $a = \cos(x)$ of P is positive when P is in quadrant I and IV, whereas the vertical coordinate $b = \sin(x)$ of P is positive when P is in quadrant I and II. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ the sign of $\tan(x)$ is determined by the signs of $\sin(x)$ and $\cos(x)$. Thus the trigonometric functions are positive/negative according to the chart:

Quadrant II	Quadrant I
$\sin(x)$ is positive	$\sin(x)$ is positive
$\cos(x)$ is negative	$\cos(x)$ is positive
$\tan(x)$ is negative	$\tan(x)$ is positive
Quadrant III	Quadrant IV
$\sin(x)$ is negative	$\sin(x)$ is negative
$\cos(x)$ is negative	$\cos(x)$ is positive
$\tan(x)$ is positive	$\tan(x)$ is negative

17.3 Exercises

Exercise 17.1

Convert from radian to degree.

a) $\frac{\pi}{4}$ b) $\frac{2\pi}{3}$ c) $\frac{5\pi}{6}$ d) $\frac{7\pi}{4}$ e) $\frac{3\pi}{2}$ f) $\frac{5\pi}{4}$ g) $\frac{13\pi}{6}$ h) $-\frac{5\pi}{3}$

Exercise 17.2

Convert from degree to radian.

a) $120°$ b) $60°$ c) $300°$ d) $135°$

e) $90°$ f) $225°$ g) $480°$ h) $-150°$

Exercise 17.3

Find $\sin(x)$, $\cos(x)$, and $\tan(x)$ for the following angles.

a) $x = 150°$ b) $x = 45°$ c) $x = 210°$ d) $x = 60°$
e) $x = 30°$ f) $x = 300°$ g) $x = 90°$ h) $x = 315°$
i) $x = 225°$ j) $x = 180°$ k) $x = 120°$ l) $x = 270°$
m) $x = 405°$ n) $x = -135°$ o) $x = -240°$ p) $x = 690°$
q) $x = \frac{5\pi}{3}$ r) $x = \frac{\pi}{6}$ s) $x = \frac{4\pi}{3}$ t) $x = \frac{5\pi}{6}$
u) $x = \frac{7\pi}{3}$ v) $x = \frac{7\pi}{4}$ w) $x = -\frac{\pi}{2}$ x) $x = \frac{13\pi}{3}$

Exercise 17.4

Find the trigonometric function values by using the addition and subtraction formulas.

a) $\sin(75°)$ b) $\cos(15°)$ c) $\tan(105°)$ d) $\sin(195°)$
e) $\cos(345°)$ f) $\sin(15°)$ g) $\cos(285°)$ h) $\tan(165°)$
i) $\cos\left(\frac{11\pi}{12}\right)$ j) $\sin\left(\frac{\pi}{12}\right)$ k) $\tan\left(\frac{13\pi}{12}\right)$ l) $\sin\left(\frac{23\pi}{12}\right)$

Exercise 17.5

Find the exact trigonometric function values by using the half-angle formulas.

a) $\cos(22.5°)$ b) $\sin(15°)$ c) $\cos(15°)$ d) $\tan(15°)$
e) $\sin(7.5°)$ f) $\tan(105°)$ g) $\sin\left(\frac{3\pi}{8}\right)$ h) $\cos\left(\frac{11\pi}{12}\right)$

Exercise 17.6

Simplify the function f using the addition and subtraction formulas.

a) $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ b) $f(x) = \cos\left(x - \frac{\pi}{4}\right)$ c) $f(x) = \tan(\pi - x)$
d) $f(x) = \sin\left(\frac{\pi}{6} - x\right)$ e) $f(x) = \cos\left(\frac{2\pi}{3} - x\right)$ f) $f(x) = \cos\left(x + \frac{11\pi}{12}\right)$