Precalculus

Third Edition (3.0)

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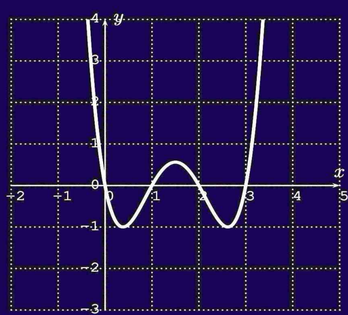
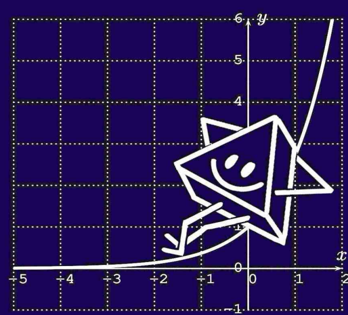
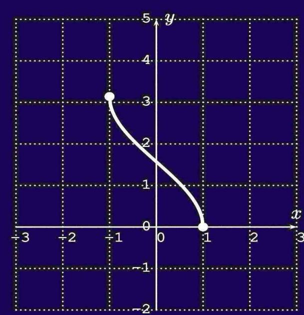
Precalculus Textbook Download Page







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Chapter 16

More applications: Compound interest and half-life

We have already encountered some applications of exponential functions in Section 15.2. In this chapter we give two more applications that come from finance (computing compound interest) and from physics (radioactive decay).

16.1 Compound interest

An important application of the exponential function is given by calculating the interest and the current value of an investment. We start with a motivating example in the following note.

Note 16.1

- We invest an initial amount of $P = \$500$ for 1 year at a rate of $r = 6\%$. The initial amount P is also called the **principal**.

  After 1 year, we receive the principal P together with the interest $r \cdot P$ generated from the principal. The final amount A after 1 year is therefore

  $$A = \$500 + 6\% \cdot \$500 = \$500 \cdot (1 + 0.06) = \$530.$$

- We change the setup of the previous example by taking a quarterly compounding. This means that instead of receiving interest on the principal once at the end of the year, we receive the interest 4 times within the year (after each quarter). However, we now receive only $\frac{1}{4}$ of the interest rate of 6%. We break down the amount received after each quarter.

$$\begin{aligned}
\text{after first quarter: } & 500 \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015 \\
\text{after second quarter: } & (500 \cdot 1.015) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^2 \\
\text{after third quarter: } & (500 \cdot 1.015^2) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^3 \\
\text{after fourth quarter: } & (500 \cdot 1.015^3) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^4 \\
& \Longrightarrow A \approx 530.68
\end{aligned}$$

Note that in the second quarter, we receive interest on the amount we had after the first quarter, and so on. So, in fact, we keep receiving interest on the interest of the interest, etc. For this reason, the final amount received after 1 year $A = \$530.68$ is slightly higher when compounded quarterly than when compounded annually (where $A = \$530.00$).

- We make yet another variation to the above setup. Instead of investing money for 1 year, we invest the principal for 10 years at a quarterly compounding. We then receive interest every quarter for a total of $4 \cdot 10 = 40$ quarters.

$$\begin{aligned}
\text{after first quarter: } & 500 \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015 \\
\text{after second quarter: } & (500 \cdot 1.015) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^2 \\
\text{after third quarter: } & (500 \cdot 1.015^2) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^3 \\
& \vdots \\
\text{after fortieth quarter: } & (500 \cdot 1.015^{39}) \cdot \left(1 + \frac{0.06}{4}\right) = 500 \cdot 1.015^{40} \\
& \Longrightarrow A \approx 907.01
\end{aligned}$$

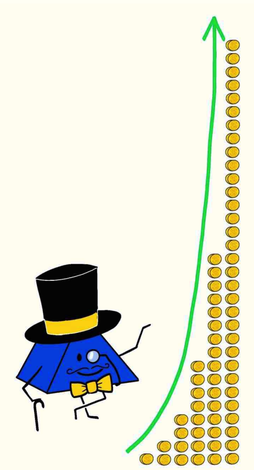
We state our observations from the previous example in the following general observation.

Observation 16.2: Value of an investment compounded n times

A principal (=initial amount) P is invested for t years at a rate r and compounded n times per year. The final amount A is given by

$$\boxed{A = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}} \quad \text{where}$$

$$\begin{cases} P = \text{principal (=initial) amount} \\ A = \text{final amount} \\ r = \text{annual interest rate} \\ n = \text{number of} \\ \qquad \text{compounding periods per year} \\ t = \text{number of years} \end{cases}$$



We can consider performing the compounding in smaller and smaller time intervals. Instead of quarterly compounding, we may take monthly compounding, or daily, hourly, secondly compounding or compounding in even smaller time intervals. Note that, in this case, the number of compounding periods n in the above formula tends to infinity. In the limit when the time intervals go to zero, we obtain what is called *continuous compounding*.

Observation 16.3: Value of an investment compounded continuously

A principal amount P is invested for t years at a rate r and with **continuous compounding**. The final amount A is given by

$$\boxed{A = P \cdot e^{r \cdot t}} \quad \text{where} \quad \begin{cases} P = \text{principal amount} \\ A = \text{final amount} \\ r = \text{annual interest rate} \\ t = \text{number of years} \end{cases}$$

Note 16.4

The reason the exponential function appears in the above formula is that the exponential is the limit of the previous formula in Observation 16.2, when n approaches infinity; compare this with Equation (13.1) on page 237.

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

A more detailed discussion of limits will be provided in a calculus course.

Example 16.5

Determine the final amount received on an investment under the given conditions.

a)	$\$700$,	compounded monthly,	at 4%,	for 3 years
b)	$\$2500$,	compounded semi-annually,	at 5.5%,	for 6 years
c)	$\$1200$,	compounded continuously,	at 3%,	for 2 years

Solution.

a) We can immediately apply the formula in which we substitute the given values of $P = 700$, $n = 12$ (because "monthly" means compounded 12 times per year), $r = 4\% = 0.04$, and $t = 3$. Therefore, we calculate

$$A = 700 \cdot \left(1 + \frac{0.04}{12}\right)^{12\cdot 3} = 700 \cdot \left(1 + \frac{0.04}{12}\right)^{36} \approx 789.09$$

b) We have $P = 2500$, $n = 2$, $r = 5.5\% = 0.055$, and $t = 6$.

$$A = 2500 \cdot \left(1 + \frac{0.055}{2}\right)^{2\cdot 6} \approx 3461.96$$

c) We have $P = 1200$, $r = 3\% = 0.03$, $t = 2$, and we use the formula for continuous compounding.

$$A = 1200 \cdot e^{0.03\cdot 2} = 1200 \cdot e^{0.06} \approx 1274.20$$

□

Instead of asking to find the final amount, we may also ask about any of the other variables in the above formulas for investments.

Example 16.6

a) Find the amount P that needs to be invested at 4.275% compounded annually for 5 years to give a final amount of $\$3000$. (This amount P is also called the **present value** of the future amount of $\$3000$ in 5 years.)

b) At what rate do we have to invest $\$800$ for 6 years compounded quarterly to obtain a final amount of $\$1200$?

c) For how long do we have to invest $\$1000$ at a rate of 2.5% compounded continuously to obtain a final amount of $\$1100$?

d) For how long do we have to invest at a rate of 3.2% compounded monthly until the investment doubles its value?

Solution.

a) We have the following data: $r = 4.275\% = 0.04275$, $n = 1$, $t = 5$, and $A = 3000$. We want to find the present value P. Substituting the given numbers into the appropriate formula, we can solve this for P.

$$3000 = P \cdot \left(1 + \frac{0.04275}{1}\right)^{1 \cdot 5} \implies 3000 = P \cdot (1.04275)^5$$

$$\overset{\text{(divide by } 1.04275^5\text{)}}{\implies} \quad P = \frac{3000}{1.04275^5} \approx 2433.44$$

Therefore, if we invest $\$2433.44$ today under the given conditions, then this will be worth $\$3000$ in 5 years.

b) Substituting the given numbers ($P = 800$, $t = 6$, $n = 4$, $A = 1200$) into the formula gives:

$$1200 = 800 \cdot \left(1 + \frac{r}{4}\right)^{4 \cdot 6} \overset{\text{(divide by 800)}}{\implies} \frac{1200}{800} = \left(1 + \frac{r}{4}\right)^{24}$$

$$\implies \left(1 + \frac{r}{4}\right)^{24} = \frac{3}{2}$$

Next, we have to get the exponent 24 to the right side. This is done by taking a power of $\frac{1}{24}$, or in other words, by taking the 24th root, $\sqrt[24]{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{1}{24}}$.

$$\left(\left(1+\frac{r}{4}\right)^{24}\right)^{\frac{1}{24}} = \left(\frac{3}{2}\right)^{\frac{1}{24}} \implies \left(1+\frac{r}{4}\right)^{24\cdot\frac{1}{24}} = \left(\frac{3}{2}\right)^{\frac{1}{24}}$$
$$\implies 1+\frac{r}{4} = \left(\frac{3}{2}\right)^{\frac{1}{24}} \implies \frac{r}{4} = \left(\frac{3}{2}\right)^{\frac{1}{24}} - 1$$
$$\implies r = 4\cdot\left(\left(\frac{3}{2}\right)^{\frac{1}{24}} - 1\right)$$

Plugging this into the calculator gives $r \approx 0.06815 = 6.815\%$. Therefore, the rate should be about 6.815%.

c) Again, we substitute the given values, $P = 1000$, $r = 2.5\% = 0.025$, $A = 1100$, but now we use the formula for continuous compounding.

$$1100 = 1000\cdot e^{0.025\cdot t} \implies \frac{1100}{1000} = e^{0.025\cdot t} \implies e^{0.025\cdot t} = 1.1$$

To solve for the variable t in the exponent, we need to apply the logarithm. Here, it is most convenient to apply the natural logarithm, because $\ln(x)$ is the inverse of the exponential e^x with base e. Thus, by applying $\ln$ to both sides, we see that

$$\ln(e^{0.025\cdot t}) = \ln(1.1) \implies 0.025\cdot t\cdot\ln(e) = \ln(1.1)$$

Note that we have used that $\log_b(x^n) = n\cdot\log_b(x)$ for any number n as we have seen in Proposition 14.2. Using that $\ln(e) = 1$ (which is the special case of the second equation in (13.3) on page 243 for the base $b = e$), the above becomes

$$0.025\cdot t = \ln(1.1) \implies t = \frac{\ln(1.1)}{0.025} \approx 3.81$$

Therefore, we have to wait 4 years until the investment is worth (more than) $\$1100$.

d) We are given that $r = 3.2\% = 0.032$ and $n = 12$, but no initial amount P is provided. We are seeking to find the time t when the investment doubles. This means that the final amount A is twice the initial amount P, or as a formula: $A = 2\cdot P$. Substituting this into the investment formula and solving gives the wanted answer.

$$2P = P \cdot \left(1 + \frac{0.032}{12}\right)^{12\cdot t} \overset{\text{(divide by } P\text{)}}{\Longrightarrow} 2 = \left(1 + \frac{0.032}{12}\right)^{12\cdot t}$$

$$\overset{\text{(apply ln)}}{\Longrightarrow} \ln(2) = \ln\left(\left(1 + \frac{0.032}{12}\right)^{12\cdot t}\right)$$

$$\Longrightarrow \ln(2) = 12 \cdot t \cdot \ln\left(1 + \frac{0.032}{12}\right)$$

$$\overset{\text{(divide by } 12\cdot\ln\left(1+\frac{0.032}{12}\right)\text{)}}{\Longrightarrow} t = \frac{\ln(2)}{12 \cdot \ln\left(1 + \frac{0.032}{12}\right)} \approx 21.69$$

So, after approximately 21.69 years, the investment will have doubled in value.

□

16.2 Half-life

Recall from Definition 15.8 on page 270 that a function with rate of growth r is an exponential function $f(x) = c \cdot b^x$ with base $b = e^r$. Instead of using the rate of growth, there are other ways to specify the base of an exponential function. One way to specify the base is given by the notion of *half-life*. We give a motivating example in the following note.

Note 16.7

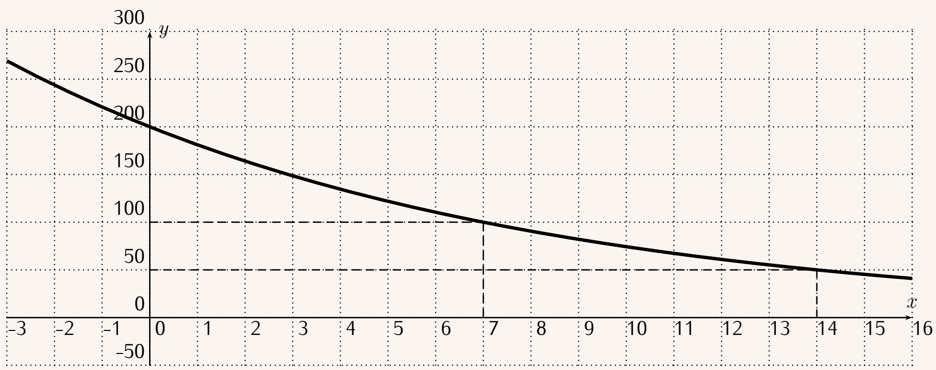
Consider the function $f(x) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{x}{7}}$. We calculate the function values $f(x)$, for $x = 0$, 7, 14, 21, and 28.

$$\begin{aligned} f(0) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{0}{7}} = 200 \cdot 1 = 200 \\ f(7) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{7}{7}} = 200 \cdot \frac{1}{2} = 100 \\ f(14) &= 200 \cdot \left(\frac{1}{2}\right)^{\frac{14}{7}} = 200 \cdot \frac{1}{4} = 50 \end{aligned}$$

$$f(21) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{21}{7}} = 200 \cdot \frac{1}{8} = 25$$

$$f(28) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{28}{7}} = 200 \cdot \frac{1}{16} = 12.5$$

From this calculation, we can see how the function values of f behave: starting from $f(0) = 200$, the function takes half of its value whenever x is increased by 7. For this reason, we say that f has a *half-life* of 7. (The general definition will be given below.) The graph of the function is displayed below.



We collect the ideas that are displayed in the above example in the definition and observation below.

Definition 16.8: Half-life

Let f be an exponential function $f(x) = c \cdot b^x$ with a domain of all real numbers, $D = \mathbb{R}$. Then we say that f has a **half-life** of h if the base is given by

$$\boxed{b = \left(\frac{1}{2}\right)^{\frac{1}{h}}} \tag{16.1}$$

Note that we can also write h in terms of b. Converting (16.1) into a logarithmic equation gives $\frac{1}{h} = \log_{\frac{1}{2}}(b) = \frac{\log b}{\log \frac{1}{2}}$, so that $h = \frac{\log \frac{1}{2}}{\log b} = \log_b\left(\frac{1}{2}\right)$.

Observation 16.9: Graphical interpretation of half-life

Let f be the exponential function given for some real constants $c > 0$ and half-life $h > 0$, that is

$$f(x) = c \cdot \left(\left(\frac{1}{2}\right)^{\frac{1}{h}}\right)^{x} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}}.$$

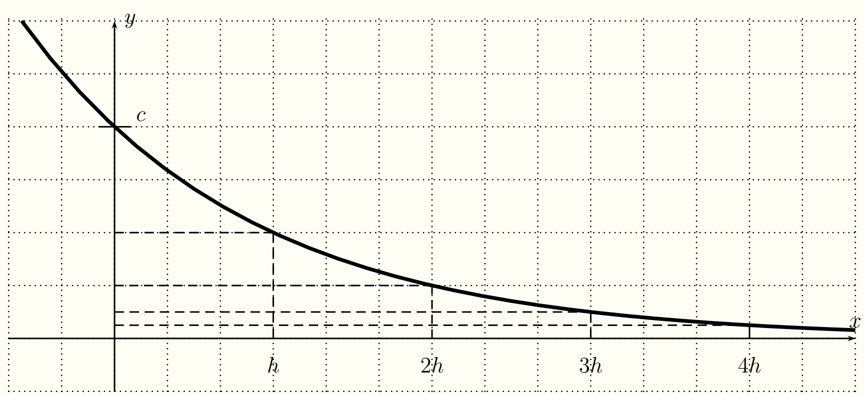
Then we can calculate $f(x+h)$ as follows:

$$\begin{aligned} f(x+h) &= c \cdot \left(\frac{1}{2}\right)^{\frac{x+h}{h}} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}+\frac{h}{h}} = c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}+1} \\ &= c \cdot \left(\frac{1}{2}\right)^{\frac{x}{h}} \cdot \left(\frac{1}{2}\right)^{1} = \frac{1}{2} \cdot f(x) \end{aligned}$$

To summarize, f has the following property:

$$\boxed{f(x+h) = \frac{1}{2} f(x)} \qquad \text{for all } x \in \mathbb{R}. \tag{16.2}$$

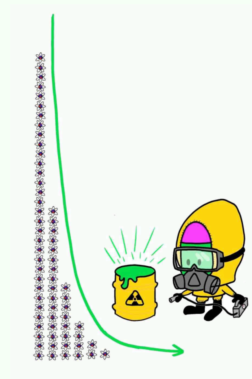
The above equation shows that whenever we add an amount of h to an input x, the effect on f is that the function value decreases by half its previous value. This is also displayed in the graph below.



We will sometimes use a different letter for the input variable. In particular, the function $f(x) = c \cdot (\frac{1}{2})^{\frac{x}{h}}$ is the same as the function $f(t) = c \cdot (\frac{1}{2})^{\frac{t}{h}}$.

Many radioactive isotopes decay with well-known half-lives.[1]

Example 16.10



a) Chromium-51 has a half-life of 27.7 days. How much of 3 grams of chromium-51 will remain after 90 days?

b) An isotope decays within 20 hours from 5 grams to 2.17 grams. Find the half-life of the isotope.

Solution.

a) We use the above formula $y = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$, where $c = 3$ grams is the initial amount of chromium-51, $h = 27.7$ days is the half-life of chromium-51, and $t = 90$ days is time that the isotope decayed. Substituting these numbers into the formula for y, we obtain:

$$y = 3 \cdot \left(\frac{1}{2}\right)^{\frac{90}{27.7}} \approx 0.316$$

Therefore, after 90 days, 0.316 grams of the chromium-51 remains.

b) We have an initial amount of $c = 5$ grams and a remaining amount of $y = 2.17$ grams after $t = 20$ hours. The half-life can be obtained as follows.

$$\begin{aligned} 2.17 = 5 \cdot \left(\frac{1}{2}\right)^{\frac{20}{h}} \quad &\overset{(\div 5)}{\Longrightarrow} \quad 0.434 = \left(\frac{1}{2}\right)^{\frac{20}{h}} \\ &\overset{(\text{apply } \ln)}{\Longrightarrow} \quad \ln(0.434) = \ln\left(0.5^{\frac{20}{h}}\right) \\ &\Longrightarrow \quad \ln(0.434) = \frac{20}{h} \cdot \ln(0.5) \\ &\overset{(\times \frac{h}{\ln(0.434)})}{\Longrightarrow} \quad h = \frac{20 \cdot \ln(0.5)}{\ln(0.434)} \approx 16.6 \end{aligned}$$

Therefore, the half-life of the isotope is approximately 16.6 hours.

□

[1] Half-lives are taken from http://en.wikipedia.org/wiki/List_of_radioactive_nuclides_by_half-life

Note 16.11: Half-life of carbon-14

An important isotope is the radioisotope carbon-14. It decays with a half-life of 5730 years with an accuracy of ± 40 years. For definiteness we will take 5730 years as the half-life of carbon-14.

The half-life of carbon-14 is 5730 years.

One can use the knowledge of the half-life of carbon-14 in dating organic materials via the so called **carbon dating method**. Carbon-14 is produced by a plant during the process of photosynthesis at a fixed level until the plant dies. Therefore, by measuring the remaining amount of carbon-14 in a dead plant, one can determine the date when the plant died. Furthermore, since humans and animals consume plants, the same argument can be applied to determine their (approximate) dates of death.

Example 16.12

a) A dead tree trunk has 86% of its original carbon-14. (Approximately) how many years ago did the tree die?

b) A dead animal at an archeological site has lost 41.3% of its carbon-14. When did the animal die?

Solution.

a) Using the function $y = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$, where c is the amount of carbon-14 that was produced by the tree until it died, y is the remaining amount to date, t is the time that has passed since the tree has died, and h is the half-life of carbon-14. Since 86% of the carbon-14 is left, we have $y = 86\% \cdot c$. Substituting the half-life $h = 5730$ of carbon-14, we can solve for t.

$$0.86 \cdot c = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} \overset{(\div c)}{\Longrightarrow} 0.86 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\overset{\text{(apply ln)}}{\Longrightarrow} \ln(0.86) = \ln\left(0.5^{\frac{t}{5730}}\right)$$

$$\Longrightarrow \ln(0.86) = \frac{t}{5730} \cdot \ln(0.5)$$

$$\overset{(\times \frac{5730}{\ln(0.5)})}{\Longrightarrow} \quad \frac{5730}{\ln(0.5)} \cdot \ln(0.86) = t$$
$$\Longrightarrow \quad t \approx 1247$$

Therefore, the tree died approximately 1247 years ago.

b) Since 41.3% of the carbon-14 is gone, $100\% - 41.3\% = 58.7\%$ remains. Using $y = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$ with $y = 58.7\% \cdot c$ and $h = 5730$, we obtain

$$0.587 \cdot c = c \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}} \quad \overset{(\div c)}{\Longrightarrow} \quad 0.587 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$\overset{(\text{apply } \ln)}{\Longrightarrow} \quad \ln(0.587) = \ln\left(0.5^{\frac{t}{5730}}\right)$$
$$\Longrightarrow \quad \ln(0.587) = \frac{t}{5730} \cdot \ln(0.5)$$
$$\overset{(\times \frac{5730}{\ln(0.5)})}{\Longrightarrow} \quad \frac{5730}{\ln(0.5)} \cdot \ln(0.587) = t$$
$$\Longrightarrow \quad t \approx 4404$$

The animal died 4404 years ago.

□

16.3 Exercises

Exercise 16.1

An investment of $\$5000$ was locked in for 30 years. According to the agreed-upon conditions, the investment will be worth $\$5000 \cdot 1.08^t$ after t years.

a) How much is the investment worth after 5 years?
b) After how many years will the investment be worth $\$20,000$?

Exercise 16.2

Determine the final amount in a savings account under the given conditions.

a)	$\$700$,	compounded quarterly,	at 3%,	for 7 years
b)	$\$1400$,	compounded annually,	at 2.25%,	for 5 years
c)	$\$1400$,	compounded continuously,	at 2.25%,	for 5 years
d)	$\$500$,	compounded monthly,	at 3.99%,	for 2 years
e)	$\$5000$,	compounded continuously,	at 7.4%,	for 3 years
f)	$\$1600$,	compounded daily,	at 3.333%,	for 1 year
g)	$\$750$,	compounded semi-annually,	at 4.9%,	for 4 years

Exercise 16.3

a) Find the amount P that needs to be invested at a rate of 5% compounded quarterly for 6 years to give a final amount of $\$2000$.

b) Find the present value P of a future amount of $A = \$3500$ invested at 6% compounded annually for 3 years.

c) Find the present value P of a future amount of $\$1000$ invested at a rate of 4.9% compounded continuously for 7 years.

d) At what rate do we have to invest $\$1900$ for 4 years compounded monthly to obtain a final amount of $\$2250$?

e) At what rate do we have to invest $\$1300$ for 10 years compounded continuously to obtain a final amount of $\$2000$?

f) For how long do we have to invest $\$3400$ at a rate of 5.125% compounded annually to obtain a final amount of $\$3700$?

g) For how long do we have to invest $\$1000$ at a rate of 2.5% compounded continuously to obtain a final amount of $\$1100$?

h) How long do you have to invest a principal at a rate of 6.75% compounded daily until the investment doubles its value?

i) A certain amount of money has tripled its value while being in a savings account that has an interest rate of 8% compounded continuously. For how long was the money in the savings account?

Exercise 16.4

An unstable element decays at a rate of 5.9% per minute. If 40mg of this element has been produced, how long will it take until 2mg of the element are left? Round your answer to the nearest thousandth.

Exercise 16.5

A substance decays radioactively with a half-life of 232.5 days. How much of 6.8 grams of this substance is left after 1 year?

Exercise 16.6

Fermium-252 decays in 10 minutes to 76.1% of its original mass. Find the half-life of fermium-252.

Exercise 16.7

How long do you have to wait until 15mg of beryllium-7 have decayed to 4mg if the half-life of beryllium-7 is 53.12 days?

Exercise 16.8

If Pharaoh Ramses II died in the year 1213 BC, then what percent of the carbon-14 was left in the mummy of Ramses II in the year 2000?

Exercise 16.9

In order to determine the age of a piece of wood, the amount of carbon-14 was measured. It was determined that the wood had lost 33.1% of its carbon-14. How old is this piece of wood?

Exercise 16.10

Archaeologists uncovered a bone at an ancient resting ground. It was determined that 62% of the carbon-14 was left in the bone. How old is the bone?