Precalculus

Third Edition (3.0)

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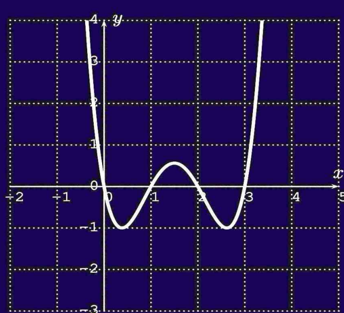

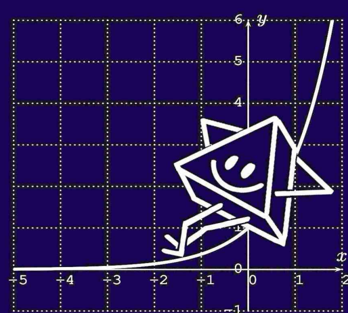

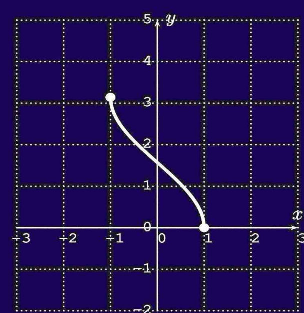

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Chapter 14

Properties of logarithms and logarithmic equations

We now study more algebraic properties of the logarithm. We then use this to solve logarithmic equations.

14.1 Algebraic properties of the logarithms

Recall the well-known identities for exponential expressions.

Review 14.1: Exponential identities

We have the following identities:

$$\boxed{\begin{aligned} b^{x+y} &= b^x \cdot b^y \\ b^{x-y} &= \tfrac{b^x}{b^y} \\ (b^x)^n &= b^{nx} \end{aligned}} \tag{14.1}$$

Writing the above identities in terms of $f(x) = b^x$, these can also be expressed as $f(x+y) = f(x)f(y)$, $f(x-y) = f(x)/f(y)$, and $f(nx) = f(x)^n$.

Since the logarithm is the inverse function of the exponential, there are some logarithmic identities that correspond to (14.1).

Proposition 14.2: Logarithmic identities

The logarithm behaves well with respect to products, quotients, and exponentiation. Indeed, for all positive real numbers $0 < b \neq 1$, $x > 0$, $y > 0$, and real numbers n, we have:

$$\boxed{\begin{gathered} \log_b(x \cdot y) = \log_b(x) + \log_b(y) \\ \log_b(\tfrac{x}{y}) = \log_b(x) - \log_b(y) \\ \log_b(x^n) = n \cdot \log_b(x) \end{gathered}} \tag{14.2}$$

In terms of the logarithmic function $g(x) = \log_b(x)$, the properties in the table above can be written: $g(xy) = g(x) + g(y)$, $g(x/y) = g(x) - g(y)$, and $g(x^n) = n \cdot g(x)$.
Furthermore, for another positive real number $0 < a \neq 1$, we have the *change of base formula*:

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}} \tag{14.3}$$

In particular, we have the formulas from Equation (13.4) on page 243 when taking the base $a = 10$ and $a = e$:

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)}} \quad \text{and} \quad \boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Proof. We start with the first formula $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$. If we call $u = \log_b(x)$ and $v = \log_b(y)$, then the equivalent exponential formulas are $b^u = x$ and $b^v = y$. With this, we have

$$x \cdot y = b^u \cdot b^v = b^{u+v}.$$

Rewriting this in logarithmic form, we obtain

$$\log_b(x \cdot y) = u + v = \log_b(x) + \log_b(y).$$

This is what we needed to show.

Next, we prove the formula $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$. We abbreviate $u = \log_b(x)$ and $v = \log_b(y)$ as before, and their exponential forms are $b^u = x$ and $b^v = y$. Therefore, we have

$$\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}.$$

Rewriting this again in logarithmic form, we obtain the desired result.

$$\log_b\left(\frac{x}{y}\right) = u - v = \log_b(x) - \log_b(y)$$

For the third formula, $\log_b(x^n) = n \cdot \log_b(x)$, we write $u = \log_b(x)$, that is in exponential form $b^u = x$. Then:

$$x^n = (b^u)^n = b^{n\cdot u} \quad\Longrightarrow\quad \log_b(x^n) = n \cdot u = n \cdot \log_b(x)$$

For the last formula (14.3), we write $u = \log_b(x)$, that is, $b^u = x$. Applying the logarithm with base a to $b^u = x$ gives $\log_a(b^u) = \log_a(x)$. As we have just shown before, $\log_a(b^u) = u \cdot \log_a(b)$. Combining these identities with the initial definition $u = \log_b(x)$, we obtain

$$\log_a(x) = \log_a(b^u) = u \cdot \log_a(b) = \log_b(x) \cdot \log_a(b)$$

Dividing both sides by $\log_a(b)$ gives the result $\frac{\log_a(x)}{\log_a(b)} = \log_b(x)$. □

Example 14.3

Combine the terms using the properties of logarithms so as to write as one logarithm.

a) $\frac{1}{2}\ln(x) + \ln(y)$

b) $\frac{2}{3}(\log(x^2y) - \log(xy^2))$

c) $2\ln(x) - \frac{1}{3}\ln(y) - \frac{7}{5}\ln(z)$

d) $5 + \log_2(a^2 - b^2) - \log_2(a+b)$

Solution.

Recall that a fractional exponent can also be rewritten with an nth root.

$$\boxed{x^{\frac{1}{2}} = \sqrt{x}} \quad\text{and}\quad \boxed{x^{\frac{1}{n}} = \sqrt[n]{x}} \quad\Longrightarrow\quad x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$$

We apply the rules from Proposition 14.2.

a) $\frac{1}{2}\ln(x) + \ln(y) = \ln(x^{\frac{1}{2}}) + \ln(y) = \ln(x^{\frac{1}{2}}y) = \ln(\sqrt{x}\cdot y)$

b) $$\frac{2}{3}(\log(x^2y) - \log(xy^2)) = \frac{2}{3}\left(\log\left(\frac{x^2y}{xy^2}\right)\right) = \frac{2}{3}\left(\log\left(\frac{x}{y}\right)\right)$$
$$= \log\left(\left(\frac{x}{y}\right)^{\frac{2}{3}}\right) = \log\left(\sqrt[3]{\frac{x^2}{y^2}}\right)$$

c) $2\ln(x) - \frac{1}{3}\ln(y) - \frac{7}{5}\ln(z) = \ln(x^2) - \ln(\sqrt[3]{y}) - \ln(\sqrt[5]{z^7}) = \ln\left(\frac{x^2}{\sqrt[3]{y}\cdot\sqrt[5]{z^7}}\right)$

d) $$5 + \log_2(a^2 - b^2) - \log_2(a+b) = \log_2(2^5) + \log_2(a^2 - b^2) - \log_2(a+b)$$
$$= \log_2\left(\frac{2^5\cdot(a^2-b^2)}{a+b}\right) = \log_2\left(\frac{32\cdot(a+b)(a-b)}{a+b}\right) = \log_2(32\cdot(a-b))$$

□

Example 14.4

Write the expressions in terms of elementary logarithms $u = \log_b(x)$, $v = \log_b(y)$, and, in part (c), also $w = \log_b(z)$. Assume that $x, y, z > 0$.

a) $\ln(\sqrt{x^5} \cdot y^2)$ b) $\log\left(\sqrt{\sqrt{x} \cdot y^3}\right)$ c) $\log_2\left(\sqrt[3]{\frac{x^2}{y\sqrt{z}}}\right)$

Solution.

In a first step, we rewrite the expression with fractional exponents, and then apply the rules from Proposition 14.2.

a)

$$\begin{aligned}\ln(\sqrt{x^5} \cdot y^2) &= \ln(x^{\frac{5}{2}} \cdot y^2) = \ln(x^{\frac{5}{2}}) + \ln(y^2)\\ &= \frac{5}{2}\ln(x) + 2\ln(y) = \frac{5}{2}u + 2v\end{aligned}$$

b)

$$\begin{aligned}\log\left(\sqrt{\sqrt{x} \cdot y^3}\right) &= \log\left(\left(x^{\frac{1}{2}}y^3\right)^{\frac{1}{2}}\right) = \frac{1}{2}\log\left(x^{\frac{1}{2}}y^3\right)\\ &= \frac{1}{2}\left(\log(x^{\frac{1}{2}}) + \log(y^3)\right) = \frac{1}{2}\left(\frac{1}{2}\log(x) + 3\log(y)\right)\\ &= \frac{1}{4}\log(x) + \frac{3}{2}\log(y) = \frac{1}{4}u + \frac{3}{2}v\end{aligned}$$

c)

$$\begin{aligned}\log_2\left(\sqrt[3]{\frac{x^2}{y\sqrt{z}}}\right) &= \log_2\left(\left(\frac{x^2}{y \cdot z^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right) = \frac{1}{3}\log_2\left(\frac{x^2}{y \cdot z^{\frac{1}{2}}}\right)\\ &= \frac{1}{3}\left(\log_2(x^2) - \log_2(y) - \log_2(z^{\frac{1}{2}})\right)\\ &= \frac{1}{3}\left(2\log_2(x) - \log_2(y) - \frac{1}{2}\log_2(z)\right)\\ &= \frac{2}{3}\log_2(x) - \frac{1}{3}\log_2(y) - \frac{1}{6}\log_2(z)\\ &= \frac{2}{3}u - \frac{1}{3}v - \frac{1}{6}w\end{aligned}$$

□

14.2 Solving logarithmic equations

We can solve exponential and logarithmic equations by applying logarithms and exponentials. Since the exponential and logarithmic functions are invertible (they are inverses of each other), these functions necessarily have to be one-to-one functions. As an algebraic expression, this means that:

Observation 14.5: $y = b^x$ and $y = \log_b(x)$ are one-to-one

The exponential and the logarithmic functions are one-to-one:

$$\boxed{b^x = b^y \quad \Leftrightarrow \quad x = y} \tag{14.4}$$

$$\boxed{\log_b(x) = \log_b(y) \quad \Leftrightarrow \quad x = y} \tag{14.5}$$

In the following examples, we use the above to solve equations that involve logarithms.

Example 14.6

Solve for x.

a) $\log_6(3x-5) = \log_6(x-1)$
b) $\log_2(x+5) = \log_2(x+3) + 4$
c) $\log(x) + \log(x+4) = \log(5)$
d) $\log_3(x-2) + \log_3(x+6) = 2$
e) $\ln(x+2) + \ln(x-3) = \ln(7)$

Solution.

a) We can use Equation (14.5) as follows.

$$\log_6(3x-5) = \log_6(x-1) \implies 3x - 5 = x - 1 \overset{(-x+5)}{\implies} 2x = 4$$
$$\implies x = 2$$

An immediate check shows $x = 2$ is indeed a solution, since $\log_6(3 \cdot 2 - 5) = \log_6(1)$ and $\log_6(2-1) = \log_6(1)$.

b) We have to solve $\log_2(x+5) = \log_2(x+3) + 4$. To combine the right-hand side, recall that 4 can be written as a logarithm, $4 = \log_2(2^4) = \log_2 16$. With this remark we can now solve the equation for x.

$$\log_2(x+5) = \log_2(x+3)+4 \implies \log_2(x+5) = \log_2(x+3)+\log_2(16)$$
$$\implies \log_2(x+5) = \log_2(16\cdot(x+3)) \implies x+5 = 16(x+3)$$
$$\implies x+5 = 16x+48 \overset{(-16x-5)}{\implies} -15x = 43 \implies x = -\frac{43}{15}$$

c) We start by combining the logarithms.

$$\begin{aligned} \log(x)+\log(x+4)=\log(5) &\implies \log(x\cdot(x+4))=\log(5) \\ &\overset{\text{remove log}}{\implies} x(x+4)=5 \\ &\implies x^2+4x-5=0 \\ &\implies (x+5)(x-1)=0 \\ &\implies x=-5 \text{ or } x=1 \end{aligned}$$

Since the equation became a quadratic equation, we ended up with two possible solutions $x = -5$ and $x = 1$. However, since $x = -5$ would give a negative value inside a logarithm in our original equation $\log(x)+\log(x+4) = \log(5)$, we need to exclude this solution. The only solution is $x = 1$.

We note that the incorrect solution $x = -5$ is introduced in the very first implication, since -5 in fact *is* a perfectly well-defined solution of the equation $\log(x\cdot(x+4)) = \log(5)$,

$$\log((-5)\cdot(-5+4)) = \log((-5)\cdot(-1)) = \log(5),$$

whereas -5 is not a solution of $\log(x)+\log(x+4) = \log(5)$, since $\log(-5)+\log(-5+4)$ is undefined.

d) Using that $2 = \log_3(3^2)$:

$$\begin{aligned} \log_3(x-2)+\log_3(x+6)=2 &\implies \log_3((x-2)(x+6))=\log_3(3^2) \\ &\implies (x-2)(x+6)=3^2 \\ &\implies x^2+4x-12=9 \\ &\implies x^2+4x-21=0 \\ &\implies (x+7)(x-3)=0 \\ &\implies x=-7 \text{ or } x=3 \end{aligned}$$

We exclude $x = -7$, since we would obtain a negative value inside a logarithm, so that the solution is $x = 3$.

e) We combine the left-hand side of $\ln(x+2)+\ln(x-3)=\ln(7)$ to get

$$\begin{aligned} \ln((x+2)\cdot(x-3))=\log(7) \quad &\Longrightarrow \quad (x+2)\cdot(x-3)=7 \\ &\Longrightarrow \quad x^2-3x+2x-6=7 \\ &\Longrightarrow \quad x^2-x-13=0 \end{aligned}$$

To solve this, we need to use the quadratic formula (8.1).

$$\begin{aligned} x^2-x-13=0 \implies x &= \frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot(-13)}}{2\cdot 1} \\ &= \frac{1\pm\sqrt{1+52}}{2} = \frac{1\pm\sqrt{53}}{2} \end{aligned}$$

To see which of these are actual solutions of $\ln(x+2)+\ln(x-3)=\ln(7)$, note that we have to plug $x=\frac{1\pm\sqrt{53}}{2}$ into $x+2$ and $x-3$ and make sure these are positive:

$$\frac{1+\sqrt{53}}{2}+2\approx 6.14>0 \text{ and } \frac{1+\sqrt{53}}{2}-3\approx 1.14>0$$
$$\frac{1-\sqrt{53}}{2}+2\approx -1.14<0 \text{ and } \frac{1-\sqrt{53}}{2}-3\approx -6.14<0$$

Thus, $\frac{1-\sqrt{53}}{2}$ is not a solution (since, for example, $\ln(\frac{1-\sqrt{53}}{2}+2)$ is undefined), and the only solution is $x=\frac{1+\sqrt{53}}{2}$.

□

In Examples 14.6 (c)–(e) our calculations showed that the given equalities had two *possible* solutions. After checking these with the original equation, we saw that one was an actual solution (making the equation true), while the other was not (and therefore was rejected). In general, it may turn out that all of the possible solutions are actual solutions, or none of the possible solutions are actual solutions. This is demonstrated in the next example.

Example 14.7

Solve for x.

a) $\log_3(x+1) + \log_3(7-x) = \log_3(12)$
b) $\log_5(x-7) + \log_5(2-x) = \log_5(4)$

Solution.

a) Combining the logarithms gives $\log_3((x+1)(7-x)) = \log_3(12)$, which implies

$$\begin{aligned}(x+1)(7-x) = 12 &\implies 7x - x^2 + 7 - x = 12 \\ &\implies 0 = x^2 - 6x + 5 \\ &\implies 0 = (x-1)(x-5) \\ &\implies x = 1, x = 5\end{aligned}$$

Since both give positive arguments in the logarithms, we have, indeed, two solutions $x = 1$ and $x = 5$.

b) We get $\log_5((x-7)(2-x)) = \log_5(4)$, and thus $(x-7)(2-x) = 4$, which can be rewritten as $2x - x^2 - 14 + 7x = 4$, and thus as $0 = x^2 - 9x + 18$. Factoring yields $0 = (x-3)(x-6)$, which has the two possible solutions $x = 3$ and $x = 6$. However, 3 is not a solution, since $2 - 3 = -1 < 0$; and 6 is not a solution since $2 - 6 = -4 < 0$. We conclude that there is no solution.

□

14.3 Exercises

Exercise 14.1

Combine the terms and write your answer as one logarithm.

a) $3\ln(x)+\ln(y)$

b) $\log(x)-\frac{2}{3}\log(y)$

c) $\frac{1}{3}\log(x)-\log(y)+4\log(z)$

d) $\log(xy^2z^3)-\log(x^4y^3z^2)$

e) $\frac{1}{4}\ln(x)-\frac{1}{2}\ln(y)+\frac{2}{3}\ln(z)$

f) $-\ln(x^2-1)+\ln(x-1)$

g) $5\ln(x)+2\ln(x^4)-3\ln(x)$

h) $\log_5(a^2+10a+9)-\log_5(a+9)+2$

Exercise 14.2

Write the expressions in terms of elementary logarithms $u=\log_b(x)$, $v=\log_b(y)$, and $w=\log_b(z)$ (whichever are applicable). Assume that $x,y,z>0$.

a) $\log(x^3\cdot y)$

b) $\log(\sqrt[3]{x^2}\cdot\sqrt[4]{y^7})$

c) $\log\left(\sqrt{x\cdot\sqrt[3]{y}}\right)$

d) $\ln\left(\frac{x^3}{y^4}\right)$

e) $\ln\left(\frac{x^2}{\sqrt{y}\cdot z^2}\right)$

f) $\log_3\left(\sqrt{\frac{x\cdot y^3}{\sqrt{z}}}\right)$

g) $\log_2\left(\frac{\sqrt[4]{x^3\cdot z}}{y^3}\right)$

h) $\log\left(\frac{100\sqrt[5]{z}}{y^2}\right)$

i) $\ln\left(\sqrt[3]{\frac{\sqrt{y}\cdot z^4}{e^2}}\right)$

Exercise 14.3

Solve for x without using a calculator.

a) $\ln(2x+4)=\ln(5x-5)$

b) $\ln(x+6)=\ln(x-2)+\ln(3)$

c) $\log_2(x+5)=\log_2(x)+5$

d) $\log(x)+1=\log(5x+380)$

e) $\log(x+5)+\log(x)=\log(6)$

f) $\log_2(x)+\log_2(x-6)=4$

g) $\log_6(x)+\log_6(x-16)=2$

h) $\log_5(x-24)+\log_5(x)=2$

i) $\log_4(x)+\log_4(x+6)=2$

j) $\log_2(x+3)+\log_2(x+5)=3$