Precalculus

Third Edition (3.0)

Thomas Tradler

Holly Carley

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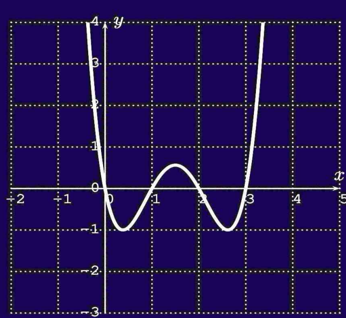
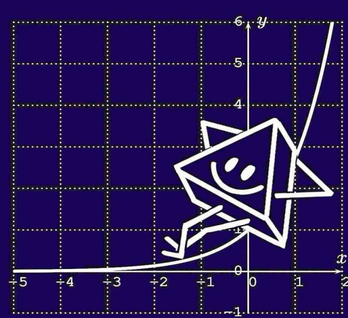
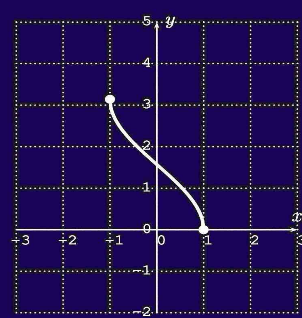
Precalculus Textbook Download Page







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The images of the Desmos graphing calculator were generated from the Desmos website at **https://www.desmos.com/calculator**

Chapter 13

Exponential and logarithmic functions

We now consider functions that differ greatly from polynomials and rational functions in their complexity. More precisely, we will explore exponential and logarithmic functions from a function theoretic point of view.

13.1 Exponential functions and their graphs

We start by recalling the definition of an exponential function and by studying its graph.

Definition 13.1: Exponential function

A function f is called an **exponential function** if it is of the form

$$f(x) = c \cdot b^x$$

for some real number c and positive real number $b > 0$. The constant b is called the **base**.
Since $f(x) = c \cdot b^x$ is defined for all real numbers, the domain of f is $D = \mathbb{R}$.

Example 13.2

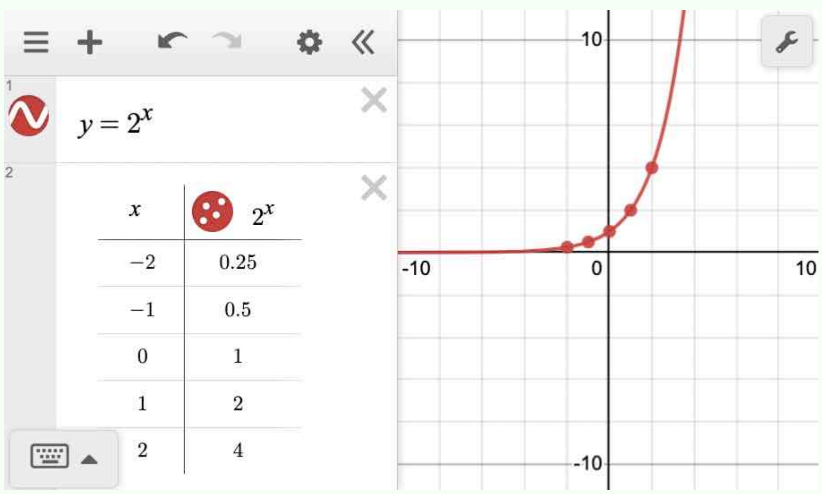
Graph the functions.

$$f(x) = 2^x, \quad g(x) = 3^x, \quad h(x) = 10^x, \quad k(x) = \left(\frac{1}{2}\right)^x, \quad l(x) = \left(\frac{1}{10}\right)^x$$

First, we will graph the function $f(x) = 2^x$ by calculating the function values in a table and then plotting the points in the x-y plane. We can calculate the values by hand, or simply use the table function of the calculator to find the function values.

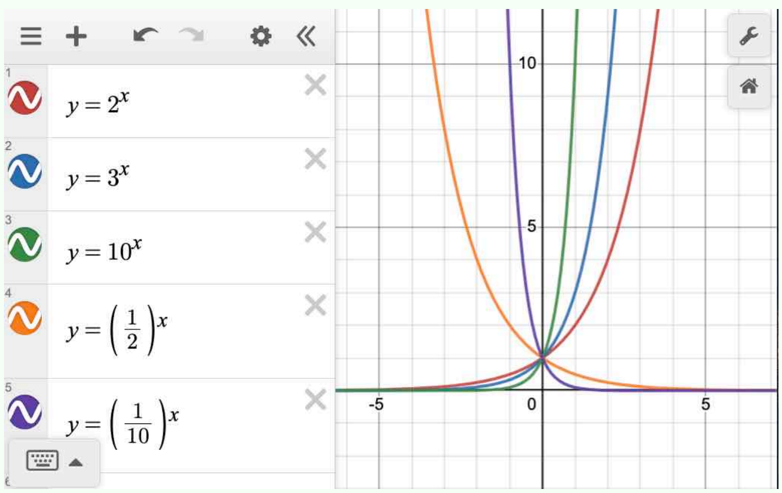
$$\begin{aligned} f(0) &= 2^0 = 1 \\ f(1) &= 2^1 = 2 \\ f(2) &= 2^2 = 4 \\ f(3) &= 2^3 = 8 \\ f(-1) &= 2^{-1} = 0.5 \\ f(-2) &= 2^{-2} = 0.25 \end{aligned}$$

We obtain the following graph.



Similarly, we can compute the table for the other functions g, h, k, and l, and plot them with the graphing calculator.

x	2^x	3^x	10^x	$\left(\frac{1}{2}\right)^x$	$\left(\frac{1}{10}\right)^x$
-2	0.25	0.11111111	0.01	4	100
-1	0.5	0.33333333	0.1	2	10
0	1	1	1	1	1
1	2	3	10	0.5	0.1
2	4	9	100	0.25	0.01
3	8	27	1000	0.125	0.001
4	16	81	10000	0.0625	1×10^{-4}



Note that the function k can also be written as

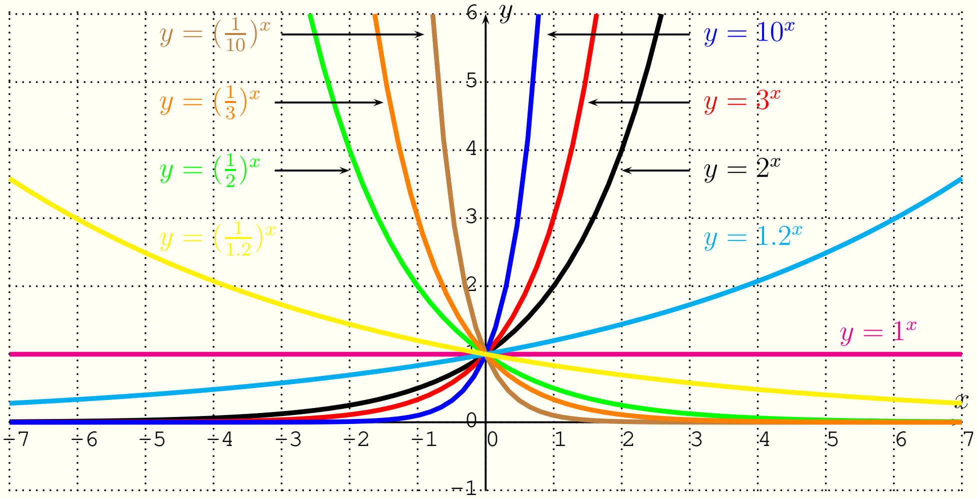
$$k(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x},$$

and similarly, $l(x) = \left(\frac{1}{10}\right)^x = 10^{-x}$.

This example shows that the exponential function has the following properties.

Observation 13.3: Graph of an exponential function

The graph of the exponential function $f(x) = b^x$ with $b > 0$ and $b \neq 1$ has a horizontal asymptote at $y = 0$.



- If $b > 1$, then $f(x)$ approaches $+\infty$ when x approaches $+\infty$, and $f(x)$ approaches 0 when x approaches $-\infty$.
- If $0 < b < 1$, then $f(x)$ approaches 0 when x approaches $+\infty$, and $f(x)$ approaches $+\infty$ when x approaches $-\infty$.

Note that all of these graphs have the horizontal asymptote $y = 0$.

An important base that we will frequently need to consider is the base of e, where e is Euler's number.

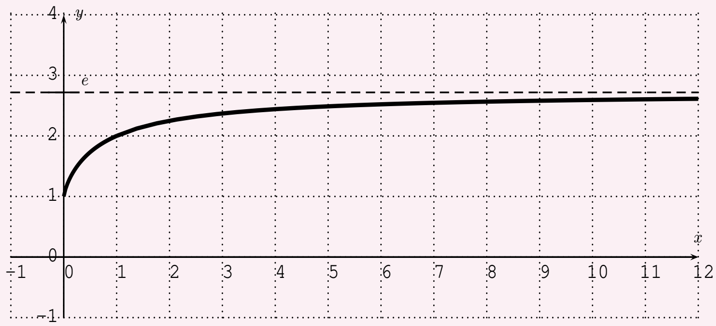
Definition 13.4: Euler's number

Euler's number e is an irrational number that is approximately

$$\boxed{e = 2.718281828459045235\ldots}$$

To be precise, we can define e as the number which is the horizontal asymptote of the function

$f(x) = \left(1+\frac{1}{x}\right)^x$ when x approaches $+\infty$.



One can show that f has, indeed, a horizontal asymptote, and this limit is defined as e.

$$e := \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Furthermore, one can show that the exponential function with base e has a similar limit expression.

$$e^r = \lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x \qquad (13.1)$$

Alternatively, Euler's number and the exponential function with base e may also be defined using an infinite series, namely, $e^r = 1+r+\frac{r^2}{1\cdot 2}+\frac{r^3}{1\cdot 2\cdot 3}+\frac{r^4}{1\cdot 2\cdot 3\cdot 4}+\dots$. These ideas will be explored further in a course in calculus.

We next graph a few functions that use Euler's number.

Example 13.5

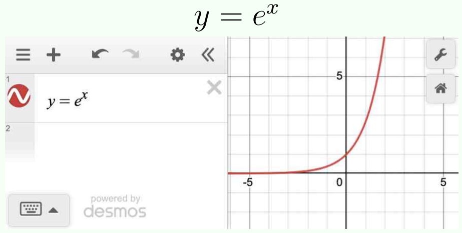
Graph the functions.

a) $y = e^x$ b) $y = e^{-x}$ c) $y = e^{-x^2}$ d) $y = \frac{e^x+e^{-x}}{2}$

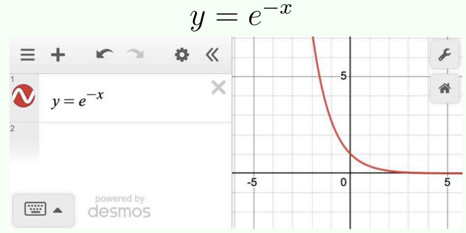
Solution.

Using the calculator, we obtain the desired graphs.

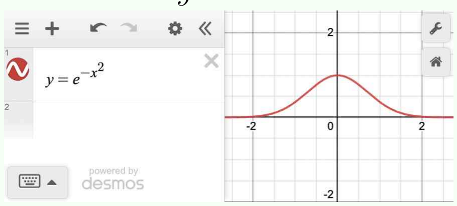
$y = e^x$



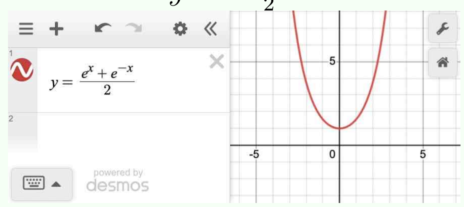
$y = e^{-x}$



$y = e^{-x^2}$



$y = \frac{e^x + e^{-x}}{2}$



The last function $y = \frac{e^x+e^{-x}}{2}$ is called the *hyperbolic cosine*, and is denoted by $\cosh(x) = \frac{e^x+e^{-x}}{2}$. (The *hyperbolic sine*, $\sinh(x) = \frac{e^x-e^{-x}}{2}$, and the *hyperbolic tangent*, $\tanh(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ will be graphed in Exercise 13.1.) □

We now study how different multiplicative factors c affect the shape of an exponential function.

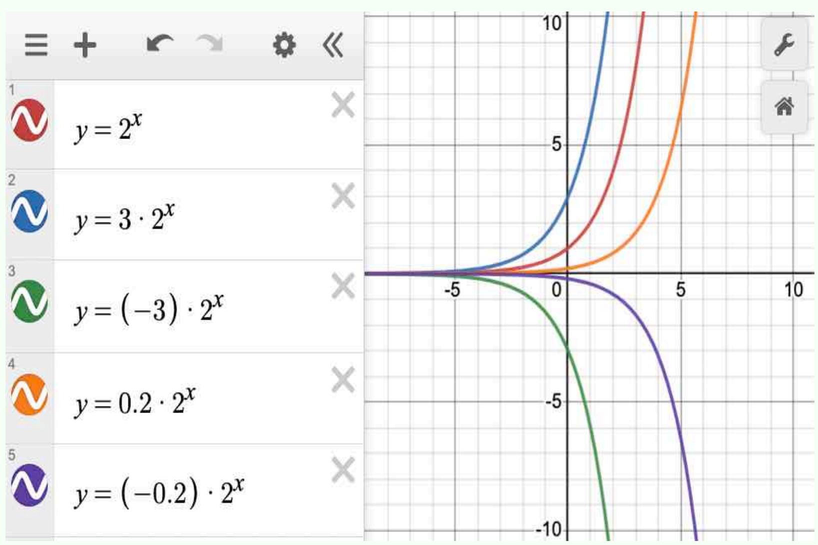
Example 13.6

Graph the functions.

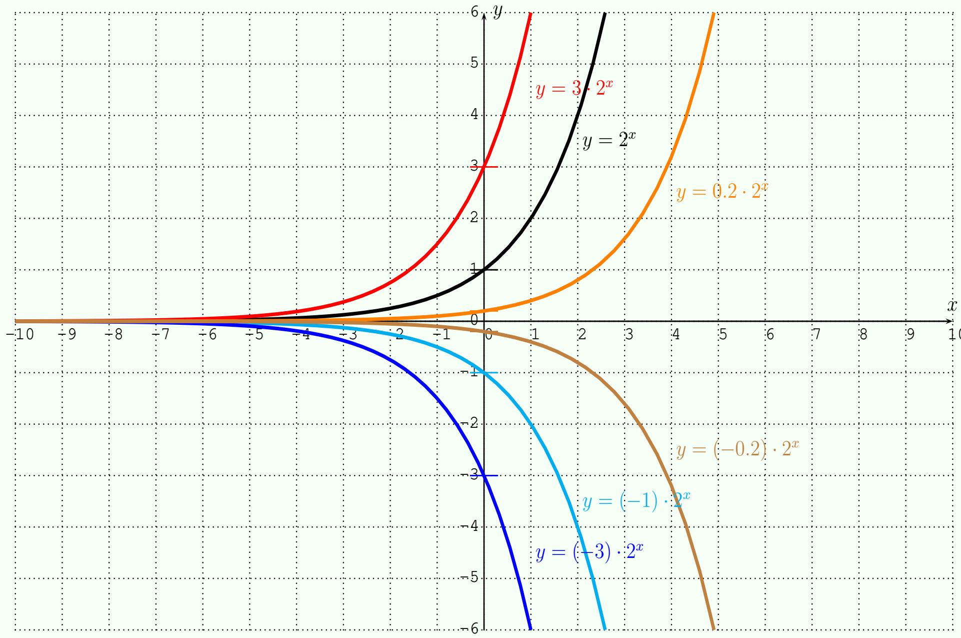
a) $y = 2^x$ b) $y = 3 \cdot 2^x$ c) $y = (-3) \cdot 2^x$
d) $y = 0.2 \cdot 2^x$ e) $y = (-0.2) \cdot 2^x$

Solution.

We graph the functions in the same viewing window.



Here are the graphs of functions $f(x) = c \cdot 2^x$ for various choices of c.



Note that for $f(x) = c \cdot 2^x$, the y-intercept is given at $f(0) = c$. □

Finally, we can combine our knowledge of graph transformations to study exponential functions that are shifted and stretched.

Example 13.7

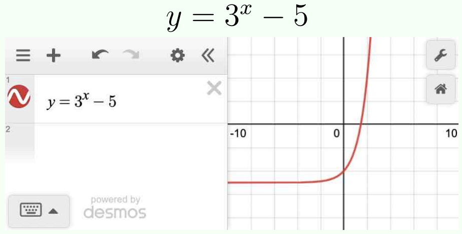
Graph the functions.

a) $y = 3^x - 5$ b) $y = e^{x+4}$ c) $y = \frac{1}{4} \cdot e^{x-3} + 2$

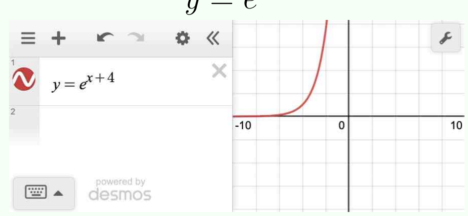
Solution.

The first two graphs are displayed below.

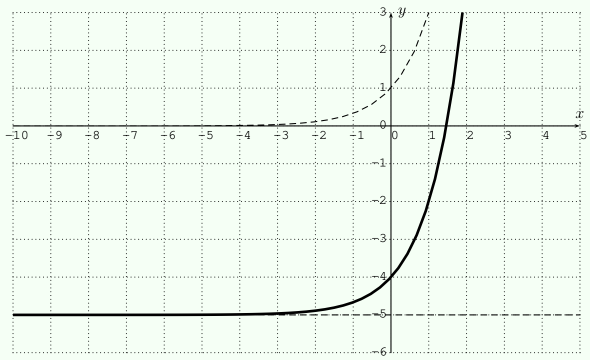
$y = 3^x - 5$



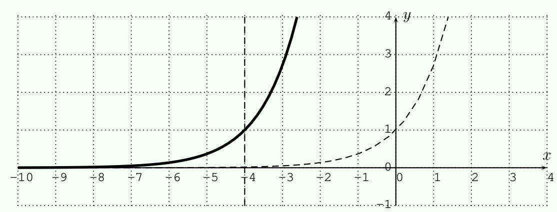
$y = e^{x+4}$



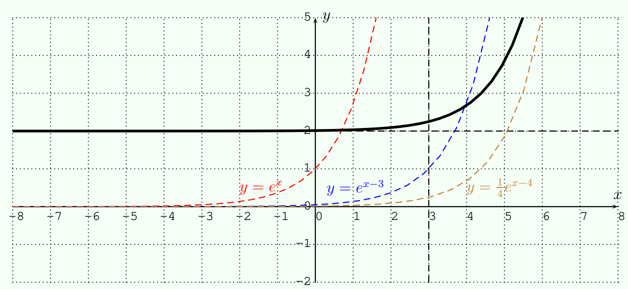
The first graph $y = 3^x - 5$ is the graph of $y = 3^x$ shifted down by 5.

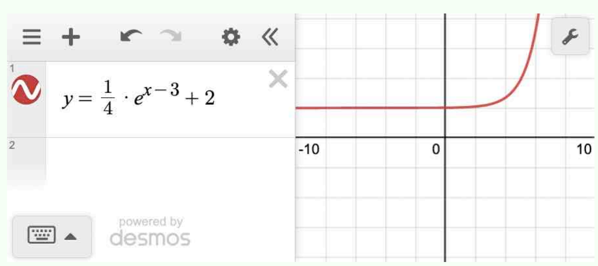


The graph of $y = e^{x+4}$ is the graph of $y = e^x$ shifted to the left by 4.



Finally, $y = \frac{1}{4}e^{x-3} + 2$ is the graph of $y = e^x$ shifted to the right by 3 (see the graph of $y = e^{x-3}$), then compressed by a factor 4 toward the x-axis (see the graph of $y = \frac{1}{4}e^{x-3}$), and then shifted up by 2.





□

13.2 Logarithmic functions and their graphs

The logarithm is defined as the inverse function of an exponential function. From the above, we can see that $y = b^x$ is one-to-one (for $0 < b$, $b \neq 1$), so that it makes sense to define the inverse of $y = b^x$. Specifically, we call the inverse function of $y = b^x$ the logarithm with base b.

Definition 13.8: Logarithmic function

Let $0 < b \neq 1$ be a positive real number that is not equal to 1. For $x > 0$, the **logarithm of** x **with base** b is defined by the equivalence

$$\boxed{x = b^y \qquad \Longleftrightarrow \qquad y = \log_b(x)} \tag{13.2}$$

Note that this computes the inverse of the exponential function $y = b^x$ with base b (that is, we exchange x and y to get $x = b^y$ and solve for y).
For the particular base $b = 10$ we use the short form

$$\boxed{\log(x) := \log_{10}(x)}$$

For the particular base $b = e$, where $e \approx 2.71828$ is Euler's number, we call the logarithm with base e the **natural logarithm**, and write

$$\boxed{\ln(x) := \log_e(x)}$$

The **logarithmic function** is the function $y = \log_b(x)$ with domain $D = \{x \in \mathbb{R} | x > 0\}$ of all positive real numbers, and range $R = \mathbb{R}$ of all real numbers.

Example 13.9

Rewrite the equation as a logarithmic equation.

a) $3^4 = 81$ b) $10^3 = 1000$ c) $e^x = 17$ d) $2^{7 \cdot a} = 53$

Solution.

We can immediately apply Equation (13.2). For part (a), we have $b = 3$, $y = 4$, and $x = 81$. Therefore we have:

$$3^4 = 81 \quad \Leftrightarrow \quad \log_3(81) = 4$$

Similarly, we obtain the solutions for (b), (c), and (d).

b) $10^3 = 1000 \quad \Leftrightarrow \quad \log(1000) = 3$

c) $e^x = 17 \quad \Leftrightarrow \quad \ln(17) = x$

d) $2^{7a} = 53 \quad \Leftrightarrow \quad \log_2(53) = 7a$ □

Example 13.10

Evaluate the expression by rewriting it as an exponential expression.

a) $\log_2(16)$ b) $\log_5(125)$ c) $\log_{13}(1)$ d) $\log_4(4)$
e) $\log(100,000)$ f) $\log(0.001)$ g) $\ln(e^7)$ h) $\log_b(b^x)$

Solution.

a) If we set $y = \log_2(16)$, then this is equivalent to $2^y = 16$. Since, clearly, $2^4 = 16$, we see that $y = 4$. Therefore, we have $\log_2(16) = 4$.

b) $\log_5(125) = y \quad \Leftrightarrow \quad 5^y = 125 \overset{\text{(since } 5^3=125)}{\Longrightarrow} 3 = y = \log_5(125)$

c) $\log_{13}(1) = y \quad \Leftrightarrow \quad 13^y = 1 \overset{\text{(since } 13^0=1)}{\Longrightarrow} 0 = y = \log_{13}(1)$

d) $\log_4(4) = y \quad \Leftrightarrow \quad 4^y = 4 \overset{\text{(since } 4^1=4)}{\Longrightarrow} 1 = y = \log_4(4)$

e) $\log(100,000) = y \quad \Leftrightarrow \quad 10^y = 100,000 \overset{\text{(since } 10^5=100{,}000)}{\Longrightarrow} 5 = y = \log(100,000)$

f) $\log(0.001) = y \quad \Leftrightarrow \quad 10^y = 0.001 \overset{\text{(since } 10^{-3}=0.001)}{\Longrightarrow} -3 = y = \log(0.001)$

g) $\ln(e^7) = y \quad \Leftrightarrow \quad e^y = e^7 \quad \Longrightarrow \quad 7 = y = \ln(e^7)$

h) $\log_b(b^x) = y \quad \Leftrightarrow \quad b^y = b^x \quad \Longrightarrow \quad x = y = \log_b(b^x)$

Note that the last example, in which we obtained $\log_b(b^x) = x$, combines all of the previous examples. □

In the previous example (in parts (c), (d), and (h)), we were able to find certain elementary logarithms. We record these in the next observation.

Observation 13.11: Basic logarithmic evaluations

We have the elementary logarithms:

$$\boxed{\log_b(b^x) = x} \qquad \boxed{\log_b(b) = 1} \qquad \boxed{\log_b(1) = 0} \tag{13.3}$$

In general, when the argument is not a power of the base, we can use the calculator to approximate the values of a logarithm via the formulas:

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)}} \quad \text{or} \quad \boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}} \tag{13.4}$$

The last two formulas will be proved in Proposition 14.2 below. For now, we want to show how they can be used to calculate any logarithmic expression with any calculator that has either the $\ln$ or the $\log$ function.

Example 13.12

Evaluate: a) $\log_3(13)$ b) $\log_{2.34}(98.765)$

Solution.

a) We calculate $\log_3(13)$ by using the first formula in (13.4).

$$\log_3(13) = \frac{\log(13)}{\log(3)} \approx 2.335$$

Alternatively, we can also calculate this with the second formula in (13.4).

$$\log_3(13) = \frac{\ln(13)}{\ln(3)} \approx 2.335$$

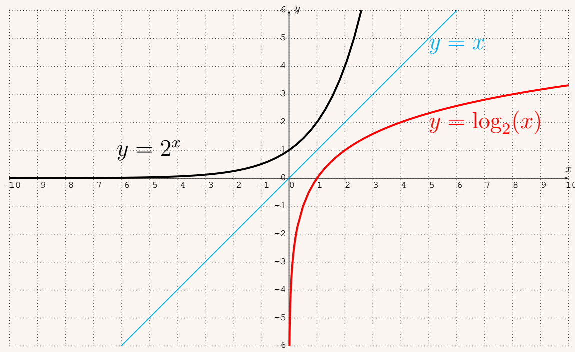
b) We compute $\log_{2.34}(98.765) = \frac{\log(98.765)}{\log(2.34)} \approx 5.402.$

□

We also study the graph of logarithmic functions.

Note 13.13

Consider the graph of $y = 2^x$ from the previous section. Recall that the graph of the inverse of a function is the reflection of the graph of the function about the diagonal line $y = x$. So in this case we have:



Note that the horizontal asymptote $y = 0$ for $y = 2^x$ becomes the vertical asymptote $x = 0$ for $y = \log_2(x)$. The x-intercept of $y = \log_2(x)$ is at $y = 0$, that is $0 = \log_2(x)$, which gives $x = 2^0 = 1$ as the x-intercept.

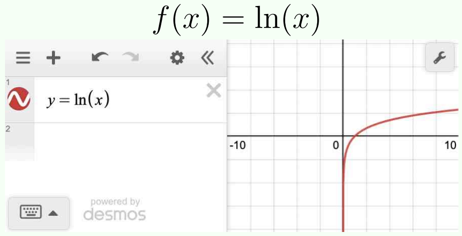
Example 13.14

Graph the functions $f(x) = \ln(x)$, $g(x) = \log(x)$, $h(x) = \log_2(x)$, and $k(x) = \log_{0.5}(x)$. What are the domains of f, g, h, and k? How do these functions differ?

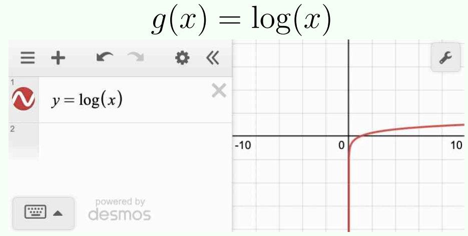
Solution.

We know from the definition that the domain of f, g, and h is all real positive numbers, $D_f = D_g = D_h = D_k = \{x | x > 0\}$. The functions f and g can immediately be entered into the calculator. The standard window gives the following graphs.

$f(x) = \ln(x)$



$g(x) = \log(x)$

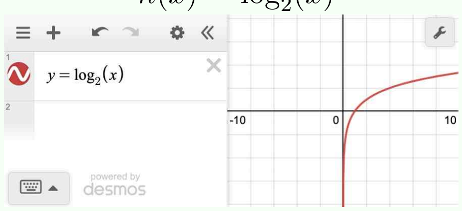


Note that we can rewrite $g(x)$, $h(x)$, and $k(x)$ as a constant times $f(x)$:

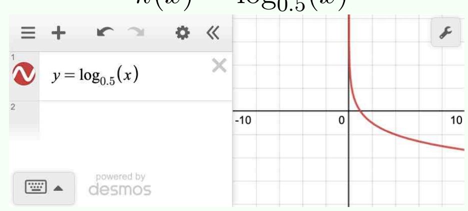
$$\begin{aligned} g(x) &= \log(x) = \log_{10}(x) = \frac{\ln(x)}{\ln(10)} = \frac{1}{\ln(10)} \cdot f(x) \\ h(x) &= \log_2(x) = \frac{\ln(x)}{\ln(2)} = \frac{1}{\ln(2)} \cdot f(x) \\ k(x) &= \log_{0.5}(x) = \frac{\ln(x)}{\ln(0.5)} = \frac{1}{\ln(0.5)} \cdot f(x) \end{aligned}$$

Since $\frac{1}{\ln(10)} \approx 0.434 < 1$, we see that the graph of g is that of f compressed toward the x-axis by a factor $\frac{1}{\ln(10)}$. Similarly, $\frac{1}{\ln(2)} \approx 1.443 > 1$, so that the graph of h is that of f stretched away from the x-axis by a factor $\frac{1}{\ln(2)}$. Finally, $\frac{1}{\ln(0.5)} \approx -1.443$, or more precisely, $\frac{1}{\ln(0.5)} = \frac{1}{\ln(2^{-1})} = -\frac{1}{\ln(2)}$, so that the graph of k is that of h reflected about the x-axis.

$h(x) = \log_2(x)$



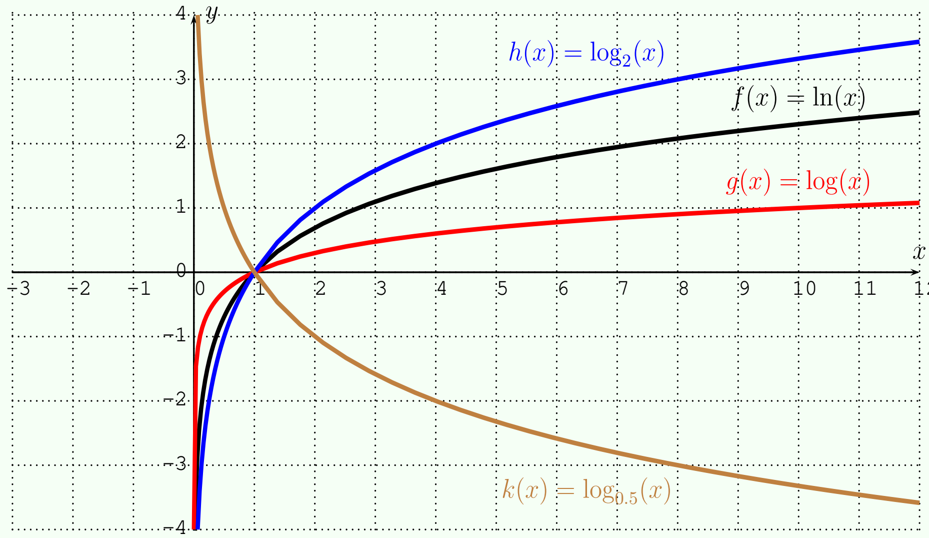
$k(x) = \log_{0.5}(x)$



Note that all these graphs have a vertical asymptote at $x = 0$. Moreover, all of the functions have an x-intercept at $x = 1$:

$$f(1) = g(1) = h(1) = k(1) = 0$$

To visualize the differences between the graphs, we graph them together in one coordinate system.



□

Example 13.15

Graph the given function. State the domain, find the vertical asymptote, and find the x-intercept of the function.

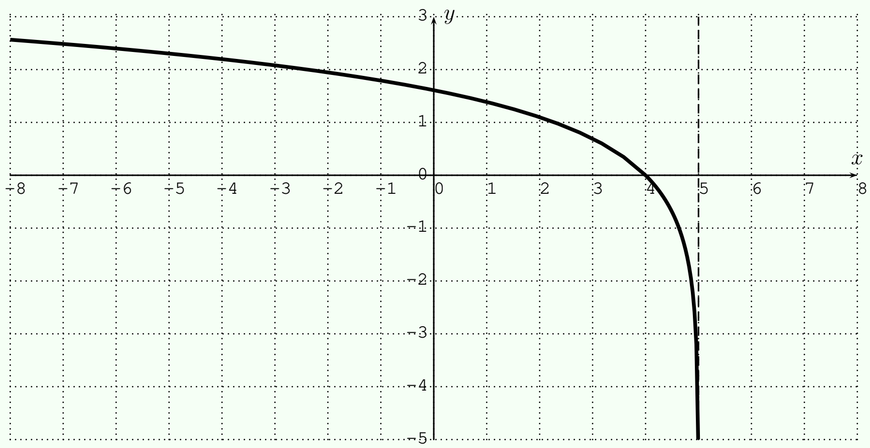
a) $f(x) = \ln(5-x)$ b) $g(x) = \log_7(2x+8)$ c) $h(x) = -3 \cdot \ln(x) + 4$

Solution.

a) To determine the domain of $f(x) = \ln(5-x)$, we have to see for which x the logarithm has a positive argument. More precisely, we need $5-x > 0$, that is, $5 > x$, so that the domain is $D_f = \{x | x < 5\}$.

The calculator displays the following graph:

Note that the graph, as displayed by the calculator, appears to end at a point that is approximately at $(5, -35)$. However, the actual graph of the logarithm *does not* stop at any point, since it has a vertical asymptote at $x = 5$, that is, the graph approaches $-\infty$ as x approaches 5. The calculator only displays an approximation, which may be misleading, since this approximation is determined by the window size and the size of each pixel. We therefore graph the function $f(x) = \ln(5 - x)$ as follows:



The x-intercept is given where $y = 0$, that is

$$0 = \ln(5 - x) \implies 5 - x = e^0 \implies 5 - x = 1 \implies x = 4$$

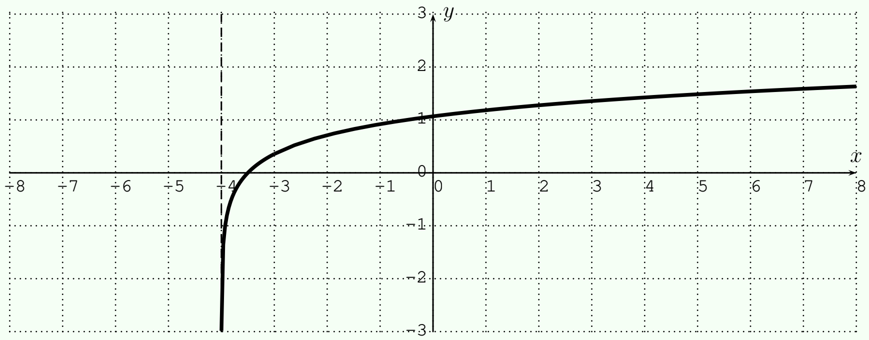
Therefore, the x-intercept is at $(4, 0)$.

b) The domain of $g(x) = \log_7(2x + 8)$ consists of those numbers x for which the argument of the logarithm is positive.

$$2x + 8 > 0 \overset{\text{(subtract 8)}}{\implies} 2x > -8 \overset{\text{(divide by 2)}}{\implies} x > -4$$

Therefore, the domain is $D_g = \{x|x > -4\}$. The graph of $g(x) =$

$\log_7(2x+8)$ is displayed below.



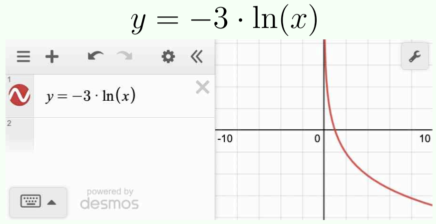
There is a vertical asymptote at $x = -4$. The x-intercept is given at $y = 0$:

$$0 = \log_7(2x+8) \implies 2x+8 = 7^0 \implies 2x+8 = 1$$
$$\implies 2x = -7 \implies x = -\frac{7}{2}$$

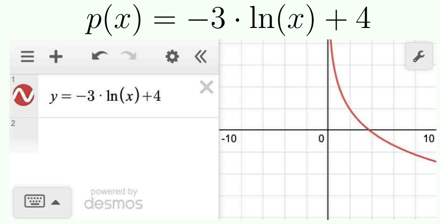
Therefore, the x-intercept is at $(-\frac{7}{2}, 0)$.

c) Using our knowledge of transformations of graphs, we expect that $h(x) = -3 \cdot \ln(x) + 4$ is that of $y = \ln(x)$ reflected and stretched away from the x-axis (by a factor 3), and then shifted up by 4. The stretched and reflected graph is on the left below, whereas the graph of the shifted function h is on the right.

$y = -3 \cdot \ln(x)$



$p(x) = -3 \cdot \ln(x) + 4$



The domain consist of numbers x for which the $\ln(x)$ is defined, that is, $D_p = \{x | x > 0\}$. The vertical asymptote is therefore also at $x = 0$. The x-intercept is computed as follows:

$$y = 0 \implies 0 = -3 \cdot \ln(x) + 4 \implies -4 = -3 \cdot \ln(x)$$

$$\implies \ln(x) = \frac{4}{3} \implies x = e^{\frac{4}{3}}$$

Therefore, the x-intercept is at $(e^{\frac{4}{3}}, 0)$.

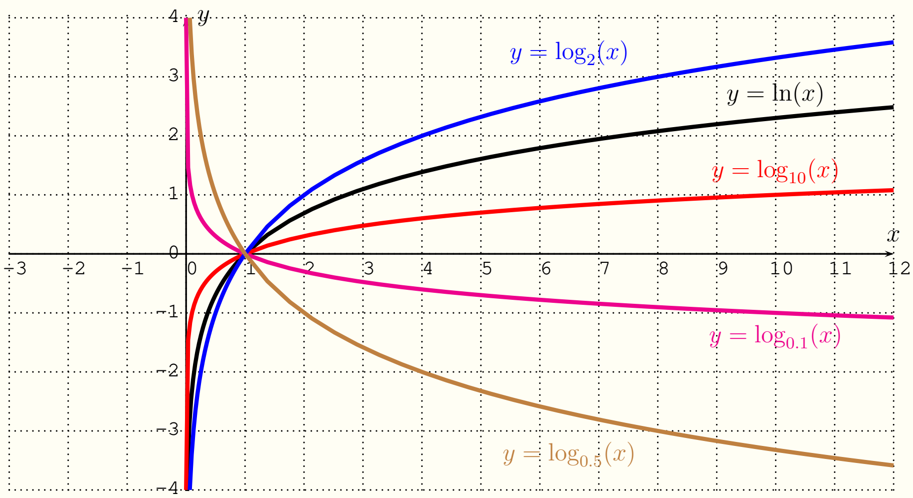
□

In the previous examples we analyzed the graphs of various logarithmic functions. The following is a summary of our findings.

Observation 13.16: Graph of a logarithmic function

The graph of a logarithmic function $y = \log_b(x)$ with base b is that of the natural logarithm $y = \ln(x)$ stretched away from the x-axis, or compressed toward the x-axis when $b > 1$. When $0 < b < 1$, the graph is furthermore reflected about the x-axis.

- The function $y = \log_b(x)$ has domain $D = \{x | x > 0\}$.
- The graph of $y = \log_b(x)$ has a vertical asymptote at $x = 0$.
- The graph of $y = \log_b(x)$ has *no* horizontal asymptote, as $f(x)$ approaches $+\infty$ when x approaches $+\infty$ for $b > 1$, and $f(x)$ approaches $-\infty$ when x approaches $+\infty$ for $0 < b < 1$.
- The x-intercept is given for $y = 0$, which for $y = \log_b(x)$ is at $x = 1$.



13.3 Exercises

Exercise 13.1

Graph the following functions with the calculator.

a) $y = 5^x$	b) $y = 1.01^x$	c) $y = (\frac{1}{3})^x$	d) $y = 0.97^x$
e) $y = 3^{-x}$	f) $y = (\frac{1}{3})^{-x}$	g) $y = e^{x^2}$	h) $y = 0.01^x$
i) $y = 1^x$	j) $y = e^x + 1$	k) $y = \frac{e^x - e^{-x}}{2}$	l) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

The last two functions are known as the *hyperbolic sine*, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and the *hyperbolic tangent*, $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Recall that the *hyperbolic cosine* $\cosh(x) = \frac{e^x + e^{-x}}{2}$ was already graphed in Example 13.5.

Exercise 13.2

Graph the given function. Describe how the graph is obtained by a transformation from the graph of an exponential function $y = b^x$ (for appropriate base b).

a) $y = 0.1 \cdot 4^x$	b) $y = 3 \cdot 2^x$	c) $y = (-1) \cdot 2^x$
d) $y = 0.006 \cdot 2^x$	e) $y = e^{-x}$	f) $y = e^{-x} + 1$
g) $y = (\frac{1}{2})^x + 3$	h) $y = 2^{x-4}$	i) $y = 2^{x+1} - 6$

Exercise 13.3

Use the definition of the logarithm to write the given equation as an equivalent logarithmic equation.

a) $4^2 = 16$	b) $2^8 = 256$	c) $e^x = 7$	d) $10^{-1} = 0.1$
e) $3^x = 12$	f) $5^{7 \cdot x} = 12$	g) $3^{2a+1} = 44$	h) $\left(\frac{1}{2}\right)^{\frac{x}{h}} = 30$

Exercise 13.4

Evaluate the following expressions *without* using a calculator.

a) $\log_7(49)$	b) $\log_3(81)$	c) $\log_2(64)$	d) $\log_{50}(2500)$
e) $\log_2(0.25)$	f) $\log(1000)$	g) $\ln(e^4)$	h) $\log_{13}(13)$
i) $\log(0.1)$	j) $\log_6(\frac{1}{36})$	k) $\ln(1)$	l) $\log_{\frac{1}{2}}(8)$

Exercise 13.5

Using a calculator, approximate the following expressions to the nearest thousandth.

a) $\log_3(50)$ b) $\log_3(12)$ c) $\log_{17}(0.44)$ d) $\log_{0.34}(200)$

Exercise 13.6

State the domain of the function f and find any vertical asymptote(s) and x-intercept(s). Use the results to sketch the graph.

a) $f(x) = \log(x)$
b) $f(x) = \log(x+7)$
c) $f(x) = \ln(x+5) - 1$
d) $f(x) = \ln(3x-6)$
e) $f(x) = 2 \cdot \log(x+4)$
f) $f(x) = -4 \cdot \log(x+2)$
g) $f(x) = \log_3(7x+5)$
h) $f(x) = \ln(-6x+14)$
i) $f(x) = \log_{0.4}(x)$
j) $f(x) = \log_3(-5x) - 2$
k) $f(x) = \log|x|$
l) $f(x) = \log|x+2|$