Precalculus

Third Edition (3.0)

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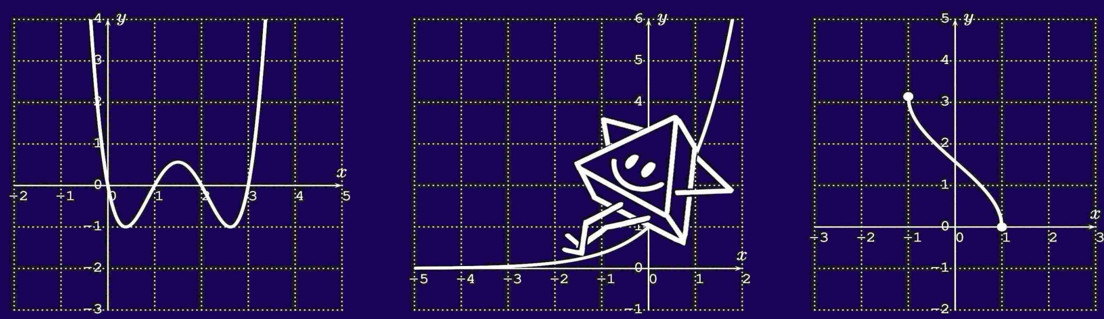
Precalculus Textbook Download Page



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The images of the Desmos graphing calculator were generated from the Desmos website at **https://www.desmos.com/calculator**

Chapter 12

Solving inequalities

In this chapter we use our knowledge of functions to solve inequalities. In Section 12.1, we study polynomial inequalities and absolute value inequalities, while in Section 12.2 we solve rational inequalities.

12.1 Polynomial and absolute value inequalities

We will develop a general strategy for solving inequalities involving non-linear functions. Linear inequalities, however, can be solved quite easily by separating the variable x, while keeping in mind that multiplying or dividing a negative number reverses the sign of the inequality.

$$\begin{array}{rcll} & -2x \leq -6 & \implies & x \geq 3 \\ \text{but} & 2x \leq 6 & \implies & x \leq 3 \end{array}$$

Example 12.1

Solve for x.

a) $-3x + 7 > 19$ b) $2x + 5 \geq 4x - 11$
c) $3 < -6x - 4 \leq 13$ d) $-2x - 1 \leq 3x + 4 < 4x - 20$

Solution.
The first three calculations are as follows:

a) $-3x + 7 > 19 \quad \stackrel{(-7)}{\implies} \quad -3x > 12 \quad \stackrel{(\div(-3))}{\implies} \quad x < -4$

b) $2x + 5 \geq 4x - 11 \quad \stackrel{(-4x-5)}{\implies} \quad -2x \geq -16 \quad \stackrel{(\div(-2))}{\implies} \quad x \leq 8$

c) $3 < -6x - 4 \le 13 \quad \overset{(+4)}{\Longrightarrow} \quad 7 < -6x \le 17$
$$\overset{(\div(-6))}{\Longrightarrow} \quad \frac{7}{-6} > x \ge \frac{17}{-6} \quad \Longrightarrow \quad -\frac{17}{6} \le x < -\frac{7}{6}$$

Here, the last implication was obtained by switching the right and left terms of the inequality. The solution set is the interval $[-\frac{17}{6}, -\frac{7}{6})$.
For part (d), it is best to consider both inequalities separately.

$$-2x - 1 \le 3x + 4 \quad \overset{(-3x+1)}{\Longrightarrow} \quad -5x \le 5 \quad \overset{(\div(-5))}{\Longrightarrow} \quad x \ge -1,$$
$$3x + 4 < 4x - 20 \quad \overset{(-4x-4)}{\Longrightarrow} \quad -x < -24 \quad \overset{(\cdot(-1))}{\Longrightarrow} \quad x > 24.$$

The solution has to satisfy *both* inequalities $x \ge -1$ *and* $x > 24$. Both inequalities are true for $x > 24$ (since then also $x \ge -1$), so that this is in fact the solution: $x > 24$. □

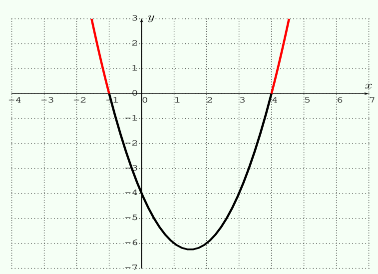
We now consider inequalities with polynomials of higher degree.

Example 12.2

Solve for x: $\quad x^2 - 3x - 4 \ge 0$

Solution.

To get an idea of where $x^2 - 3x - 4 \ge 0$, we graph the left-hand side function $f(x) = x^2 - 3x - 4$.



Note that the output $x^2 - 3x - 4$ is greater or equal to zero when the graph of $f(x)$ is above or on the x-axis, which is marked in red. Since the graph is a parabola, the graph can only switch from above to below the x-axis (and the same from below to above the x-axis) when it intersects the x-axis. These are the roots of the function.
So, we first find the roots of the polynomial, which, in this case, can be

done by factoring.

$$x^2 - 3x - 4 = 0 \implies (x-4)(x+1) = 0 \implies x = 4 \text{ or } x = -1$$

From the graph we see that $f(x) \geq 0$ when $x \leq -1$ or when $x \geq 4$ (the parts of the graph above the x-axis). To show this without using the calculator, we can check one point in each of the intervals $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$:

Check $x = -2$**:**	**Check** $x = 0$**:**	**Check** $x = 5$**:**
$f(-2) = (-2)^2 - 3 \cdot (-2) - 4$	$f(0) = 0^2 - 3 \cdot 0 - 4$	$f(5) = 5^2 - 3 \cdot 5 - 4$
$= 4 + 6 - 4$	$= 0 - 0 - 4$	$= 25 - 15 - 4$
$= 6 \geq 0$	$= -4 \not\geq 0$	$= 6 \geq 0$
TRUE	FALSE	TRUE

The solution set S is therefore

$$S = \{x | x \leq -1 \text{ or } x \geq 4\} = (-\infty, -1] \cup [4, \infty).$$

The numbers -1 and 4 are included in the solution set since this is where we have equality $x^2 - 3x - 4 = 0$, and the original inequality $x^2 - 3x - 4 \geq 0$ includes the equality. □

Note 12.3: Solving inequalities

Analyzing the previous example, we use a three-step approach when dealing with inequalities.

- In *step one* we find the x where the left-hand side and the right-hand side of the inequality change from "$>$" to "$<$" and vice versa. In particular, we check where the two sides are equal.

- In *step two* we check one x in each of the subintervals from step one to decide whether they satisfy the original inequality or not.

For steps one and two we may also use the graphing calculator to gain further insights.

- In *step three* we check which of the endpoints of the intervals are included in the solution set.

Example 12.4

Solve for x.

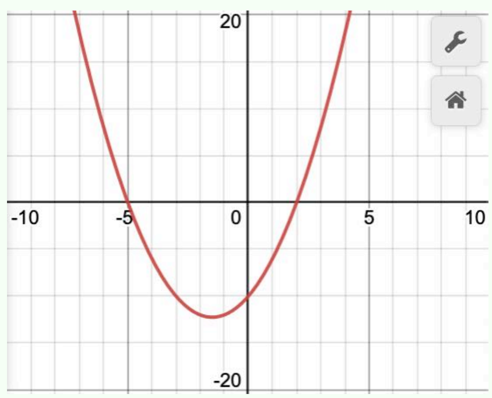
a) $x^2 + 3x - 10 < 0$ b) $x^3 - 9x^2 + 23x - 15 \leq 0$
c) $x^3 + 15x > 7x^2 + 9$ d) $x^4 - x^2 \geq 5(x^3 - x)$

Solution.

a) We can find the roots of the polynomial on the left by factoring.

$$x^2 + 3x - 10 = 0 \implies (x+5)(x-2) = 0 \implies x = -5 \text{ or } x = 2$$

To see where $f(x) = x^2 + 3x - 10$ is < 0, we graph it with the calculator and check numbers in each interval where $f(x) \neq 0$.

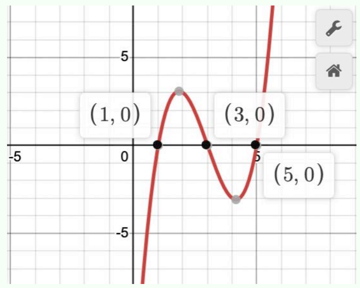


-5 2

Check $x = -6$:	Check $x = 0$:	Check $x = 3$:
$f(-6)$	$f(0)$	$f(3)$
$= (-6)^2 + 3 \cdot (-6) - 10$	$= 0^2 + 3 \cdot 0 - 10$	$= 3^2 + 3 \cdot 3 - 10$
$= 36 - 18 - 10$	$= 0 + 0 - 10$	$= 9 + 9 - 10$
$= 8 \not< 0$	$= -10 < 0$	$= 8 \not< 0$
FALSE	TRUE	FALSE

We see that $f(x) < 0$ when $-5 < x < 2$. The numbers -5 and 2 are not included because the inequality "$<$" does not include equality. The solution set is therefore $S = \{x| -5 < x < 2\} = (-5, 2)$.

b) Here is the graph of the function $f(x) = x^3 - 9x^2 + 23x - 15$ from the graphing calculator.



This graph shows that there are two intervals where $f(x) \leq 0$ (the parts of the graph below the x-axis). To determine the exact intervals, we calculate where $f(x) = x^3 - 9x^2 + 23x - 15 = 0$. The graph suggests that the roots of $f(x)$ are at $x = 1$, $x = 3$, and $x = 5$. This can be confirmed by a calculation:

$$\begin{aligned} f(1) &= 1^3 - 9 \cdot 1^2 + 23 \cdot 1 - 15 = 1 - 9 + 23 - 15 = 0, \\ f(3) &= 3^3 - 9 \cdot 3^2 + 23 \cdot 3 - 15 = 27 - 81 + 69 - 15 = 0, \\ f(5) &= 5^3 - 9 \cdot 5^2 + 23 \cdot 5 - 15 = 125 - 225 + 115 - 15 = 0. \end{aligned}$$

Since f is a polynomial of degree 3, the roots $x = 1, 3, 5$ are all of the roots of f. (Alternatively, we could have divided $f(x)$, for example, by $x - 1$ and used this to completely factor f and with this obtain all the roots of f.) We next check each interval.

1	3	5	
Check $x = 0$:	**Check** $x = 2$:	**Check** $x = 4$:	**Check** $x = 6$:
$f(0) = -15 \leq 0$	$f(2) = 3 \not\leq 0$	$f(4) = -3 \leq 0$	$f(6) = 16 \not\leq 0$
TRUE	FALSE	TRUE	FALSE

With this, we can determine the solution set to be the set:

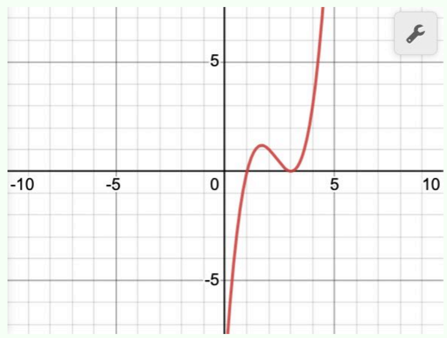
$$\begin{aligned} \text{solution set } S &= \{x \in \mathbb{R} | x \leq 1, \text{ or } 3 \leq x \leq 5\} \\ &= (-\infty, 1] \cup [3, 5]. \end{aligned}$$

Note that we include the roots 1, 3, and 5 in the solution set since the original inequality was "$\leq$" (and not "$<$"), which includes the solutions of the corresponding equality.

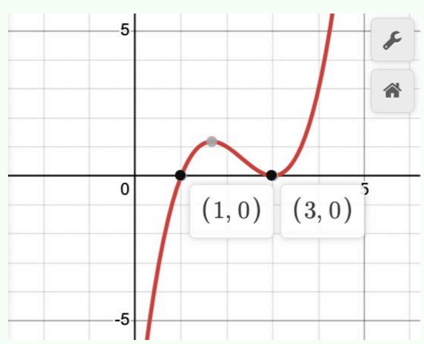
c) We rewrite the inequality in a way that has zero on one side so that we can get a better view of where the corresponding equality holds.

$$x^3 + 15x > 7x^2 + 9 \implies x^3 - 7x^2 + 15x - 9 > 0$$

(Here it does not matter whether we bring the terms to the right or the left side of the inequality sign! The resulting inequality is different, but the solution to the problem is the same.) With this, we now use the graphing calculator to find the graph of the function $f(x) = x^3 - 7x^2 + 15x - 9$.



The graph suggests at least one root (the left-most intersection point), but possibly one or two more roots. To gain a better understanding of whether the graph intersects the x-axis on the right, we rescale the window size of the previous graph.



This viewing window suggests that there are two roots $x = 1$ and $x = 3$. We confirm that these are the only roots with an algebraic computation. First, we check that $x = 1$ and $x = 3$ are indeed roots:

$$f(1) \quad = \quad 1^3 - 7 \cdot 1^2 + 15 \cdot 1 - 9 = 1 - 7 + 15 - 9 = 0,$$

$$f(3) = 3^3 - 7 \cdot 3^2 + 15 \cdot 3 - 9 = 27 - 63 + 45 - 9 = 0.$$

To confirm that these are the *only* roots (and we have not just missed one of the roots that might possibly become visible after sufficiently zooming into the graph), we factor $f(x)$ completely. We divide $f(x)$ by $x-1$:

$$\begin{array}{r|rrrr}
 & & x^2 & -6x & +9 \\ \hline
x-1 & x^3 & -7x^2 & +15x & -9 \\
 & -(x^3 & -x^2) & & \\ \cline{2-3}
 & & -6x^2 & +15x & -9 \\
 & & -(-6x^2 & +6x) & \\ \cline{3-4}
 & & & 9x & -9 \\
 & & & -(9x & -9) \\ \cline{4-5}
 & & & & 0
\end{array}$$

and use this to factor f:

$$\begin{aligned} f(x) &= x^3 - 7x^2 + 15x - 9 = (x-1)(x^2 - 6x + 9) \\ &= (x-1)(x-3)(x-3) \end{aligned}$$

This shows that 3 is a root of multiplicity 2, and so f has no other roots than $x = 1$ and $x = 3$. The solution set consists of those numbers x for which $f(x) > 0$. We check points in each interval.

$x < 1$	$1 < x < 3$	$x > 3$
Check $x = 0$**:**	**Check** $x = 2$**:**	**Check** $x = 4$**:**
$f(0) = -9 \not> 0$	$f(2) = 1 > 0$	$f(4) = 3 > 0$
FALSE	TRUE	TRUE

From this calculation, as well as from the graph, we see that $f(x) > 0$ when $1 < x < 3$ and when $x > 3$ (the roots $x = 1$ and $x = 3$ are not included as solutions). We can write the solution set in several different ways:

$$\text{solution set } S = \{x | 1 < x < 3 \text{ or } x > 3\} = \{x | 1 < x\} - \{3\},$$

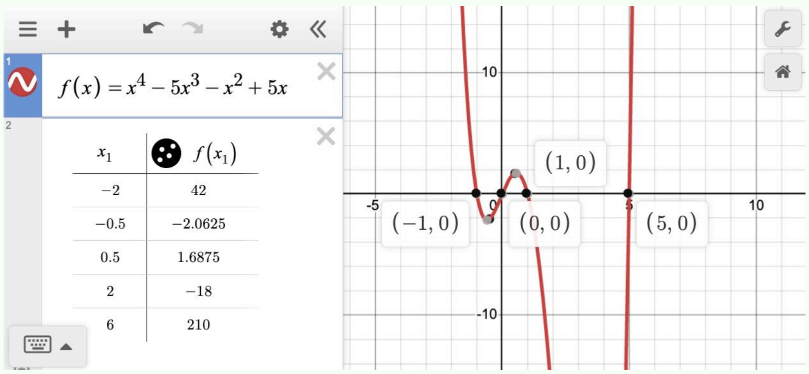
or in interval notation:

$$\text{solution set } S = (1,3) \cup (3, \infty) = (1, \infty) - \{3\}.$$

d) Again, we move all terms to one side:

$$
\begin{aligned}
x^4 - x^2 \geq 5(x^3 - x) \quad \text{(distribute 5)} \quad &\implies \quad x^4 - x^2 \geq 5x^3 - 5x \\
\text{(subtract } 5x^3\text{, add } 5x\text{)} \quad &\implies \quad x^4 - 5x^3 - x^2 + 5x \geq 0.
\end{aligned}
$$

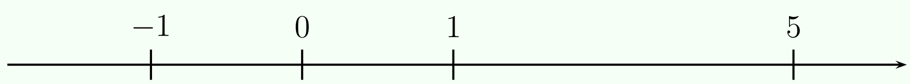
We graph $f(x) = x^4 - 5x^3 - x^2 + 5x$ with the graphing calculator.



The graph suggests the roots $x = -1$, 0, 1, and 5. This can be confirmed by a straightforward calculation.

$$
\begin{aligned}
f(-1) &= (-1)^4 - 5 \cdot (-1)^3 - (-1)^2 + 5 \cdot (-1) = 1 + 5 - 1 - 5 = 0, \\
f(0) &= 0^4 - 5 \cdot 0^3 - 0^2 - 5 \cdot 0 = 0, \\
f(1) &= 1^4 - 5 \cdot 1^3 - 1^2 + 5 \cdot 1 = 1 - 5 - 1 + 5 = 0, \\
f(5) &= 5^4 - 5 \cdot 5^3 - 5^2 + 5 \cdot 5 = 125 - 125 - 25 + 25 = 0.
\end{aligned}
$$

The roots $x = -1$, 0, 1, and 5 are the only roots, since f is of degree 4. We check points in each interval.



Check $x = -2$**:** $f(-2) = (-2)^4 - 5 \cdot (-2)^3 - (-2)^2 + 5 \cdot (-2)$
$= 42 \geq 0$ TRUE

Check $x = -0.5$**:** $f(-0.5) = (-0.5)^4 - 5 \cdot (-0.5)^3 - (-0.5)^2 - 5 \cdot (-0.5)$
$\approx -2.1 \not\geq 0$ FALSE

Check $x = 0.5$**:** $f(0.5) = 0.5^4 - 5 \cdot 0.5^3 - 0.5^2 + 5 \cdot 1$
$\approx 1.7 \geq 0$ TRUE

Check $x = 2$:	$f(2) = 2^4 - 5 \cdot 2^3 - 2^2 + 5 \cdot 5 = -18 \ngeq 0$	**FALSE**
Check $x = 6$:	$f(6) = 6^4 - 5 \cdot 6^3 - 6^2 + 5 \cdot 6 = 210 \geq 0$	**TRUE**

Since the inequality we want to solve is $f(x) \geq 0$, which includes equality, the zeros of f are included in the solution, and so the solution set is:

$$S = (-\infty, -1] \cup [0, 1] \cup [5, \infty)$$

□

Polynomial inequalities come up, for example, when finding the domain of functions involving a square root, as we will show in the next example.

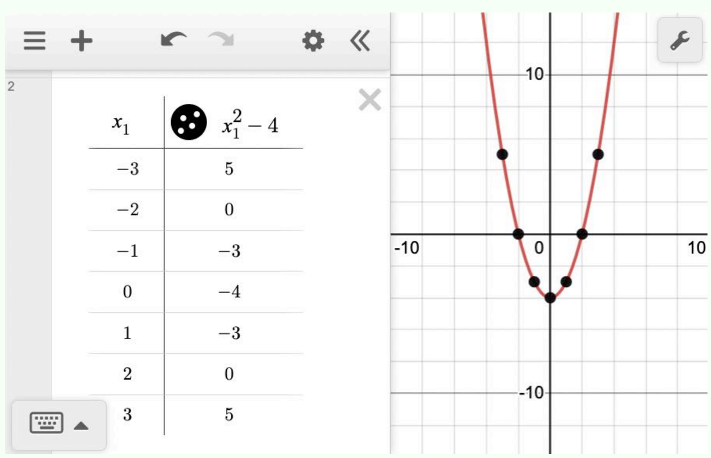
Example 12.5

Find the domain of the given functions.

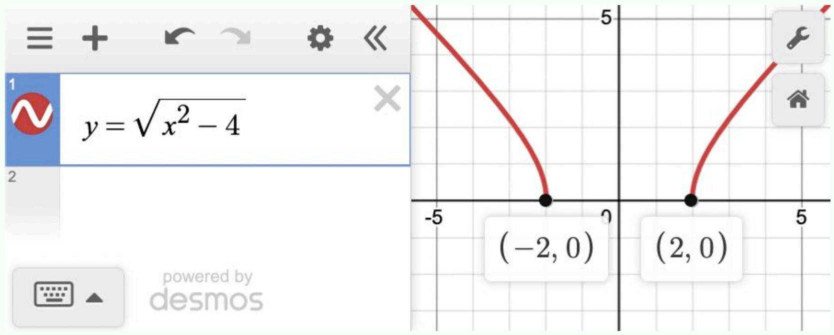
a) $f(x) = \sqrt{x^2 - 4}$ b) $g(x) = \sqrt{x^3 - 5x^2 + 6x}$

Solution.

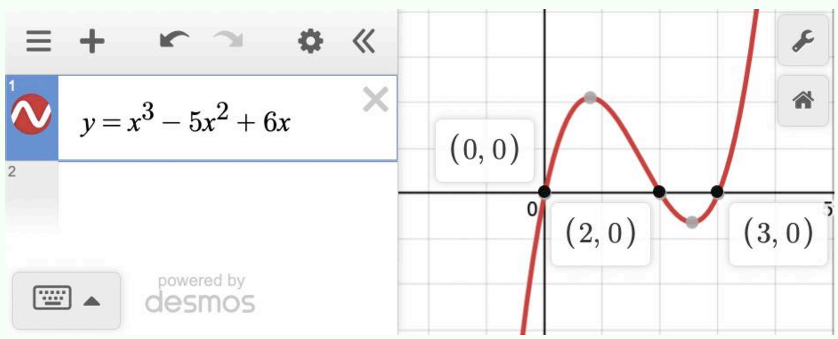
a) The domain of $f(x) = \sqrt{x^2 - 4}$ is given by all x for which the square root is non-negative. In other words, the domain is given by numbers x with $x^2 - 4 \geq 0$. Graphing the function $y = x^2 - 4 = (x+2)(x-2)$, we see that this is precisely the case when $x \leq -2$ or $x \geq 2$.



Therefore, the domain is $D_f = (-\infty, -2] \cup [2, \infty)$. This is also confirmed by the graph of f, which is shown below.



b) For the domain of $g(x) = \sqrt{x^3 - 5x^2 + 6x}$, we need find those x with $x^3 - 5x^2 + 6x \geq 0$. To this end, we graph $y = x^3 - 5x^2 + 6x$ and check for its roots.



From the graph above, we calculate the roots of $y = x^3 - 5x^2 + 6x$ at $x = 0$, $x = 2$, and $x = 3$. Furthermore, the graph shows that $x^3 - 5x^2 + 6x \geq 0$ precisely when $0 \leq x \leq 2$ or $3 \leq x$. The domain is therefore $D_g = [0, 2] \cup [3, \infty)$.

$\square$

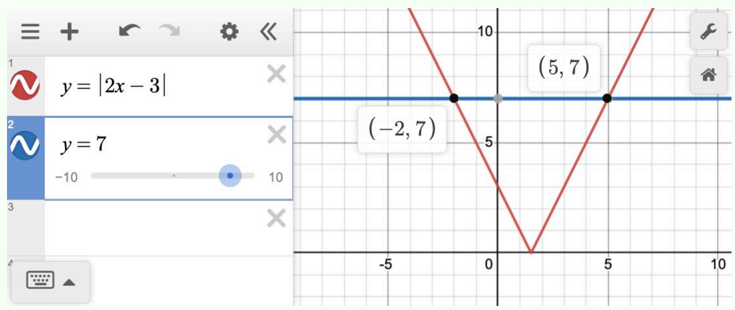
A similar computation to that for polynomial inequalities also applies to absolute value inequalities, which we show in the next example.

Example 12.6

Solve for x: $|2x-3| \geq 7$

Solution.

To analyze $|2x-3| \geq 7$, we graph the function $f(x) = |2x-3|$, as well as the function $g(x) = 7$.



To see where $|2x-3| \geq 7$, we first find the values where $|2x-3| = 7$. Note that $2x-3$ has an absolute value of 7 exactly when $2x-3$ is either 7 or -7.

$$|2x-3| = 7 \quad \Longrightarrow \quad 2x-3 = \pm 7$$

	$\Longrightarrow$ $2x-3=7$	$\Longrightarrow$ $2x-3=-7$
(add 3)	$\Longrightarrow$ $2x=10$	(add 3) $\Longrightarrow$ $2x=-4$
(divide by 2)	$\Longrightarrow$ $x=5$	(divide by 2) $\Longrightarrow$ $x=-2$

We next check in each interval whether $|2x-3| \geq 7$:

$x < -2$	$-2 < x < 5$	$x > 5$
Check $x = -3$:	**Check $x = 0$:**	**Check $x = 6$:**
$\lvert 2\cdot(-3)-3\rvert \overset{?}{\geq} 7$	$\lvert 2\cdot 0-3\rvert \overset{?}{\geq} 7$	$\lvert 2\cdot 6-3\rvert \overset{?}{\geq} 7$
$\lvert -9\rvert \overset{?}{\geq} 7$	$\lvert -3\rvert \overset{?}{\geq} 7$	$\lvert 9\rvert \overset{?}{\geq} 7$
$9 \overset{?}{\geq} 7$	$3 \overset{?}{\geq} 7$	$9 \overset{?}{\geq} 7$
TRUE	FALSE	TRUE

Since the values at $x=-2$ and $x=5$ give equality, the solution set for $|2x-3| \geq 7$ is given by $S = (-\infty, -2] \cup [5, \infty)$. $\square$

12.2 Rational inequalities

Rational inequalities are solved with a similar three-step process that was used to solve the polynomial and absolute value inequalities before (see Note 12.3 page 212). That is, in step 1, we find possible inputs where the inequality may change its sign (for example at the x-intercepts). In step 2, we check in which of the intervals the given inequality is true, and which are thus part of the solution set. Finally, in step 3, we determine which of the endpoints of the intervals should be included in the solution set.

Example 12.7

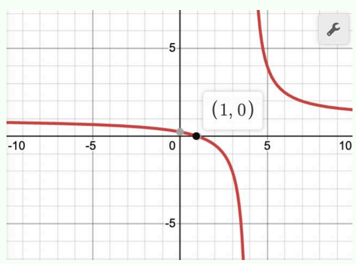
Solve for x.

a) $\frac{x-1}{x-4} \leq 0$ b) $\frac{7x-3}{6x+5} > 0$ c) $\frac{x^2-5x+6}{x^2-5x} \geq 0$

d) $\frac{5}{x-2} \leq 3$ e) $\frac{4}{x+5} < \frac{3}{x-3}$

Solution.

a) We first graph the function $f(x) = \frac{x-1}{x-4}$.



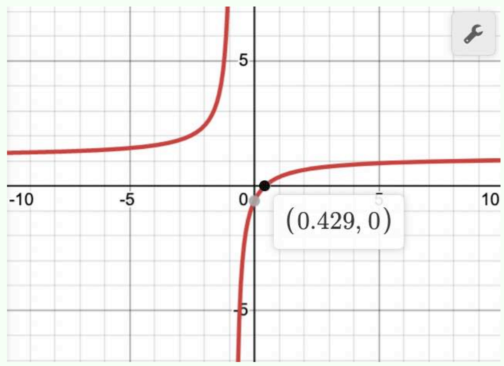
As shown above, the graph changes from above to below the x-axis at the x-intercept $x = 1$, and then changes from below to above the x-axis at the vertical asymptote at $x = 4$. Using both $x = 1$ and $x = 4$, we get the three intervals $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, which we will check as to whether $f(x) \leq 0$ or not.

1 4

Check $x = 0$**:**	**Check** $x = 2$**:**	**Check** $x = 5$**:**
$\frac{0-1}{0-4} \overset{?}{\leq} 0$	$\frac{2-1}{2-4} \overset{?}{\leq} 0$	$\frac{5-1}{5-4} \overset{?}{\leq} 0$
$\frac{-1}{-4} = \frac{1}{4} \overset{?}{\leq} 0$	$\frac{1}{-2} \overset{?}{\leq} 0$	$\frac{4}{1} \overset{?}{\leq} 0$
FALSE	TRUE	FALSE

Since the inequality is $\frac{x-1}{x-4} \leq 0$, we include the root at $x = 1$ in the solution set. However, we do not include $x = 4$, since this is a vertical asymptote of f and would not give a solution of the inequality, but would rather give an undefined expression on the left-hand side of the inequality. The solution set is therefore $S = [1, 4)$.

b) To solve $\frac{7x-3}{6x+5} > 0$, we graph the function $f(x) = \frac{7x-3}{6x+5}$.



$$x\text{-intercept: } 7x - 3 = 0 \implies 7x = 3 \implies x = \frac{3}{7}$$

$$\text{vertical asymptote: } 6x + 5 = 0 \implies 6x = -5 \implies x = -\frac{5}{6}$$

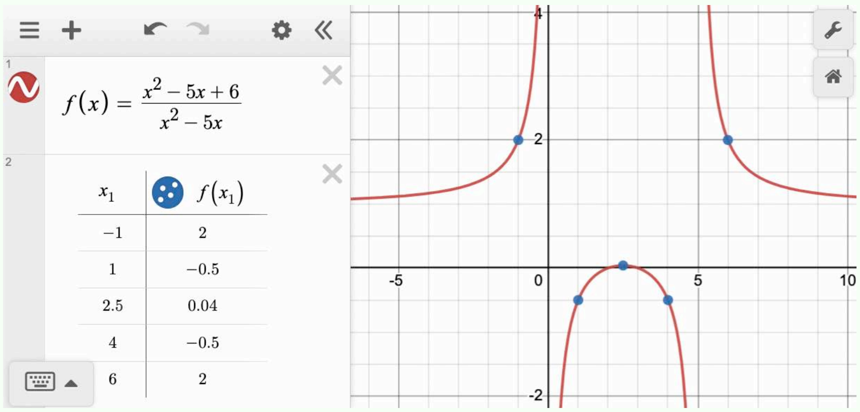
Note that $\frac{3}{7} \approx 0.429$ is indicated in the graph. The vertical asymptote is approximately at $-\frac{5}{6} \approx -0.833$. We can therefore use -1, 0, and 1 to check the inequality $f(x) = \frac{7x-3}{6x+5} > 0$ on the corresponding intervals $(-\infty, -\frac{5}{6})$, $(-\frac{5}{6}, \frac{3}{7})$ and $(\frac{3}{7}, \infty)$.

	$-\frac{5}{6}$	$\frac{3}{7}$
Check $x = -1$:	**Check $x = 0$:**	**Check $x = 1$:**
$\frac{7\cdot(-1)-3}{6\cdot(-1)+5} \overset{?}{>} 0$	$\frac{7\cdot 0-3}{6\cdot 0+5} \overset{?}{>} 0$	$\frac{7\cdot 1-3}{6\cdot 1+5} \overset{?}{>} 0$
$\frac{-10}{-1} = 10 \overset{?}{>} 0$	$\frac{-3}{5} \overset{?}{>} 0$	$\frac{4}{11} \overset{?}{>} 0$
TRUE	FALSE	TRUE

For the solution set, we do not include the root of f since the inequality is strict $f(x) > 0$, and we never include the vertical asymptote of f. The solution set is therefore

$$S = \big(-\infty, -\tfrac{5}{6}\big) \cup \big(\tfrac{3}{7}, \infty\big)$$

c) Here is the graph of $\frac{x^2-5x+6}{x^2-5x}$ in the standard window.



Factoring numerator and denominator, we can determine vertical asymptotes, holes, and x-intercepts.

$$\frac{x^2-5x+6}{x^2-5x} = \frac{(x-2)(x-3)}{x(x-5)}$$

The vertical asymptotes are at $x=0$ and $x=5$, the x-intercepts are at $x=2$ and $x=3$. To see where $\frac{x^2-5x+6}{x^2-5x} \geq 0$, we check numbers in each of the corresponding intervals.

$$\begin{array}{cccc} 0 & 2 & 3 & 5 \end{array}$$

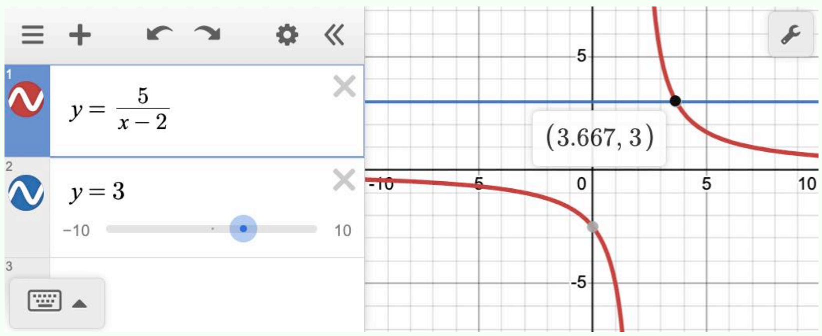
Check $x=-1$:	$f(-1) = \frac{(-1)^2-5\cdot(-1)+6}{(-1)^2-5\cdot(-1)} = \frac{12}{6} \geq 0$	TRUE
Check $x=1$:	$f(1) = \frac{1^2-5\cdot 1+6}{1^2-5\cdot 1} = \frac{2}{-4} \not\geq 0$	FALSE
Check $x=2.5$:	$f(2.5) = \frac{2.5^2-5\cdot 2.5+6}{2.5^2-5\cdot 2.5} = \frac{-0.25}{-6.25} \geq 0$	TRUE
Check $x=4$:	$f(4) = \frac{4^2-5\cdot 4+6}{4^2-5\cdot 4} = \frac{2}{-4} \not\geq 0$	FALSE
Check $x=6$:	$f(6) = \frac{6^2-5\cdot 6+6}{6^2-5\cdot 6} = \frac{12}{6} \geq 0$	TRUE

Combining all of the above information, we obtain the solution set:

$$\text{solution set } S = (-\infty,0) \cup [2,3] \cup (5,\infty)$$

Notice that the $x-$intercepts $x = 2$ and $x = 3$ are included in the solution set, whereas the values $x = 0$ and $x = 5$ associated with the vertical asymptotes are not included, since the fraction is not defined for $x = 0$ and $x = 5$.

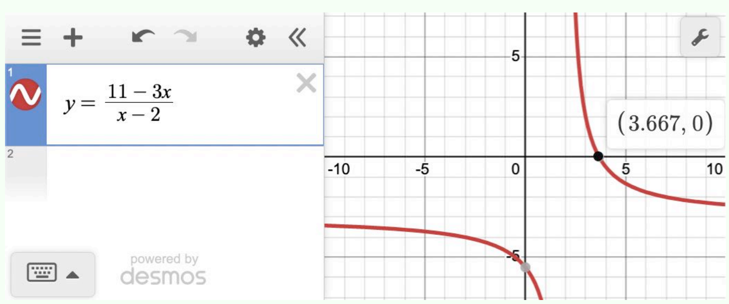
d) To find the numbers x where $\frac{5}{x-2} \leq 3$, we can graph the two functions on the left- and right-hand side of the inequality.



However, this can sometimes be confusing, and we recommend rewriting the inequality so that one side becomes zero. Then, we graph the function on the other side of the new inequality.

$$\frac{5}{x-2} \leq 3 \iff \frac{5}{x-2} - 3 \leq 0 \iff \frac{5-3(x-2)}{x-2} \leq 0$$
$$\iff \frac{5-3x+6}{x-2} \leq 0 \iff \frac{11-3x}{x-2} \leq 0$$

Therefore, we graph the function $f(x) = \frac{11-3x}{x-2}$.



The vertical asymptote is $x = 2$, and the x-intercept found is thus

$$11 - 3x = 0 \implies 11 = 3x \implies x = \frac{11}{3} \approx 3.667.$$

We check the inequality $\frac{11-3x}{x-2} \le 0$ at 0, 3, and 4.

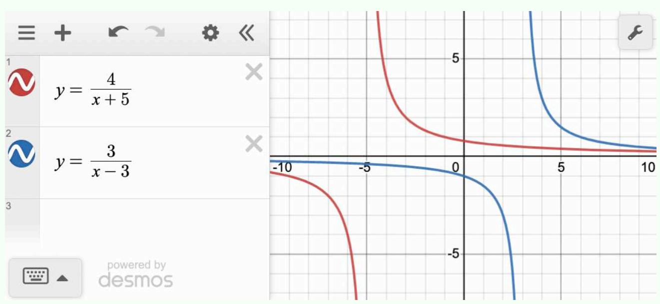
$$2 \qquad\qquad \frac{11}{3}$$

Check $x = 0$:	Check $x = 3$:	Check $x = 4$:
$\frac{11-3\cdot 0}{0-2} \overset{?}{\le} 0$	$\frac{11-3\cdot 3}{3-2} \overset{?}{\le} 0$	$\frac{11-3\cdot 4}{4-2} \overset{?}{\le} 0$
$\frac{11}{-2} \overset{?}{\le} 0$	$\frac{2}{1} \overset{?}{\le} 0$	$\frac{-1}{2} \overset{?}{\le} 0$
TRUE	FALSE	TRUE

This, together with the fact that f is undefined at 2 and $f(\frac{11}{3}) = 0$, gives the following solution set:

$$S = \Big(-\infty, 2\Big) \cup \Big[\frac{11}{3}, \infty\Big)$$

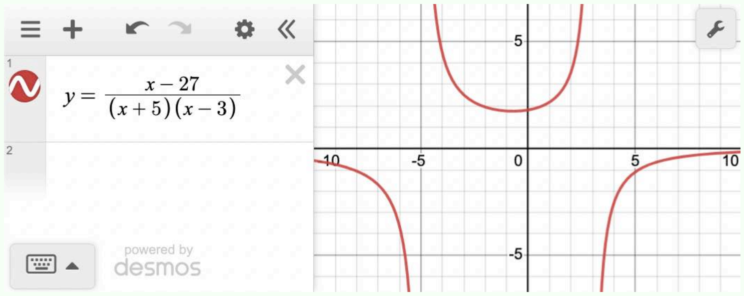
e) We want to find those numbers x for which $\frac{4}{x+5} < \frac{3}{x-3}$. One way to do this is given by graphing both functions $f_1(x) = \frac{4}{x+5}$ and $f_2(x) = \frac{3}{x-3}$, and by trying to determine where $f_1(x) < f_2(x)$. The graphs of f_1 and f_2 are displayed below. Note that it may sometimes not be completely obvious to determine in which intervals f_1 is greater than f_2.



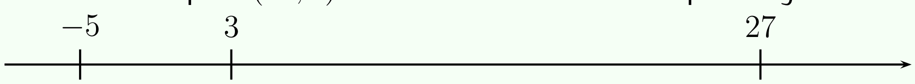
As before, we recommend rewriting the inequality so that one side of the inequality becomes zero:

$$\begin{aligned}
\frac{4}{x+5} < \frac{3}{x-3} \quad &\iff \quad \frac{4}{x+5} - \frac{3}{x-3} < 0 \\
&\iff \quad \frac{4(x-3) - 3(x+5)}{(x+5)(x-3)} < 0 \\
&\iff \quad \frac{4x - 12 - 3x - 15}{(x+5)(x-3)} < 0
\end{aligned}$$

Simplifying this, we get the inequality: $\frac{x-27}{(x+5)(x-3)} < 0$. We therefore graph the function $f(x) = \frac{x-27}{(x+5)(x-3)}$.

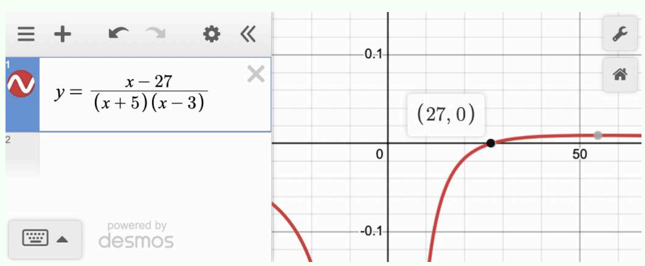


The vertical asymptotes of $f(x) = \frac{x-27}{(x+5)(x-3)}$ are $x = -5$ and $x = 3$. The x-intercept is $(27, 0)$. We next check the corresponding intervals.



Check $x = -6$:	$f(-6) = \frac{(-6)-27}{((-6)+5)\cdot((-6)-3)} = \frac{-33}{9} < 0$		TRUE
Check $x = 0$:	$f(0) = \frac{0-27}{(0+5)\cdot(0-3)} = \frac{-27}{-15} = \frac{27}{15} \not< 0$		FALSE
Check $x = 4$:	$f(4) = \frac{4-27}{(4+5)\cdot(4-3)} = \frac{-23}{9} < 0$		TRUE
Check $x = 28$:	$f(28) = \frac{28-27}{(28+5)\cdot(28-3)} = \frac{1}{825} \not< 0$		FALSE

Note that the graph of f is indeed above the x-axis for $x > 27$.



Therefore, the solution set is

$$\text{solution set } S = \{x | x < -5, \text{ or } 3 < x < 27\} = (-\infty, -5) \cup (3, 27).$$

Here, the x-intercept $x = 27$ is not included in the solution set since the inequality had a "$<$" and not "$\leq$" sign.

□

12.3 Exercises

Exercise 12.1

Solve for x.

a) $5x+6 \leq 21$
b) $3+4x > 10x$
c) $2x+8 \geq 6x+24$
d) $9-3x < 2x-13$
e) $-5 \leq 2x+5 \leq 19$
f) $15 > 7-2x \geq 1$
g) $3x+4 \leq 6x-2 \leq 8x+5$
h) $5x+2 < 4x-18 \leq 7x+11$

Exercise 12.2

Solve for x.

a) $x^2-5x-14 > 0$
b) $x^2-2x \geq 35$
c) $x^2-4 \leq 0$
d) $x^2+3x-3 < 0$
e) $2x^2+2x \leq 12$
f) $3x^2 < 2x+1$
g) $x^2-4x+4 > 0$
h) $x^3-2x^2-5x+6 \geq 0$
i) $x^3+4x^2+3x+12 < 0$
j) $-x^3-4x < -4x^2$
k) $x^4-10x^2+9 \leq 0$
l) $x^4-5x^3+5x^2+5x < 6$
m) $x^4-5x^3+6x^2 > 0$
n) $x^5-6x^4+x^3+24x^2-20x \leq 0$
o) $x^5-15x^4+85x^3-225x^2+274x-120 \geq 0$,
p) $x^{11}-x^{10}+x-1 \leq 0$

Exercise 12.3

Find the domain of the functions below.

a) $f(x) = \sqrt{x^2-8x+15}$
b) $f(x) = \sqrt{9x-x^3}$
c) $f(x) = \sqrt{(x-1)(4-x)}$
d) $f(x) = \sqrt{(x-2)(x-5)(x-6)}$
e) $f(x) = \frac{5}{\sqrt{6-2x}}$
f) $f(x) = \frac{1}{\sqrt{x^2-6x-7}}$

Exercise 12.4

Solve for x.

a) $|2x+7|>9$ b) $|6x+2|<3$

c) $|5-3x|\geq 4$ d) $|-x-7|\leq 5$

e) $|1-8x|\geq 3$ f) $1>\left|2+\frac{x}{5}\right|$

Exercise 12.5

Solve for x.

a) $\frac{x+2}{x+4}\geq 0$ b) $\frac{x-5}{2-x}>0$ c) $\frac{9x-11}{7x+15}\leq 0$ d) $\frac{13x+4}{6x-1}\geq 0$

e) $\frac{7x-2}{3x+8}<0$ f) $\frac{4x-4}{x^2-4}\geq 0$ g) $\frac{x-2}{x^2-4x-5}<0$ h) $\frac{x^2-9}{x^2-4}\geq 0$

i) $\frac{x-3}{x+3}\leq 4$ j) $\frac{1}{x+10}>5$ k) $\frac{2}{x-2}\leq\frac{5}{x+1}$ l) $\frac{x^2}{x+4}\leq x$