Precalculus

Third Edition (3.0)

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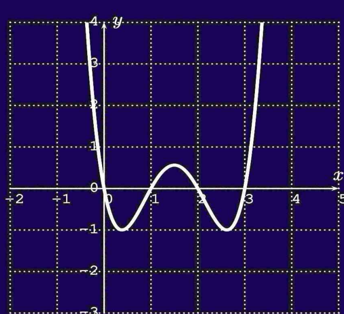

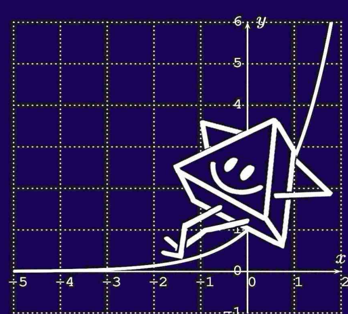

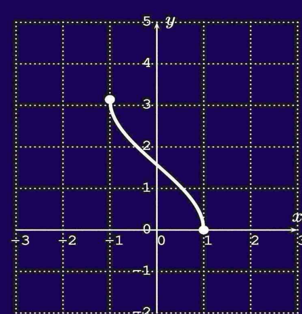

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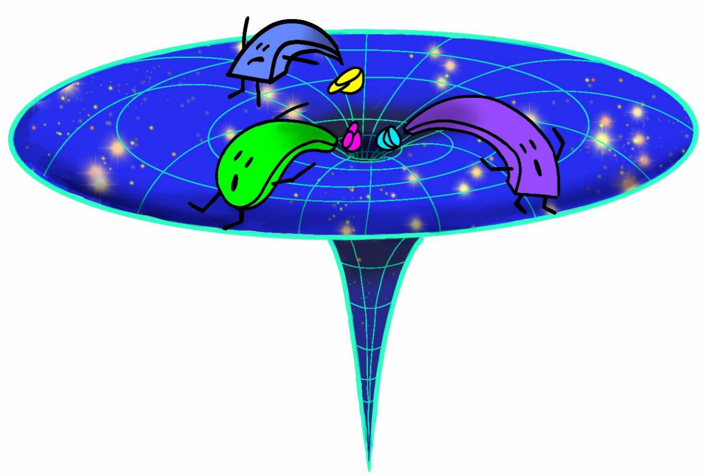
Chapter 11

Exploring discontinuities and asymptotes

We have seen that rational functions have certain features that were not present in polynomials, such as discontinuities. These discontinuities can be removable ("holes"), or non-removable (at vertical asymptotes), where the function can become arbitrarily large, and thus approaches infinity.

It may be perplexing to think about functions approaching an infinite value, as we do not experience infinities in everyday life. Indeed, a quantity that approaches an infinite value would probably come with some strange side effects. For example, it is theorized that gravity approaches an infinite value at the center of a black hole (often called the singularity), and we definitely do not recommend to get anywhere near such an object!



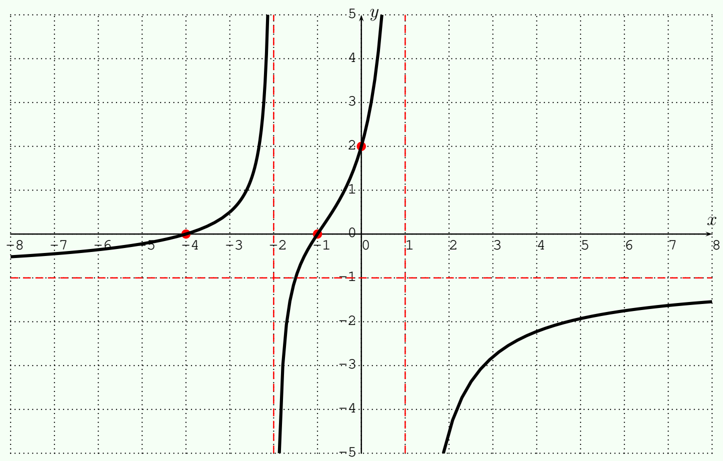
In this chapter, we will explore the behavior of functions near an input x_0. In Section 11.1 we look at asymptotes and removable singularities for rational functions, while we look at the general case in Section 11.2.

11.1 More on rational functions

We now explore the asymptotic behavior of rational functions near a discontinuity, as well as the behavior at infinity in more detail. First, we review how one can recover the formula of a rational function $f(x) = \frac{p(x)}{q(x)}$ from its asymptotes and roots.

Example 11.1

The graph of the rational function $f(x) = \frac{p(x)}{q(x)}$ is displayed below, where p and q are polynomials of degree 2. Assuming that all intercepts and asymptotes are at integer values as indicated (in red), find these intercepts and asymptotes. Use this information to find a formula for $f(x)$.



Solution.
The intercepts and asymptotes can be read off from the graph:

x-intercepts :	$(x, y) = (-4, 0)$ and $(x, y) = (-1, 0)$
y-intercept :	$(x, y) = (0, 2)$
vertical asymptotes :	$x = -2$ and $x = 1$
horizontal asymptote :	$y = -1$

Since the x-intercepts determine the roots of the numerator of f, and the vertical asymptotes determine the roots of the denominator of f, we see that f must be of the form

$$f(x) = a \cdot \frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$$

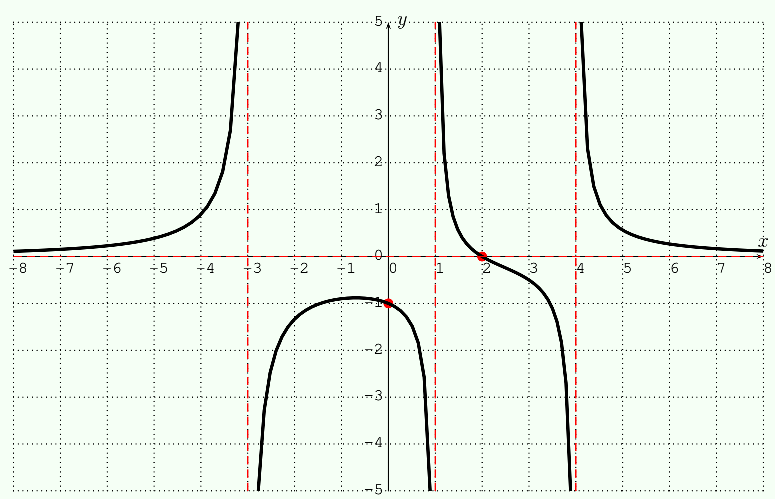
where a is some overall coefficient. The coefficient a can be determined either via the horizontal asymptote or via the y-intercept. In the first case, note that the horozintal asymptote of $f(x) = a \cdot \frac{x^2+5x+4}{x^2+x-2}$ is $y = a$, so that we conclude $a = -1$. For the latter case, note that the y-intercept of $f(x) = a \cdot \frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$ is at $y = f(0) = a \cdot \frac{4\cdot 1}{2\cdot(-1)} = a \cdot (-2)$, which, according to the graph, has to be equal to 2:

$$2 = -2a \implies a = -1$$

Therefore, $f(x) = (-1) \cdot \frac{(x+4)\cdot(x+1)}{(x+2)\cdot(x-1)}$. □

Example 11.2

The graph of the rational function $f(x) = \frac{p(x)}{q(x)}$ is displayed below, where p is a polynomial of degree 1 and q is a polynomial of degree 3. Assuming that all intercepts and asymptotes are at integer values as indicated (in red), find these intercepts and asymptotes. Use this information to find a formula for $f(x)$.



Solution.
From the graph, we see that the intercepts and asymptotes are:

x-intercept :	$(x, y) = (2, 0)$
y-intercept :	$(x, y) = (0, -1)$
vertical asymptotes :	$x = -3$ and $x = 1$ and $x = 4$
horizontal asymptote :	$y = 0$

Using the x-intercept and the vertical asymptotes, we see that f is of the form

$$f(x) = a \cdot \frac{(x-2)}{(x+3)\cdot(x-1)\cdot(x-4)}$$

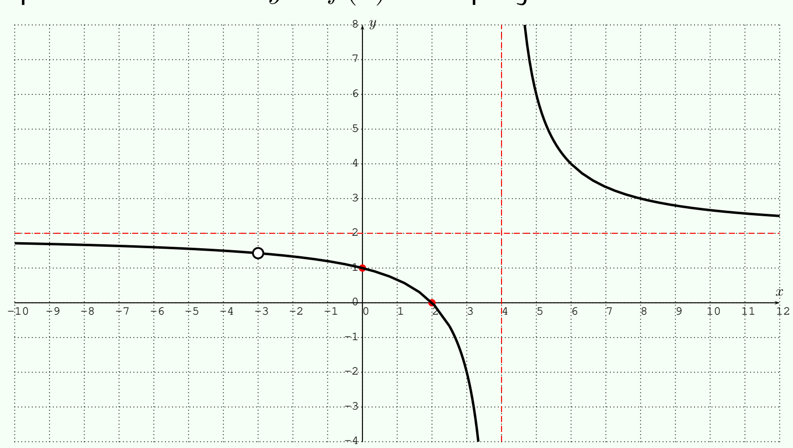
for some coefficient a. Note that the horizontal asymptote is automatically $y = 0$, since the denominator has a higher degree than the numerator. To find the coeffient a, we use the y-intercept, which in this formula is given by $y = f(0) = a \cdot \frac{-2}{3\cdot(-1)\cdot(-4)} = a \cdot \frac{-2}{12} = a \cdot \frac{-1}{6}$. Since the graph shows a y-intercept at $y = -1$, it follows that

$$-1 = a \cdot \frac{-1}{6} \implies a = 6$$

Therefore, $f(x) = 6 \cdot \frac{(x-2)}{(x+3)\cdot(x-1)\cdot(x-4)}$. □

Example 11.3

The graph of the function $y = f(x)$ is displayed below.



Assume that $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where p and q are polynomials of degree 2; that all intercepts and asymptotes are at integer values (indicated in red); and that f has a removable discontinuity at $x = -3$.

a) Use this information to find a formula for $f(x)$.

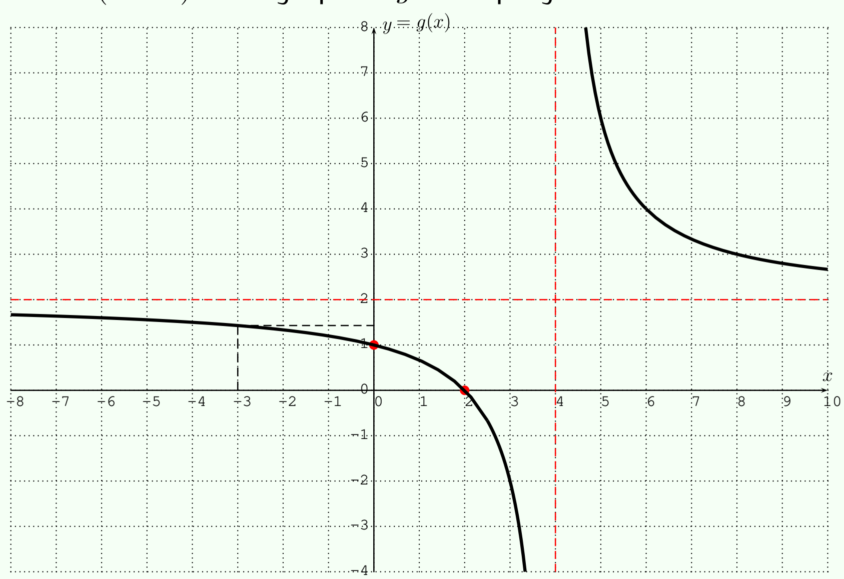
b) Find the coordinates of the removable discontinuity.

Solution.

a) The x-intercept is at $(2, 0)$, the y-intercept is at $(0, 1)$, the vertical asymptote is at $x = 4$, and the horizontal asymptote is at $y = 2$. The discontinuity at $x = -3$ requires a factor of $(x + 3)$ in the numerator and the denominator, so that $f(x) = a \cdot \frac{(x-2)\cdot(x+3)}{(x-4)\cdot(x+3)}$. From the horizontal asymptote $y = 2$ we see that $a = 2$, so that

$$f(x) = 2 \cdot \frac{(x-2)\cdot(x+3)}{(x-4)\cdot(x+3)}$$

b) To find the y-coordinate of the discontinuity, we may try to plug $x = -3$ into the function f. Unfortunately, this does not lead to an answer, since $f(-3)$ is undefined. However, we can see that for all $x \neq -3$, the function f coincides with $g(x) = 2 \cdot \frac{x-2}{x-4}$ after canceling the factor $(x + 3)$. The graph of g is displayed below.



Evaluating g at $x = -3$ gives

$$g(-3) = 2 \cdot \frac{-3-2}{-3-4} = 2 \cdot \frac{-5}{-7} = \frac{10}{7}$$

This shows that the coordinates of the discontinuity of f are $(x, y) = (-3, \frac{10}{7})$. We say that, "as x approaches -3, $f(x)$ approaches $\frac{10}{7}$", which is written as:

$$\text{as } x \to -3, \quad f(x) \to \frac{10}{7}$$

This is also called a limit, written as: $\lim_{x \to -3} f(x) = \frac{10}{7}$.

□

We also want to study a rational function as x becomes arbitrarily large (having large positive or large negative values), or, saying it differently, when x approaches ±infinity ($x \to \infty$ or $x \to -\infty$). We already saw the behavior of some rational functions for this case when we considered horizontal asymptotes. More generally, we can describe the behavior of a rational function when $x \to \pm\infty$ as follows.

Observation 11.4: Asymptotic behavior, slant asymptote

The asymptotic behavior of a rational function $f(x) = \frac{p(x)}{q(x)}$, in the case where the degree of p is greater than the degree of q, can be calculated by performing a long division. If the long division has a quotient $g(x)$ and a remainder $r(x)$, then

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}.$$

Now, since deg(r) <deg(q), the fraction $\frac{r(x)}{q(x)}$ approaches zero as x approaches $\pm\infty$, so that $f(x) \approx g(x)$ for large $|x|$. Thus, $f(x)$ is approximately $g(x)$ for large $|x|$.
If $g(x)$ is a linear function (that is, a polynomial of degree 1), then g is called the **slant asymptote** of f.

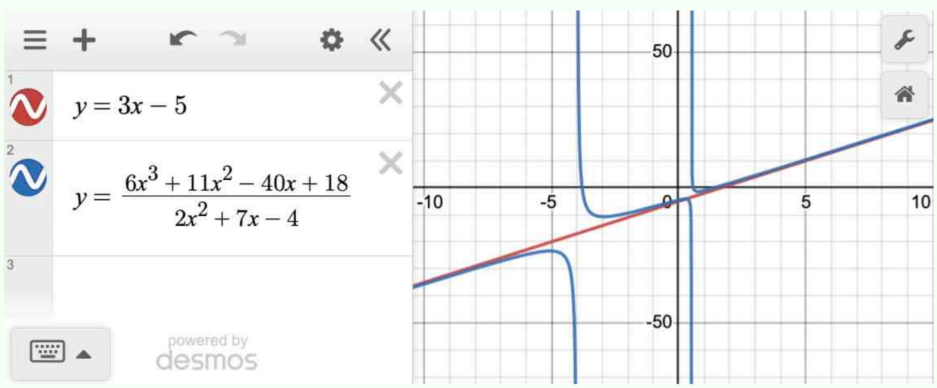
Example 11.5

Find the slant asymptote of $f(x) = \frac{6x^3+11x^2-40x+18}{2x^2+7x-4}$.

Solution.
We divide the polynomials via a long division:

$$\begin{array}{r|rrrr}
 & & & 3x & -5 \\ \hline
2x^2+7x-4 & 6x^3 & +11x^2 & -40x & +18 \\
 & -(6x^3 & +21x^2 & -12x) & \\ \hline
 & & -10x^2 & -28x & +18 \\
 & & -(-10x^2 & -35x & +20) \\ \hline
 & & & 7x & -2
\end{array}$$

Therefore, $f(x) = 3x - 5 + \frac{7x-2}{2x^2+7x-4}$, so that for large $|x|$, we have $f(x) \approx 3x - 5$. Thus, the slant asymptote of $f(x)$ is $y = 3x - 5$.



Note that the slant asymptote has positive slope, so that it goes to $+\infty$ when x becomes large (that is $x \to \infty$), while it goes to $-\infty$ when $x \to -\infty$.
We also write this as $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$. □

11.2 *Optional section:* Limits

In Example 11.3 we implicitly calculated the *limit* as the y-coordinate of the removable discontinuity. A full treatment of limits is the subject of a course in calculus, which will provide many more tools to evaluate limits. For now, we will only explore some intuition regarding limits mainly stemming from studying graphs of functions.

Definition 11.6: Limit

Let $y = f(x)$ be a function, which is defined near a number a. We write

$$\boxed{\text{as } x \to a, \quad f(x) \to L}$$

if $f(x)$ approaches L as x approaches a. Alternatively, we also write

$$\boxed{\lim_{x \to a} f(x) = L}$$

and we call L the **limit** of f as x approaches a.

This definition contains several concepts that were not made precise, such as what it means to be "near a", what it means to "approach a number a", or what it means to "approach L as x approaches a".

A precise version of a limit will formally specify that $f(x)$ will be within an arbitrarily small distance from L for all x close enough to a. This is what is done, for example, in the ϵ-δ definition of a limit. The details are topics of a course in calculus, and are beyond the scope of this text.

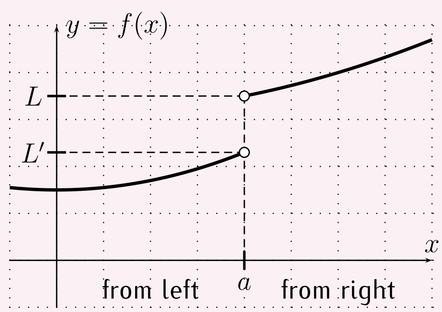
We also consider the case when x approaches a number from one side only, that is, from the right or from the left:

- if $f(x)$ approaches L as x approaches a from the *right*, then we write

$$\boxed{\text{as } x \to a^+, \quad f(x) \to L} \qquad \text{or} \qquad \boxed{\lim_{x \to a^+} f(x) = L}$$

- if $f(x)$ approaches L' as x approaches a from the *left*, then we write

$$\boxed{\text{as } x \to a^-, \quad f(x) \to L'} \qquad \text{or} \qquad \boxed{\lim_{x \to a^-} f(x) = L'}$$



Note 11.7

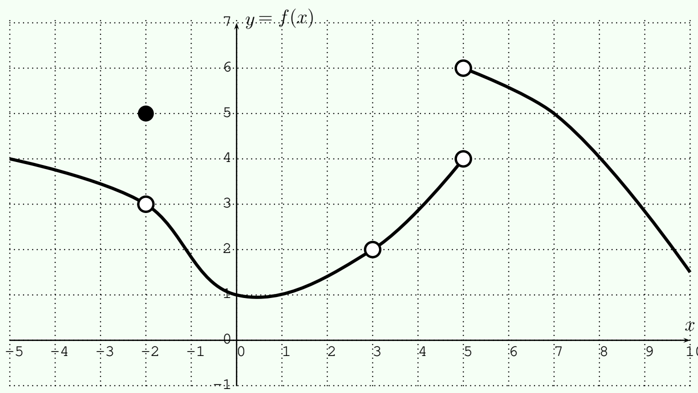
We note that $f(x)$ approaches L when x approaches a, precisely when $f(x)$ approaches L when x approaches a from the right and from the left.

$$\left(\lim_{x\to a} f(x) = L\right) \quad \Leftrightarrow \quad \left(\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = L\right)$$

We explore these concepts in the next examples.

Example 11.8

The graph of $y = f(x)$ is shown below.



Find the limit from the right, the limit from the left, and the (two-sided) limit as x approaches the following numbers.

a) as $x \to 3$ b) as $x \to 5$ c) as $x \to 7$ d) as $x \to -2$

Solution.

a) The limits from the right and from the left approaching 3 are $\lim_{x\to 3^+} f(x) = 2$ and $\lim_{x\to 3^-} f(x) = 2$. Therefore, we also have $\lim_{x\to 3} f(x) = 2$.

b) The limits from the right and left approaching 5 are $\lim_{x\to 5^+} f(x) = 6$ and $\lim_{x\to 5^-} f(x) = 4$. Since these limits differ, the two-sided limit $\lim_{x\to 5} f(x)$ does not exist.

c) The limits approaching 7 are $\lim_{x\to 7^+} f(x) = 5$ and $\lim_{x\to 7^-} f(x) = 5$, and therefore also $\lim_{x\to 7} f(x) = 5$. Note that in this case $f(7)$ is also defined and $f(7) = 5$ coincides with the limit. (We say that f is *continuous at* 7.)

d) The limits approaching -2 are equal, $\lim_{x\to -2^+} f(x) = 3$ and $\lim_{x\to -2^+} f(x) = 3$, and therefore $\lim_{x\to -2} f(x) = 3$. In this case $f(-2)$ is also defined but does not coincide with the limit.

□

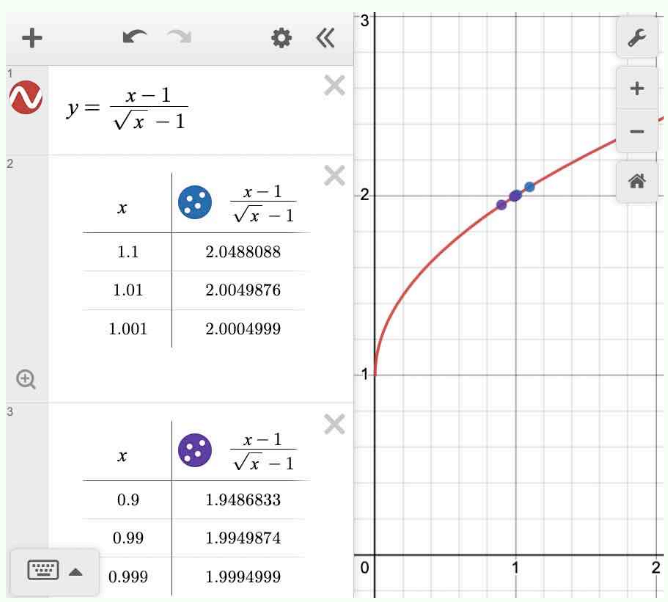
Example 11.9

Use the graphing calculator to identify the stated limits.

a) $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$ b) $\lim_{x\to 2} \frac{x\cdot|x-2|}{x-2}$ c) $\lim_{x\to -3} \frac{x+5}{x+3}$ d) $\lim_{x\to 0^+} x^2 \cdot \ln(x)$

Solution.

a) We plug numbers into the graphing calculator that approach 1 from the right and from the left. Suitable numbers from the right are: 1.1, 1.01, 1.001, etc. Suitable numbers from the left are: 0.9, 0.99, 0.999, etc.



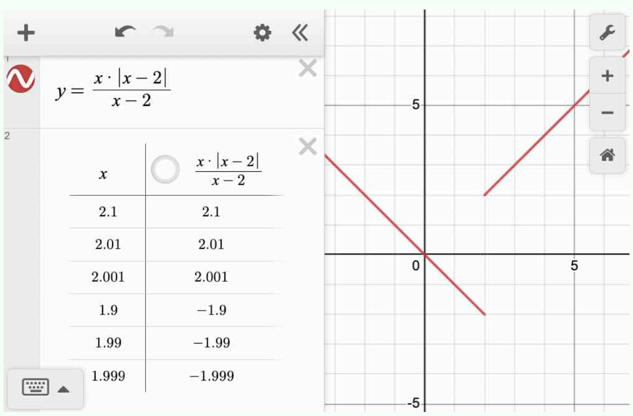
These evaluations appear to indicate that $f(x)$ approaches 2 as x approaches 1.

Note that we only used the graphing calculator to get an idea what the limit might be. A precise evaluation of the limit will require a more thorough analysis of the function at $x = 1$. Such an analysis is beyond the scope of this exercise. Nevertheless, we will provide this analysis for one case (that is for part (a) of this example) in the hope that it might help some readers, but will not do so for parts (b) and (c). Note that for $x \neq 1$:

$$f(x) = \frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1$$

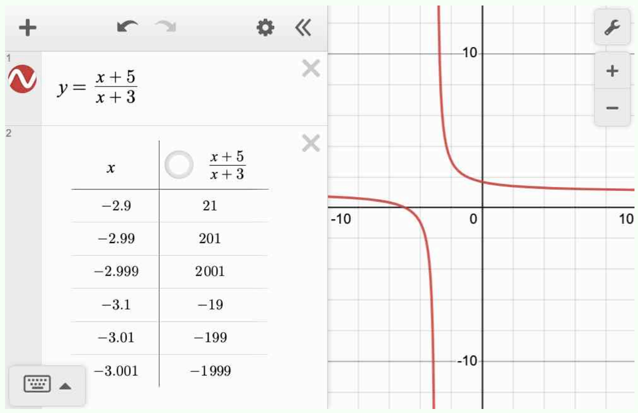
Therefore, the function $y = \sqrt{x}+1$ coincides with $y = f(x)$ for $x \neq 1$, but does not have a discontinuity at $x = 1$. Evaluating $y = \sqrt{x}+1$ at $x = 1$ gives $y = \sqrt{1}+1 = 1+1 = 2$, which shows that $\lim\limits_{x\to 1} f(x) = 2$.

b) Evaluating $\frac{x\cdot|x-2|}{x-2}$ to the right and left of 2 gives:



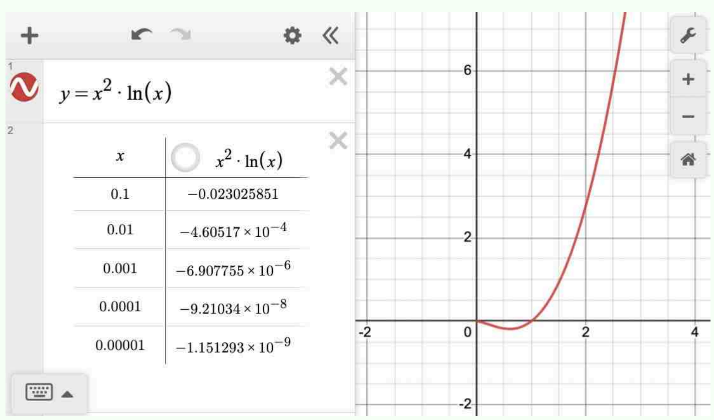
Therefore, $\lim\limits_{x\to 2^+} \frac{x\cdot|x-2|}{x-2} = 2$ and $\lim\limits_{x\to 2^-} \frac{x\cdot|x-2|}{x-2} = -2$. As these limits differ, $\lim\limits_{x\to 2} \frac{x\cdot|x-2|}{x-2}$ does not exist.

c) We evaluate $\frac{x+5}{x+3}$ close to -3.



From this, it is reasonable to conclude that $\lim_{x\to -3^+} \frac{x+5}{x+3} = +\infty$ and $\lim_{x\to -3^-} \frac{x+5}{x+3} = -\infty$, so that $\lim_{x\to -3} \frac{x+5}{x+3}$ does not exist. Note that this also aligns with our knowledge about the vertical asymptote at $x = -3$.

d) Note that $y = x^2 \cdot \ln(x)$ is only defined for $x > 0$, since this is where $\ln(x)$ is defined.



Moreover, when $x \to 0^+$, we know that x^2 approaches 0, but $\ln(x)$ approaches $-\infty$. Interestingly, from the values shown the calculators, it appears that this product approaches $\lim_{x\to 0^+} x^2 \cdot \ln(x) = 0$.

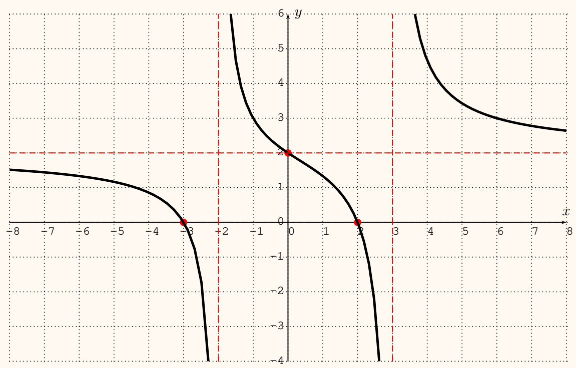
□

11.3 Exercises

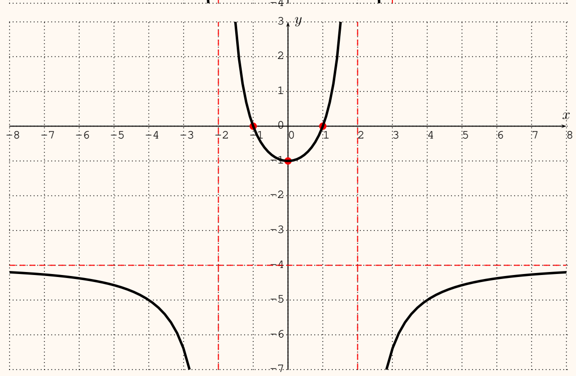
Exercise 11.1

Below are the graphs of rational functions whose numerators and denominators are polynomials of degree 2. All intercepts and asymptotes are at integer values, indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

a)



b)

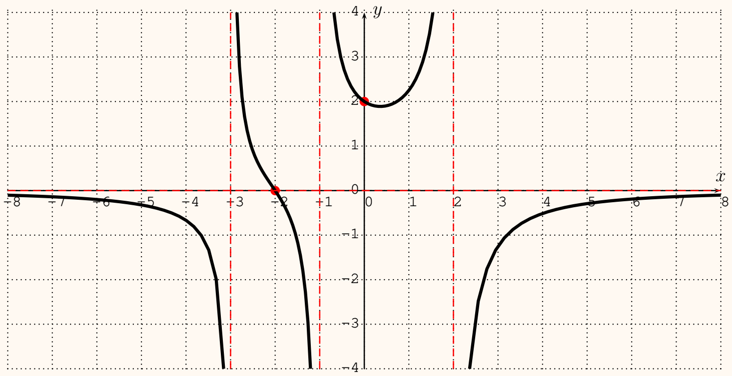


c)

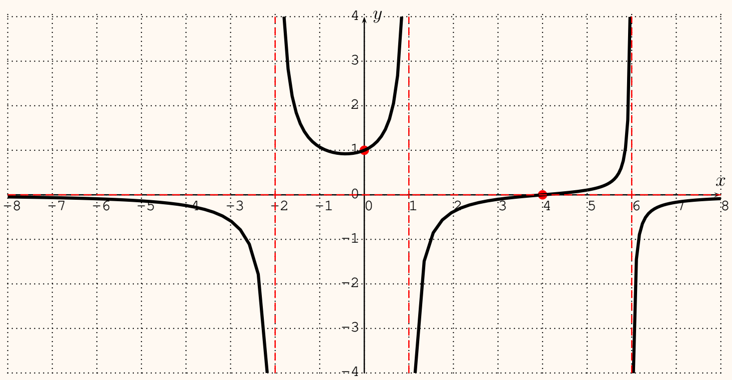
Exercise 11.2

Below are the graphs of rational functions whose numerators are polynomials of degree 1 and whose denominators are polynomials of degree 3. All intercepts and asymptotes are at integer values indicated in red. Find all intercepts and asymptotes, and find a formula for each function.

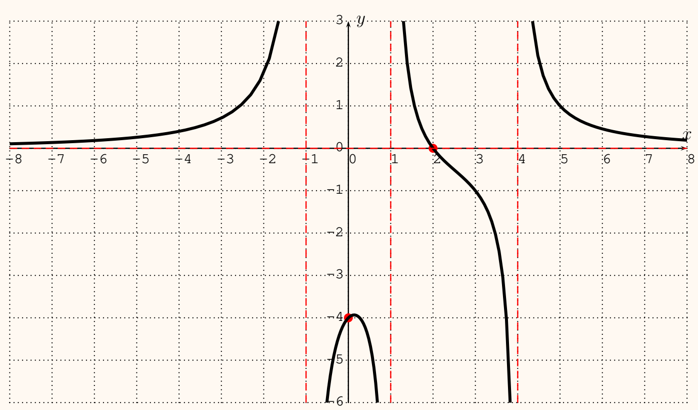
a)



b)



c)



Exercise 11.3

Find the domain of each rational function below. Identify the removable discontinuities and find their x- and y-coordinates.

a) $f(x) = \frac{(x-3)(x-4)}{(x+5)(x-4)}$

b) $f(x) = \frac{3(x+2)(x-5)}{(x+3)(x-5)}$

c) $f(x) = \frac{7(x-2)}{(x+3)(x-2)(x-6)}$

d) $f(x) = \frac{x^2+6x+8}{x^2+x-12}$

e) $f(x) = \frac{x^2-9}{x^2-x-6}$

f) $f(x) = \frac{x^2-4x+3}{x^3+x^2-2x}$

Exercise 11.4

Find the slant asymptote of the rational function.

a) $f(x) = \frac{2x^3+9x^2-20x-21}{2x^2-3x-4}$

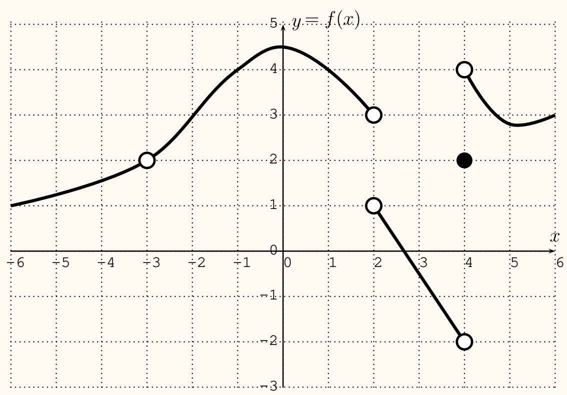
b) $f(x) = \frac{2x^3-13x^2+35x-26}{x^2-4x+6}$

c) $f(x) = \frac{12x^3+10x^2-4x-9}{3x^2+x-2}$

d) $f(x) = \frac{-3x^3-4x^2+20x-16}{x^2+2x-5}$

Exercise 11.5

The graph of the function $y = f(x)$ is shown below.



Find the limits of $f(x)$ as x approaches the values indicated below.

a) $x \to 2^+$

b) $x \to 2^-$

c) $x \to 2$

d) $x \to -3^+$

e) $x \to -3^-$

f) $x \to -3$

g) $x \to -1^+$

h) $x \to -1^-$

i) $x \to -1$

j) $x \to 4^+$

k) $x \to 4^-$

l) $x \to 4$

Exercise 11.6

Choose inputs that approach the given value from the indicated side (right or left). (Note that there is not just one unique answer for this part of the problem!)
Then, use the graphing calculator to compute the corresponding output values and guess what the limit might be.

a) $\lim_{x\to 3^-} \frac{x-3}{|x-3|}$

b) $\lim_{x\to 1^+} \frac{x^3-1}{\sqrt{x}-1}$

c) $\lim_{x\to 2^-} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

d) $\lim_{x\to -5^+} \frac{x^3+5x^2}{|x+5|}$

e) $\lim_{x\to -5^-} \frac{x^3+5x^2}{|x+5|}$

f) $\lim_{x\to 4^-} \frac{x-1}{x-4}$