Precalculus

Third Edition (3.0)

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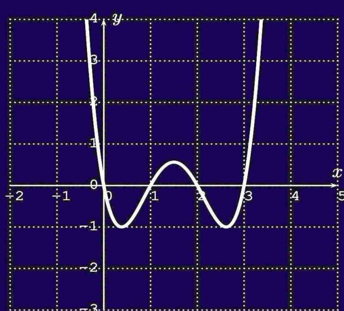

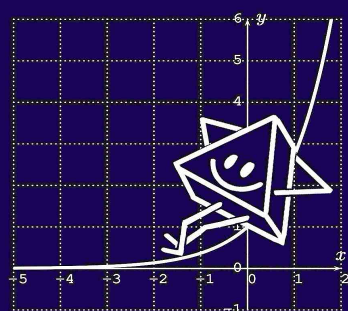

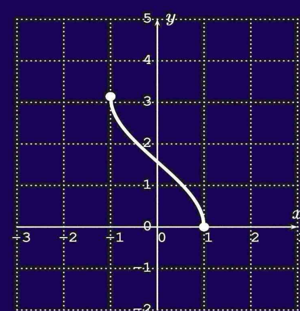

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Chapter 10

Rational functions

Recall that a rational function is a fraction of polynomials:

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

In this chapter, we will study some of the characteristics of graphs of rational functions.

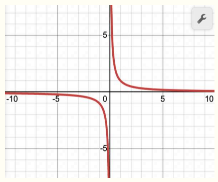
10.1 Graphs of rational functions

The graph of rational functions can have new features that were not present in the graph of polynomials, such as, for example, *asymptotes*. An *asymptote* is a line that is approached by the graph of a function: $x = a$ is a *vertical asymptote* if $f(x)$ approaches $\pm\infty$ as x approaches a from either the left or from the right, and $y = b$ is a *horizontal asymptote* if $f(x)$ approaches b as x approaches ∞ or $-\infty$.

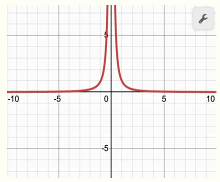
In order to get an idea of some of the features of graphs of rational functions, we look at various sample graphs. First, we graph the basic functions $y = \frac{1}{x^n}$.

Observation 10.1: $f(x) = \frac{1}{x^n}$

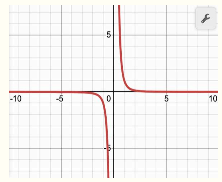
Graphing $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^4}$, we obtain:



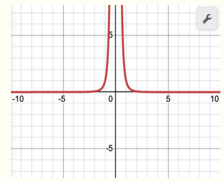
$y = \frac{1}{x}$



$y = \frac{1}{x^2}$

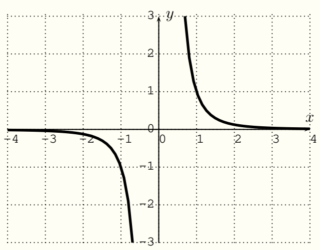


$y = \frac{1}{x^3}$

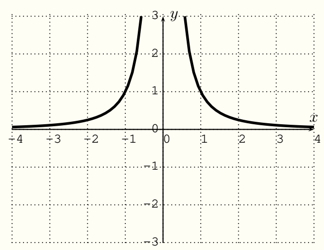


$y = \frac{1}{x^4}$

In general, we see that $x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote. The shape of $y = \frac{1}{x^n}$ depends on n being even or odd. We have:



$y = \frac{1}{x^n}$, n odd

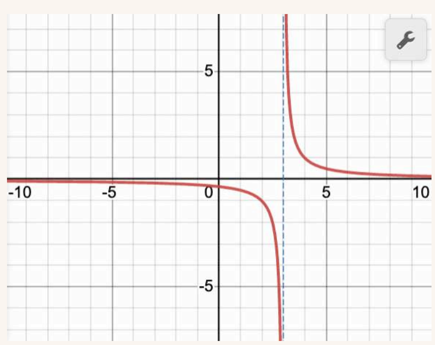


$y = \frac{1}{x^n}$, n even

In the next note, we graph four sample rational functions. These examples will help us understand many general aspects of graphs of rational functions.

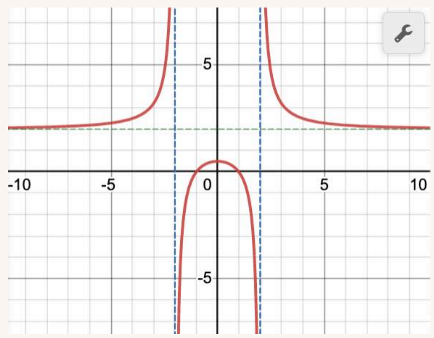
Note 10.2

(1) Our first graph is $f(x) = \frac{1}{x-3}$.



Here, the domain is all numbers where the denominator is not zero, that is $D = \mathbb{R} - \{3\}$. There is a vertical asymptote, $x = 3$. Furthermore, the graph approaches 0 as x approaches $\pm\infty$. Therefore, f has a horizontal asymptote, $y = 0$. Indeed, whenever the denominator has a higher degree than the numerator, the line $y = 0$ will be the horizontal asymptote.

(2) Next, we graph $f(x) = \frac{8x^2-8}{4x^2-16}$.



Here, the domain is all x for which $4x^2 - 16 \neq 0$. To see where this occurs, calculate

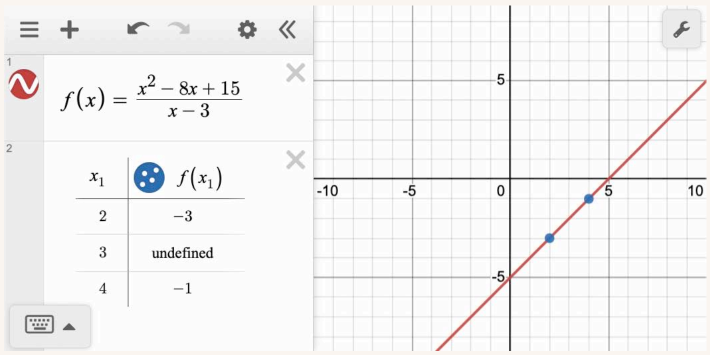
$$4x^2 - 16 = 0 \implies 4x^2 = 16 \implies x^2 = 4 \implies x = \pm 2.$$

Therefore, the domain is $D = \mathbb{R} - \{-2, 2\}$. As before, we see from the graph that the domain reveals the vertical asymptotes $x = 2$ and $x = -2$ (the vertical dashed lines). To find the horizontal asymptote (the horizontal dashed line), we note that, when x becomes very large, the highest terms of both numerator and denominator dominate the function value, so that

$$\text{for } |x| \text{ very large} \implies f(x) = \frac{8x^2 - 8}{4x^2 - 16} \approx \frac{8x^2}{4x^2} = 2$$

Therefore, when x approaches $\pm\infty$, the function value $f(x)$ approaches 2, and therefore the horizontal asymptote is at $y = 2$ (the horizontal dashed line).

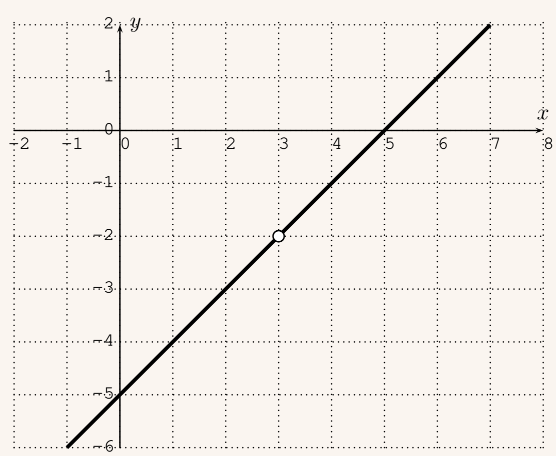
(3) Our next graph is $f(x) = \frac{x^2-8x+15}{x-3}$.



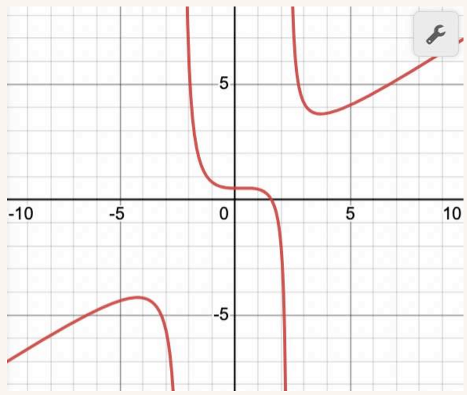
We see that there does not appear to be any vertical asymptote, despite the fact that 3 is not in the domain. The reason for this is that we can "remove the singularity" by canceling the troubling term $x-3$ as follows:

$$f(x) = \frac{x^2-8x+15}{x-3} = \frac{(x-3)(x-5)}{(x-3)} = \frac{x-5}{1} = x-5, \quad x \neq 3$$

Therefore, the function f reduces to $x-5$ for all values where it is defined. However, note that $f(x) = \frac{x^2-8x+15}{x-3}$ is not defined at $x = 3$. We denote this in the graph by an open circle at $x = 3$, and call this a removable singularity (or a hole).



(4) Our fourth and last graph before stating the rules in full generality is $f(x) = \frac{2x^3-8}{3x^2-16}$.



The graph indicates that there is no horizontal asymptote, as the graph appears to increase toward ∞ and decrease toward $-\infty$. To make this observation precise, we calculate the behavior when x approaches $\pm\infty$ by ignoring the lower terms in the numerator and denominator

$$\text{for } |x| \text{ very large} \quad \Longrightarrow \quad f(x) = \frac{2x^3-8}{3x^2-16} \approx \frac{2x^3}{3x^2} = \frac{2x}{3}$$

Therefore, when x becomes very large, $f(x)$ behaves like $\frac{2}{3}x$, which approaches ∞ when x approaches ∞, and approaches $-\infty$ when x approaches $-\infty$. (In fact, after performing a long division we obtain $\frac{2x^3-8}{3x^2-16} = \frac{2}{3} \cdot x + \frac{r(x)}{3x^2-16}$, which would give rise to what is called a *slant asymptote* $y = \frac{2}{3} \cdot x$; see also Observation 11.4 below.) Indeed, whenever the degree of the numerator is greater than the degree of the denominator, we find that there is no horizontal asymptote, but the graph blows up to $\pm\infty$. (Compare this also with example (c) above).

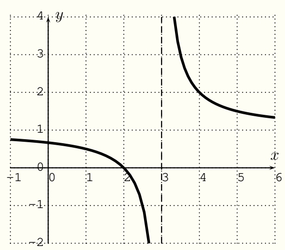
We summarize the observations from the above examples in the following observation.

Observation 10.3: Rational functions

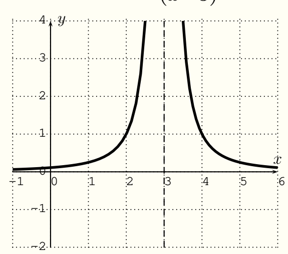
Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function with polynomials $p(x)$ and $q(x)$ in the numerator and denominator, respectively.

- The **domain** of f is all real numbers x for which the denominator is not zero,
$$D = \{ \quad x \in \mathbb{R} \quad | \quad q(x) \neq 0 \quad \}$$
- Assume that $q(x_0) = 0$, so that f is not defined at x_0. If x_0 is not a root of $p(x)$, or if x_0 is a root of $p(x)$ but of a lesser multiplicity than the root in $q(x)$, then f has a **vertical asymptote** $x = x_0$.

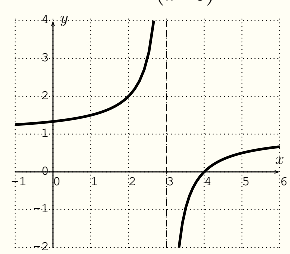
$f(x) = \frac{x-2}{x-3}$



$f(x) = \frac{1}{(x-3)^2}$

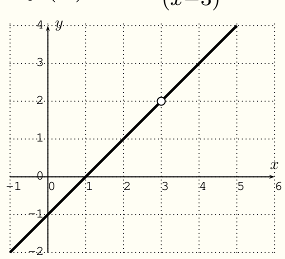


$f(x) = \frac{(x-3)(x-4)}{(x-3)^2}$

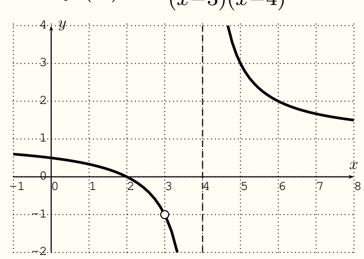


- If $p(x_0) = 0$ and $q(x_0) = 0$, and the multiplicity of the root x_0 in $p(x)$ is at least the multiplicity of the root in $q(x)$, then these roots can be canceled, and we say that there is a **removable discontinuity** (also called a **hole**) at $x = x_0$.

$f(x) = \frac{(x-1)(x-3)}{(x-3)}$



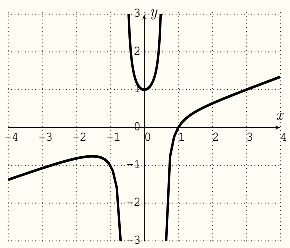
$f(x) = \frac{(x-2)(x-3)}{(x-3)(x-4)}$



- To find the **horizontal asymptotes**, we need to distinguish the cases where the degree of $p(x)$ is less than, equal to, or greater than $q(x)$.

deg(p) >deg(q)

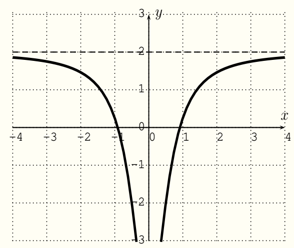
$f(x) = \frac{x^3-1}{3x^2-1}$



no horizontal asymptote

deg(p) =deg(q)

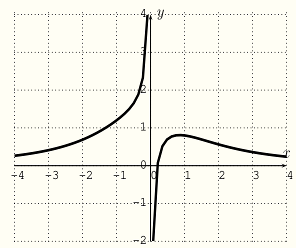
$f(x) = \frac{6x^2-5}{3x^2+1}$



asymptote: $y = \frac{\text{highest coeff. of } p}{\text{highest coeff. of } q}$

deg(p) <deg(q)

$f(x) = \frac{5x-1}{x^3+4x}$

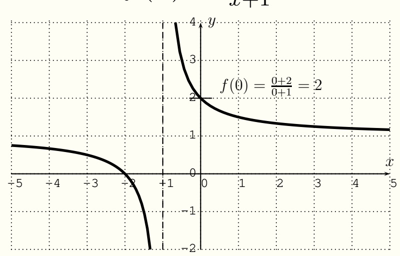


asymptote: $y = 0$

In addition, it is also useful to determine the x– and y–intercepts.

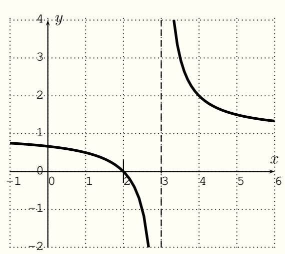
- If 0 is in the domain of f, then the y**–intercept** is $(0, f(0))$.

$f(x) = \frac{x+2}{x+1}$

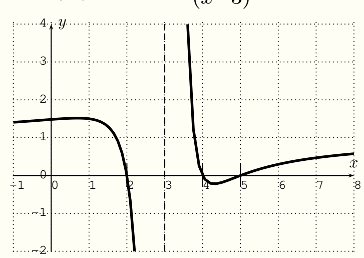


- If $p(x_0) = 0$ but $q(x_0) \neq 0$, then $f(x_0) = \frac{p(x_0)}{q(x_0)} = \frac{0}{q(x_0)} = 0$, so that $(x_0, 0)$ is an x**–intercept**, that is, the graph intersects with the x–axis at x_0.

$f(x) = \frac{x-2}{x-3}$



$f(x) = \frac{(x-2)(x-4)(x-5)}{(x-3)^3}$



Example 10.4

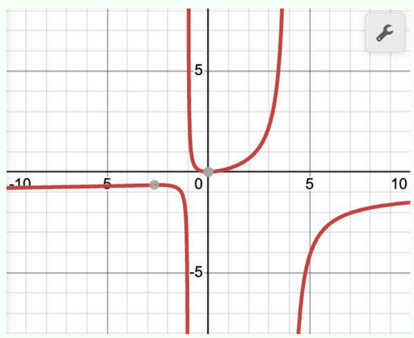
Find the domain, all horizontal asymptotes, vertical asymptotes, removable singularities, and x- and y-intercepts. Use this information together with the graph of the calculator to sketch the graph of f.

a) $f(x) = \frac{-x^2}{x^2-3x-4}$ b) $f(x) = \frac{5x}{x^2-2x}$ c) $f(x) = \frac{x^3-9x^2+26x-24}{x^2-x-2}$

d) $f(x) = \frac{x-4}{(x-2)^2}$ e) $f(x) = \frac{3x^2-12}{2x^2+1}$

Solution.

a) We combine our knowledge of rational functions and its algebra with the particular graph of the function. The graphing calculator shows the following graph:



To find the domain of f we only need to exclude from the real numbers those x that make the denominator zero. Since $x^2-3x-4=0$ exactly when $(x+1)(x-4)=0$, which gives $x=-1$ or $x=4$, we have the domain:

$$\text{domain } D = \mathbb{R} - \{-1, 4\}$$

The numerator has a root exactly when $-x^2 = 0$, that is $x = 0$. Therefore, $x=-1$ and $x=4$ are vertical asymptotes, and since we cannot cancel terms in the fraction, there is no removable singularity. Furthermore, since $f(x)=0$ exactly when the numerator is zero, the only x-intercept is $(0,0)$.

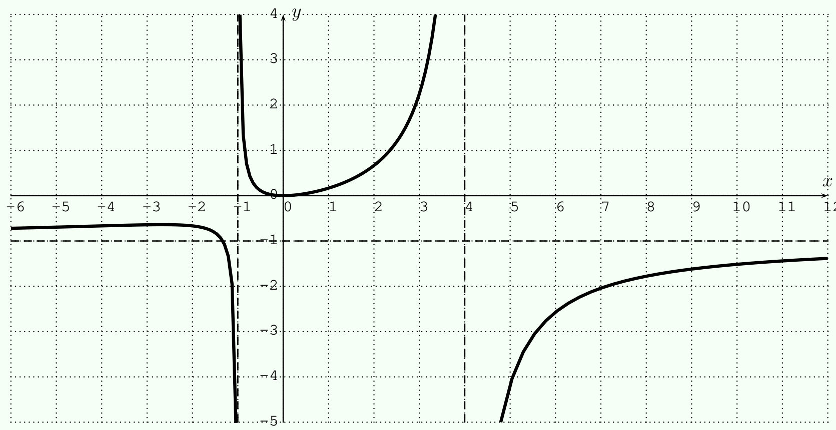
To find the horizontal asymptote, we consider $f(x)$ for large values of x by ignoring the lower order terms in numerator and denominator,

$$|x| \text{ large} \implies f(x) \approx \frac{-x^2}{x^2} = -1$$

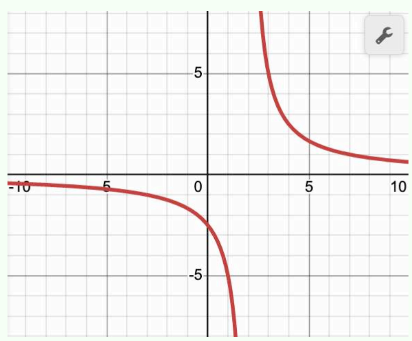
We see that the horizontal asymptote is $y = -1$. Finally, for the y-intercept, we calculate $f(0)$:

$$f(0) = \frac{-0^2}{0^2 - 3 \cdot 0 - 4} = \frac{0}{-4} = 0.$$

Therefore, the y-intercept is $(0, 0)$. The function is then graphed as follows:



b) The graph of $f(x) = \frac{5x}{x^2-2x}$ as drawn with the graphing calculator is shown below.



For the domain, we find the roots of the denominator,

$$x^2 - 2x = 0 \implies \quad x(x-2) = 0 \implies \quad x = 0 \text{ or } x = 2.$$

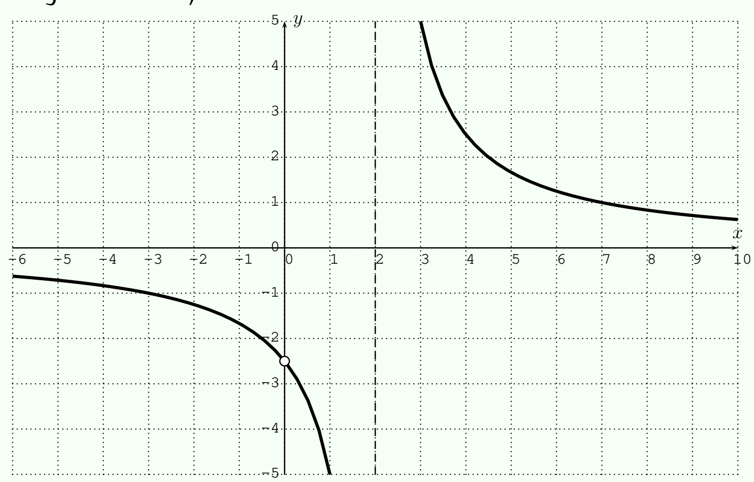
The domain is $D = \mathbb{R} - \{0, 2\}$. For the vertical asymptotes and removable singularities, we calculate the roots of the numerator,

$$5x = 0 \implies \quad x = 0.$$

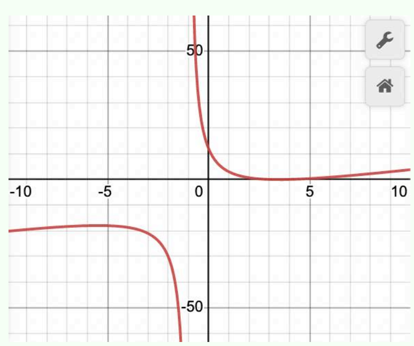
Therefore, $x = 2$ is a vertical asymptote, and $x = 0$ is a removable singularity. Furthermore, the denominator has a higher degree than the numerator, so that $y = 0$ is the horizontal asymptote. For the y-intercept, we calculate $f(0)$ by evaluating the fraction $f(x)$ at 0

$$\frac{5 \cdot 0}{0^2 - 2 \cdot 0} = \frac{0}{0},$$

which is undefined. Therefore, there is no y-intercept (we, of course, already noted that there is a removable singularity when $x = 0$). Finally, for the x-intercept, we need to analyze where $f(x) = 0$, that is where $5x = 0$. The only candidate is $x = 0$ for which f is undefined. Again, we see that there is no x-intercept. The function is then graphed as follows. (Notice in particular the removable singularity at $x = 0$.)



c) We start again by graphing the function $f(x) = \frac{x^3 - 9x^2 + 26x - 24}{x^2 - x - 2}$ with the calculator. After zooming to an appropriate window, we get:



To find the domain of f, we find the zeros of the denominator

$$x^2 - x - 2 = 0 \implies (x+1)(x-2) = 0 \implies x = -1 \text{ or } x = 2.$$

The domain is $D = \mathbb{R} - \{-1, 2\}$. The graph suggests that there is a vertical asymptote $x = -1$. However the $x = 2$ appears not to be a vertical asymptote. This would happen when $x = 2$ is a removable singularity, that is, $x = 2$ is a root of both numerator and denominator of $f(x) = \frac{p(x)}{q(x)}$. To confirm this, we calculate the numerator $p(x)$ at $x = 2$:

$$p(2) = 2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 8 - 36 + 52 - 24 = 0$$

Therefore, $x = 2$ is indeed a removable singularity. To analyze f further, we also factor the numerator. Using the factor theorem, we know that $x-2$ is a factor of the numerator. Its quotient is calculated via long division.

$$\begin{array}{r|rrrr}
 & & x^2 & -7x & +12 \\ \hline
x-2 & x^3 & -9x^2 & +26x & -24 \\
 & -(x^3 & -2x^2) & & \\ \cline{2-3}
 & & -7x^2 & +26x & -24 \\
 & & -(-7x^2 & +14x) & \\ \cline{3-4}
 & & & 12x & -24 \\
 & & & -(12x & -24) \\ \cline{4-5}
 & & & & 0
\end{array}$$

With this, we obtain:

$$f(x) = \frac{(x-2)(x^2 - 7x + 12)}{x^2 - x - 2} = \frac{(x-2)(x-3)(x-4)}{(x+1)(x-2)}$$

Therefore, we conclude that $x = -1$ is a vertical asymptote and $x = 2$ is a removable singularity. We also see that the x-intercepts are $(3, 0)$ and $(4, 0)$ (that is $x-$ values where the numerator is zero).

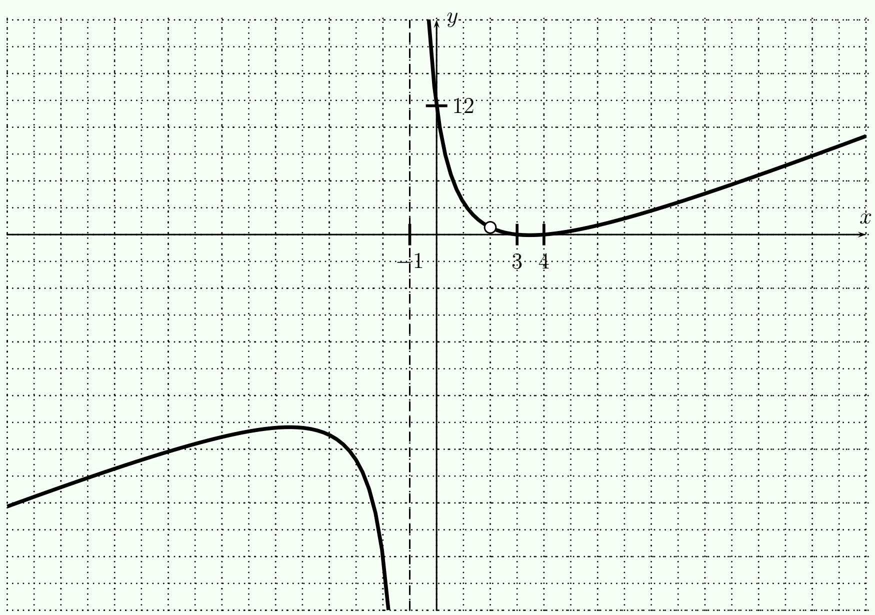
Now, the long range behavior is determined by ignoring the lower terms in the fraction,

$$|x| \text{ large} \implies f(x) \approx \frac{x^3}{x^2} = x \implies \text{no horizontal asymptote}$$

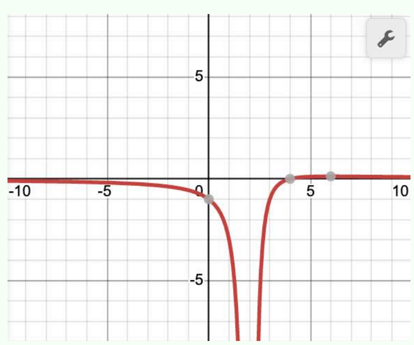
Finally, the y-coordinate of the y-intercept is given by

$$y = f(0) = \frac{0^3 - 9 \cdot 0^2 + 26 \cdot 0 - 24}{0^2 - 0 - 2} = \frac{-24}{-2} = 12.$$

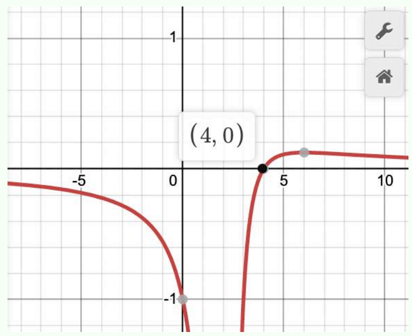
We draw the graph as follows:



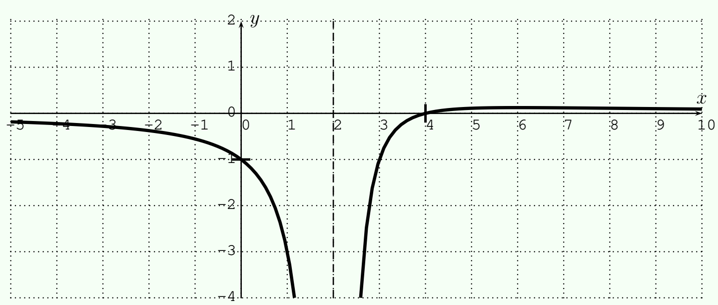
d) We first graph $f(x) = \frac{x-4}{(x-2)^2}$.



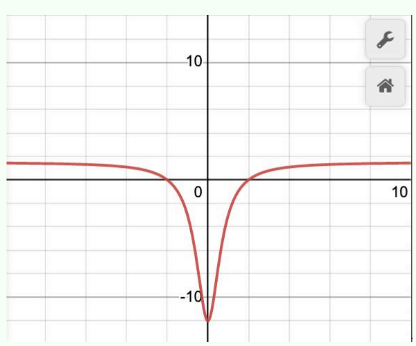
The domain is all real numbers except where the denominator becomes zero, that is, $D = \mathbb{R} - \{2\}$. The graph has a vertical asymptote $x = 2$ and no hole. The horizontal asymptote is at $y = 0$, since the denominator has a higher degree than the numerator. The y-intercept is at $y_0 = f(0) = \frac{0-4}{(0-2)^2} = \frac{-4}{4} = -1$. The x-intercept is where the numerator is zero, $x - 4 = 0$, that is at $x = 4$. Since the above graph did not show the x-intercept clearly, we can observe it better by zooming into the graph vertically (see Note 4.4):



Note in particular that the graph intersects the x-axis at $x = 4$ and then changes its direction to approach the x-axis from above. A graph of the function f which includes all these features is displayed below.



e) We graph $f(x) = \frac{3x^2-12}{2x^2+1}$.



For the domain, we determine the zeros of the denominator.

$$2x^2 + 1 = 0 \implies \quad 2x^2 = -1 \implies \quad x^2 = -\frac{1}{2}.$$

The only solutions of this equation are given by complex numbers, but not by any real numbers. In particular, for any real number x, the denominator of $f(x)$ is not zero. The domain of f is all real numbers, $D = \mathbb{R}$. This implies in turn that there are no vertical asymptotes, and no removable singularities.

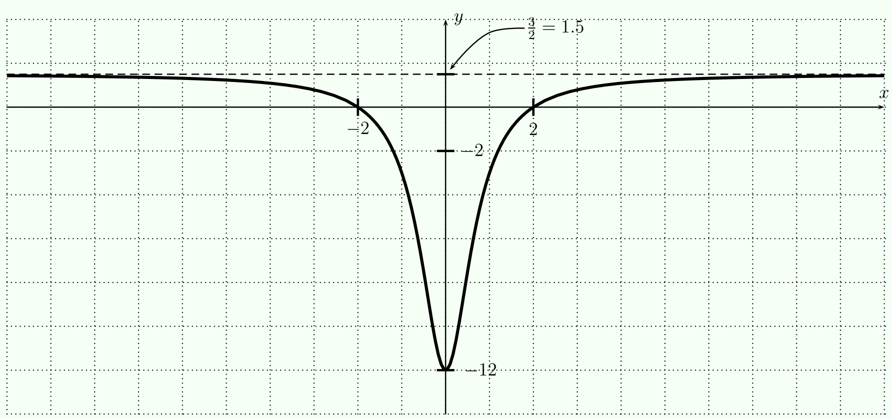
The x-intercepts are determined by $f(x) = 0$, that is where the numerator is zero,

$$3x^2 - 12 = 0 \implies \quad 3x^2 = 12 \implies \quad x^2 = 4 \implies \quad x = \pm 2.$$

The horizontal asymptote is given by $f(x) \approx \frac{3x^2}{2x^2} = \frac{3}{2}$, that is, it is at $y = \frac{3}{2} = 1.5$. The y-intercept is at

$$y = f(0) = \frac{3 \cdot 0^2 - 12}{2 \cdot 0^2 + 1} = \frac{-12}{1} = -12.$$

We sketch the graph as follows:



Since the graph is symmetric with respect to the y-axis, we can make one more observation, namely that the function f is even (see Observation 4.24 on page 74):

$$f(-x) = \frac{3(-x)^2 - 12}{2(-x)^2 + 1} = \frac{3x^2 - 12}{2x^2 + 1} = f(x)$$

□

10.2 *Optional section:* Graphing rational functions by hand

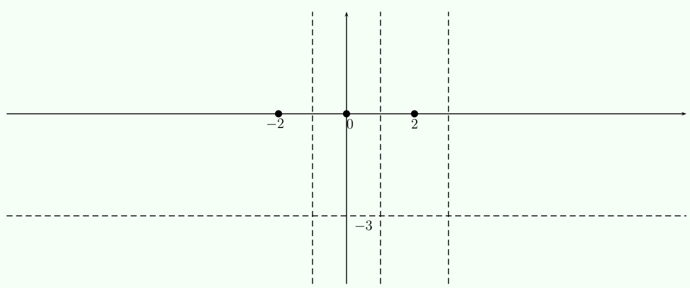
In this section we will show how to sketch the graph of a factored rational function without the use of a calculator. It will be helpful to the reader to have read Section 8.3 on graphing a polynomial by hand before continuing in this section. In addition to having the same difficulties as polynomials, calculators often have difficulty graphing rational functions near an asymptote.

Example 10.5

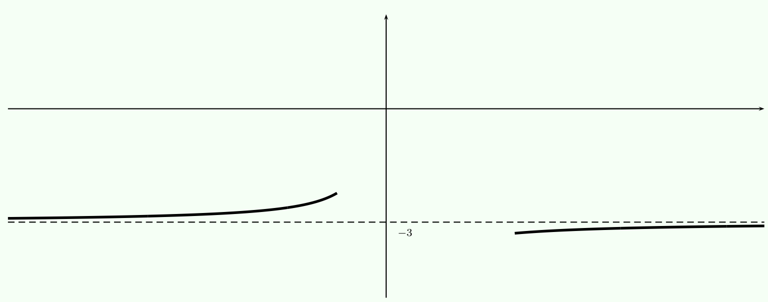
Graph the function $p(x) = \frac{-3x^2(x-2)^3(x+2)}{(x-1)(x+1)^2(x-3)^3}$.

Solution.

We can see that p has zeros at $x = 0, 2,$ and -2 and vertical asymptotes $x = 1$, $x = -1$ and $x = 3$. Also note that for large $|x|$, $p(x) \approx -3$. So there is a horizontal asymptote $y = -3$. We indicate each of these facts on the graph:



We can in fact get a more precise statement by performing a long division and writing $p(x) = \frac{n(x)}{d(x)} = -3 + \frac{r(x)}{d(x)}$. If we drop all but the leading order terms in the numerator and the denominator of the second term, we see that $p(x) \approx -3 - \frac{12}{x}$, whose graph for large $|x|$ looks like

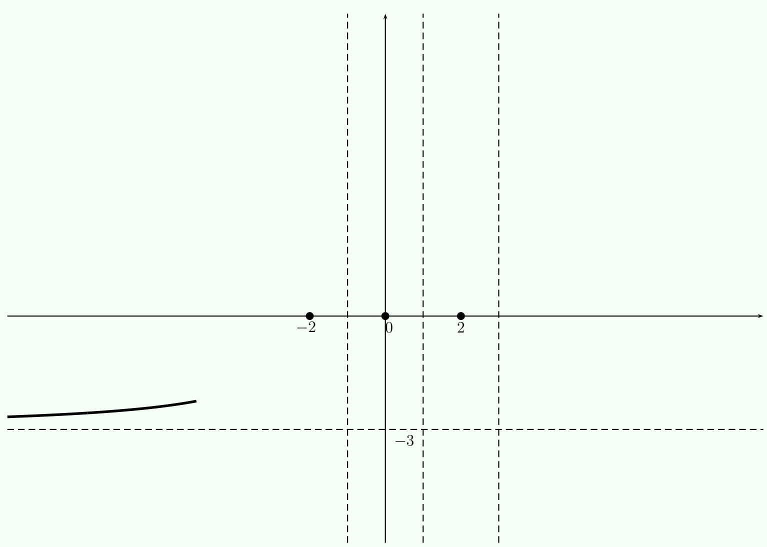


This sort of reasoning can make the graph a little more accurate but is not necessary for a sketch.

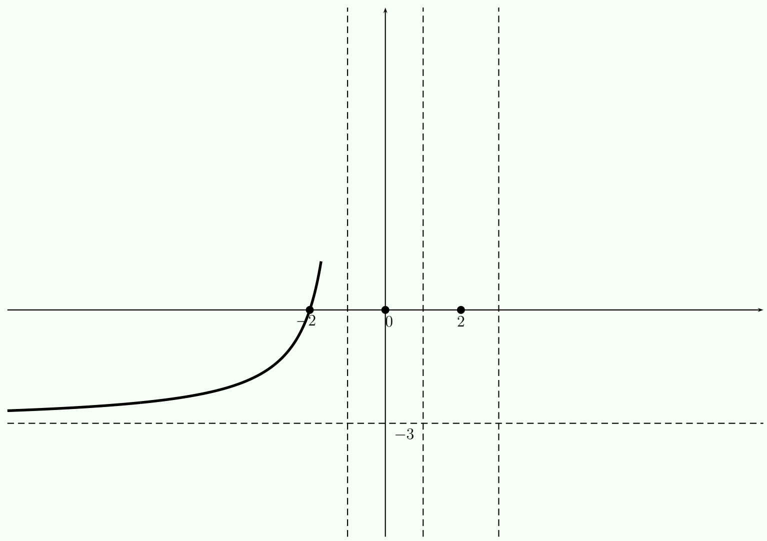
We also have the following table:

for a	near a, $p(x) \approx$	type	sign change at a
-2	$C_1(x+2)$	linear	changes
-1	$C_2/(x+1)^2$	asymptote	does not change
0	$C_3\ x^2$	parabola	does not change
1	$C_4/(x-1)$	asymptote	changes
2	$C_5(x-2)^3$	cubic	changes
3	$C_6/(x-3)^3$	asymptote	changes

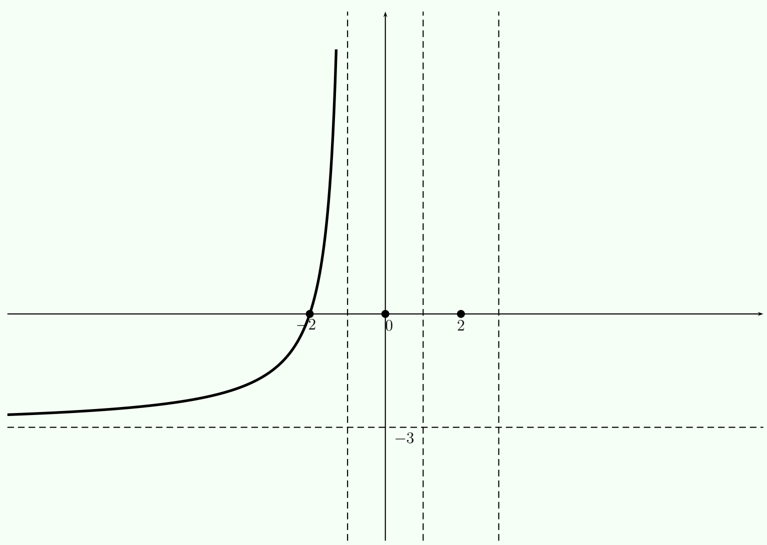
Note that, if the power appearing in the second column is even, then the function does not change from one side of a to the other. If the power is odd, the sign changes (either from positive to negative or from negative to positive).
Now we move from large negative x values toward the right, taking into account the above table. For large negative x, we start our sketch as follows:



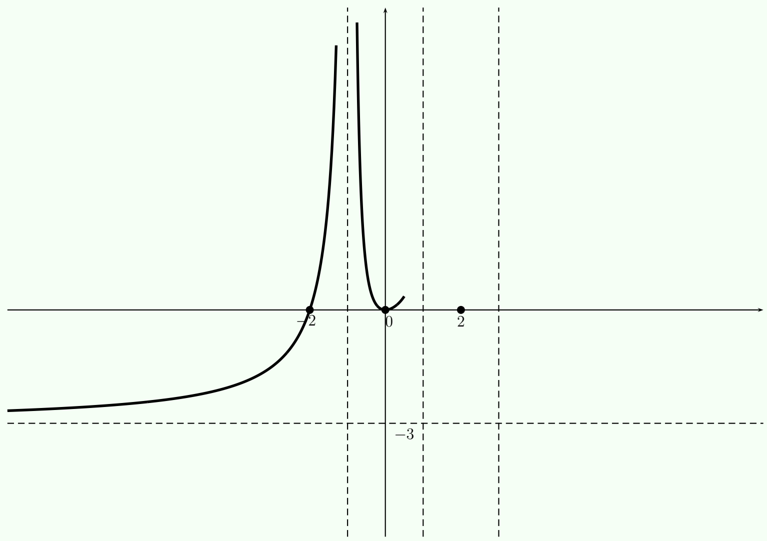
And noting that near $x = -2$ the function $p(x)$ is approximately linear, we have



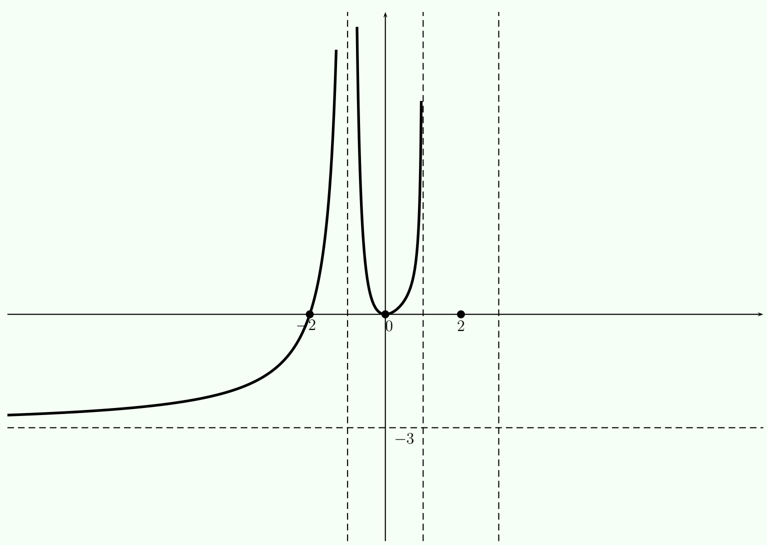
Then noting that we have an asymptote (noting that we cannot cross the x-axis without creating an x-intercept) we have



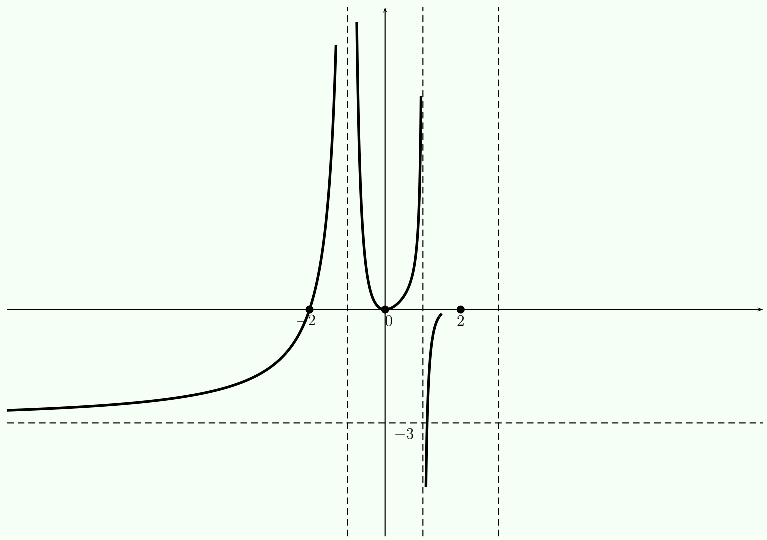
Now, from the table we see that there is no sign change at -1 so we have



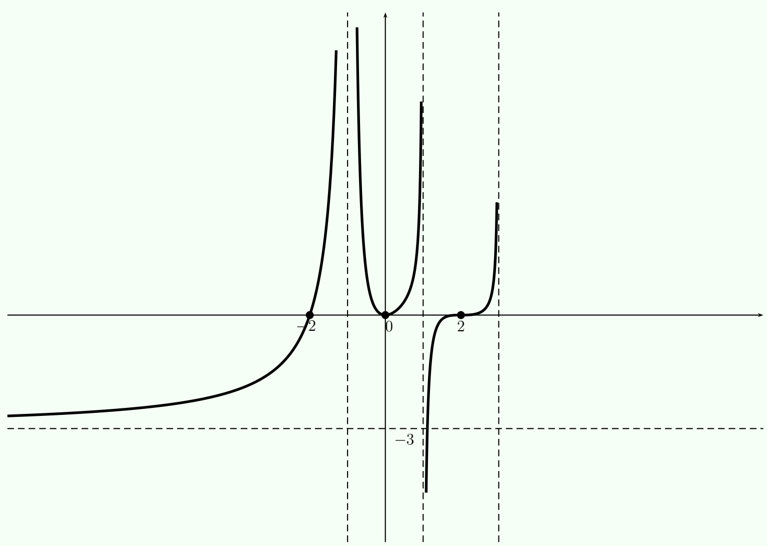
and from the table we see that near $x = 0$ the function $p(x)$ is approximately quadratic and therefore the graph looks like a parabola. This together with the fact that there is an asymptote at $x = 1$ gives



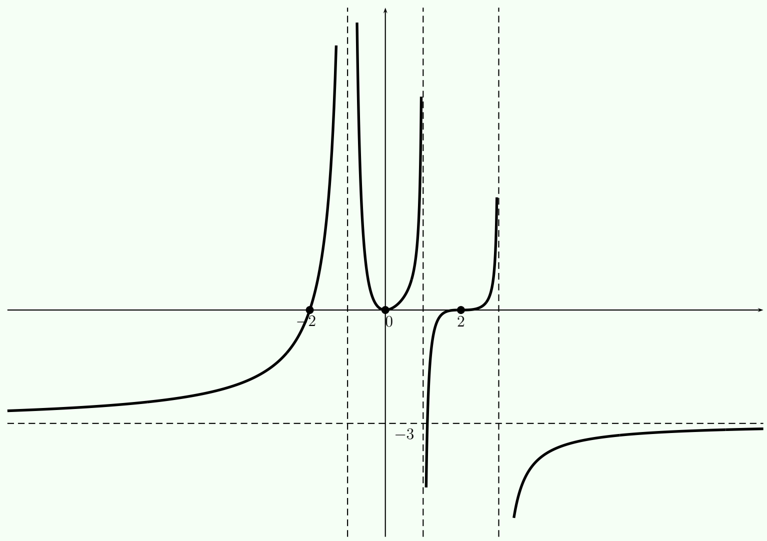
Now, from the table we see that the function changes sign at the asymptote, so while the graph "hugs" the top of the asymptote on the left-hand side, it "hugs" the bottom on the right-hand side giving



Now, from the table we see that near $x = 2$, $p(x)$ is approximately cubic. Also, there is an asymptote $x = 3$ so we get



Finally, we see from the table that $p(x)$ changes sign at the asymptote $x = 3$ and has a horizontal asymptote $y = -3$, so we complete the sketch:



Note that if we had made a mistake somewhere there is a good chance that we would have not been able to get to the horizontal asymptote on the right side without creating an additional x-intercept.
What can we conclude from this sketch? This sketch exhibits only the general shape which can help decide on an appropriate window size if we want to investigate details using technology. Furthermore, we can infer where $p(x)$ is positive and where $p(x)$ is negative. However, it is important to notice that there may be wiggles in the graph that we have not included in our sketch. □

We now give one more example of graphing a rational function where the horizontal asymptote is $y = 0$.

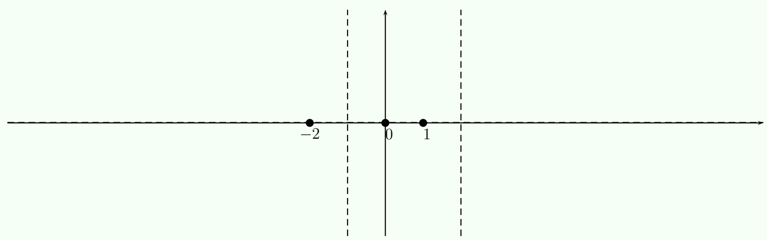
Example 10.6

Sketch the graph of

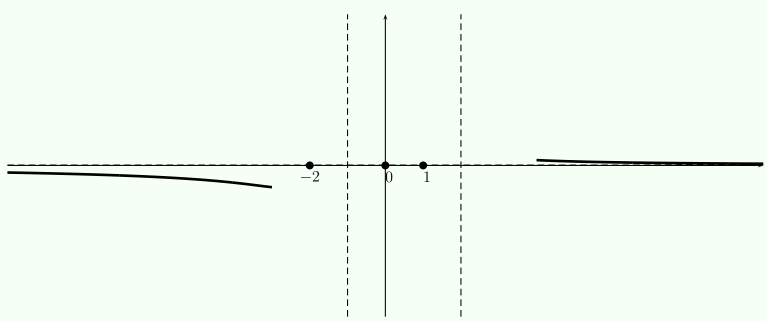
$$r(x) = \frac{2x^2(x-1)^3(x+2)}{(x+1)^4(x-2)^3}.$$

Solution.

Here we see that there are x-intercepts at $(0,0)$, $(0,1)$, and $(0,-2)$. There are two vertical asymptotes: $x = -1$ and $x = 2$. In addition, there is a horizontal asymptote at $y = 0$. (Why?) Putting this information on the graph gives



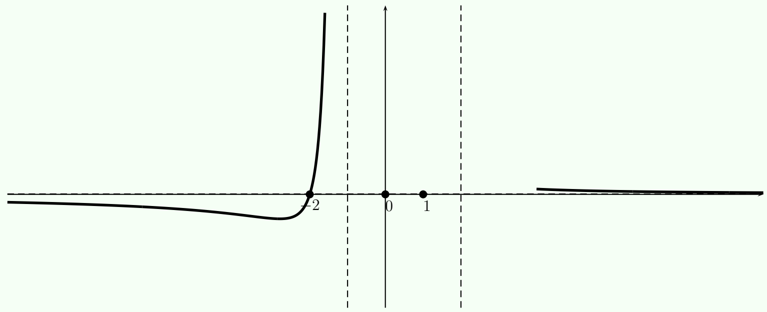
In this case, it is easy to get more information for large $|x|$ that will be helpful in sketching the function. Indeed, when $|x|$ is large, we can approximate $r(x)$ by dropping all but the highest order term in the numerator and denominator which gives $r(x) \approx \frac{2x^6}{x^7} = \frac{2}{x}$. So for large $|x|$, the graph of r looks like



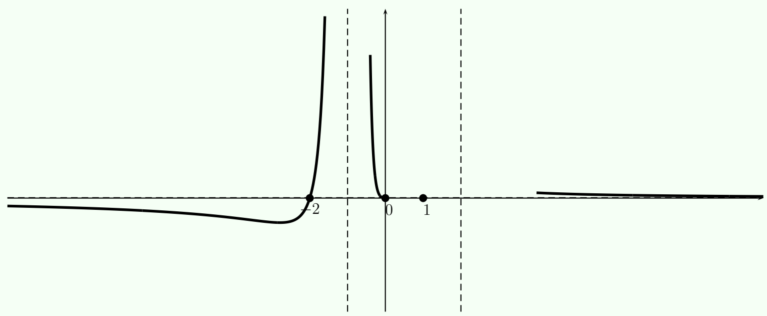
The function gives the following table:

for a	near a, $p(x) \approx$	type	sign change at a
-2	$C_1(x+2)$	linear	changes
-1	$C_2/(x+1)^4$	asymptote	does not change
0	$C_3\ x^2$	parabola	does not change
1	$C_4(x-1)^3$	cubic	changes
2	$C_5/(x-2)^3$	asymptote	changes

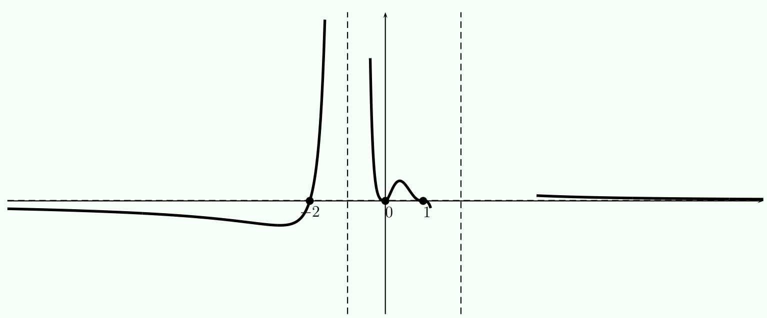
Looking at the table for this function, we see that the graph should look like a line near the zero $(0, -2)$ and since it has an asymptote $x = -1$, the graph looks something like:



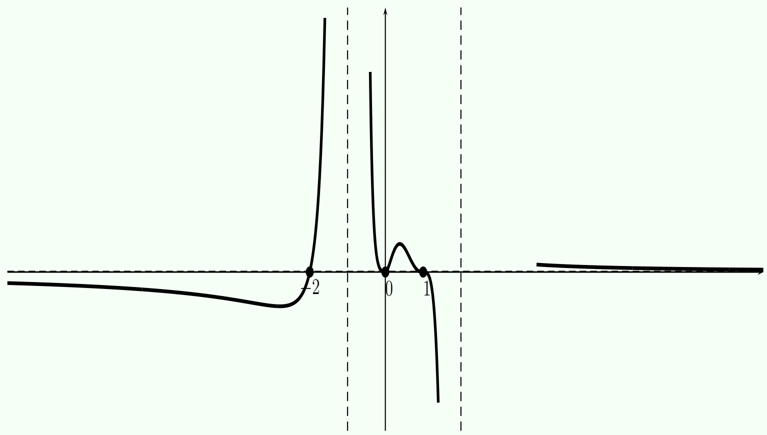
Then, looking at the table, we see that $r(x)$ does not change its sign near $x = -1$, so that we obtain:



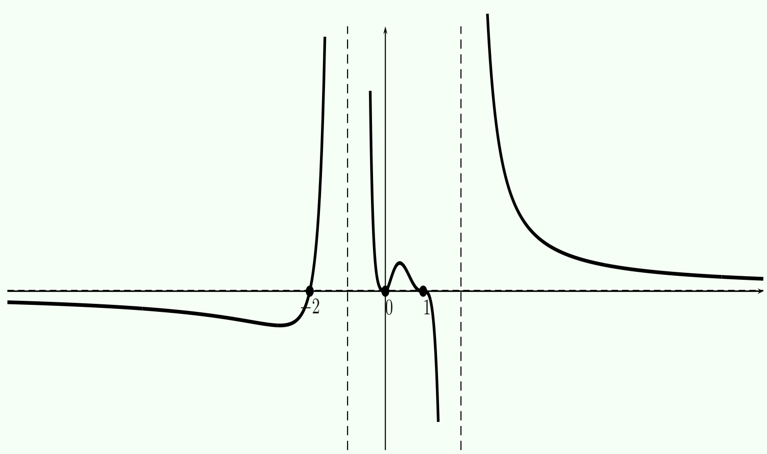
Now, the function is approximately quadratic near $x = 0$, so the graph looks like:



Next, looking at the root at 1, we note that the function is approximately cubic there, and that there is an asymptote at $x = 2$. We draw this as follows:



Finally, we see that the function changes sign at $x = 2$ (see the table). So, since the graph "hugs" the asymptote near the bottom of the graph on the left side of the asymptote, it will "hug" the asymptote near the top on the right side. So this, together with the fact that $y = 0$ is an asymptote, gives the sketch (perhaps using an eraser to match the part of the graph on the right that uses the large x):



Note that if the graph could not be matched at the end without creating an extra x-intercept, then a mistake has been made. □

10.3 Exercises

Exercise 10.1

Find the domain, the vertical asymptotes, and removable discontinuities of the functions:

a) $f(x) = \frac{2}{x-2}$ b) $f(x) = \frac{x^2+2}{x^2-6x+8}$

c) $f(x) = \frac{3x+6}{x^3-4x}$ d) $f(x) = \frac{(x-2)(x+3)(x+4)}{(x-2)^2(x+3)(x-5)}$

e) $f(x) = \frac{x-1}{x^3-1}$ f) $f(x) = \frac{2}{x^3-2x^2-x+2}$

Exercise 10.2

Find the horizontal asymptotes of the functions:

a) $f(x) = \frac{8x^2+2x+1}{2x^2+3x-2}$ b) $f(x) = \frac{1}{(x-3)^2}$

c) $f(x) = \frac{x^2+3x+2}{x-1}$ d) $f(x) = \frac{12x^3-4x+2}{-3x^3+2x^2+1}$

Exercise 10.3

Find the x- and y-intercepts of the functions:

a) $f(x) = \frac{x-3}{x-1}$ b) $f(x) = \frac{x^3-4x}{x^2-8x+15}$

c) $f(x) = \frac{(x-3)(x-1)(x+4)}{(x-2)(x-5)}$ d) $f(x) = \frac{x^2+5x+6}{x^2+2x}$

Exercise 10.4

Sketch a complete graph of the function f. To this end, calculate the domain of f, the horizontal and vertical asymptotes, the removable singularities, the x- and y-intercepts of the function, and graph the function with the graphing calculator.

a) $f(x) = \frac{7x+2}{3x-5}$ b) $f(x) = \frac{x^2-x-2}{x^2+2x-3}$

c) $f(x) = \frac{3x^2-7x+2}{x^2-3x-10}$ d) $f(x) = \frac{x^2+7x+12}{x^2+6x+8}$

e) $f(x) = \frac{x-3}{x^3-3x^2-6x+8}$ f) $f(x) = \frac{x^3-3x^2-x+3}{x^3-2x^2}$

Exercise 10.5

Find a rational function f that satisfies all the given properties.

a) vertical asymptote at $x = 4$ and horizontal asymptote at $y = 0$

b) vertical asymptotes at $x = 2$ and $x = 3$ and horizontal asymptote at $y = 5$

c) removable singularity at $x = 1$ and no horizontal asymptote